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THE USE OF THE HOMOGENEOUS FORM IN THE STUDY OF SINGULAR SYSTEMS*

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Abstract. The objective of this paper is to present an algebraic theory for analyzing the infinite structures of singular systems. The key concept used in the development is the so-called homogeneous form for polynomial and rational matrices, which can be seen as a generalization of the well-known notion of homogeneous form for matrix pencils. We show how familiar concepts such as matrix fraction description, infinite poles and zeros, irreducibility, and cancellation at infinity appear naturally in this framework. The characterizations for some of these concepts are derived in terms of homogeneous forms. Considering singular systems to be alternative representations for (possibly nonproper) rational matrices, we are able to relate the fundamental concepts of controllability and reconstructibility for singular systems to the concept of complete coprimeness for rational matrices, and to relate the concept of minimal order of singular systems to the concept of the McMillan degree of rational matrices. As an important result, the equivalence relations among the concepts of joint controllability and reconstructibility, minimal order, and complete irreducibility are established. Moreover, we show that for any rational matrix factorized in matrix fraction description, the determinantal degree of the denominator matrix is the McMillan degree of the rational matrix if and only if the matrix fraction description is completely irreducible.

1. Introduction

The theory of linear time-invariant multivariable systems developed in the past decades has mainly been concerned with causal dynamic systems. In the frequency domain, this type of dynamic systems can be described globally by *proper* rational matrices. Therefore the algebraic aspects of proper rational matrices often represent some significant dynamic properties of the systems.

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To illustrate this point, let us recall that any proper rational matrices can be alternatively represented (or to be precise, decomposed) as fractions of polynomial matrices, the so-called matrix fractions, and the zeros of the denominator matrices are known to be the dynamical modes of the corresponding systems. It is also known that some of these matrix fractions have direct physical interpretation. For example, ARMA modeling will result in a fraction of the form $D(s)^{-1}N(s)$, while the well-known state-space description can simply be viewed as a two-sided matrix fraction $C(Is - A)^{-1}B$. Algebraically, there may exist cancellation between the numerator and the denominator of a matrix fraction. This cancellation represents the cancellation of the dynamic modes of the underlying systems. When applied to the state-space representations, the absence of the cancellation is termed *controllability* and *observability*, and plays a fundamental role in the whole system theory [6], [10], [5].

Recent interests in linear system theory have been extending beyond causal systems [11], [17]. Even for causal systems, a great deal of attention has been paid to the detailed local connections of the components which constitute the entire system [9]. Such extensions of the scope, as can be expected, make the proper rational matrix representation inadequate. It becomes necessary to consider nonproper rational matrix representations. In fact, this extension is also conceivable from a purely mathematical point of view.

The systems represented as nonproper rational matrices have been shown to possess uncommon dynamics. One particular example is the existence of the impulsive modes which are so called in contrast with the more familiar exponential modes. These exotic modes have been shown by Verghese *et al.* [17] to relate to the infinite algebraic structure of the nonproper rational matrices. Actually Rosenbrock [11] for quite some time had already made the important observation that when dealing with the systems representable in nonproper rational matrices, account has to be taken of the property at infinity of the underlying rational matrices. It therefore raises the need to examine the algebraic aspect of rational matrices at infinity, and a thorough understanding of this aspect is really crucial in both understanding and control of these uncommon dynamics.

Despite the fact that extensive theories for handling the finite aspect of rational functions and matrices exist [1], [16], [3], [4], there have been relatively fewer efforts made in dealing with infinity for rational matrices. One of the well-known conventional approaches is to apply a suitably chosen conformal mapping to bring the infinity to some finite point. This approach has severe limitations, since it does not lead to a natural description of the redundancy at infinity. The purpose of this paper is, therefore, to present an algebraic theory of this nature. Unlike the conformal mapping-based approaches, the key idea of our theory is to use *the homogeneous variable to represent infinity*, thus putting all the finite points and infinity on an equal

footing. The idea of using the homogeneous variable to preserve infinity was conceived a long time ago (see, for example, [15]) and has been applied successfully to the case of matrix pencils. However, extension to the more general case, especially to rational matrices, has not been pursued. Our results can thus be seen as one of the attempts in this direction.

It is shown how the familiar concepts such as matrix fraction description, infinite poles and zeros, irreducibility, and cancellation at infinity appear naturally in this framework. The characterizations for some of these concepts are derived in terms of homogeneous forms. Considering singular system to be an alternative representation for nonproper rational matrices, we are able to relate the fundamental concepts of controllability and reconstructibility for singular systems to the concept of complete coprimeness for rational matrices, and to relate the concept of minimal order of singular systems to the concept of the McMillan degree of rational matrices. As an important result, the equivalence relations among the concepts of joint controllability and reconstructibility, minimal order, and complete irreducibility are established. Moreover, we show that for any rational matrix factorized in matrix fraction description (MFD), the determinantal degree of the denominator matrix is the McMillan degree of the rational matrix if and only if the MFD is completely irreducible.

The results in this paper combine and streamline new results described by Tan [12] and Tan and Vandewalle [14]. The paper is structured as follows. Section 2 is devoted to polynomial matrices, the notions of coprimeness at infinity and complete coprimeness are introduced, three important types of polynomial matrices are also defined. Rational matrices are considered in Section 3 in which the central concepts of irreducibility and complete irreducibility are introduced along with some interesting characterizations. Finally, Section 4 contains the conclusions.

2. Polynomial matrices at infinity

2.1. *Basic definitions*

Polynomial matrices are simply matrices with polynomial entries. Here we allow the entries of polynomial matrices to have zero leading coefficients. Note that the zero leading coefficients of these polynomials are considered to be significant, they actually define the infinite zeros for the corresponding polynomials. In order to treat these in a unified way with their finite counterparts, we have to modify some familiar definitions concerning the algebraic operations for polynomials.

First let us define what is meant by the *homogeneous form* of a polynomial. If a polynomial $p(s)$ is given by

$$p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_n, \tag{1}$$

where each coefficient p_i $(i = 0, \ldots, n)$ belongs to the real field R (it is certainly possible to consider other fields than R; for simplicity of the discussion, we confine ourselves to the real field), then the following (two-variable) homogeneous polynomial, denoted by $p(s, r)$,

$$p(s, r) = p_0 s^n + p_1 s^{n-1} r + \cdots + p_n r^n, \tag{2}$$

is called the *homogeneous form of* $p(s)$. We emphasize that $p_0 \neq 0$ is not required. The n is called the *degree* of the polynomial (1), and p_0 is the *leading coefficient* of $p(s)$. Moreover, if $p_i = 0$ $(i = 0, \ldots, n)$, then $p(s)$ is said to be a zero polynomial (so is $p(s, r)$), and can be thought to have *any* degree. Another special case is $n = 0$, this time both the polynomial $p(s)$ and its homogeneous form are simply the same real number. It should be observed that, for a given polynomial, since its degree is fixed, its homogeneous form is therefore unique. Also the concept of degree defined here should not be confused with the conventional concept which is only defined for nonzero leading coefficients.

The importance of the homogeneous form is that it forms a basis to define the set of algebraic operations among all the polynomials with, possibly, zero leading coefficients.

Definition 1. All the algebraic operations among the polynomials with, possibly, zero leading coefficients are defined as the same algebraic operations among their corresponding homogeneous forms provided that the operations result in a homogeneous polynomial.

Observe that the above definition effectively implies that polynomials of different degrees can never be considered equivalent nor can they be added.

Once the homogeneous form of a polynomial $p(s)$ is introduced we can naturally define the infinite zeros of $p(s)$ as follows. Polynomial $p(s)$ is said to have α zeros at infinity if $p(s, r) = \bar{p}(s, r) \cdot r^{\alpha}$, where $\bar{p}(s, 0) \neq 0$.

Let us now turn to polynomial matrices. The homogeneous form $P(s, r)$ for a polynomial matrix $P(s)$ is defined by replacing all its entries with their corresponding homogeneous forms. Analogously, the algebraic operations among polynomial matrices are defined as the same operations among their corresponding homogeneous forms provided that the outcomes of the operations are matrices with homogeneous entries.

The determinant of a square matrix $P(s)$ $(m \times m)$ is defined to be $\det P(s, r)$ if it is a homogeneous polynomial, where $P(s, r)$ is the homogeneous form of $P(s)$. If the result is not a homogeneous polynomial, then $P(s)$ is said to have no determinant. Obviously, not every square polynomial matrix has a determinant as a result of Definition 1. If the determinant of a polynomial matrix $P(s)$ is zero, it is said to be singular. Otherwise, it is nonsingular. For nonsingular polynomial matrices we have the following definition concerning their zeros.

Definition 2. Let the determinant of a nonsingular matrix $P(s)$ be defined. Then the zeros (finite and infinite) of $P(s)$ are defined as those of its determinant.

Nonsingular polynomial matrices with a constant determinant are called *unimodular*. A unimodular polynomial matrix apparently has no zeros. Thus multiplication with a unimodular will not change the zeros of polynomial matrices. Moreover, the inverse of a unimodular polynomial matrix is also a unimodular polynomial matrix.

Zeros can also be defined for singular, or even nonsquare, polynomial matrices. The definition, however, is not as straightforward as for nonsingular ones. Since this paper does not involve these zeros, they are discussed here.

We remark that the above definitions are only meaningful when zeros at infinity for polynomials and polynomial matrices are of concern. Actually these definitions do not contradict the familiar concepts and conventions which have been established already. In our framework, the conventional operations can be viewed as an arbitrary modification to zeros at infinity by adding or deleting *zero leading coefficient* terms in such a way that the preceding definitions can apply, since infinite zeros for polynomials (or equivalently, the zero leading coefficients) are of no importance there. For example, if infinity is not concerned, any square polynomial matrix can have a determinant by properly adjusting the zero leading coefficients of the entries.

2.2. *Three special classes of polynomial matrices*

The preceding subsection has only been concerned with arbitrary polynomial matrices. As we shall see in Section 3, only three special classes of polynomial matrices especially deserve our attention, namely, *matrix polynomials*, *column-based polynomial matrices*, and *row-based polynomial matrices*. In this subsection we describe these three classes of polynomial matrices in detail.

Matrix polynomials. If all entries of a polynomial matrix $P(s)$ $(p \times m)$ have the same degree, then $P(s)$ is called a *matrix polynomial*. It is evident that the homogeneous form $P(s, r)$ of $P(s)$ can be written as

$$P(s, r) = P_0 s^n + P_1 s^{n-1} r + \cdots + P_n r^n, \tag{3}$$

where n is the degree which is common for all entries of $P(s)$; P_i $(i = 0, \ldots, n)$ are $p \times m$ constant matrices. On setting r to 1, (3) becomes

$$P(s, 1) = P_0 s^n + P_1 s^{n-1} + \cdots + P_n. \tag{4}$$

We note that (4) can actually be looked upon as a polynomial with constant matrix coefficients. This also explains the term "matrix polynomial." In (3) or (4) P_0 is called the *leading coefficient matrix* of $P(s)$, and n is called the degree of $P(s)$.

Column-based polynomial matrices. If the degrees of entries of a polynomial matrix $Q(s)$ $(p \times m)$ in the same column are the same, then $Q(s)$ is called a *column-based polynomial matrix.* For such a matrix, the ith column $(i = 1, \ldots, m)$ of its homogeneous form $Q(s, r)$ is representable in the following form:

$$\alpha_0^i s^{k_i} + \alpha_1^i s^{k_i - 1} r + \cdots + \alpha_{k_i}^i r^{k_i}, \tag{5}$$

where, in (5), each column vector α_j^i is a constant p-vector; k_i is the degree for all entries in the ith column, and is called the ith *column degree.*

The homogeneous form of a column-based polynomial matrix can always be decomposed into the form

$$Q(s, r) = Q_{\text{hc}} S_Q(s) + Q_{\text{lc}} T_Q(s, r), \tag{6}$$

where

$$Q_{\text{hc}} = [\alpha_0^1 \quad \alpha_0^2 \quad \cdots \quad \alpha_0^m],$$

$$S_Q(s) = \text{diag}\{s^{k_1}, s^{k_2}, \ldots, s^{k_m}\},$$

$$T_Q^T(s, r) = \text{block diag}\{[s^{k_i - 1} r \quad s^{k_i - 2} r^2 \quad \cdots \quad r^{k_i}], i = 1, \ldots, m\},$$

and Q_{lc} is an appropriate constant matrix containing all coefficients of lower powers with respect to s. Matrix Q_{hc} is called the *leading coefficient matrix* of $Q(s)$. We see later that Q_{hc} plays an important role in characterizing some specific properties of the matrix.

Row-based polynomial matrices. If the degrees of the entries of a polynomial matrix $W(s)$ $(p \times m)$ in the same row are the same, $W(s)$ is called a *row-based polynomial matrix.* It is easy to see that a row-based polynomial matrix is the transpose of a column-based one. This fact enables us to decompose the homogeneous form of $W(s)$ as follows:

$$W(s, r) = S_W(s) W_{\text{hc}} + T_W^T(s, r) W_{\text{lc}}, \tag{7}$$

which can be regarded as the transpose of (6). Matrix W_{hc} is similarly defined to be the *leading coefficient matrix* of $W(s)$. We note that k_i in $S_W(s)$ or $T_W^T(s, r)$ is the ith row degree of $W(s)$.

Some relationships among these three classes of polynomial matrices can easily be established. It turns out that a matrix polynomial is a special column- (or row-) based polynomial matrix, while a row-based polynomial matrix is the transpose of a column-based one. Therefore, in most cases, it suffices to consider only column-based polynomial matrices.

An important feature associated with these three classes of matrices is that, if they are square, their determinant is well defined. Moreover, let a square nonsingular polynomial matrix $P(s)$ in one of these three classes be given, and denote by deg det $P(s, r)$ the determinantal degree of $P(s)$ and denote by

$k_i = \deg p_i(s)$ (where $p_i(s)$ is the ith column (row) of $P(s)$) $(i = 1, \ldots, m)$ the ith column (row) degree, then the following relationship must always hold:

$$\deg \det P(s, r) = \sum_{i=1}^{m} k_i. \tag{8}$$

It may be of interests to draw a comparison between (8) and the well-known inequality given, for example, on p. 384 of [5]. We can now interpret that inequality as follows. The conventional definition of the determinant of a polynomial matrix only takes the finite zeros into consideration, whereas the sum of column degrees reflects both the number of finite and the number of infinite zeros of a polynomial matrix.

Other features of these three classes of polynomial matrices can also be seen when they undergo algebraic operations. The following two lemmas present some interesting factorizations of these polynomial matrices.

Lemma 1. *If a matrix polynomial $P(s)$ can be factorized into*

$$P(s) = Q(s)R(s), \tag{9}$$

where $R(s)$ is a square matrix, then $Q(s)$ must be column-based and $R(s)$ must be row-based.

Lemma 2. *If a column-based polynomial matrix $P(s)$ can be factorized into*

$$P(s) = Q(s)R(s), \tag{10}$$

where $R(s)$ is a square matrix, then $Q(s)$ must be column-based and $R(s)$ is an arbitrary polynomial matrix whose determinant is defined.

These lemmas follow directly from the definitions of operations. To show, for example, Lemma 2, we simply recall the relation $\det P(s, r) = \det Q(s, r) \cdot \det R(s, r)$. Since both $\det P(s, r)$ and $\det Q(s, r)$ are homogeneous, $\det R(s, r)$ must also be homogeneous, which implies that $R(s)$ has the determinant.

We shall see in Section 3 that polynomial matrices of these classes have a special significance in the context of rational matrices. Therefore, we hereby assume that all the polynomial matrices considered fall into these classes.

2.3. *Coprimeness at infinity and complete coprimeness*

Given two polynomial matrices $D(s)$ $(p \times p)$, $N(s)$ $(m \times p)$, and their respective homogeneous forms $D(s, r)$, $N(s, r)$, it is assumed that the corresponding column degrees of the two polynomial matrices are the same. If $D(s, r) = \bar{D}(s, r)R(s, r)$ and $N(s, r) = \bar{N}(s, r)R(s, r)$, where $\bar{D}(s, r)$, $\bar{N}(s, r)$ are column-based polynomial matrices, and $R(s, r)$ is a square nonsingular polynomial matrix, then $R(s, r)$ is called a *common right divisor* of $D(s)$ and $N(s)$.

If we are only considering column-based polynomial matrices, then the determinant of a right common divisor for such matrices exists (recall Lemma 2). We call such a divisor *nontrivial*, if its determinant is not a constant.

Definition 3. A nonsingular matrix $R(s, r)$ is called a common right divisor at infinity for polynomial matrices $D(s)$ and $N(s)$, if

(1) it is a common right divisor of $D(s)$ and $N(s)$;
(2) $\det R(s, r) = \alpha \cdot r^{\sigma}$, where α is a constant scalar different from zero, and σ is a positive integer.

The above definition allows us to introduce the notion of *coprimeness at infinity* for two polynomial matrices.

Definition 4. Two polynomial matrices with the same number of columns are said to be right coprime at infinity if they have no common right divisors at infinity.

An interesting characterization of this concept can be stated as a rank condition for a constant matrix constructed with the leading coefficient matrices of both $D(s)$ and $N(s)$.

Theorem 1. *Let $D(s)$ and $N(s)$ be two column-based polynomial matrices of dimensions $p \times p$ and $m \times p$, respectively. It is further assumed that they have the same corresponding column degrees. Then $D(s)$ and $N(s)$ are right coprime at infinity if and only if*

$$\operatorname{rank}\begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p, \tag{11}$$

where D_{hc} and N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

The proof for the theorem is lengthy and thus omitted. Interested readers are referred to [12] and [14].

The notion of (finite) coprimeness for two polynomial matrices is well known (see, e.g., [10]). Here we simply recall that two polynomial matrices are coprime if their greatest common divisor is a unimodular matrix (it is assumed that common divisors are always nonsingular). One characterization of this concept can be given as a rank condition of an appropriate matrix. A combination of this concept together with the concept of coprimeness at infinity leads to the following definition.

Definition 5. Two polynomial matrices are said to be completely right coprime if they are both right coprime and right coprime at infinity.

Based on Theorem 1 and the characterization of the right coprimeness, an interesting characterization of the complete coprimeness can be stated.

Theorem 2. *Let $D(s)$ and $N(s)$ be two column-based polynomial matrices. Then $D(s)$ and $N(s)$ are completely right coprime if and only if*

$$\operatorname{rank}\begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p$$

and

$$\operatorname{rank}\begin{bmatrix} D(s, r) \\ N(s, r) \end{bmatrix} = p, \qquad \textit{for all finite } s,$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

3. Rational matrices at infinity

3.1 *Basic definitions*

A rational matrix $G(s)$ of dimension $m \times p$ is simply a matrix whose entries are rational functions (to be defined next).

Definition 6. A rational function $g(s)$ is defined to be $b(s)/a(s)$, where $a(s)$, $b(s)$ are polynomials satisfying $\deg a(s) = \deg b(s)$, and $a(s) \neq 0$. The poles of $g(s)$ finite and infinite) are the zeros (finite and infinite) of $a(s)$; the zeros of $g(s)$ (finite and infinite) are the zeros (finite and infinite) of $b(s)$.

The definition of rational functions is based on a simple fact that when taking infinity into account, any rational function will have the same number of poles and zeros. This fact is reflected by the degree constraint in Definition 6. A point which should be emphasized is that a polynomial is not considered to be a rational function. A polynomial $p(s) = a_0 s^n + \cdots + a_n$ should not be confused with the rational function $g(s) = (a_0 s^n + \cdots + a_n)/(0 \cdot s^n + \cdots + 1)$.

It is easy to show that a rational matrix can in fact be represented alternatively as a right MFD of two column-based polynomial matrices which have the same corresponding column degrees. A rational matrix represented in terms of MFD is more appropriate for introducing poles of rational matrices.

Definition 7. Let a rational matrix $G(s)$ be given in right MFD

$$G(s) = N(s)D^{-1}(s).$$

Then its poles (finite and infinite) are defined to be the zeros of $\det D(s)$.

The notion of zeros entails a bit more thought. In fact, if $N(s)$ is a nonsingular matrix, then the zeros of $G(s)$ are defined to be those of $\det N(s, r)$. If it is singular or nonsquare, then the definition of zeros will involve the concept of generalized invariant polynomials. We shall, however, not go any further in this direction. The above definition of rational matrices is only stated for the right matrix fraction and the column-based polynomial matrices. We can analogously give a definition based on the left matrix fraction and the row-based polynomial matrices. We remark that the poles for any right MFD of a rational matrix basically consist of two sets: those which can be cancelled by the zeros and those which cannot be cancelled. The former set of poles is subject to cancellation, and when realized into the so-called *singular system representation* (to be introduced in the next subsection), they are precisely the set of decoupling zeros defined by Verghese *et al.* [17]. The second set of poles determines the nature of the rational matrix, and is thus called *the essential poles of the rational matrix*. It will be shown in the last section that the number of essential poles for a rational matrix is in fact the McMillan degree of that rational matrix.

When a rational matrix $G(s)$ is represented in right MFD, say $N(s)D^{-1}(s)$, the numerator $N(s)$ and the denominator $D(s)$ may have common right factors at infinity. The absence of such factors is termed *irreducibility at infinity*. Moreover, the absence of any common divisor (finite and infinite) is called *complete irreducibility*. We now state the characterizations for each of the two concepts.

Theorem 3. *Let an $m \times p$ rational matrix $G(s)$ be given by*

$$G(s) = N(s)D^{-1}(s), \tag{12}$$

where $N(s)$, $D(s)$ are column-based polynomial matrices. Then $G(s)$ is irreducible at infinity if and only if

$$\operatorname{rank}\begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p, \tag{13}$$

in which D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

Theorem 4. *Let $G(s)$ be given by* (12). *It is completely irreducible if and only if, in addition to* (13), *the following is also true*:

$$\operatorname{rank}\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = p, \qquad \textit{for all finite } s.$$

3.2. *Singular system representations*

It is well known that a proper rational matrix in MFD can always be realized into the state-space description. An equivalent statement in the framework of nonproper rational matrices is that any MFD defined in the previous subsection can be realized into the so-called *singular system representation.*

Let us start with the general form of singular system representation

$$G(s) = C(Es - A)^{-1}B + D, \tag{14}$$

where, as usual, $x \in R^n$, $u \in R^p$, and $y \in R^m$; E and A are matrices of dimensions $n \times n$, B and C are matrices of dimensions $n \times p, m \times n$, respectively. Moreover, n is defined as the dimension of the system, and $f = \text{rank } E$ is called the order of the system. We can see that this representation is simply a two-sided matrix fraction.

As an alternative of (14), the following system matrix introduced by Rosenbrock [10] is often used:

$$\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix}. \tag{15}$$

With (15) we define two singular systems to be *strongly equivalent* if one can be transformed to the other by any combination of the following three operations [17]:

1. *Left and right multiplication*

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix}\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix}\begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix}, \tag{16}$$

   where $QE = 0$ and $ER = 0$.
2. *Trivial augmentation* from (15) to

$$\begin{bmatrix} Es - A & 0 & B \\ 0 & I & 0 \\ -C & 0 & D \end{bmatrix}. \tag{17}$$

3. *Trivial deflation*, namely from (17) to (15).

It is easy to see that the transformations defined above will not change the order $f(= \text{rank}E)$ of a singular system.

Here we introduce a few more basic definitions concerning the singular system representation. To begin with, consider the following matrix:

$$[Es - A \quad B]. \tag{18}$$

By *str. eq.* operations, (18) can be brought to

$$\begin{bmatrix} E_\varepsilon s - A_\varepsilon & B_\varepsilon \\ A_\pi & B_\pi \end{bmatrix} \tag{19}$$

with E_ε of full row rank. Similarly, the matrix

$$\begin{bmatrix} Es - A \\ -C \end{bmatrix} \tag{20}$$

can be brought, by *str. eq.* operations, to

$$\begin{bmatrix} E_\eta s - A_\eta & A_\xi \\ -C_\eta & -C_\xi \end{bmatrix} \tag{21}$$

with E_η of full column rank. We now state the following definition.

Definition 8. Singular system (14) is said to be finitely controllable if

$$\operatorname{rank}[Es - A \quad B] = n \tag{22}$$

for all finite s, and controllable at infinity if

$$\operatorname{rank}\begin{bmatrix} E_\varepsilon & 0 \\ A_\pi & B_\pi \end{bmatrix} = n, \tag{23}$$

where the block matrix in (23) is built with the corresponding block elements in (19). It is said to be controllable if it is both finitely controllable and controllable at infinity.

Matrix (15) is said to be finitely reconstructible if

$$\operatorname{rank}\begin{bmatrix} Es - A \\ -C \end{bmatrix} = n \tag{24}$$

for all finite s, and reconstructible at infinity if

$$\operatorname{rank}\begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} = n, \tag{25}$$

where the block matrix in (25) is built with the corresponding block elements in (21). It is said to be reconstructible if it is both finitely reconstructible and reconstructible at infinity.

These definitions have been developed by Verghese *et al.* [17] along with the dynamical implications of controllability and reconstructibility. More complete treatments of these concepts can also be found, for example, in [2], [8], and [7]. In fact, as we see later, these are precisely the algebraic conditions which guarantee the absence of the common factors (finite and at infinity).

3.3 *General properties of completely irreducible MFDs*

We present a method to realize a right MFD into the singular system representation. Before doing so, however, let us define what is meant by a *realization.*

First, apply a left and right multiplication (16) such that the nonsingular matrix M row-compresses E, $Q = R = 0$, and $N = I$. The system of (14) is transformed into

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_\varepsilon s - A_\varepsilon & B_\varepsilon \\ -A_\pi & B_\pi \\ -C & D \end{bmatrix}, \tag{26}$$

where E_ε is of full row rank. Then a given MFD $G(s) = N(s)D^{-1}(s)$ is said to be realized by (14) if

$$N(s, r)D^{-1}(s, r) = C\begin{bmatrix} E_\varepsilon s - A_\varepsilon r \\ -A_\pi \end{bmatrix}^{-1}\begin{bmatrix} B_\varepsilon \\ B_\pi \end{bmatrix} + D. \tag{27}$$

The above definition is based on row-compression. It is, of course, possible to present the definition with respect to column-compression. It should be noted that (27) also defines the *transfer matrix* of a singular system.

To start with the realization algorithm, let a column-based right MFD be given and let k_i $(i = 1, \ldots, p)$ be the ith column degree of $N(s)$ $(m \times p)$ and $D(s)$ $(p \times p)$.

As the first step of the construction, define the following matrix:

$$\Gamma^{\mathrm{T}}(s, r) = \{[s^{k_i} \quad s^{k_i-1}r \quad \cdots \quad sr^{k_i-1} \quad r^{k_i}],\ i = 1, \ldots, p\}, \tag{28}$$

and write $D(s, r)$ and $N(s, r)$ in the form

$$\begin{aligned} D(s, r) &= D_{\mathrm{c}}\Gamma(s, r), \\ N(s, r) &= N_{\mathrm{c}}\Gamma(s, r), \end{aligned} \tag{29}$$

where D_{c} and N_{c} are appropriate constant matrices.

Secondly, construct the *core realization* of the form

$$E_{\mathrm{c}}s - A_{\mathrm{c}} = \text{block diag}\left\{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix},\ (k_i + 1) \times (k_i + 1),\ i = 1, \ldots, p\right\}, \tag{30}$$

$$B_{\mathrm{c}}^{\mathrm{T}} = \text{block diag}\{[1 \quad 0 \quad \cdots \quad 0],\ 1 \times (k_i + 1),\ i = 1, \ldots, p\},$$

$$C_{\mathrm{c}} = I_n, \qquad n = \sum_{i=1}^{p} k_i + p.$$

Finally, the desired realization follows from (29) and (30):

$$E = E_c, \qquad A = A_c - BD_c, \qquad B = B_c, \qquad C = N_c \cdot C_c = N_c. \quad (31)$$

Next, we prove that (31) indeed realizes the given right MFD. Then some interesting properties of the realization are shown.

Theorem 5. *The singular system in* (31) *realizes the given column-based MFD.*

Proof. It is easily checked that

$$(Es - A_c r + BD_c)\Gamma(s, r) = BD_c\Gamma(s, r) = BD(s, r)$$

or, equivalently,

$$(Es - A_c r + BD_c)^{-1}B = \Gamma(s, r)D^{-1}(s, r). \quad (32)$$

(The nonsingularity of $Es - A_c r + BD_c$ is proved later.) The theorem is proved by premultiplying (32) with $C\ (= N_c)$ and by observing that $N(s, r) = N_c\Gamma(s, r)$. □

Theorem 6. *The realization* (31) *has the following properties*:

(1) $\det(Es - A_c r + BD_c) = \det D(s, r)$;
(2) *it is controllable.*

These two properties can be readily checked by direct computation. We omit the details. Observe that the second property guarantees nonsingularity of the matrix $Es - A_c r + BD_c$, which is needed in the proof of Theorem 5. Moreover, this property also implies that the order of the realization is equal to the determinantal degree of the denominator of the transfer matrix.

It has been shown by Verghese *et al.* [17] that the order f attains its minimal value if and only if the system is jointly controllable and reconstructible. Here we wish to show that the concept of complete irreducibility and joint controllability and reconstructibility are also linked.

We make use of the realization (31) as a primary vehicle to establish the relationship. First we show that the desired relationship holds for this particular realization.

Theorem 7. *The realization* (31) *is reconstructible if and only if the MFD* $N(s)D^{-1}(s)$ *is completely irreducible.*

Proof. Only a proof for the equivalence between the reconstructibility at infinity of the realization and the irreducibility at infinity of an MFD is given. The finite part is well known in the literature and thus omitted.

From Definition 8, the realization (31) is reconstructible at infinity if and

only if the matrix

$$\begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} \tag{33}$$

has full column rank, where the block elements in (33) are defined in (25). From (31), it is easily seen that (33) can be transformed, by elementary transformations, to the following matrix:

$$\begin{bmatrix} D_{hc} & 0 \\ 0 & I \\ N_{hc} & 0 \end{bmatrix},$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively. The above matrix is of full column rank if and only if $D(s)$ and $N(s)$ are irreducible at infinity. The theorem thus follows. □

With Theorem 7 in mind, the rest of the development follows quite straightforwardly.

Theorem 8. *If a column-based MFD has one controllable and reconstructible fth-order realization, then all its fth-order realization must also be controllable and reconstructible.*

Proof. Otherwise, by using the fourfold decomposition developed in [17], this would imply that f is not the minimal order, which clearly contradicts the conclusion in the preceding subsection. This proves the theorem. □

We are now in a position to state and prove the main theorem of this subsection.

Theorem 9. *A column-based right MFD is completely irreducible if and only if all of its nth-order realizations are controllable and reconstructible, where n is the determinantal degree of the denominator $D(s)$, i.e., $n = \deg \det D(s, r)$.*

Proof. In view of Theorem 8, it suffices to show that the theorem holds for a special nth-order realization. But Theorem 7 indicates that the realization (31) satisfies this condition. □

It has been shown in [17] that the order of a singular system will equal the McMillan degree of the underlying transfer matrix if and only if the singular system is controllable and reconstructible. Combining this fact with Theorem 9 we obtain the following important result.

Theorem 10. *A column-based right MFD is completely irreducible if and only if the determinantal degree of its denominator is equal to the McMillan degree of the MFD.*

This theorem is very significant since it provides an alternative, yet the simplest, interpretation for the well-known McMillan degree of rational matrices, namely, the McMillan degree only describes the completely irreducible part of rational matrices. It is now clear that when a rational matrix is factorized in, say MFD, some redundancy (finite and infinite) may be introduced. We can then work out the cancellation. As a consequence, the determinantal degree of the denominator of the MFD will decrease. As soon as the MFD becomes completely irreducible the determinantal degree attains the minimal value, which is precisely the McMillan degree of the MFD.

4. Conclusions

A general theory is presented which relates the infinite structure of rational matrices to that of singular systems. The results obtained appear to have generalized some well-known facts on proper rational matrices. The theory also sets up a general framework for the interpretation of some of our early results [13].

The emphasis of the development is placed on the descriptions of the so-called irreducibility at infinity and complete irreducibility. It is shown that these new concepts are consistent with the conventional definition of the McMillan degree. The algebraic framework developed here is believed to be able to provide a better understanding of the dynamical behavior at infinity of singular systems. It is also noted that the entire treatment in this paper is confined to the framework set up by Verghese *et al.* [17]. In fact, a parallel development within the framework of Rosenbrock can be found in [12] in which the dynamical concepts of reachability and observability are interpreted in terms of the pseudoproperness concept.

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