Canonical Form under Strong Equivalence Transformations and Controllability Indexes in Singular Systems

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Abstract—In this paper, the singular controller canonical form for singular systems is presented. Such a canonical form appears to extend both the well-known controller canonical form and the generalized controller canonical form which was developed recently. It is shown that any singular system is strongly equivalent to such a canonical form if and only if it is controllable. Another result of the paper is to establish a connection between the minimal indexes of a certain matrix pencil and the controllability indexes defined via the canonical form.

I. Introduction

A well-known fact concerning the dynamic systems representable in the state-space form is that the group of *similarity transformations* can be used to reveal the invariant structures (e.g., the controllability indexes) of the systems while preserving the input–output relation [4], [10]. Central to this discussion is the introduction of the canonical forms for controllable (or observable) systems [4], [9], [10]. It is well understood that some appropriate choice of the similarity transformation will bring the state-space form of a controllable system to certain canonical structures, for example, the controller canonical form [10], which exhibits clearly the controllability indexes. More importantly, canonical forms are of interest in its own right, since they can be looked upon as parameterizations of underlying systems, and these parameterizations often lead to straightforward design and synthesis procedures. As an example, the pole placement problem can be easily solved when a system is put into the controller canonical form (see, e.g., [10]).

There has been a lot of interest recently in the so-called *singular systems* [2], [3], [5], [6] i.e., systems of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, and $y \in R^p$. E, A are constant square matrices of order n, and B, C, D are constant matrices of appropriate sizes. In particular, E may be singular. Furthermore, the pencil $Es - A$ is regular, viz., $\det(Es - A)$ does not vanish for all $s \in C$. Such systems naturally arise in a number of practical situations [11], and have been shown to possess some distinct dynamic properties [2], [3]. In fact, the concepts such as controllability, observability, etc., which are of prominent importance in state-space systems have been generalized into the framework of singular systems. Another important development is the invention of some proper equivalence transformations in place of the similarity transformations, since the similarity transformation can no longer apply to (1). The *restricted system equivalence transformations* and the *strong system equivalence transformations* have been introduced as a result (see [2], [3] for detailed discussions).

The primary purpose of the present paper is to present a canonical form, the so-called *singular controller canonical form*, for singular systems. We shall show that any controllable singular system is strongly equivalent to such a canonical form, and that the converse statement is also true. Moreover, the canonical form will induce a set of what we call controllability indexes which is shown to be linked to the minimal indexes of the matrix pencil $[Es - A \quad B]$. Our results provide a nice singular system counterpart of the relationships among the controllability, the similarity transformation, the controller canonical form, the controllability indexes and the minimal indexes of the pencil $[Is - A \quad B]$ in state-space systems.

II. Preliminaries

Let a singular system be given by (1). It is well-known that (1) can be alternatively represented, in the frequency domain, by the *system matrix* [4]

$$P(s) = \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix}. \tag{2}$$

In fact, the equivalence transformations can be conveniently defined upon the system matrix.

First, if system matrices of two singular systems S and S_1 are related by

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} \tag{3}$$

where M, N are nonsingular, and the matrices with subscript 1 describe the system S_1, then these two systems are said to be *restricted system equivalent* (*RSE*) [2].

Secondly, the *strong equivalence transformations* [3] consist of any combination of the following three operations defined on the system matrix.

(i) The left and right multiplication:

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} \tag{4}$$

where M, N are nonsingular, $QE = 0$ and $ER = 0$;

(ii) The trivial augmentation, namely, from (2) to

$$\begin{bmatrix} Es - A & 0 & B \\ 0 & I & 0 \\ -C & 0 & D \end{bmatrix}; \tag{5}$$

(iii) The trivial deflation, namely, from (5) to (2).

Given two singular systems. If one of them can be transformed to the other by a sequence of strong equivalence transformations, then these two systems will be called strongly equivalent (STR. EQ.).

We observe that the RSE transformations are a subset of the STR. EQ. transformations.

System (1) is said to be *reachable* [2], [5], [6] if

$$\operatorname{rank}[Es - A \quad B] = n \tag{6}$$

for all finite s and if

$$\operatorname{rank}[E \quad B] = n. \tag{7}$$

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The system is called *controllable* [3], [6] if, in addition to (6), the following is also true:

$$\operatorname{rank}\begin{bmatrix} E_1 & 0 \\ A_2 & B_2 \end{bmatrix} = n \tag{8}$$

where the block-partitioned matrix in (8) is obtained as follows. Let a nonsingular matrix M row-compress E. In other words, premultiplied by M, the pencil $[Es - A \quad B]$ is converted into the partitioned-matrix below

$$M[Es - A \quad B] = \begin{bmatrix} E_1 s - A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \tag{9}$$

where E_1 is of full row rank. Then the matrix in (8) is built from block matrices in (9).

It is readily checked that the RSE transformations preserve the reachability and STR. EQ. transformations preserve the controllability [3].

As the last point of this section, we recall the *generalized controller canonical form* for singular systems stated in [1]. It is shown that a singular system (1) is reachable if and only if $[Es - A \quad B]$ can be brought to the following generalized controller canonical form by the RSE transformations:

$$\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1\kappa} & B_{c1} \\ R_{21} & R_{22} & \cdots & R_{2\kappa} & B_{c2} \\ \vdots & \vdots & & \vdots & \vdots \\ R_{\kappa 1} & R_{\kappa 2} & \cdots & R_{\kappa\kappa} & B_{c\kappa} \end{bmatrix} \tag{10}$$

where κ is defined as the number of diagonal blocks; and for $i = 1, \cdots, \kappa$

$$R_{ii} = E_{ii}s - A_{ii} = \begin{bmatrix} a_i s + X & X & \cdots & X & X \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}$$

in which the a_i is 1 or zero; the X's denote possible nonzero values. While the R_{ij} $(i \neq j)$ have all rows zero except possibly the first one. Further, each B_{ci} $(i = 1, \cdots, \kappa)$ has the form

$$B_{ci} = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad b_i = 1;\ b_1 = b_2 = \cdots b_{i-1} = 0.$$

Here we have assumed that B has full column rank.

We note that the structures of C and D are completely arbitrary in the generalized controller canonical form.

III. The Canonical Form Under Strong Equivalence Transformations

Definition 1: A singular system (1) is said to be in the singular controller canonical form if its matrix pencil $[Es - A \quad B]$ assumes the form

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1\kappa} & B_1 \\ T_{21} & T_{22} & \cdots & T_{2\kappa} & B_2 \\ \vdots & \vdots & & \vdots & \vdots \\ T_{\kappa 1} & T_{\kappa 2} & \cdots & T_{\kappa\kappa} & B_\kappa \end{bmatrix} \tag{11}$$

where κ is defined in (11) as the number of diagonal blocks; and for $i = 1, \cdots, \kappa$

$$T_{ii} = E_i s - A_i = \begin{bmatrix} X & X & \cdots & X & X \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}$$

while the T_{ij} $(i \neq j)$ have all rows zero except possibly the first one. Moreover, each B_i $(i = 1, \cdots, \kappa)$ has the form

$$B_i = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad b_i = 1;\ b_1 = b_2 = \cdots b_{i-1} = 0.$$

Here we again assume that B has full column rank.

It is noted that the matrices C and D have no specific structures, thus they are not concerned in the above definition. We observe that the obvious difference between (10) and (11) is that the coefficient rows in (11) are all constants. The theorem below links the concept of controllability with the canonical form (11).

Theorem 1: Any singular system is strongly equivalent to the singular controller canonical form (11) if and only if it is controllable.

The necessity of the theorem is easy to establish since a system in the singular controller canonical form is controllable in nature (this can be shown by using the definition given in Section II) and since the STR. EQ. transformations can never alter the controllability of a singular system [3]. Here we shall concentrate on the sufficiency proof of the theorem.

We first recall, from the theory of matrix pencil, that for every regular pencil $Es - A$, there exist two nonsingular matrices M and N such that [7]

$$M(Es - A)N = \begin{bmatrix} Is - J_1 & 0 \\ 0 & I - J_2 s \end{bmatrix} \tag{12}$$

where J_1, J_2 are nilpotent and in Jordan form, i.e., both are block diagonal matrices whose diagonal blocks have the form

$$J_d = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}. \tag{13}$$

Then the singular system (1) is said to have no nondynamic variables if J_2 contains no 1×1 Jordan diagonal blocks [3].

Now we state a lemma which is essential to the subsequent derivation.

Lemma 1: If singular system (1) has no nondynamic variables, then the system is controllable if and only if it is reachable.

This result is a direct consequence of the theory expounded in [3]. The reader can also find simple algebraic proofs based on the definitions. We shall nonetheless omit the details.

We are now ready to prove the sufficiency of Theorem 1.

Sufficiency proof of Theorem 1

Let us assume that the singular system (1) has no nondynamic variables. If this is not the case, it can always be made so by STR. EQ. transformations [3]. Then we know by Lemma 1 that if the system is controllable, it will be reachable as well. Therefore,

the system is transformable by RSE operations (which are special type of STR. EQ. operations) to the generalized controller canonical form.

Suppose now that (1) is already put into the generalized controller canonical form (10), we show that further STR. EQ. transformations can reduce the system to the singular controller canonical form (11).

To be specific, we start from the ith block row ($i=1,\cdots,r$), namely (in the matrices which follow, X's denote possibly nonzero values, entries not shown are zeros)

$$\begin{bmatrix} X & X & \cdots & X & a_i s+X & X & \cdots & X & X & X & \cdots & X & 1 & X & \cdots & X \\ & & & & -1 & s & \cdots & 0 & 0 & & & & & & & \\ & & & & \vdots & \vdots & & \vdots & \vdots & & & & & & & \\ & & & & 0 & 0 & \cdots & -1 & s & & & & & & & \end{bmatrix}. \tag{14}$$

If $a_i=0$, we will do nothing but simply move to the next block row. Otherwise, by a trivial augmentation, (14) is brought to

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ X & X & \cdots & X & 0 & s+X & X & \cdots & X & X & X & \cdots & X & 1 & X & \cdots & X \\ & & & & 0 & -1 & s & \cdots & 0 & 0 & & & & & & & \\ & & & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & & & & & \\ & & & & 0 & 0 & 0 & \cdots & -1 & s & & & & & & & \end{bmatrix}. \tag{15}$$

Adding, in (15), the first row multiplied by -1 to the second one, we have

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ X & X & \cdots & X & -1 & s+X & X & \cdots & X & X & X & \cdots & X & 1 & X & \cdots & X \\ & & & & 0 & -1 & s & \cdots & 0 & 0 & & & & & & & \\ & & & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & & & & & \\ & & & & 0 & 0 & 0 & \cdots & -1 & s & & & & & & & \end{bmatrix}. \tag{16}$$

Eliminating all the X's in the second row by -1 in the same row, we arrive at

$$\begin{bmatrix} X & X & \cdots & 1 & X & X & X & \cdots & X & X & X & \cdots & X & 1 & X & \cdots & X \\ 0 & 0 & \cdots & 0 & -1 & s & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & & & 0 & -1 & s & \cdots & 0 & 0 & & & & & & & \\ & & & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & & & & & \\ & & & & 0 & 0 & 0 & \cdots & -1 & s & & & & & & & \end{bmatrix}. \tag{17}$$

Note that the above manipulations are simply a sequence of STR. EQ. operations. By undergoing these operations (17) is brought into the desired structure. When i goes from 1 to κ, the system will be in the desired canonical form (11). This completes the proof.

IV. Controllability Indexes and Minimal Indexes

For a given pencil of matrices, say, $Fs-L$, its minimal indexes for columns are well defined (see, for example, [4, p. 96] for the standard definitions). In this section, we first define the set of controllability indexes via the singular controller canonical form (11), we shall then establish a link between these indexes and the minimal indexes for columns of the matrix pencil $[Es-A \quad B]$.

Definition: 2: Let a controllable singular system (1) be transformed, by STR. EQ. operations, to the singular controller canonical form (11). Then a set of controllability indexes $\{\gamma_i\}$ for (1) is defined by

$$\gamma_i = \text{rank } E_i, \qquad i=1,\cdots,\kappa. \tag{18}$$

It is easy to see from the definition of strong equivalence that strongly equivalent singular systems will have the same set of controllability indexes. The following theorem relates the set $\{\gamma_i\}$ to the set of minimal indexes for the columns of the pencil $[Es-A \quad B]$:

Theorem 2: If a singular system is controllable, then the set $\{\gamma_i\}$ is a rearrangement of the set $\{\alpha_i\}$ which is the set of minimal indexes for the columns of $[Es-A \quad B]$.

Proof: Start with the pencil $[Es-A \quad B]$. Since the STR. EQ. transformation will not change the minimal indexes for the columns we can assume that $[Es-A \quad B]$ is in the singular controller canonical form.

Make use of 1's in the matrix B to zero out all constants in those rows where 1 lies, then multiply all the 1's by -1. The resulting pencil is

$$\begin{bmatrix} Q_1 & 0 & \cdots & 0 & P_1 \\ 0 & Q_2 & \cdots & 0 & P_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & Q_\kappa & P_\kappa \end{bmatrix} \tag{19}$$

where

$$Q_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix} = E_{ci}s - J_{ci}$$

and

$$P_i = \begin{bmatrix} g_1 & g_2 & \cdots & g_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad g_i = -1;\ g_j = 0, \qquad j \neq i.$$

By properly rearranging the columns in (19) we can arrive at the form

$$\begin{bmatrix} L_1(s) & 0 & \cdots & 0 \\ 0 & L_2(s) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & L_\kappa(s) \end{bmatrix} \tag{20}$$

where for $i = 1, \cdots, \kappa$

$$L_i(s) = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & s \end{bmatrix} = \begin{bmatrix} -e_1 & E_{ci}s - J_{ci} \end{bmatrix}$$

in which $e_1^T = [1 \quad 0 \quad \cdots \quad 0]$. So that the equation

$$L_i(s)\, v_i(s) = 0 \tag{21}$$

admits the solution

$$v_i(s) = \begin{bmatrix} 0 \\ s^{\gamma_i} d(s) \\ s^{\gamma_i - 1} d(s) \\ \vdots \\ sd(s) \\ d(s) \end{bmatrix} \tag{22}$$

in which $d(s)$ is an arbitrary polynomial. Obviously, γ_i is a minimal index for the columns.

Since there only exist κ linearly independent solutions for the equation

$$[Es - A \quad B]\, v(s) = 0$$

the number of minimal indexes is precisely κ. The theorem is therefore proven.

V. Conclusions

A so-called singular controller canonical form is presented in this paper. We have shown that every controllable singular system is STR. EQ. to such a form and vice versa. Based on this canonical form the set of controllability indexes can be naturally defined. An interesting connection between the set of controllability indexes for a controllable singular system and the set of minimal indexes for the pencil $[Es - A \quad B]$ is established.

In an algebraically dual way, the singular observer canonical form and the set of reconstructibility indexes can also be derived for reconstructable singular systems.

Although this present note only puts emphasis on the structural aspect of the introduced canonical form, it is believed that this canonical form will also enable many elegant design methods to be devised, for example, the design of the feedback gain which is used to place all dynamical modes of the system in a desirable way. Another interesting point is that such a canonical structure precisely coincides with the one which is used to realize an arbitrary transfer matrix in right MFD [8].

References

[1] S. Tan and J. Vandewalle, "Canonical forms for singular systems," in *Proc. 25th IEEE Conf. Decision Contr.*, pp. 2144–2149, 1986.

[2] H. H. Rosenbrock, "Structural properties of linear dynamical systems," *Int. J. Contr.*, vol. 20, pp. 191–202, 1974.

[3] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 811–830, 1981.

[4] H. H. Rosenbrock, *State-Space and Multivariable Systems*. New York: Wiley, 1970.

[5] F. L. Lewis, "Fundamental, reachability and observability matrices for discrete descriptor systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 502–505, 1985.

[6] F. L. Lewis and K. Ozcaldiran, "Reachability and controllability for descriptor systems," in *Proc. 27th Midwest Symp. Circuits Systems*, pp. 690–695, 1984.

[7] F. R. Gantmacher, *The Theory of Matrices*. vol. 2, New York: Chelsea, 1960.

[8] S. Tan and J. Vandewalle, "A singular system realization for arbitrary matrix fraction descriptions," in *Proc. 1988 IEEE Int. Symp. Circuits Systems*, 1988.

[9] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automat. Control*, vol. AC-12, pp. 290–293, 1967.

[10] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[11] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 312–321, 1977.

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