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PROPERTIES OF MINIMAL FIRST DEGREE FACTORIZATIONS FOR ARBITRARY RATIONAL MATRICES

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Abstract

In [1] a state space technique is used in order to prove that any complex square or rectangular rational matrix $G(s)$ of Smith-McMillan degree n can always be factorized as a product of n first degree rational matrices, the so-called minimal first degree factorization. The procedure is constructive in the sense that the extraction of a first degree factor can be obtained by performing some linear operations on the system matrix of a state space description of $G(s)$. However there is still much freedom in the factor which is being extracted, e.g., the number of intermediate interconnections may vary depending on the rank of a submatrix of the system matrix; and certain classes of rational matrices may have special factorizations.

It is the aim of this paper to study the properties of minimal first degree factorizations with the emphasis on the reduction of the number of interconnections, since this reduction issue is not only structurally interesting, but also important for implementations (VLSI). It is shown that the number of intermediate connections does not increase when the A-matrix has the simple structure (no Jordan-blocks). In the general case, some upper bounds are derived on the number of interconnections. By exploring some aspects of similarity transformations, we are able to present a modified factorization algorithm which will be shown to produce a minimal factorization with smaller number of interconnections.

1 Introduction

In a recent publication [1], it is shown that any rational matrix $G(s)$ $(m \times p)$ of degree n always admits a *minimal first degree factorization* of the form

$$G(s) = G_1(s) \cdots G_n(s) \tag{1}$$

where each $G_i(s)$ $(i = 1, \ldots, n)$ is of degree 1. While the theory developed there has mainly focussed on splitting the degree of $G(s)$ into the first degree factors, no attention has been paid to the dimensions of the factors generated via the scheme. In this note, we continue the investigation by showing that these dimensions are always upper-bounded by the number $(n + m + p)/2$. When the condition $m - p \geq n - 1$ $(m \geq p)$ holds, the factorization procedure presented in [1] will generate a minimal factorization (1) with the dimensions of the factors less than $m \times m$. It is generally known that the dimensions of the factors also depend upon choices of similarity transformations. Therefore we shall provide a modification using this principle to the original procedure in order to reduce the dimensions of the factors. We show, among other things, that with this modification the dimensions of the factors will generically not exceed m.

2 Factorization of rational matrices via the state space description

To facilitate the ensuing discussion we first lay down a few conventions, and then restate the factorization procedure presented in [1] for easy reference.

Let an $m \times p$ rational matrix $G(s)$ of degree n be given, we shall always assume, throughout this note, that $m \geq p$. If this is not the case, its transpose will be considered. When $G(s)$ is factorized into (1), the dimensions of the factors can be viewed as a sequence of $n-1$ integers, each representing the number of interconnections between two adjacent factors. This sequence is called the connection sequence, and all the results concerning the dimensions of the factors will be stated in terms of such a sequence.

To factorize an arbitrary $G(s)$, let us first realize it in its minimal state-space form, then apply a similarity transformation to bring the system matrix of $G(s)$ into the following form

$$P(s) = \begin{bmatrix} s-\alpha & -h^T & \bar{b}^T \\ 0 & I_{n-1}s - A_r & B_r \\ -c_1 & -\bar{c} & D \end{bmatrix} \tag{2}$$

where the superscript T denotes transpose. Note in particular that c_1, h, b, are vectors. Construct from (2) the following $(m+1) \times (n+p-1)$ submatrix

$$\pi = \begin{bmatrix} -h^T & \bar{b}^T \\ -\bar{c} & D \end{bmatrix}. \tag{3}$$

Then π can be decomposed into

$$\pi = \begin{bmatrix} b_1^T \\ D_1 \end{bmatrix} [-C_r \;\; D_r] \tag{4}$$

where the matrices on the right hand side of (4) have dimensions $(m+1) \times f, f \times (n+p-1)$ respectively, and $f = rank\ \pi$.

Thus a first degree factor of $G(s)$ can be extracted from the left based on (4), i.e.

$$G(s) = G_1(s)G_r(s) \tag{5}$$

and the system matrices for $G_1(s), G_r(s)$ are given respectively by

$$P_1(s) = \begin{bmatrix} s-\alpha & b_1^T \\ -c_1 & D_1 \end{bmatrix}, \quad P_r(s) = \begin{bmatrix} I_{n-1}s - A_r & B_r \\ C_r & D_r \end{bmatrix}$$

The same procedure will be applied to $P_r(s)$ and so on in order to achieve a complete first degree factorization.

It is obvious from the above factorization procedure that the dimensions of the factors are determined by the rank of the matrix π, which is influenced by choices of similarity transformation and its actual structure as depicted in (3). Our intention, therefore, is to explore the structure of π so as to reveal the tendency and the bound of the connection sequence generated by the factorization procedure. Moreover, we shall explore the properties of similarity transformations in relation to the rank of the matrix π, and shall provide two tricks to reduce the dimensions of the factors. These tricks constitute a modification to the original factorization scheme.

3 Simple bounds for the minimal first degree factorization procedure

It is clear from the preceding discussion that the connection sequence is generated as ranks of a sequence of matrices of the form (3). We observe from (3) that the dimension of this matrix is simply $(m_i + 1) \times (n + p - (i+1))$ at each stage of the factorization. Here $m_0 = m$, and $\{m_i\}$, $i = 1, 2, \ldots, n-1$ is the connection sequence. If this matrix is such that the number of its rows exceeds the number of its columns right at the start of the factorization, the connection sequence generated by the factorization procedure will never be greater than m, or more precisely we have

Observation 1 *For a given $G(s)$, if $m - p \geq n - 1$ (recall that $m \geq p$) then the connection sequence generated by the factorization procedure will be upperbounded by a descending sequence*

$$\{m - (i-1)\} \quad (i = 1, \ldots, n-1)$$

Proof:
To show this, simply notice that if $m - p \geq n - 1$, then $f = rank\ \pi \leq n + p - 1 \leq m$, moreover, at each next factorization stage the number of the columns of the matrix π is reduced by 1.

If the condition $m - p \geq n - 1$ is not met, then the factorization procedure may produce a connection sequence in which some of the elements will exceed m. However, we can derive an upper-bound in this case.

Observation 2 *For any $G(s)$ with $m - p < n - 1$, the connection sequence produced by a factorization scheme is upper-bounded by $(n + m + p)/2$*

Proof:
It is clear that the number of rows in (3) after the ith factorization is given by $1 + m_i \leq 1 + m_{i-1} + 1 \leq 1 + m + i$, whereas the number of columns decreases, being given by $n + p - i - 1$. Now define

$$r := min\{i \mid m + i \geq n + p - i - 1\}. \tag{6}$$

Then $m_i \leq m + r$ for each i: If $i > r$, then $m_i \leq n + p - i \leq n + p - r - 1 \leq m + r$, and for $i \leq r$ one has $m_i \leq m + i \leq m + r$. Now $m + r = n + p - r - 1$ or $m + r = n + p - r$. In both cases one has

$$r \leq [n - (m - p)]/2, \tag{7}$$

and hence $m + r \leq (m + n + p)/2$. This proves the observation.

Let us consider an example in order to illustrate the implication of the above observation. Consider a 3×2 matrix $G(s)$ with degree 5. Then the factorization scheme will produce five first degree factors with the dimensions no greater than 5×5. One should realize that this is simply an upper-bound; hence the factors actually obtained may have far less dimensions than 5×5.

4 Modifications on the factorization procedure in order to reduce the interconnectivity

The upper bounds given above may not seem to be satisfactory if the dimensions of factors need to be seriously limited. This happens, for example, in the VLSI filter implementation in which an increase in dimensions of the factors would mean a considerable increase in the complexity. Recall that the connection sequence is determined as ranks of the matrix (3) constructed at each stage of the factorization and the rank of this matrix depends on the applied similarity transformation. The main idea is therefore to apply suitably chosen similarity transformations to (2) in such a way that the rank of (3) will not increase at all. Applying these additional similarity transformations to $P(s)$ to induce certain structures before going from (2) to (3) is a modification to the original procedure. Though the general implication of this modification may yield further results, some useful results have already been obtained.

Theorem 1 *If the matrix A in a minimal state-space realization of $G(s)$ has the simple structure (no Jordan-blocks), then the connection sequence produced by the modified scheme will not exceed m.*

Proof:
For square $(m \times m)$ rational matrix $G(s)$, the theorem has been shown in, for example, [2, 3]. Here we only consider the case $m > p$.

Start with (2) and assume, without loss of generality, that the first row of D is zero. If this is not the case, we shall apply a nonsingular K to $G(s)$ such that $KG(s)$ has this property, this will have no effect on the factorizability of $G(s)$. Now consider the first row of $[-c_1 \ -\bar{c}]$. Obviously, if A is first put in the diagonal form, we can always find a similarity transformation such that the system matrix $P(s)$ is in the form (2) and the first row of $[-c_1 \ -\bar{c}]$ will be of the form $[\lambda \ 0 \cdots 0]$. Consequently, the rank of the matrix (3) can at most be m. The same argument can be applied to $G_r(s)$ till the desired factorization is obtained. The proof is complete.

The above result does not seem to carry over to the general case, at least we know of examples which can not be factorized this way [2]. In spite of this fact, we can still come up with a few local tricks to be applied at each stage of the factorization. Consequently, the connection sequence obtained using these tricks may ensure that at least locally the sequence is upper-bounded by m.

Trick 1. Start with the system matrix (2), if it is possible to induce a $1 \times (1+n-1+p)$ row of the form $[\lambda \ e^T \ 0]$ $(\lambda \neq 0)$ in the matrix $[-c_1 \ -\bar{c} \ D]$ by using the row operations only, then the dimension of the factorization at this stage will not exceed m. If this is not possible for a pole α, we can shuffle all the other poles of $G(s)$ to check if it is possible.

To prove the assertion, we note that there exists a similarity transformation in this case so as to bring this row into $[\lambda \ 0 \cdots 0]$ while keeping the structure of A

unchanged. As a result, the matrix (3) will not have rank greater than m. To implement this trick, one simply row-compresses D first, then check if the elements in C, corresponding to the zero rows of D are non-zero. If so, pick up one of these rows.

Trick 2. Start with the system matrix of the following form

$$\begin{bmatrix} I_{n-1}s - A_1 & u & B_1 \\ 0 & s-\alpha & b_2^T \\ c_1 & c_2 & D \end{bmatrix} \tag{8}$$

If it is possible to induce a $(n-1+1+p) \times 1$ column of the form $[e^T \ \lambda \ 0]^T$ in the matrix $[B_1^T \ b_2 \ D^T]^T$ by using the column operations only, then the dimension of the factorization at this stage will not exceed p. If this is not possible for a pole α, all the poles of $G(s)$ can be shuffled to check if it is possible.

Note that Trick 2 is not a simple algebraic dual of Trick 1. It is a property of the matrix $[B_1^T \ b_2 \ D^T]^T$, and works independently from Trick 1.

Combining these two tricks with our original factorization procedure, we arrive at the following modified version of the scheme, which will produce a minimal first degree factorization with limited number of interconnections:

Modified Algorithm

- Step 1 : minimally realized $G(s)$ into $P(s)$;
- Step 2 : Apply a similarity transformation to bring A into upper triangular.
- Step 3 : Apply Trick 1 to the transformed system matrix. If successful, go to step 5; otherwise go to the next step;
- Step 4 : Apply Trick 2. If successful go to step 5; otherwise conclude that at this stage the dimension of the factorization may be greater than m, and go to the next step;
- Step 5 : Factorize the matrix (3) so as to obtain the factors;
- Step 6 : If n is 1, then stop; otherwise replace $G(s)$ by $G_r(s)$ and go to Step 2.

Our computation experiences with the above algorithm have shown that it almost always gives a limited interconnection factorization. It especially works well for rectangular and singular square rational matrices.

5 Conclusions

By analysing in detail the factorization procedure presented in [1], we have derived an upper-bound for the connection sequence. We have also shown that under various conditions the connect sequence will actually be less than m.

Based on the observation that similarity transformations will change the rank of the matrix (3), a modification to the original procedure is also devised. By applying this modified scheme in dealing with factorization problems, one may find it a very attractive algorithm.

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Polynomial matrix factorizations via pole-zero cancelation

P. Van Dooren

Abstract

In this paper we give an algorithm for obtaining factorizations $P(\lambda) = P_1(\lambda) \cdot P_2(\lambda)$, where $P_1(\lambda)$ and/or $P_2(\lambda)$ are polynomial matrices and $P_1(\lambda)$ is regular. Moreover the factors $P_1(\lambda)$ and $P_2(\lambda)$ are such that either the *poles* of $[P_1(\lambda)]^{-1}$ and $P_2(\lambda)$ are in a prescribed set Γ of the extended complex plane, or their *zeros*. Such factorizations cover e.g. the specific cases of coprime factorization, polynomial factor extraction and GCD extraction. The algorithm works on the state space (or generalized state space) realization of $P(\lambda)$ and derives the corresponding realizations of the factors.

1 Introduction

Several problems occurring in the literature of linear systems theory can be rephrased as a factorization problem of some rational matrix $R(\lambda)$. We consider a certain class of such factorizations, namely where the $p \times m$ rational matrix $R(\lambda)$ is factored into a product of two rational matrices :

$$R(\lambda) = R_1(\lambda) \cdot R_2(\lambda), \tag{1}$$

where $R_1(\lambda)$ is $p \times p$ regular and of minimal degree and where, given a set Γ of the extended complex plane, one of the following two conditions is satisfied :

a) the *poles* of $R_1^{-1}(\lambda)$ and $R_2(\lambda)$ lie in Γ (2.a)

b) the *zeros* of $R_1^{-1}(\lambda)$ and $R_2(\lambda)$ lie in Γ (2.b)

We give a general algorithm for obtaining such factorizations using the state space (or generalized state space) realization of $R(\lambda)$ and yielding the corresponding realizations of the factors. Applying this to problems involving polynomial matrices, one solves the following standard polynomial factorization problems :

1) Coprime factorization

Given a $p \times m$ rational matrix $R(\lambda)$, one wants to find polynomial matrices $D(\lambda)$ and $N(\lambda)$ with $D(\lambda)$ regular, such that $R(\lambda) = D^{-1}(\lambda) \cdot N(\lambda)$. This fits into the above formulation with condition (2.a), where $\Gamma = \{\infty\}$.

2) GCD extraction

Let $P_i(\lambda)$, $i, 1, \cdots, k$ be a set of polynomial matrices of dimensions $p \times m_i$, then their greatest common (left) divisor (GCD) is defined as the *regular* polynomial matrix $D(\lambda)$ such that : $P(\lambda) = D(\lambda) \cdot Q(\lambda)$ where $P(\lambda) = [P_1(\lambda), \cdots, P_k(\lambda)]$ and $Q(\lambda) = [Q_1(\lambda), \cdots, Q_k(\lambda)]$ and where the *quotients* $Q_i(\lambda)$ form together a polynomial matrix $Q(\lambda)$ with *no* Smith zeros anymore. Hence, both $D^{-1}(\lambda)$ and $Q(\lambda)$ have all their McMillan zeros at ∞, which is thus condition (2.b) with $\Gamma = \{\infty\}$.

3) Polynomial factor extraction

Given a $p \times m$ polynomial matrix $P(\lambda)$, one wants to find a $p \times p$ *regular divisor* $P_1(\lambda)$ which contains all the s of $P(\lambda)$ inside a set Γ, *not* containing the point at infinity : $P(\lambda) = P_1(\lambda) \cdot P_2(\lambda)$. The factors $P_1^{-1}(\lambda)$ and $P_2(\lambda)$ have thus their zeros in the complement