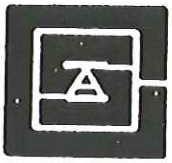
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ON PRIME FACTORS OF NONSINGULAR RATIONAL MATRICES

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Abstract

We examine the minimal factorization problem for square nonsingular rational matrices with an additional constraint that the dimensions of the factors are required to be the same as the original matrix. This dimensional constraint poses certain difficulties, and in fact it is known that the first degree minimal factorization is no longer possible in this case. In other words, the prime factors (matrices which can not be factorized) can re degrees greater than 1. The main result of the paper is to present a necessary condition for prime factor of nonsingular rational matrices with disjoint sets of poles and zeros. There are indications that this condition may also be sufficient, but it can not be shown at present.

1 Introduction

The minimal factorization problem for rational matrices is often stated in the following way: given a rational matrix $G(s)$ of degree n find two factors $G_1(s)$ of degree n_1 and $G_2(s)$ of degree n_2 such that $G(s) = G_1(s)G_2(s)$ with $n = n_1 + n_2$. This type of factorization stems from the circuit analysis and is also useful in a number of fields (see [1-3] and the references therein).

The question arises whether it is possible to factorize an arbitrary rational matrix $G(s)$ this way. This question in its general form has been answered recently in [4]. We now know that any $G(s)$ can always be minimally factorized provided that its degree is greater than 1 (rational matrices of degree 1 may still be further factorized trivially as product of two factors of degrees 1 and 0 respectively, but this is considered uninteresting since it involves nonsingular constant matrix as a factor). Hence, if we define a *prime* rational matrix to be the on which can not be minimally factorized, then a prime rational matri an only have degree 1 (here we exclude the trivial case).

Sometimes it is of interest to consider the minimal factorization problem with certain constraints. A prominent problem of this nature is the following: given a *nonsingular* rational matrix $G(s)$ of degree n and dimension $m \times m$, find two rational matrices $G_1(s), G_2(s)$ of *the same* dimension (i.e., $m \times m$ for each of the two matrices) and of degrees n_1, n_2 respectively, such that $G(s) = G_1(s)G_2(s)$ and $n = n_1 + n_2$. Needless to say, the two factors $G_1(s)$, $G_2(s)$ should be nonsingular as well. One can easily realize that this restricted minimal factorization problem is much more involved, and has actually been a subject of intensive research [1-2]. Interpreted in terms of poles and zeros, we can see that while the general minimal factorization problem is only concerned with splitting the poles of rational matrices, the restricted minimal factorization problem decomposes both the poles and the zeros into the factors.

Due to the restrictions on the dimensions of the factors, the minimal factorization problem is more complex. It is known, by way of examples, that not every nonsingular rational matrix of degree greater than 1 can be minimally factorized in this manner. In fact, a prime factor of degree 2 has been constructed in [1]. We shall see in the subsequent section that there exist prime rational rational matrices of *any* degree. Moreover, there is no general way to factorize minimally a given nonsingular $G(s)$ to a product of prime factors.

As an attempt to understand the nature of the factorization, we try to characterize the prime rational matrices in this paper. Confining to nonsingular rational matrices with disjoint sets of poles and zeros, we derive, among other things, a necessary condition for such a matrix to be prime. It will be seen that two known minimally factorizable classes, namely, scalar rational functions and the rational matrices with simple structure (the matrix A in the minimal realization is diagonalizable), are naturally excluded by the condition. This condition also suggests ways to find prime factors of higher degree. By our experiences with some examples constructed along the line, we expect that this condition is also sufficient, but this is a moot-point at present and yet to be proven.

2 Preliminaries

In this section, some conventions and basic facts are presented to facilitate the subsequent development. Let us start with an $m \times m$ rational matrix $G(s)$ of McMillan degree n, we shall assume throughout the paper that it is nonsingular and proper (For nonproper $G(s)$, a conformal mapping is applied to make it proper before the realization shown below can be carried out). Note that this conformal mapping will not change the minimal factorizability, so that the properness assumption here is not at all essential). Then $G(s)$ can be minimally realized into the state-space representation written below in the system matrix form

$$P(s) = \begin{bmatrix} Is - A & -B \\ C & D \end{bmatrix} \tag{1}$$

where A, B, C, D are constant matrices of dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. Once $P(s)$ is known, any similarity transformation of the form

$$\tilde{P}(s) = \begin{bmatrix} Is - TAT^{-1} & -TB \\ CT^{-1} & D \end{bmatrix} \tag{2}$$

where T is an arbitrary nonsingular constant matrix, is also a minimal realization for $G(s)$. Hence there will be no loss of generality if we assume the matrix A in (1) to be in Jordan form. The similarity transformation will be used from time to time to bring $P(s)$ into some desirable form in our development.

Sometimes we need to apply a conformal mapping of the form: $s = 1/\lambda + p_0$ to bring the point p_0 to infinity. This type of transformation will certainly change $G(s)$, but it will never modify the number of poles and zeros of $G(s)$. Thus, if $G(s)$ is first brought to $\bar{G}(\lambda)$ by some conformal mapping and $\bar{G}(\lambda)$ is minimally factorized into $\bar{G}(\lambda) = \bar{G}_1(\lambda)\bar{G}_2(\lambda)$, we can always derive a minimal factorization $G(s) = G_1(s)G_2(s)$ by applying the inverse of the conformal mapping. In this sense we say that the minimal factorizability of $G(s)$ is not affected by the conformal mappings. It will be seen later that the conformal mapping is essential to induce the minimally factorizable structures for $G(s)$.

Another type of transformations we are concerned with is the so-called input and output transformations defined in the following way:

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Input transformation : $G(s) \Rightarrow G(s)N$, where N is constant and nonsingular;

Output transformation : $G(s) \Rightarrow MG(s)$, where M is constant and nonsingular.

Like the conformal mapping, the input and output transformations do change $G(s)$, but they will never change the minimal factorizability of $G(s)$. To see this, we simply note that if $MG(s)N$ is minimally factorizable into $MG(s)N = G_1(s)G_2(s)$, then $(M^{-1}G_1(s))(G_2(s)N^{-1})$ is a minimal factorization for $G(s)$. When $G(s)$ is in the state-space form, i.e., $G(s) = C(Is - A)^{-1}B + D$, the input and output transformations for $G(s)$ can be interpreted as matrix operations upon the matrix D, or upon the matrices $[C\ D]$, $[B^T\ D^T]^T$. This interpretation will be used quite liberally in the subsequent development.

3 A necessary condition for prime rational matrices

We shall assume through out the section that the set of poles of $G(s)$ is disjoint from the set of zeros. The reason to consider this special case will become clear when we proceed to introduce the condition. Let a minimal realization (1) for $G(s)$ be given, and let the matrix A be in the Jordan form shown below

$$A = \text{block diag}\ \{A_1, A_2, \ldots, A_r\} \tag{3}$$

where each of the $n_i \times n_i$ blocks A_i, $(i = 1, \ldots, r)$ is given by

$$A_i = \begin{bmatrix} \alpha_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_i \end{bmatrix}$$

Note that we are only interested in identifying the Jordan blocks not its eigenvalues, so that it is possible that $\alpha_i = \alpha_j$ for $i \neq j$. The corresponding partitions of the matrices B and C with respect to (3) are given by

$$C = [C_1\ \ C_2\ \cdots\ C_r], \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix} \tag{4}$$

The matrix D will not assume any partition because it is not dimensionally related to A. Let the rank of D be p $(p \leq m)$, then there exists an input transformation matrix N such that $DN = [D_1\ 0]$, where D_1 is of dimension $m \times p$. We note that the matrix BN will have the same row partition as B, and moreover, the structure of DN induces a column partition of this matrix as follows

$$\begin{bmatrix} BN \\ DN \end{bmatrix} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \\ \vdots & \vdots \\ \bar{B}_{r1} & \bar{B}_{r2} \\ D_1 & 0 \end{bmatrix} \tag{5}$$

Based on (5), we can define the following $r \times (m-p)$ constant matrix Ψ

$$\Psi = \begin{bmatrix} \bar{b}_{n_1 2} \\ \bar{b}_{n_2 2} \\ \vdots \\ \bar{b}_{n_r 2} \end{bmatrix} \tag{6}$$

in which the vector $\bar{b}_{n_i 2}$ is the *last row* of the submatrix $\bar{B}_{i2}$, $(i = 1, \ldots, r)$. It is apparent that Ψ is well defined if and only if the matrix D is singular. When the singularity of D is interpreted in terms of zeros, we see that Ψ is meaningful if the rational matrix $G(s)$ has at least one zero at infinity.

In an similar fashion, we can define a Φ matrix by looking at the row-compression of D, and the corresponding partition of the matrix MC of the following form

$$[MC\ \ MD] = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \cdots & \bar{C}_{1r} & \bar{D}_1 \\ \bar{C}_{21} & \bar{C}_{22} & \cdots & \bar{C}_{2r} & 0 \end{bmatrix} \tag{7}$$

where M is nonsingular, and $\bar{D}_1$ has full row rank. Based on (7), the matrix Φ is defined by

$$\Phi = [\bar{c}_{2n_1}\ \ \bar{c}_{2n_2}\ \cdots\ \bar{c}_{2n_r}] \tag{8}$$

where the vector $\bar{c}_{2n_i}$ is the *first column* of the submatrix $\bar{C}_{2i}$, $i = 1, \ldots, r$.

Clearly, the matrices Ψ and Φ so-defined have a definite relation to the infinite zero(s) of $G(s)$. It can be expected that similar matrices for the *finite* zeros of $G(s)$ can also be defined. This is done as follows. Suppose that $G(s)$ has a zero at some finite point z_0 (note that z_0 may be a multiple zero for $G(s)$), then we can apply a conformal mapping of the form $s = 1/\lambda + z_0$ to move the zero to infinity. This mapping is always possible since it is assumed that z_0 can not be a pole of $G(s)$ (which is one of the reasons why the sets of poles and zeros of $G(s)$ are required to be disjoint). Once $G(s)$ is brought to $\tilde{G}(\lambda)$, we can define the matrices Ψ, Φ for it in the way described above. To indicate the explicit dependence on z_0, we shall denote these two matrices as Ψ_{z_0} and Φ_{z_0}. In general, if $G(s)$ has k distinct zeros $z_1, \ldots, z_k$ each of which may be a multiple zero, then $2k$ matrices $\Psi_{z_1}, \ldots, \Psi_{z_k}$ and $\Phi_{z_1}, \ldots, \Phi_{z_k}$ can be unambiguously defined. The following theorem shows that these matrices are closely related to the minimal factorizability of $G(s)$.

Theorem 1 *A necessary condition for $G(s)$ to be prime is that the matrices Ψ_{z_i}, Φ_{z_i}, $(i = 1, \ldots, k)$ are zero.*

Proof

By contradiction, suppose that one of the matrices, say Ψ_{z_q}, $(1 \leq q \leq k)$ is nonzero, we shall show that $G(s)$ is always minimally factorizable.

To begin with, let us assume, without loss of generality, that z_q is at infinity, so that we can examine $G(s)$ in its original from (otherwise $\tilde{G}(s)$ is considered instead, which is obtained by applying the conformal mapping given before). Since Ψ_{z_q} is nonzero, we shall more specifically assume that its entry in the (ℓ, j) position is nonzero.

Consider the matrix $G(s)N$ as given by (3), (5), and C. Obviously, the (ℓ, j) (nonzero) entry of the matrix Ψ_{z_q} corresponds to the entry $\beta_{d_\ell, m-p+j}$, where $d_j = \sum_{i=1}^{j} n_i$ (recall that n_i is the dimension of the i-th Jordan block in A) in the matrix (5). Hence, we can use this nonzero entry to annihilate the other nonzero entries in the d_ℓ row and the $m - p + j$ column of the matrix BN. Written in matrix notation, this annihilation can be described as BN pre- and post-multiplied by nonsingular matrices T, R which leads to the matrix $TBNR$. If the matrix T is used to form a similarity transformation, the transformed matrix TAT^{-1} will be such that except the d_ℓ column which might be changed (the element at (ℓ, ℓ) will remain the same) the Jordan structure of A is not affected. Due to the structure of TAT^{-1}, it is then possible to extract from $Is - TAT^{-1}$ a factor of the form

$$F(s) = \text{diag}\ \{1, \cdots, 1,\ s - \alpha_\ell,\ 1, \cdots, 1\}$$

where the nonunity element is in the d_ℓ-th position. Consequently,

$$\begin{aligned} G(s)NR &= CT^{-1}(Is - TAT^{-1})^{-1}TBNR \\ &= CT^{-1}[F(s)(Is - A_{n-1})]^{-1}TBNR \\ &= CT^{-1}(Is - A_{n-1})^{-1}F^{-1}(s)TBNR \\ &= CT^{-1}(Is - A_{n-1})^{-1}TBNRF^{-1}(s) \end{aligned}$$

Note that the interchange performed at the last step is validated by the special structures of the matrices $F(s)$ and $TBNR$. The rational matrix $G_1(s) = CT^{-1}(Is - A_{n-1})^{-1}TB$ obviously has degree $n-1$. Therefore, if we define $G_2(s) = NRF^{-1}(s)(NR)^{-1}$, then $G_1(s)G_2(s)$ is a minimal factorization for $G(s)$.

Similarly, if one of the Φ_{z_i}'s is nonzero, a minimal factorization can also be obtained for $G(s)$ following the same idea. This contradiction shows the theorem. □

We note that the essence of the proof is to check if there is a minimally factorizable match between a pole and a zero, and such a check is facilitated by bringing the corresponding zero(s) to infinity. One might doubt whether the condition is tight enough, since it seemingly *only* implies the extraction of the first degree factors. This is not true in fact, the detailed discussion in this vain will be provided in the next section.

4 Discussion

We begin our discussion by showing that two classes of nonsingular rational matrices known to be minimally factorizable are naturally excluded by the condition of the theorem. They are scalar rational functions, and nonsingular rational matrices of a simple structure (i.e., the A matrix in its minimal state-space realization has a simple structure.

First, let us examine an arbitrary rational function $g(s)$. If $g(s)$ has a zero at infinity (otherwise a conformal mapping would be applied), then its minimal realization is given by

$$p(s) = \begin{bmatrix} Is - A & -b \\ c & 0 \end{bmatrix} \tag{9}$$

in which A is in the Jordan from (3); b and c are column and row vectors respectively. According to the theorem, in order that $g(s)$ be a prime, at least one entry in b, say β_{nj}, should be zero. This is impossible since a zero entry in b would imply that the realization (9) is nonminimal. The contradiction shows that a scalar rational function can never be a prime unless it has degree one. This result seems to be trivial, because it is known already from the fundamental theorem of polynomials. The point is, however, that we can show this without resorting to the fundamental theorem. As a contrast, the fact that a $g(s)$ of degree n is factorizable into n scalar rational functions of the first degree, which is known via the fundamental theorem of polynomials, can even not be shown (trivially or not) by some well-developed state-space techniques.

By extending the preceding discussions, it is easy to show that if a nonsingular rational matrix $G(s)$ (nonscalar) can be brought to the strictly proper (i.e., $G(\infty) = 0$) rational matrix by some conformal mapping, then $G(s)$ can never be a prime.

A less trivial case is the class of nonsingular rational matrices of simple structure. If $G(s)$ is such a matrix, its minimal realization can be brought by the input transformation to the following form

$$\tilde{P}(s) = \begin{bmatrix} Is - A & -\tilde{B}_1 & -\tilde{B}_2 \\ C & D_1 & 0 \end{bmatrix} \tag{10}$$

Since A is a diagonal, the matrix Ψ for (10) is easily seen to be equal to $\tilde{B}_2$. Therefore, the primeness of $G(s)$ would imply that $\Psi = \tilde{B}_2 = 0$ according to Theorem 1. Consequently, $G(s)$ would not be nonsingular. This contradiction shows that $G(s)$ can never be prime. Actually, the extraction of a degree one factor from $G(s)$ as indicated in the proof of the theorem will not destroy the simple structure, as a result, $G(s)$ can always be minimally factorized into a product of n factors of degree one and dimension $m \times m$. Thus, by following the non-constructive arguments we have arrived at the well-known fact proved in [1], [5], [6].

It may be argued that the condition in Theorem 1 only excludes the possibility of extracting the first degree factors, but not the higher degree factors. The idea is in fact not true. We can see from (3) (5) that the block matrix $\tilde{B}_{i2}$ corresponding to the i-th Jordan block $J_i(s) = Is - A_i$ must be nonsingular (or contain a nonsingular matrix formed from its last row upwards) in order to extract a high degree factor of the form $G_f(s) =$ block diag$\{I, \cdots, I, J_i^{-1}(s), I, \cdots, I\}$ (or $G_f(s) =$ block diag$\{I, \cdots, I, J_{pi}^{-1}(s), I, \cdots, I\}$, where J_{pi} is a part of $J_i(s)$ formed from its last row upwards). The required nonsingularity can not be guaranteed if the condition of Theorem 1 holds. Therefore, the condition of the theorem also excludes the possibility of extracting higher degree factors.

It can be realized from the preceding discussions that the necessary condition is tight enough to rule out the known classes of minimally factorizable nonsingular rational matrices, and to rule out the extraction of higher degree factors. This has led us to believe that the condition in the theorem may also be sufficient, but we have not been able to prove it at present.

In what follows, we shall discuss a few examples in order to gain some insights into the nature of prime factors. The first example is the well-known 2×2 prime rational matrix $G_1(s)$ of degree 2, a transpose of which first appeared in [1]

$$G_1(s) = \begin{bmatrix} \frac{s-1}{s+1} & -\frac{2s}{(s+1)^2} \\ 0 & \frac{s-1}{s+1} \end{bmatrix} \tag{11}$$

It is easy to know that $G_1(s)$ has two poles at $s = -1$ and two zeros at $s = 1$. If the conformal mapping of the form $s - 1 = 1/\lambda$ is applied, $G(s)$ can be brought to the following form

$$\tilde{G}_1(\lambda) = \begin{bmatrix} \frac{1}{1+2\lambda} & -\frac{-2\lambda(1+\lambda)}{(1+2\lambda)^2} \\ 0 & \frac{1}{1+2\lambda} \end{bmatrix} \tag{12}$$

A minimal realization of (12) is computed to be

$$\tilde{P}(\lambda) = \begin{bmatrix} \lambda + \frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & \lambda + \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{13}$$

Without further transformation, we can see from (13) that $\Psi_{-1} = \Phi_{-1} = 0$, as is predicted by the theorem. Note that what is shown in (13) represents a typical structure of a prime factor.

The above rational matrix may suggest that the matrix A in the minimal realization of a prime factor can only have a single Jordan block. This is not necessarily true as is seen in the next example.

The rational matrix is given by

$$G_2(s) = \begin{bmatrix} \frac{1}{s^2} + \frac{1}{(s-1)^2} + 1 & \frac{1}{s} + \frac{1}{s-1} \\ \frac{1}{s} + \frac{1}{s-1} & 0 \end{bmatrix} \tag{14}$$

It can be minimally realized into

$$P(s) = \begin{bmatrix} s & -1 & 0 & 0 & 0 & -1 \\ 0 & s & 0 & 0 & -1 & 0 \\ 0 & 0 & s-1 & -1 & 0 & -1 \\ 0 & 0 & 0 & s-1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{15}$$

We know that this $G_2(s)$ has four poles at 0, 0, 1, 1, and four zeros at ∞, ∞, 1/2, 1/2, and moreover its A matrix has two Jordan blocks. We know of no way to factorize minimally in whatsoever the form. One may wish to check that $G_2(s)$ satisfies the necessary condition to be a prime.

This second example indicates that the number of Jordan blocks is not at all related to the primeness of rational matrices. More generally, the dimension and the degree also have nothing to do with the primeness. In fact, if we do not insist on the disjointness of the sets of poles and zeros of $G(s)$, we shall be able to see more examples such as the following

$$G_3(s) = \begin{bmatrix} 1 & 0 \\ \frac{1}{(s-\alpha)^n} & 1 \end{bmatrix}$$

We know of no way to factorize $G_3(s)$ minimally into a product of two 2×2 rational matrices. $G_3(s)$ obviously has all its poles and zeros at the same location, which is sometimes called a unimodular. A unimodular of the same nature but of arbitrary dimension is given by

$$G_4(s) = I_m + \frac{1}{(s-\alpha)^n} F$$

where all the entries of F $(m \times m)$ are zero except the one at, say $(1, m)$. One should note that unimodulars do not necessarily cause problems, an example of minimally factorizable unimodular is found to be

$$G_5(s) = \begin{bmatrix} 1 & 0 \\ \frac{2}{(s+1)(s-1)} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{s-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{s+1} & 1 \end{bmatrix}$$

Therefore, prime rational matrices are completely irregular with respect to the number and size of Jordan blocks, the dimension and the degree. We suspect that the condition stated in Theorem 1 is bound to be the only nature of prime rational matrices.

The irregularity of prime factors makes the minimal prime decomposition completely irregular. It is known already that an arbitrary nonsingular $G(s)$ may be minimally factorized into a product of prime (nonsingular) factors in more than one way [1] with unmatchable degrees. There have been no general assertion concerning the nature of these different prime factorizations. However, if the condition in Theorem 1 turns out to be sufficient, then we could deduce that all the minimal prime factorization for any $G(s)$ with disjoint sets of poles and zeros can only contain a single prime factor of degree greater than one. More generally, an arbitrary rational matrix $G(s)$ might first be minimally decomposed into a product of a unimodular and a rational matrix with disjoint sets of poles and zeros, the latter part can subsequently be minimally factorized into a product of first degree factors and a *single* higher degree factor. These statements are simply conjectures at present and should nonetheless be substantiated.

5 Conclusions

We have presented a necessary condition of the primeness for rational matrices with disjoint sets of poles and zeros. The underlying idea of the condition is to reveal a special structure in the minimal state-space realization of the rational matrices, which prevents extraction of first or higher degree factors. This necessary condition is shown to be sharp enough to exclude some well-known classes of minimally factorizable rational matrices.

A few examples have been provided to illustrate the condition, some of which suggest that the primeness is related neither to the number and size of the Jordan blocks in the matrix A, nor to the dimension and the degree of rational matrices.

It is expected that the condition given is also sufficient which leads to the conjecture that any nonsingular rational matrix is minimally factorizable into a product of one (or more) unimodular and a sequence of first degree factors and a single higher degree factor. But this conjecture is yet to be proven.

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