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FUNDAMENTAL FACTORIZATION THEOREMS FOR RATIONAL MATRICES OVER COMPLEX OR REAL FIELDS

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Abstract

We show in this paper that any complex rational matrix $G(s)$ can be factorized over the complex field as a minimal product of n complex rational matrices each of which has McMillan degree 1, where n is the McMillan degree of $G(s)$. When confined to the real field, $G(s)$ with real coefficient can be factorized as a minimal product of a sequence of real rational matrices with McMillan degree 1 or 2, where the McMillan degrees of the factors add up to n. Our results thus settle a long-standing conjecture over the factorization of rational matrices. The proofs of these fundamental factorization theorems are constructive. Therefore, by following the suggested numerical procedures, all the factors over the real or the complex fields can be easily obtained.

1 Introduction

A well-know fact concerning scalar rational functions is that any complex rational function (i.e., the rational functions with complex coefficients), say, $g(s)$ can always be factorized over the complex field as

$$g(s) = g_1(s)g_2(s)\cdots g_\alpha(s) \tag{1}$$

where the $g_i(s)$ is a complex rational function with only one pole ($i = 1,\ldots,\alpha$) and α is the total number of poles of $g(s)$. Moreover, the order by which all the poles of $g(s)$ are distributed in $g_i(s)$ ($i = 1,\ldots,\alpha$) can be specified arbitrarily. When $g(s)$ is a real rational function, it can also be factorized as a product of a sequence of real rational functions with at most two poles. Moreover, the total number of poles of all the factors add up to α. One should note that here the zeros of $g(s)$ are not of concern, although the factors are often required to be proper.

In a number of practical situations, we often encounter, instead of rational functions, rational matrices. For example, linear time-invariant control systems, linear time-invariant multi-port circuits, linear digital shift-invariant multi-channel filters, etc. can always be conveniently modelled by rational matrices. An interesting question naturally arises: can any rational matrix $G(s)$ of dimension $m \times p$ be factorized in the same fashion as rational functions? This factorization problem is not merely of theoretical interests. In fact, it is really a key to a number of practical analysis and design problems. For instance, if the factorization is possible, then any multi-port circuit (rational, of course) can be synthesized as a cascade of subsystems with least complexity, so that only local feedback of length one needs to be used. This kind of realization has been known to possess many favorable properties such as requiring least number of components, localizing the parameter perturbation, etc.

This factorization problem has been investigated extensively in the past [1-7]. The classical results are limited to rational matrices of particular structures (see [1] for an exposition). The problem in its general form first appeared in [2], although the attention is only paid to nonsingular rational matrices. By using the so-called "degree reduction theorem" [8], Dewilde and Vandewalle [2] devised a procedure to extract the "elementary factors" (defined in [1]) from a nonsingular rational matrix. Their procedure, however, did not always yield the desired factorization since essentially the prime factors (the one which can not be further factorized) may not be the elementary factors. The real factorization treated in [3], again by means of elementary factor extraction, therefore, did not lead to the most general form of the factorization. Under the term "minimal factorization", Van Dooren et al. [4,5] and Bart et al. [6,7] have approached the factorization problem in a different way. By using a state-space formulation, they have obtained a number of important observations. However, since their formulation implicitly sets bounds on the number of inputs and outputs of the factors, they did not obtain the most general result. Thus, the factorization problem remained open.

In this paper, we show that any rational matrix can be minimally factorized in exactly the same way as rational functions thereby settling this long-standing problem. Our proofs of the factorization theorems are derived from a state-space formulation and allow easy numerical construction of the factors. Our approach clearly demonstrates how the eigenstructure of the state transition matrix of $G(s)$ is disassembled into individual factors.

The paper is organized as follows. In section 2, a review of a few concepts and techniques is provided. These will serve as a basis to state and prove the factorization theorems. Section 3, 4 present, respectively, the factorization theorems over the complex and real fields. Section 5 discusses the numerical techniques which might be used to compute the factors, and an illustrative example is also provided. Section 6 contains the discussions of the theorems and a general conclusion.

2 Preliminaries

2.1 Facts Concerning the State-space Representation

Consider a rational matrix $G(s)$ of dimension $m \times p$ with coefficients in the complex field. Its degree is defined to be the Smith-McMillan degree, i.e., the number of poles of $G(s)$, multiplicities counted (see [9] for an extensive discussion). Without loss of generality, we shall restrict the attention to the proper rational matrices. In fact, if $G(s)$ is non-proper a conformal mapping $s \to 1/s$ can first be applied to make $G(s)$ proper. When the factorization is performed, we can apply this conformal mapping again to obtain the desired factorization of the original rational matrix.

Every proper $G(s)$ can always be minimally realized in the following state-space representation (see, for example, [9,10,11])

$$G(s) = C(I_n s - A)^{-1}B + D \tag{2}$$

where A, B, C, D are complex constant matrices of dimensions $n \times n$, $n \times p$, $m \times n$ and $m \times p$, respectively. It is well-known that for any minimal realization (2) of $G(s)$, all the poles of $G(s)$ are actually the eigenvalues of the matrix A, so that n (which is the dimension of A) is precisely the degree of $G(s)$.

A minimal realization of $G(s)$ can be alternatively represented as the following system matrix form [9]

$$P(s) = \begin{bmatrix} I_n s - A & B \\ -C & D \end{bmatrix} \tag{3}$$

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The matrix representation (3) of the minimal realization is advantageous since a number of operations such as similarity transformation can simply be treated as matrix algebraic operations applied to (3). To see this, we note that the following matrix operation

$$\tilde{P}(s) = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_n s - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \tag{4}$$

is in fact a similarity transformation, where T is a nonsingular matrix. Obviously, the system matrix $\tilde{P}(s)$ provides a different but equivalent (in the system theoretical sense) minimal realization of $G(s)$. As a matter of fact, all the minimal realizations of $G(s)$ are related by similarity transformations.

The system matrix is also convenient when we are going to build up a minimal realization for $G(s)$ from those of its factors. Indeed, if $G(s) = G_1(s)G_2(s)$, and the minimal realizations for $G_1(s), G_2(s)$ are given, respectively, by

$$P_1(s) = \begin{bmatrix} I_{r_1} s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} I_{r_2} s - A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} \tag{5}$$

then a minimal realization for $G(s)$ can be readily constructed to be [11]

$$P(s) = \begin{bmatrix} I_{r_1} s - A_1 & -B_1 C_2 & B_1 D_2 \\ 0 & I_{r_2} s - A_2 & B_2 \\ -C_1 & -D_1 C_2 & D_1 D_2 \end{bmatrix} \tag{6}$$

Conversely, if a minimal realization of $G(s)$ is given by (6), we can easily conclude that $G(s)$ is factorable into $G(s) = G_1(s)G_2(s)$, with $G_1(s), G_2(s)$ being obtained from (5).

2.2 Facts Concerning Matrix Algebra

We shall only state two simple facts in matrix algebra which will be used in the development in the ensuring sections, simple proofs for these facts can be found, for example, in [12, 13]. First, consider a complex matrix A of dimension $n \times n$. Then we can always find a unitary matrix U such that

$$U^* A U = \begin{bmatrix} \mu & h^* \\ 0 & A_r \end{bmatrix} \tag{7}$$

where the superscript $*$ denotes the conjugate transpose; h is an $(n-1)$-vector and A_r is an $(n-1) \times (n-1)$ matrix. When A is a real matrix, we can always find real orthogonal matrices K, N such that for any real eigenvalue λ of A

$$K^T A K = \begin{bmatrix} \lambda & g^T \\ 0 & A_{n-1} \end{bmatrix} \tag{8}$$

and for any complex conjugate pair of eigenvalues $\sigma \pm j\omega$ of A ($\omega \neq 0$)

$$N^T A N = \begin{bmatrix} \sigma & -\omega & H \\ \omega & \sigma & \\ 0 & & A_{n-2} \end{bmatrix} \tag{9}$$

where the superscript T denotes the transpose; the matrices A_{n-1}, A_{n-2} are real matrices of dimensions $(n-1) \times (n-1), (n-2) \times (n-2)$, and g, H are real matrices of appropriate dimensions.

As the second fact, if the rank of a $m \times n$ real (complex) matrix E is r, then E can always be factored as

$$E = F \cdot R \tag{10}$$

where F, R are real (complex) matrices of dimensions $m \times r$, $r \times n$, respectively.

These two facts will be used in the next two sections to prove the factorization theorems.

3 Complex Factorization

In this section we present a complex factorization theorem. It is assumed throughout the section that the rational matrix has complex coefficients, thus the real rational matrices reduce to a special case.

To begin with, let an $m \times p$ rational matrix $G(s)$ of degree n be given. Then $G(s)$ can always be realized minimally in the state-space form (3). Upon choosing a unitary matrix U such that (7) holds, where μ is an eigenvalue of A, we can apply the similarity transformation with U (as in (4)) to bring $P(s)$ into the form

$$\tilde{P}(s) = \begin{bmatrix} s - \mu & -h^* & \tilde{B} \\ 0 & I_{n-1} s - A_r & B_r \\ -C_1 & -\tilde{C} & D \end{bmatrix} \tag{11}$$

Note that (11) has the same transfer matrix $G(s)$ as (3). Construct from (11) the following constant $(m+1) \times (n+p-1)$ matrix

$$\begin{bmatrix} -h^* & \tilde{B} \\ -\tilde{C} & D \end{bmatrix} \tag{12}$$

If the rank of the matrix (12) is q, then (12) can be decomposed into (see (10))

$$\begin{bmatrix} -h^* & \tilde{B} \\ -\tilde{C} & D \end{bmatrix} = \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} \begin{bmatrix} -C_r & D_r \end{bmatrix} \tag{13}$$

where the matrix $[B_1^T \quad D_1^T]^T$ and $[-C_r \quad D_r]$ are of dimensions $(m+1) \times q$, $q \times (n+p-1)$, respectively. Substitute (13) into (11) we have

$$\tilde{P}(s) = \begin{bmatrix} s - \mu & -B_1 C_r & B_1 D_r \\ 0 & I_{n-1} s - A_r & B_r \\ -C_1 & -D_1 C_r & D_1 D_r \end{bmatrix} \tag{14}$$

By comparing (14) with (6) , we know that $G(s)$ can be factored into $G(s) = G_1(s)G_r(s)$, where $G_1(s)$ ($m \times q$) and $G_r(s)$ ($q \times p$) are rational matrices of degree 1 and $n-1$, respectively; moreover, the system matrices for $G_1(s)$ and $G_r(s)$ are given respectively by

$$P_1(s) = \begin{bmatrix} s - \mu & B_1 \\ -C_1 & D_1 \end{bmatrix}, \quad P_r(s) = \begin{bmatrix} I_{n-1} s - A_r & B_r \\ -C_r & D_r \end{bmatrix} \tag{15}$$

If $n > 2$, the above procedure can be applied again to factorize $G_r(s)$. Since n is finite, the procedure will eventually stop for some $G_r(s)$ of degree 1. We have thus arrived at the following complex factorization theorem:

Theorem 1 *Any complex rational matrix $G(s)$ of degree n can always be factorized as*

$$G(s) = G_1(s)G_2(s)\cdots G_n(s) \tag{16}$$

where the $G_i(s)$ are proper rational matrices of degree 1 ($i = 1, \cdots, n$).

An important fact which should be observed is that the dimensions of the factors $G_i(s)$ ($i = 1, \cdots, n$) play no role in the above factorization theorem. This implies that the primary concern of the factorization problem is to split minimally the *pole structure* of rational matrices into its factors. Thus, the zero structure and the so-called singular structure (i.e., the minimal column and row indices) are of no importance (see [10] for standard definitions for these structures). In the factorization (16), apart from the properness, nothing is known of the relationship between the zero as well as the singular structures of $G(s)$ and those of its factors. This also explains why the methods based on the elementary factor extraction [2, 3] can not arrive at the most general factorization (16), since this method implicitly requires the split of the zero structure of rational matrices. (in fact, the elementary factors are always square matrices of a single zero). [4, 6, 7] suffer from the same drawback, although a state-space formulation for solving the factorization problem is proposed for the first time.

Another observation concerning (16) is that each of the poles of $G(s)$ is disjoint from the others as the unique pole for each of the factors. It follows from the derivation that the order by which all the poles of $G(s)$ appear in the factorization (16) can be specified at will. The reader may find that (16) achieves a pleasing symmetry between the factorization for rational functions and that for rational matrices.

4 Real Factorization

It is often desirable to factorize a real rational matrix over the real field. In this section, we show that if this is the case the prime factors (i.e., the factors which can not be further factorized) may have degree 1 or 2.

Again we start with the minimal realization (3) of an $m \times p$ real rational matrix $G(s)$ with degree n. Note that this time A, B, C, D are constant real matrices. For a real pole λ of $G(s)$, an orthogonal real matrix K can always be found such that (8) holds. Let us apply the similarity transformation formed with K to bring (3) into

$$\tilde{P}(s) = \begin{bmatrix} s-\lambda & -g^T & \tilde{B} \\ 0 & I_{n-1}s - A_{n-1} & B_{n-1} \\ -C_1 & -\tilde{C} & D \end{bmatrix} \tag{17}$$

Now we perform the following decomposition

$$\begin{bmatrix} -g^T & \tilde{B} \\ -\tilde{C} & D \end{bmatrix} = \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} \begin{bmatrix} -C_{n-1} & D_{n-1} \end{bmatrix} \tag{18}$$

where q is the rank of the matrix on the left side of (18); $[B_1^T \quad D_1^T]^T$, $[-C_{n-1} \quad D_{n-1}]$ are real matrices of dimensions $(m+1)\times q$, $q\times(n+p-1)$, respectively.

On substituting (18) into (17), we obtain

$$\begin{bmatrix} s-\lambda & -B_1C_{n-1} & B_1D_{n-1} \\ 0 & I_{n-1}s - A_{n-1} & B_{n-1} \\ -C_1 & -D_1C_{n-1} & D_1D_{n-1} \end{bmatrix} \tag{19}$$

It is easy to see from (19) that $G(s)$ can be factorized as $G(s) = G_1(s)G_{n-1}(s)$, where $G_1(s), G_{n-1}(s)$ are real rational matrices of dimensions $m \times q$, $q \times p$, respectively; furthermore, $G_1(s)$ is of degree 1, and has the unique pole λ, $G_{n-1}(s)$ has degree $n-1$; and the minimal realizations for $G_1(s)$ and $G_{n-1}(s)$ are respectively

$$P_1(s) = \begin{bmatrix} s-\lambda & B_1 \\ -C_1 & D_1 \end{bmatrix}, \quad P_r(s) = \begin{bmatrix} I_{n-1}s - A_{n-1} & B_{n-1} \\ -C_{n-1} & D_{n-1} \end{bmatrix} \tag{20}$$

Similarly, for a complex conjugate pair of poles of $G(s)$, say $\sigma \pm j\omega$, $(\omega > 0)$ we can find an orthogonal matrix N such that (9) holds. Apply the similarity transformation with N to (3) resulting in

$$\begin{bmatrix} s-\sigma & \omega & -H & \tilde{B} \\ -\omega & s-\sigma & & \\ 0 & & I_{n-2}s - A_{n-2} & B_{n-2} \\ -C_1 & & -\tilde{C} & D \end{bmatrix} \tag{21}$$

Carrying out the following decomposition

$$\begin{bmatrix} -H & \tilde{B} \\ -\tilde{C} & D \end{bmatrix} = \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} \begin{bmatrix} -C_{n-2} & D_{n-2} \end{bmatrix} \tag{22}$$

where q is the rank of the matrix on the left side of (22); $[B_1^T \quad D_1^T]^T$, $[-C_{n-2} \quad D_{n-2}]$ are real matrices of dimensions $(m+1)\times q$, $q\times(n+p-1)$, respectively. Substituting (22) into (21), we have

$$\begin{bmatrix} s-\sigma & \omega & -B_1C_{n-2} & B_1D_{n-2} \\ -\omega & s-\sigma & & \\ 0 & & I_{n-2}s - A_{n-2} & B_{n-2} \\ -C_1 & & -D_1C_{n-2} & D_1D_{n-2} \end{bmatrix} \tag{23}$$

(23) suggests that $G(s)$ can be factorized into $G(s) = G_1(s) \cdot G_{n-2}(s)$ where $G_1(s)$, $G_{n-2}(s)$ are real rational matrices of dimensions $m \times q$, $q \times p$, respectively; $G_1(s)$ only has a pair of conjugate poles, $G_{n-2}(s)$ has degree $n-2$; and the minimal realizations of $G_1(s)$ and $G_{n-2}(s)$ are given respectively by

$$\begin{bmatrix} s-\sigma & \omega & B_1 \\ -\omega & s-\sigma & \\ -C_1 & & D_1 \end{bmatrix}, \quad \begin{bmatrix} I_{n-2}s - A_{n-2} & B_{n-2} \\ -C_{n-2} & D_{n-2} \end{bmatrix} \tag{24}$$

Now we can apply the above procedures to $G(s)$ repeatedly until no further factorization is possible. To summarize we state the following theorem:

Theorem 2 *Any $m \times p$ real rational matrix $G(s)$ of degree n can always be factorized as*

$$G(s) = G_1(s)G_2(s)\cdots G_k(s) \tag{25}$$

where the $G_i(s)$ are real proper rational matrices of degree at most two. For any factors of degree two, it will have a pair of complex conjugate poles. Moreover, all the poles of the factors add up to n.

Theorem 1, 2 have in fact completely answered the question of minimal factorization. We now know that if the factorization is only aimed at splitting the degree, any rational matrix can be factorized in precisely the same fashion as rational functions in the complex as well as real fields. Therefore, the prime factor of rational matrix is of degree one over the complex field, and at most of degree two over the real field. These facts put to the end the conjecture that there might exist prime factors of degree higher than 1 in the complex field, and higher than 2 in the real field [2, 4]. For any given rational matrix, since its poles are uniquely defined, the sequence of degrees of prime factors is unique. Moreover, if a rational matrix is minimally factored in two different ways, the degrees of the factors of these two factorizations will match with each other although the factors may not at all be the same, which is the typical situation for the minimal factorization over ring or module.

Another advantage is that the methods which we have used to derive the factorizations are constructive, and allow numerically stable methods to be applied to actually obtain the factors. In the next section, we shall consider this problem is some detail.

5 Numerical Implementation and Illustrative Examples

As we have seen from the foregoing two sections that the construction of minimal factors mainly involves two steps of computation, namely, reducing A to (7), (8) or (9) by applying the similarity transformation with unitary or orthogonal matrices, and performing the appropriate matrix decompositions. We shall discuss in this section how these computations can be performed easily and accurately.

To carry out the decomposition (10), we simply compute the singular value decomposition of E as follows

$$E = U\Lambda V^* \tag{26}$$

where U, V are unitary matrices (or orthogonal matrices in case E is real), Λ is a diagonal matrix with non-negative diagonal entries. Partition (26) as

$$E = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} \tag{27}$$

where Σ is a diagonal matrix with positive diagonal entries. The desired decomposition follows from (27)

$$E = F \cdot R$$

where $F = U_1\Sigma$, $R = V_1^*$.

To perform the similarity transformation, we proceed by first decomposing A into upper Schur form, i.e.,

$$U^*AU = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} \tag{28}$$

where N is upper triangular; U is a unitary matrix. (3) can then be transformed into

$$\begin{bmatrix} s-a_{11} & -a_{12} & \cdots & -a_{1n} & B_1 \\ 0 & s-a_{22} & \cdots & -a_{2n} & B_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s-a_{nn} & B_n \\ -C_1 & -C_2 & \cdots & -C_n & D \end{bmatrix} \tag{29}$$

where $C_1, \cdots, C_n, B_1, \cdots, B_n$ are the proper partitions of the matrices CU and U^*B. From (29), it is obvious that we can start to decompose the matrix

$$\begin{bmatrix} -a_{12} & \cdots & -a_{1n} & B_1 \\ -C_2 & \cdots & -C_n & D \end{bmatrix} \tag{30}$$

and so on without performing further similarity transformation. In case of real factorization, (28) is replaced by the real Schur decomposition

$$Q^T A Q = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1d} \\ 0 & R_{22} & \cdots & R_{2d} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & R_{dd} \end{bmatrix} \tag{31}$$

where each R_{ii} is either a real scalar or a 2 by2 real matrix having complex conjugate eigenvalues; Q is a real orthogonal matrix. The remaining steps are the same as for the complex factorization.

In practice, either SVD or Schur decomposition can be computed in great precision by using the famous QR algorithm or its extended versions [12, 13], so that the numerical methods outlined above offer great advantages in the computational aspect. To end this section we provide a simple example to illustrate the above computational procedure.

Example Consider

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & -\frac{2s}{(s+1)^2} \\ 0 & \frac{s-1}{s+1} \end{bmatrix} \tag{32}$$

The transpose of this rational matrix appeared in [2] as a prime rational matrix of degree 2. Here we show that (32) can be further factorized into two prime factors of degree 1 over the real field.

First, a minimal realization of (32) is found to be

$$\begin{bmatrix} Is-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} 1+s & 1 & 1 & 0 \\ 0 & 1+s & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Since the matrix A is already in the desired Schur form, we only need to perform the following decomposition

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the factors can be seen to be

$$G_1(s) = \frac{\begin{bmatrix} -2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}{1+s} + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

$$G_2(s) = \frac{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{1+s} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reader may wish to check that $G(s) = G_1(s) \cdot G_2(s)$ indeed holds.

6 Conclusions

A relatively complete theory for prime factorizations of rational matrices over complex and real fields is presented. The factorization is aimed at splitting minimally the pole structure of rational matrices into factors, so that the zero structure and other structural information (e.g., the left and right minimal indices) are not involved. Our approach is constructive, and the computation steps allow numerically stable and efficient algorithms to be applied.

Our result has a number of important applications. In fact, we now know that any multi-port or multi-channel filter can be realized in a cascade of the subsystems of degree 1. The modularity of these subsystems is well-suited for VLSI implementation. Moreover, the cascade nature of the factorization reduces the sensitivity to the parameter variation significantly. Finally, we believe that our result has important implications which may render itself very well for further investigation.

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