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An Algebraic Approach to Polynomial Matrices and Rational Matrices at Infinity

Shaohua Tan and Joos Vandewalle

E.S.A.T. Laboratory, Department of Electrical Engineering, K. U. Leuven, Kardinaal Mercierlaan 94, B-3000 Heverlee, Belgium.

Abstract

The objective of this paper is to present a novel algebraic theory which deals with polynomial and rational matrices at infinity. Unlike the conventional approach, the primary treatment here is based on the concept of divisor at infinity for polynomial matrices. The notions of infinite zeros, coprimeness and cancellation at infinity are introduced. General conditions for coprimeness at infinity for two polynomial matrices are derived for some specific classes of polynomial matrices, which naturally arise as the numerator and the denominator matrices of any rational matrix in MFD. We show that for any rational matrix factorized in MFD, the determinantal degree of the denominator matrix is the well-known Smith-McMillan form of the rational matrix if and only if the MFD is both finitely coprime and coprime at infinity.

1 Introduction

The theory of linear time-invariant multivariable systems developed in the past decades has mainly been concerned with causal dynamic systems. In the frequency domain, this type of dynamic systems is globally describable by *proper* rational matrices, therefore the algebraic aspects of proper rational matrices often represent some significant dynamic properties of the systems. To illustrate this point, let us recall that any proper rational matrices can be alternatively represented (or to be precise, decomposed) as fractions of polynomial matrices, the so-called matrix fractions. It is known that some of these matrix fractions have direct physical interpretation. For example, the ARMA modelling will result in a fraction of the form $D(s)^{-1}N(s)$, while the well-known state-space description can simply be viewed as a two-sided matrix fraction $C(Is-A)^{-1}B$. Algebraically, there may exist cancellation between the numerator and the denominator of a matrix fraction. This cancellation represents the cancellation of the dynamic modes of the underlying systems. When specialized to the state-space representations, the absence of the cancellation is termed *controllability* and *observability*, and plays a fundamental role in the whole system theory (Kalman 1963, Rosenbrock 1970, Kailath 1980).

Recent interests in linear system theory have been extending beyond causal systems (Rosenbrock 1974, Verghese et. al. 1981). Even for causal systems, a great deal of attention has been paid to the detailed local connections of the components which constitute the entire system (Luenberger 1977). Such extensions of the scope, as can be expected, make the proper rational matrix representation inadequate. It becomes necessary to consider non-proper rational matrix representations. In fact, this extension is also conceivable from a purely mathematical point of view.

The systems representable in non-proper rational matrices have been shown to possess uncommon dynamics. The existence of the impulsive modes as so called in contrast with the more familiar exponential modes is one particular example. These exotic modes have been shown by Verghese et. al. (1981) to relate to the infinite algebraic structure of the non-proper rational matrices. Actually, an important observation that when dealing with the systems representable in non-proper rational matrices one has to account for the property at infinity of the underlying rational matrices has been made by Rosenbrock (1974) for quite some time already. It therefore raises the need to examine the algebraic aspect of rational matrices at infinity, and a thorough understanding of this aspect is really crucial in both the understanding and the control of these uncommon dynamics.

Despite the fact that there exist extensive theories for handling the finite aspect of rational functions and matrices (Bocher 1907, Van der Waerden 1953, Gantmacher 1960a, b), there have been relatively less efforts made in dealing with infinity for rational matrices. The conventional approach is often to apply a suitably chosen conformal mapping to bring the infinity to some finite point. This approach has severe limitations, since it does not lead to a natural description of the redundancy at infinity. The purpose of the present paper is, therefore, to present a novel algebraic theory of this nature. Unlike the conformal mapping based approaches, the key idea of our theory is to use the homogenous variable to represent infinity, thus putting all the finite points and the infinity on a equal footing. After laying down a few basic concepts, we shall be especially concentrated on the definitions and the characterizations of the coprimeness at infinity and complete coprimeness. As an important consequence, we show that for any rational matrix factorized in MFD, the determinantal degree of the denominator matrix is the well-known Smith-McMillan form of the rational matrix if and only if the MFD is both finitely coprime and coprimeness at infinity.

The paper is structured as follows. Sec.2 is devoted to polynomial matrices, the notions of coprimeness at infinity and complete coprimeness are introduced, three important types of polynomial matrices are also defined. Rational matrices are considered in Sec.3 in which the central concepts of irreducibility and complete irreducibility are introduced along with some interesting characterizations. Finally, Sec.4 contains the conclusions.

2 Polynomial Matrices at Infinity

2.1 Basic Definitions

Polynomial matrices are simply matrices with polynomial entries. Here we allow the entries of polynomial matrices to have zero leading coefficient. Note that the zero leading coefficients of these polynomials are considered to be significant, they actually define the infinite zeros for the corresponding polynomials. In order to treat these in a unifying way with their finite counterparts, we have to modify some familiar definitions concerning the algebraic operations for polynomials.

First let us define what is meant by the *homogeneous form* of a polynomial. If a polynomial $p(s)$ is given by

$$p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_n \qquad (1)$$

where each coefficient p_i $(i = 0, \ldots, n)$ belongs to the real field R (it is certainly possible to consider other fields than R. For simplicity of the discussion, we shall confine ourselves in the real

field), then the following (two-variable) homogeneous polynomial, denoted by $p(s,r)$

$$p(s,r) = p_0 s^n + p_1 s^{n-1} r + \cdots + p_n r^n \tag{2}$$

will be called the *homogeneous form* of $p(s)$. We emphasize that $p_0 \neq 0$ is not required. n will be called the *degree* of the polynomial (1), and p_0 the *leading coefficient* of $p(s)$. Moreover, if $p_i = 0$ $(i = 0,\ldots,n)$, then $p(s)$ is said to be a zero polynomial (so is $p(s,r)$), and can be thought to have *any* degree. Another special case is $n = 0$, this time both the polynomial $p(s)$ and its homogeneous form are simply the same real number.

The importance of the homogeneous form is that it forms a basis to define the set of algebraic operations among all the polynomials with possibly zero leading coefficient.

Definition 1 *All the algebraic operations among the polynomials with, possibly, zero leading coefficient are defined as the same algebraic operations among their corresponding homogeneous forms provided that the operations result in a homogeneous polynomial.*

Observe that the above definition effectively implies that polynomials of different degrees can never be considered equivalent nor can they be added.

Once the homogeneous form of a polynomial $p(s)$ is introduced one can naturally define the infinite zeros of $p(s)$ as follows. $p(s)$ is said to have α zeros at infinity if $p(s,r) = \bar{p}(s,r) \cdot r^\alpha$, where $\bar{p}(s,0) \neq 0$.

Let us now turn to polynomial matrices. The homogeneous form $P(s,r)$ for a polynomial matrix $P(s)$ is defined by replacing all its entries with their corresponding homogeneous forms. Analogously, the algebraic operations among polynomial matrices will be defined as the same operations among their corresponding homogeneous forms provided that the outcomes of the operations are matrices with homogeneous entries.

The determinant of a square matrix $P(s)$ $(m \times m)$ is defined to be $det\, P(s,r)$ if it is a homogeneous polynomial, where $P(s,r)$ is the homogeneous form of $P(s)$. If the result is not a homogeneous polynomial, then $P(s)$ is said to have no determinant. Obviously, not every square polynomial matrix has a determinant as a result of Definition 1. If the determinant of a polynomial matrix $P(s)$ is zero, it is said to be singular. Otherwise, it is nonsingular. For nonsingular polynomial matrices we have the following definition concerning their zeros

Definition 2 *Let the determinant of a nonsingular matrix $P(s)$ be defined. Then the zeros (finite and infinite) of $P(s)$ are defined as those of its determinant.*

Nonsingular polynomial matrix with constant determinant are called *unimodular*. A unimodular polynomial matrix apparently has no zeros. Thus multiplication with a unimodular will not change the zeros of polynomial matrices. Moreover, the inverse of a unimodular polynomial matrix is also a unimodular polynomial matrix.

Zeros can also be defined for singular, or even non-square polynomial matrices. The definition, however, is not as straightforward as for the nonsingular ones. Since the present paper will not involve these zeros, they will not be discussed here.

2.2 Three Special Classes of Polynomial Matrices

The preceding subsection has only been concerned with arbitrary polynomial matrices. As we shall see in Sec.3, only three special classes of polynomial matrices especially deserve our attention, namely, *matrix polynomials, column-based polynomial matrices and row-based polynomial matrices.* In this subsection, we shall describe these three classes of polynomial matrices in detail.

Matrix Polynomials If all entries of a polynomial matrix $P(s)$ $(p \times m)$ have the same degree, then $P(s)$ will be called a *matrix polynomial.* It is evident that the homogeneous form $P(s,r)$ of $P(s)$ can be written as

$$P(s,r) = P_0 s^n + P_1 s^{n-1} r + \cdots + P_n r^n \tag{3}$$

where n is the degree which is common for all entries of $P(s)$; P_i $(i = 0,\ldots,n)$ are $p \times m$ constant matrices. On setting r to 1, (3) becomes

$$P(s,1) = P_0 s^n + P_1 s^{n-1} + \cdots + P_n \tag{4}$$

We note that (4) can actually be looked upon as a polynomial with constant matrix coefficients. This also explains the term "matrix polynomial". P_0 in (3) or (4) will be called the *leading coefficient matrix* of $P(s)$, and n will be called the degree of $P(s)$.

Column-based Polynomial Matrices If the degrees of entries of a polynomial matrix $Q(s)$ $(p \times m)$ in the same column are the same, then $Q(s)$ will be called a *column-based polynomial matrix.* For such a matrix, the ith column $(i = 1,\ldots,m)$ of its homogeneous form $Q(s,r)$ is representable in the following form

$$\alpha_0^i s^{k_i} + \alpha_1^i s^{k_i-1} r + \cdots + \alpha_{k_i}^i r^{k_i} \tag{5}$$

where in (5), each column vector α_j^i is a constant p-vector; k_i is the degree for all entries in the ith column, and will be called the ith *column degree.*

The homogeneous form of a column-based polynomial matrix can always be decomposed into the form

$$Q(s,r) = Q_{hc} S_Q(s) + Q_{lc} T_Q(s,r) \tag{6}$$

where

$$Q_{hc} = [\alpha_0^1 \;\; \alpha_0^2 \;\; \cdots \;\; \alpha_0^m]$$

$$S_Q(s) = diag\{s^{k_1}, s^{k_2}, \ldots, s^{k_m}\}$$

$$T_Q{}^T(s,r) = block\,diag\{[s^{k_i-1} r \;\; s^{k_i-2} r^2 \;\; \cdots \;\; r^{k_i}], \quad i = 1,\ldots,m\}$$

and Q_{lc} is an appropriate constant matrix containing all coefficients of lower powers with respect to s. Q_{hc} will be called the *leading coefficient matrix* of $Q(s)$. We shall see later that Q_{hc} will play an important role in characterizing some specific properties of the matrix.

Row-based Polynomial Matrices If the degrees of the entries of a polynomial matrix $W(s)$ $(p \times m)$ in the same row are the same, $W(s)$ is called a *row-based polynomial matrix.* It is easy to see that a row-based polynomial matrix is the transpose of a column-based one. This fact enables to decompose the homogeneous form of $W(s)$ as follows

$$W(s,r) = S_W(s) W_{hc} + T_W{}^T(s,r) W_{lc} \tag{7}$$

which can be regarded as the transpose of (6). W_{hc} is similarly defined to be the *leading coefficient matrix* of $W(s)$. We note that k_i in $S_W(s)$ or $T_W{}^T(s,r)$ is the ith row degree of W(s).

The relationships among these three classes of polynomial matrices can easily be established. It turns out that a matrix polynomial is a special column (or row)-based polynomial matrix, while a row-based polynomial matrix is the transpose of a column-based one. Therefore, in most cases, it suffices to consider only column-based polynomial matrices.

An important feature associated with these three classes of matrices is that if they are square their determinant is well defined. Moreover, let a square nonsingular polynomial matrix $P(s)$ in one of these three classes be given, and denote by $deg\, det P(s,r)$ the determinantal degree of $P(s)$, and by $k_i = deg\, p_i(s)$ (where $p_i(s)$

is the ith column (row) of $P(s)$) ($i = 1, \ldots, m$) the ith column (row) degree, the following relationship must always hold

$$deg\, det P(s,r) = \sum_{i=1}^{m} k_i \tag{8}$$

The reader may wish to draw a comparison between Eq. (8) and the well-known inequality given, for example, in Kailath 1980, p.384. We can now interpret that inequality as follows. The conventional definition of the determinant of a polynomial matrix only takes the finite zeros into consideration, whereas the sum of column degrees reflects both the number of finite and the number of infinite zeros of a polynomial matrix.

Other features of these three classes of polynomial matrices can also be seen when they undergo algebraic operations. The following two lemmas present some interesting factorizations of these polynomial matrices.

Lemma 1 *If a matrix polynomial $P(s)$ can be factorized into*

$$P(s) = Q(s)R(s) \tag{9}$$

where $R(s)$ is a square matrix, then $Q(s)$ must be column-based and $R(s)$ must be row-based.

Lemma 2 *If a column-based polynomial matrix $P(s)$ can be factorized into*

$$P(s) = Q(s)R(s) \tag{10}$$

where $R(s)$ is a square matrix, then $Q(s)$ must be column-based and $R(s)$ is an arbitrary polynomial matrix whose determinant is defined.

The reader should have no difficulty in carrying over Lemma 2 to row-based polynomial matrices. These lemmas follow directly from the definitions of operations. To show, for example, Lemma 2 , we simply recall the relation $det P(s,r) = det\, Q(s,r) \cdot det R(s,r)$. Since both $det P(s,r)$ and $det\, Q(s,r)$ are homogeneous, $det R(s,r)$ must also be homogeneous, which implies that $R(s)$ has the determinant.

We shall see in Sec.3 that polynomial matrices of these classes have special significance in the context of rational matrices. Therefore, we shall hereby assume that all the polynomial matrices considered fall into these classes.

2.3 Coprimeness at Infinity and Complete Coprimeness

Given two polynomial matrices $D(s)$ $(p \times p)$, $N(s)$ $(m \times p)$, and their respective homogeneous forms $D(s,r), N(s,r)$. It is assumed that the corresponding column degrees of the two polynomial matrices are the same. If $D(s,r) = \bar{D}(s,r)R(s,r)$ and $N(s,r) = \bar{N}(s,r)R(s,r)$, where $R(s,r)$ is a square nonsingular polynomial matrix, then $R(s,r)$ is called a *common right divisor* of $D(s)$ and $N(s)$.

If we are only considering column-based polynomial matrices, then the determinant of a right common divisor for such matrices exists (recall Lemma 2). We shall call such a divisor *non-trivial*, if its determinant is not a constant.

Definition 3 *A nonsingular matrix $R(s,r)$ is called a common right divisor at infinity for polynomial matrices $D(s)$ and $N(s)$, if*

1. *it is a common right divisor of $D(s)$ and $N(s)$;*
2. *$det\ R(s,r) = \alpha \cdot r^{\sigma}$, where α is a constant scalar different from zero, and σ is a positive integer.*

The above definition allows to introduce the notion of *coprimeness at infinity* for two polynomial matrices.

Definition 4 *Two polynomial matrices with the same number of columns are said to be right coprime at infinity if they have no common right divisors at infinity.*

A interesting characterization of this concept can be stated as a rank condition for a constant matrix constructed with the leading coefficient matrices of both $D(s)$ and $N(s)$, as shown next

Theorem 1 *Let $D(s)$ and $N(s)$ be two column-based polynomial matrices of dimensions $p \times p$ and $m \times p$, respectively. It is further assumed that they have the same corresponding column degrees. Then $D(s)$ and $N(s)$ are right coprime at infinity if and only if*

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p \tag{11}$$

where D_{hc} and N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

The proof for the theorem is lengthy and thus omitted. The interested readers are referred to Tan 1987.

The notion of (finite) coprimeness for two polynomial matrices is well-known (see, e.g., Rosenbrock 1970). Here we simply recall that two polynomial matrices are coprime if their greatest common divisor is a unimodular matrix (it is assumed that common divisors are always nonsingular). One characterization of this concept can be given as a rank condition of an appropriate matrix. A combination of this concept together with the concept of copimeness at infinity leads to the following definition.

Definition 5 *Two polynomial matrices are said to be completely right coprime if they are both right coprime and right coprime at infinity.*

Based on Theorem 1 and the characterization of the right coprimeness, an interesting characterization of the complete coprimeness can be stated.

Theorem 2 *Let $D(s)$ and $N(s)$ be two column-based polynomial matrices. Then $D(s)$ and $N(s)$ are completely right coprime if and only if*

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p$$

and

$$rank \begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} = p, \quad \textit{for all finite } s.$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

3 Rational Matrices at Infinity

3.1 Basic Definitions

A rational matrix $G(s)$ of dimension $m \times p$ is simply defined as a ratio of two column-based polynomial matrices which have the same corresponding column degrees. One can easily show that this definition is equivalent to saying that a rational matrix is a matrix whose entries are rational functions to be defined next.

Definition 6 *A rational function $g(s)$ is defined to be $b(s)/a(s)$, where $a(s), b(s)$ are polynomials satisfying $deg\ a(s) = deg\ b(s)$, and $a(s) \neq 0$. The poles of $g(s)$ (finite and infinite) are the zeros (finite and infinite) of $a(s)$; the zeros of $g(s)$ (finite and infinite) are the zeros (finite and infinite) of $b(s)$.*

The definition of rational functions is based on a simple fact that when taking infinity into account, any rational function will have the same number of poles and zeros. This fact is reflected by the degree constraint in Definition 6. A point which should be emphasized is that a polynomial is not considered to be a rational function. One should not confuse a polynomial $p(s) = a_0 s^n + \cdots + a_n$ with the rational function $g(s) = (a_0 s^n + \cdots + a_n)/(0 \cdot s^n + \cdots + 1)$.

A rational matrix defined in terms of MFD is more appropriate for introducing poles and zeros of rational matrices.

Definition 7 *Let a rational matrix $G(s)$ be given by*

$$G(s) = N(s)D^{-1}(s)$$

Then its poles (finite and infinite) are defined to be the zeros of $detD(s)$.

If $N(s)$ is a square matrix, then the zeros of $G(s)$ are defined to be those of $detN(s,r)$. If it is a non-square, then the definition of zeros will involve the concept of generalized invariant polynomial. We shall, however, not go any further in this direction.

The above definition of rational matrices is only stated for the right matrix fraction and the column-based polynomial matrices. One can perfectly give a definition based on the left matrix fraction and the row-based polynomial matrices.

When a rational matrix $G(s)$ is represented in right MFD, say $N(s)D^{-1}(s)$, the numerator $N(s)$ and the denominator $D(s)$ may have common right factors at infinity. The absence of such factors is termed *irreducibility at infinity*. Moreover, the absence of any common divisor (finite and infinite) is called *complete irreducibility*. We now state the characterizations for each of the two concepts.

Theorem 3 *Let an $m \times p$ rational matrix $G(s)$ be given by*

$$G(s) = N(s)D^{-1}(s) \tag{12}$$

where $N(s), D(s)$ are column-based polynomial matrices. Then $G(s)$ is irreducible at infinity if and only if

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p \tag{13}$$

in which D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

Theorem 4 *Let $G(s)$ be given by (12). It is completely irreducible if and only if in addition to (13), the following is also true*

$$rank \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = p, \quad for\ all\ finite\ s$$

3.2 Singular System Representations

It is well-known that a proper rational matrix in MFD can always be realized into the state-space description. An equivalent statement in the framework of non-proper rational matrices is that any MFD defined in the previous subsection can be realized into the so-called *singular system representation.*

Let us start with the general form of singular system representation

$$G(s) = C(Es - A)^{-1}B + D \tag{14}$$

where, as usual, $x \in R^n, u \in R^p$ and $y \in R^m$; E, A are matrices of dimension $n \times n$, B and C are matrices of dimensions $n \times p, m \times n$, respectively. Moreover, n is defined as the dimension of the system, and f = rank E is called the order of the system. One can see that this representation is simply a two-sided matrix fraction.

As an alternative of (14), the following system matrix introduced by Rosenbrock (1970) is often used

$$\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \tag{15}$$

With (15) we shall define two singular systems to be *strongly equivalent* if one can be transformed to the other by any combination of the following three operations (Verghese et. al. 1981):

1. *The left and right multiplication*

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} \tag{16}$$

where $QE = 0$ and $ER = 0$;

2. *the trivial augmentation* from (15) to

$$\begin{bmatrix} Es - A & 0 & B \\ 0 & I & 0 \\ -C & 0 & D \end{bmatrix} \tag{17}$$

3. *the trivial deflation*, namely from (17) to (15).

It is easy to see that the above defined transformation will not change the order f ($= rankE$) of a singular system.

Here we shall introduce a few more basic definitions concerning the singular system representation. To begin with, consider the following matrix

$$[Es - A \quad B] \tag{18}$$

By *str. eq.* operations, (18) can be brought to

$$\begin{bmatrix} E_\epsilon s - A_\epsilon & B_\epsilon \\ A_\pi & B_\pi \end{bmatrix} \tag{19}$$

with E_ϵ of full row rank. Similarly, the matrix

$$\begin{bmatrix} Es - A \\ -C \end{bmatrix} \tag{20}$$

can be brought, by *str. eq.* operations, to

$$\begin{bmatrix} E_\eta s - A_\eta & A_\xi \\ -C_\eta & -C_\xi \end{bmatrix} \tag{21}$$

with E_η of full column rank. We now state the following definition.

Definition 8 *Singular system (14) is said to be finitely controllable if*

$$rank[Es - A \quad B] = n \tag{22}$$

for all finite s; and controllable at infinity if

$$rank \begin{bmatrix} E_\epsilon & 0 \\ A_\pi & B_\pi \end{bmatrix} = n \tag{23}$$

where the block matrix in (23) is built with the corresponding block elements in (19). It is said to be controllable if it is both finitely controllable and controllable at infinity.

(15) is said to be finitely reconstructible if

$$rank \begin{bmatrix} Es - A \\ -C \end{bmatrix} = n \tag{24}$$

for all finite s; and reconstructible at infinity if

$$rank \begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} = n \tag{25}$$

where the block matrix in (25) is built with the corresponding block elements in (21). It is said to be reconstructible if it is both finitely reconstructible and reconstructible at infinity.

These definitions have been developed by Verghese et al. (1981) along with the dynamical implications of the controllability and reconstructibility. The more complete treatments of these concepts can also be found, for example, in Cobb 1984, Lewis and Ozcaldiran 1984, and Lewis 1985. In fact, as we shall see later, these are precisely the algebraic conditions which guarantee the absence of the common factors (finite and at infinity).

3.3 General Properties of Completely Irreducible MFD's

We shall present a method to realize a right MFD into the singular system representation. Before doing so, however, let us define what we mean by *realization.*

First, apply a left and right multiplication (16) such that the

nonsingular matrix M row-compresses E, $Q = R = 0$, and $N = I$. The system matrix of (14) is transformed into

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_c s - A_c & B_c \\ -A_\pi & B_\pi \\ -C & D \end{bmatrix} \quad (26)$$

where E_c is of full row rank. Then a given MFD $G(s) = N(s)D^{-1}(s)$ is said to be realized by (14) if

$$N(s,r)D^{-1}(s,r) = C \begin{bmatrix} E_c s - A_c r \\ -A_\pi \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ B_\pi \end{bmatrix} + D \quad (27)$$

The above definition is based on the row-compression. It is, of course, possible to present the definition with respect to the column-compression. It should be noted that (27) also defines the *transfer matrix* of a singular system.

To start with the realization algorithm, let a column-based right MFD be given and k_i $(i = 1, \cdots, p)$ be the ith column degree of $N(s)$ $(m \times p)$ and $D(s)$ $(p \times p)$.

As the first step of the construction, define the following matrix

$$\Gamma^T(s,r) = \{[\ s^{k_i} \quad s^{k_i-1}r \quad \cdots \quad sr^{k_i-1} \quad r^{k_i}\ ], \quad i = 1, \ldots, p\} \quad (28)$$

and write $D(s,r)$ and $N(s,r)$ in the form

$$\begin{aligned} D(s,r) &= D_c\Gamma(s,r) \\ N(s,r) &= N_c\Gamma(s,r) \end{aligned} \quad (29)$$

where D_c and N_c are appropriate constant matrices.

Secondly, construct the *core realization* of the form

$$E_c s - A_c = block\ diag\{ \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}}_{(k_i+1)\times(k_i+1),\ i=1,\cdots,p} \},$$

$$B_c^T = block\ diag\{[1\ 0\ \cdots\ 0],\ 1 \times (k_i+1),\ i = 1, \ldots, p\}$$

$$C_c = I_n, \quad n = \textstyle\sum_{i=1}^{p} k_i + p \quad (30)$$

Finally, the desired realization follows form (29) and (30)

$$E = E_c, \quad A = A_c - BD_c, \quad B = B_c, \quad C = N_c \cdot C_c = N_c \quad (31)$$

Next, we prove that (31) indeed realizes the given right MFD. Then some interesting properties of the realization are shown.

Theorem 5 *The singular system in (31) realizes the given column-based MFD.*

proof It is easily checked that

$$(Es - A_c r + BD_c)\Gamma(s,r) = BD_c\Gamma(s,r) = BD(s,r)$$

or equivalently

$$(Es - A_c r + BD_c)^{-1}B = \Gamma(s,r)D^{-1}(s,r) \quad (32)$$

(The nonsingularity of $Es - A_c r + BD_c$ will be proved later). By premultiplying (32) with C $(= N_c)$ and by observing that $N(s,r) = N_c\Gamma(s,r)$, the theorem is proved.

Theorem 6 *The realization (31) has the properties below*

1. $det(Es - A_c r + BD_c) = detD(s,r)$;
2. *it is controllable.*

These two properties can readily be checked by direct computation. We shall omit the detail. Observe that the second property guarantees the nonsingularity of the matrix $Es - A_c r + BD_c$, which is needed in the proof of Theorem 5. Moreover, this property also implies that the order of the realization is equal to the determinantal degree of the denominator of the transfer matrix.

It has been shown by Verghese et al. (1981) that that the order f attains its minimal value if and only if the system is jointly controllable and reconstructible. Here we wish to show that the concept of complete irreducibility and joint controllability and reconstructibility are also linked

We shall be making use of the realization (31) as a primary vehicle to establish the relationship. First we show that the desired relationship holds for this particular realization.

Theorem 7 *The realization (31) is reconstructible if and only if the MFD $N(s)D^{-1}(s)$ is completely irreducible.*

proof Only a proof for the equivalence between the reconstructibility at infinity of the realization and the irreducibility at infinity of a MFD is given. The finite part is well-known in the literature and thus omitted.

From Definition 8, the realization (31) is reconstructible at infinity if and only if the matrix

$$\begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} \quad (33)$$

has full column rank, where the block elements in (33) are defined in (25). From (31), it is easily seen that (33) can be transformed, by elementary transformations, to the following matrix

$$\begin{bmatrix} D_{hc} & 0 \\ 0 & I \\ N_{hc} & 0 \end{bmatrix}$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively. The above matrix is of full column rank if and only if $D(s)$ and $N(s)$ are irreducible at infinity. The theorem thus follows.

With Theorem 7 in hand, the rest of the development follows quite straightforwardly.

Theorem 8 *If a column-based MFD has one controllable and reconstructible fth-order realization, then all its fth-order realization must also be controllable and reconstructible.*

proof Otherwise by using the fourfold decomposition developed in Verghese et. al. 1981, this would imply that f is not the minimal order, which clearly contradicts the conclusion in the preceding subsection. This proves the theorem.

We are now in the position to state and prove the main theorem of this subsection.

Theorem 9 *A column-based right MFD is completely irreducible if and only if all of its nth-order realizations are controllable and reconstuctible; where n is the determinantal degree of the denominator $D(s)$, i.e., $n = deg\ detD(s,r)$.*

proof In view of Theorem 8, it suffices to show that the theorem holds for a special nth order realization. But Theorem 7 indicates that the realization (31) can serve this purpose. The theorem is proven.

It has been shown in Verghese et al. 1981 that the order of a singular system will equal the McMillan degree of the underlying transfer matrix if and only if the singular system is controllable and reconstructible. Combining this fact with Theorem 9 we obtain the following important result.

Theorem 10 *A column-based right MFD is completely irreducible if and only if the determinantal degree of its denominator is equal to the McMillan degree of the MFD.*

This theorem is very significant since it provides an alternative, yet the simplest interpretation for the well-known McMillan degree of rational matrices, viz., the McMillan degree only describes the completely irreducible part of rational matrices. It is now clear that when a rational matrix is factorized in, say MFD, some redundancy (finite and infinite) may be introduced. We can then perform the cancellation. As a consequence, the determinantal degree of the denominator of the MFD will decrease. As soon as the MFD becomes completely irreducible the determinantal degree attains the minimal value which is precisely the McMillan degree of the MFD.

4 Conclusions

A general theory for dealing with polynomial and rational matrices is presented. The results obtained appear to have generalized some well-known facts in proper rational matrices. The theory also sets up a general framework for the interpretation of some of our early results (Tan and Vandewalle 1987).

The emphasis of the development is placed on the descriptions of the so-called irreducibility at infinity and complete irreducibility. It is shown that these new concepts are consistent with the conventional definition of the McMillan degree. The algebraic framework developed here is believed to be able to provide a better understanding of the dynamical behavious at infinity for non-causal systems, or systems with detailed specification of the connections.

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