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A SINGULAR SYSTEM REALIZATION FOR ARBITRARY MATRIX FRACTION DESCRIPTIONS

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Abstract

A realization algorithm is presented which determines a singular system realization from an arbitrary right MFD. Neither the properness nor the column-reducedness of the right MFD is required. The constructed realization also has some nice properties.

1 Introduction

In this paper, a realization procedure is presented which determines a singular system

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

from a given rational matrix in right MFD, i.e.,

$$G = N(s)D^{-1}(s) \tag{2}$$

In the above, $x \in R^n, u \in R^m, y \in R^p$. E, A are square constant matrices of order n, and B, C are constant matrices of appropriate sizes; in particular, E may be singular. Moreover, $N(s)$ and $D(s)$ in (2) are polynomial matrices of dimensions $p \times m, m \times m$, respectively.

Unlike existing realization algorithms, our construction assumes neither the properness nor the column-reducedness of the rational matrix $G(s)$.

The procedure presented here is also an interesting extension of the well-known realization algorithm given by Wolovich [1]. However, it is more general than an earlier construction [2] in the sense that the latter assumes the condition of the so-called pseudo-properness.

2 Preliminaries

We recall two concepts from singular systems, and introduce some necessary notations and definitions which will be used in the ensuing development. First, the singular system (1) is said to be controllable [3-5] if

$$\text{rank } [Es - A \quad B] = n \tag{3}$$

for all finite s and if

$$\text{rank } \begin{bmatrix} E_1 & 0 \\ A_2 & B_2 \end{bmatrix} = n \tag{4}$$

where the matrix in (4) is obtained as follows.

By premultiplying the matrix pencil $[Es - A \quad B]$ with a nonsingular matrix M the rows of E are compressed. This results in the following partitioned matrix

$$M[Es - A \quad B] = \begin{bmatrix} E_1 s - A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \tag{5}$$

where E_1 is of full row rank. Then the matrix in (4) is constructed from the block elements in (5).

Secondly, the order of the singular system (1) is defined to be the rank of E (see, for example, [3]).

Let an arbitrary right MFD (2) be given. Denote by $n_i(s)$ and $d_i(s)$ $(i = 1, \ldots, m)$ the ith columns of the polynomial matrices $N(s)$ and $D(s)$, respectively, and $r_i = \deg n_i(s), k_i = \deg d_i(s)$ are the column degrees of their ith columns. Then we assume throughout the paper that $r_i = k_i$ $(i = 1, \ldots, m)$. If this is not the case, by properly adding some leading terms with zero column coefficient to the one of the smaller degree, such a column degree balance can be achieved.

3 The Construction

Let (2) be given and r_i, k_i $(i = 1, \ldots, m)$ be the column degrees of $N(s)$ and $D(s)$, respectively (note that $r_i = k_i$).

As the first step of the construction, define the following

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matrix

$$S^T(s) = \begin{bmatrix} s^{k_1}\ s^{k_1-1}\cdots s\ 1 & 0 & \cdots & 0 \\ 0 & s^{k_2}\ s^{k_2-1}\cdots s\ 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{k_m}\ s^{k_m-1}\cdots s\ 1 \end{bmatrix} \tag{6}$$

and write $D(s)$ and $N(s)$ in the form

$$\begin{aligned} D(s) &= D_c S(s) \\ N(s) &= N_c S(s) \end{aligned} \tag{7}$$

where D_c and N_c are constant matrices.

Secondly, construct a so-called *core realization* which assumes the form

$$\begin{aligned} &E_c s - A_c = \\ &\text{block diag}\left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}, \quad \begin{matrix} (k_i+1)\times(k_i+1) \\ i = 1,\ldots,m \end{matrix} \right\} \\ &B_c^T = \text{block diag}\ \{[1\ 0\cdots\ 0],\ 1\times(k_i+1),\ i=1,\ldots,m\} \\ &C_c = I_n,\ \ n = \textstyle\sum_{i=1}^m k_i + m \end{aligned} \tag{8}$$

Finally, the desired realization follows form (7) and (8)

$$\begin{aligned} E &= E_c, \quad A = A_c + BD_c, \\ B &= B_c, \quad C = N_c C_c = N_c \end{aligned} \tag{9}$$

4 Proof and Properties of the Realization

In this section we first prove that (9) indeed realizes the given right MFD. Then some interesting properties of the realization are shown.

Theorem 1 *The system (9) realizes the given right MFD in the sense that*

$$C(Es - A)^{-1}B = N(s)D^{-1}(s) \tag{10}$$

proof

It is easily checked that

$$(Es - A)S(s) = BD_c S(s) = BD(s)$$

or equivalently

$$(Es - A)^{-1}B = S(s)D^{-1}(s) \tag{11}$$

(The nonsingularity of Es-A will be proved later). By premultiplying (11) with C $(= N_c)$ and by observing that $N(s) = N_c S(s)$, (10) is proved.

Theorem 2 *The realization (9) has the properties below*

(i). The order of the realization (9) is equal to the sum of the column degrees of $D(s)$ (or $N(s)$);

(ii). The realization (9) is controllable;

(iii). $det(Es - A) = detD(s)$.

proof

(i). This property is obvious since by construction

$$\text{rank } E = \sum_{i=1}^{m} k_i$$

(ii). The controllability is easily checked by recalling the definition in Sec. 2.

(iii). We first notice that this last property implies that $Es - A$ is nonsingular, which is needed in the proof of Theorem 1. Although a direct verification can be used to prove this property, we choose a more compact way to establish it.

By observing that $Es - A$ and B are left coprime and that $S(s)$ and $D(s)$ are right coprime, there exist polynomial matrices $U(s), V(s), X(s), Y(s)$ such that

$$\begin{aligned} (Es - A)X(s) + BY(s) &= I \\ U(s)S(s) + V(s)D(s) &= I \end{aligned} \tag{12}$$

where the matrix I are unit matrices of appropriate dimensions. Combine (12) and (11) in matrix form

$$\begin{bmatrix} Es - A & B \\ -U(s) & V(s) \end{bmatrix} \begin{bmatrix} X(s) & -S(s) \\ Y(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ Q(s) & I \end{bmatrix} \tag{13}$$

where

$$Q(s) \stackrel{def}{=} -U(s)X(s) + V(s)Y(s)$$

is a polynomial matrix. Since all the three block-partitioned matrices in (13) are square, on taking the determinants of the matrices on both sides of (13), we conclude that the matrix

$$K(s) \stackrel{def}{=} \begin{bmatrix} X(s) & -S(s) \\ Y(s) & D(s) \end{bmatrix}$$

is unimodular.

On the other hand, from (11) and (12) we may write

$$\begin{bmatrix} Es-A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X(s) & -S(s) \\ Y(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y(s) & D(s) \end{bmatrix} \tag{14}$$

By taking the determinants of both sides of (14) and by observing that $K(s)$ is a unimodular matrix, we have

$$\det(Es-A) = q \cdot \det D(s)$$

Where q is a nonzero constant. Since no scaling is used in the construction, we must have $q = 1$. This proves our assertion.

5 Illustrative Examples

Example 1. Let a transfer function be given by

$$G(s) = \frac{2s^3+s+1}{s^2+2s+2} \tag{15}$$

One should note first that the degree of the denominator is 3 rather than 2. This becomes natural when we count the extra one pole at infinity. The subsequent calculation is straightforward.

$$D(s) = D_c S(s) = [0 \;\; 1 \;\; 2 \;\; 2] \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

$$N(s) = N_c S(s) = [2 \;\; 0 \;\; 1 \;\; 1] \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

The core realization is

$$E_c s - A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & s \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the final realization is

$$Es - A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & s \end{bmatrix}$$

$$B = B_c$$

and $C = [2 \;\; 0 \;\; 1 \;\; 1]$. The reader can easily check that the transfer function of the above realization is indeed (15).

Example 2. The transfer matrix is

$$G(s) = \begin{bmatrix} s^2 & s+1 \\ s & s \end{bmatrix} \begin{bmatrix} s^2+s+1 & 1 \\ s & 1 \end{bmatrix}^{-1}$$

Hence

$$D(s) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

$$N(s) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

The core realization is

$$E_c s - A_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & s \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, the final realization turns out to be is

$$Es - A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & s \end{bmatrix}$$

$$B = B_c$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

6 Conclusions

We have developed an algorithm which realizes a singular system from a given right MFD. There should be no additional difficulties in adapting the algorithm to realize a left MFD.

The distinct features of our construction are that no assumptions like properness and column-reducedness are required (these are essential assumptions for Wolovich's algorithm). In fact, an arbitrary right (left) MFD can be realized this way.

Our realization also possesses some important properties which, although in a generalized sense, are similar to those of Wolovich's realization.

It should be observed, however, that the property (iii) in Theorem 2 only shows the link between the finite modes of the realization and those of the given right MFD. Actually, a similar link between the infinite modes of the two can also be established. This is, however, beyond the scope of this paper.

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