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ON FACTORIZATIONS OF RATIONAL MATRICES

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Abstract

We show in this paper that any complex rational matrix $G(s)$ can be factorized over the complex field as a product of n complex rational matrices each of which has McMillan degree 1, where n is the McMillan degree of $G(s)$. When confined to the real field, $G(s)$ with real coefficient can be factorized as a product of a sequence of real rational matrices with McMillan degree 1 or 2, where the McMillan degrees of the factors add up to n.

1 Introduction

In a number of situations, the following problems concerning rational matrix factorizations often arise. Can any complex rational matrix $G(s)$ $(m \times p)$ of McMillan degree n be factorized into a product of n complex rational matrices of McMillan degree 1? Alternatively, if $G(s)$ only has real coefficients, can it be factorized into a product of real rational matrices of degree at most 2 such that the degrees of the factors add up to n? These factorization problems are not merely of theoretical interests. In fact, they are really keys to a number of practical analysis and design problems. For instance, if the real factorization is possible, then any multi-port circuit can be synthesized as a cascade of subsections with least complexity, so that only local feedback of length one needs to be used. This kind of realization has been known to possess many favorable properties. Because of the cascade nature, the influence due to the parameter variations of a section will not propogate, and further, the minimality guarantees that no redundant dynamic element (e.g., op. amp.) is necessary in order to realize each section.

The aforementioned factorizations for rational matrices were termed "minimal factorization" in the literature [2-4,6], since an arbitrary factorization of a rational matrix often leads to the fact that the sum of the degrees of the factors can only be equal to or greater than the degree of that rational matrix. This minimal factorization problem has been investigated extensively in the past [1-7]. However, there are two notable differences between the problem considered in [1-7] and the one considered here. Firstly, the previous works often implicitly (or explicitly, see, for example, [5]) take into consideration the zero structure and minimal indices of rational matrices, apart from the minimality constraint on the degrees. Secondly, the rational matrices are often confined to be proper.

In fact, it is one of the purposes of the paper to show that factorization problems with minimal degree, without bothering about the zero structure and minimal indices, can be solved in a straightforward way.

The main results of the paper consist in the exsistence of the minimal factorizability even for non-proper rational matrices. Moreover, the proofs of the factorization theorems are derived from a state-space formulation and allow easy numerical construction of the factors.

The paper is organized as follows. In section 2, a review of a few concepts and techniques is provided. These will serve as a basis to state and prove the factorization theorems. Section 3 present the factorization theorems for complex and real rational matrices together with an

illustrative example. Section 4 contains the discussions of the theorems and a general conclusion.

2 Preliminaries

2.1 Facts concerning the state-space representation

Consider a rational matrix $G(s)$ of dimension $m \times p$ with coefficients in the complex field. Its degree is defined to be the McMillan degree, i.e., the sum of the number of poles of $G(s)$, multiplicities counted (see [9] for an extensive discussion). Without loss of generality, we shall restrict the attention to the proper rational matrices. In fact, if $G(s)$ is non-proper a conformal mapping $s \rightarrow 1/(s - s_0)$ can first be applied to make $G(s)$ proper, where $G(s)$ has no pole at s_0. When the factorization is performed, we can apply the reverse conformal mapping $s \rightarrow s_0 + 1/s$ to obtain the desired factorization of the original rational matrix. Recall that conformal mapping of this kind will preserve the McMillan degree of rational matrices.

Every proper $G(s)$ can always be minimally realized in the following state-space representation (see, for example, [9,10,11])

$$G(s) = C(I_n s - A)^{-1} B + D \tag{1}$$

where A, B, C, D are complex constant matrices of dimensions $n \times n$, $n \times p$, $m \times n$ and $m \times p$, respectively. It is well-known that for any minimal realization (1) of $G(s)$, all the poles of $G(s)$ are actually the eigenvalues of the matrix A, so that n (which is the dimension of A) is precisely the (McMillan) degree of $G(s)$.

A minimal realization of $G(s)$ can be alternatively represented as the following system matrix form [9]

$$P(s) = \begin{bmatrix} I_n s - A & B \\ -C & D \end{bmatrix} \tag{2}$$

The matrix representation (2) of the minimal realization is advantageous since a number of operations such as similarity transformation can simply be treated as matrix algebraic operations applied to (2). To see this, we note that the following matrix operation

$$\tilde{P}(s) = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_n s - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \tag{3}$$

is in fact a similarity transformation, where T is a nonsingular matrix. Obviously, the system matrix $\tilde{P}(s)$ provides a different but equivalent (in the system theoretical sense) minimal realiza-

tion of $G(s)$. As a matter of fact, all the minimal realizations of $G(s)$ are related by similarity transformations.

The system matrix is also convenient when we are going to build a realization for $G(s)$ from that of its factors. Indeed, if $G(s) = G_1(s)G_2(s)$, and the realizations for $G_1(s), G_2(s)$ are given, respectively, by

$$P_1(s) = \begin{bmatrix} I_\tau s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} I_\gamma s - A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} \tag{4}$$

then a realization for $G(s)$ can be readily constructed to be [11]

$$P(s) = \begin{bmatrix} I_\tau s - A_1 & -B_1 C_2 & B_1 D_2 \\ 0 & I_\gamma s - A_2 & B_2 \\ -C_1 & -D_1 C_2 & D_1 D_2 \end{bmatrix} \tag{5}$$

Conversely, if a minimal realization of $G(s)$ is given by (5), we can easily conclude that $G(s)$ is factorable into $G(s) = G_1(s)G_2(s)$, with $G_1(s), G_2(s)$ being obtained from (4). We shall see that this fact is crucial in the proofs of the factorization theorems to be stated later.

2.2 Facts concerning matrix algebra

We shall only state two simple facts in matrix algebra which will be used in the development in the ensuing sections, simple proofs for these facts can be found, for example, in [12, 13]. First, consider a complex matrix A of dimension $n \times n$. Then we can always find a unitary matrix U such that

$$U^* A\, U = \begin{bmatrix} \mu & h^* \\ 0 & A_r \end{bmatrix} \tag{6}$$

where the superscript $*$ denotes the conjugate transpose; h is an $(n-1)$-vector and A_r is an $(n-1) \times (n-1)$ matrix. When A is a real matrix, we can always find real orthogonal matrices K, N such that for any real eigenvalue λ of A

$$K^T A\, K = \begin{bmatrix} \lambda & g^T \\ 0 & A_{n-1} \end{bmatrix} \tag{7}$$

and for any complex conjugate pair of eigenvalues $\sigma \pm j\omega$ of A $(\omega > 0)$

$$N^T A\, N = \begin{bmatrix} A_2 & H \\ 0 & A_{n-2} \end{bmatrix} \tag{8}$$

where the superscript T denotes the transpose; the matrices A_{n-1}, A_{n-2} are real matrices of dimensions $(n-1)\times(n-1), (n-2)\times(n-2)$, g, H are real matrices of appropriate dimensions; A_2 is a real matrix of dimension 2×2, and has the pair of eigenvalues $\sigma \pm j\omega$.

As the second fact, if r is the rank of an $m \times n$ real (complex) matrix E, then E can always be factored as

$$E = F \cdot R \tag{9}$$

where F, R are real (complex) matrices of dimensions $m \times r$, $r \times n$, respectively.

These two facts will be used in the next two sections to prove the factorization theorems.

3 Factorization Theorems

In this section we present both the complex and the real factorization theorems. First consider complex rational matrices.

To begin with, let an $m \times p$ rational matrix $G(s)$ of degree n be given. Then $G(s)$ can always be realized minimally in the state-space form (2). Upon choosing a unitary matrix U such that (6) holds, where μ is an eigenvalue of A, we can apply the following similarity transformation to $P(s)$

$$\tilde{P}(s) = \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_n s - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} s-\mu & -h^* & \bar{B} \\ 0 & I_{n-1}s - A_r & B_r \\ -C_1 & -\bar{C} & D \end{bmatrix} \tag{10}$$

Note that (10) has the same transfer matrix $G(s)$ as (2). Construct from (10) the following constant $(m+1)\times(n+p-1)$ matrix

$$\begin{bmatrix} -h^* & \bar{B} \\ -\bar{C} & D \end{bmatrix} \tag{11}$$

If the rank of the matrix (11) is q, then (11) can be decomposed into (see (9))

$$\begin{bmatrix} -h^* & \bar{B} \\ -\bar{C} & D \end{bmatrix} = \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} \begin{bmatrix} -C_r & D_r \end{bmatrix} \tag{12}$$

where the matrix $[B_1^T \quad D_1^T]^T$ and $[-C_r \quad D_r]$ are of dimensions $(m+1)\times q$, $q \times (n+p-1)$,

respectively. Substitute (12) into (10) we have

$$\tilde{P}(s) = \begin{bmatrix} s-\mu & -B_1C_r & B_1D_r \\ 0 & I_{n-1}s - A_r & B_r \\ -C_1 & -D_1C_r & D_1D_r \end{bmatrix} \tag{13}$$

By comparing (13) with (5) , we know that $G(s)$ can be factored into $G(s) = G_1(s)G_r(s)$, where $G_1(s)$ $(m \times q)$ and $G_r(s)$ $(q \times p)$ are rational matrices of degree 1 and $n-1$, respectively; moreover, the system matrices for $G_1(s)$ and $G_r(s)$ are given respectively by

$$P_1(s) = \begin{bmatrix} s-\mu & B_1 \\ -C_1 & D_1 \end{bmatrix}, \quad P_r(s) = \begin{bmatrix} I_{n-1} - A_r & B_r \\ -C_r & D_r \end{bmatrix} \tag{14}$$

If $n > 2$, the above procedure can be applied again to factorize $G_r(s)$. Since n is finite, the procedure will eventually stop for some $G_r(s)$ of degree 1. We have thus arrived at the following complex factorization theorem:

Theorem 1 *Any complex proper rational matrix $G(s)$ of degree n can always be factorized as*

$$G(s) = G_1(s)G_2(s)\cdots G_n(s) \tag{15}$$

where the $G_i(s)$ are proper rational matrices of degree 1 $(i = 1, \cdots, n)$.

The reader may have observed that the same procedure can be applied to the real factorization with only a minor modification, i.e., the similarity transformation should account for real as well as a pair of complex conjugate poles, and should replace (7) by (8) or (9). We can then state the following real factorization theorem:

Theorem 2 *Any $m \times p$ real proper rational matrix $G(s)$ of degree n can always be factorized as*

$$G(s) = G_1(s)G_2(s)\cdots G_k(s) \tag{16}$$

where the $G_i(s)$ are real proper rational matrices of degree at most two. For any factors of degree two, it will have a pair of complex conjugate poles. Moreover, the sum of the McMillan degrees of the factors equals the McMillan degree of $G(s)$.

An important fact which should be observed is that the dimensions of the factors $G_i(s)$ $(i = 1, \cdots, n)$ play no role in the above factorization theorems. This implies that we are only interested

in spliting minimally the *pole structure* of rational matrices into its factors. Thus, the zero structure and the so-called singular structure (i.e., the minimal column and row indices) are of no importance (see [10] for standard definitions for these structures). In the factorizations (15), (16) apart from the properness, nothing is known of the relationship between the zero as well as the singular structures of $G(s)$ and those of its factors.

Another observation concerning (15) and (16) which follows from the derivation is that the order by which all the poles of $G(s)$ appear in the factorization (15) can be specified at will.

Theorem 1, 2 have in fact completely answered the question of minimal factorization. We now know that if the factorization is only aimed at splitting the degree, then the prime factor of a rational matrix is of degree one over the complex field, and at most of degree two over the real field. As another consequence, for any given rational matrix, since its poles are uniquely defined, the sequence of degrees of prime factors is unique. Moreover, if a rational matrix is minimally factored in two different ways, the degrees of the factors of these two factorizations will match with each other although the factors may not at all be the same, which is the typical situation for the minimal factorization over ring or module.

Another advantage is that the methods which we have used to derive the factorizations are constructive, and allow numerically stable methods to be applied to actually obtain the factors.

It is also of interest to note that the complex factorization problem has been considered in the past by Cohen [14] under the term "complete factorization". In fact, based on a similar technique a special case of Theorem 1 is derived in [14], namely, all the square proper rational matrices with the system matrix A (as in their state-space realization) having simple eigenstructure admit the factorization (15). An essential difference between the approach taken in [14] and the one in the present paper lies again in that the factors in [14] are implicitly assumed to be square, which implies that the zero structure is taken into consideration. In our approach, however, the zero structure is ignored.

To conclude this section, let us consider the following illustrative example.

Example 2 Consider

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & -\frac{2s}{(s+1)^2} \\ 0 & \frac{s-1}{s+1} \end{bmatrix} \tag{17}$$

The transpose of this rational matrix appeared in [2] as a prime rational matrix of degree 2 when

square factors are considered. Here we show that (17) can be further factorized into two prime factors of degree 1 over the real field.

First, a minimal realization of (17) is found to be

$$\begin{bmatrix} Is-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} 1+s & 1 & 1 & 0 \\ 0 & 1+s & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Since the matrix A is already in the desired Schur form, we only need to decompose the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \tag{18}$$

in order to obtain the factors. Since the rank of (18) is 3, it can be decomposed as

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the factors can be seen to be

$$G_1(s) = \frac{\begin{bmatrix} -2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}{1+s} + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad G_2(s) = \frac{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{1+s} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reader may wish to check that $G(s) = G_1(s) \cdot G_2(s)$ indeed holds.

4 Conclusions

A relatively complete theory for prime factorizations of rational matrices over complex and real fields is presented. The factorization is aimed at splitting minimally the pole structure of rational matrices into the factors, without imposing conditions on the zero structure or other structural information (e.g., the left and right minimal indices). Our approach is constructive, and the computation steps allow numerically stable and efficient algorithms to be applied.

Our result has a number of important design and synthesis applications in signal processing, circuit theory, and control. In fact, we now know that any multi-port or multi-channel filter can be realized in a cascade of the subsystems of degree 1. The modularity of these subsystems is well-suited for chip implementation. Moreover, the cascade nature of the factorization reduces the sensitivity to the parameter variation significantly.

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