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Abstract

It is shown how to generalize the *Ordinary Singular Value Decomposition* of one matrix into a combined factorization of any number of matrices. We propose to call these factorizations *Generalized Singular Value Decompositions.* For two matrices, this reduces to the *Product* and *Quotient Singular Value Decomposition.* One of the factorizations for three matrices is the *Restricted Singular Value Decomposition.* These generalizations form a tree of factorizations, where at level k, for k matrices, there are 2^k factorizations, not all of which are independent. The different levels are related to each other in a recursive fashion. Any *Generalized Singular Value Decomposition* for k matrices, can be constructed from a decomposition for $k-1$ matrices. This results in an inductive proof which only uses the *Ordinary Singular Value Decomposition.* Several examples are analysed in detail.

Keywords: Singular Value Decomposition, Product SVD, Quotient SVD, Restricted SVD.

1 Introduction

The *ordinary singular value decomposition* (**OSVD**) has become an important tool in the analysis and numerical solution of numerous problems (see e.g. [6] [8] for properties and applications.) Not only does it allow for an elegant problem formulation, but at the same time it provides geometrical and algebraic insight together with an immediate numerically robust implementation [8]. In [11, p.78], credit for the first proofs of the **OSVD** is given to Beltrami [2], Jordan [10], Sylvester [13] and Autonne [1].
Recently, several generalizations to the **OSVD** have been proposed and their properties analysed. The best known example is the *Generalized SVD* for two matrices, as introduced in [14] and refined in [12], which we propose to rename as the *Quotient SVD* (**QSVD**) [4]. A specific reason for this name is the relation of this matrix factorization to the SVD of the 'quotient' of two matrices while the main motivation is of course the fact that there are several other similar generalizations. For instance, a *Product induced SVD*, also for two matrices, was proposed in [7], where it was called the ΠSVD. It was a formalization of ideas in [9]. We shall refer to it as the **PSVD** (see [4]). In [15], another generalization, this time for three matrices, was proposed. In [3] we have called it the *Restricted SVD* (**RSVD**) and analysed its properties in detail.

One of the main results of this paper is the fact that all of these decompositions can be organised into a tree of generalizations of the **OSVD**. The **OSVD** resides at the top of this tree. The **PSVD** and **QSVD** are two different decompositions for two matrices at the next level of generality. The **RSVD** is one of the four possible factorizations for three matrices. It will be shown that only three of these are theoretically essential. Similar generalizations for 4, 5, etc ... matrices are obtained in a completely structured manner.

The main idea can be easily understood from the special case of *real nonsingular* $n \times n$

matrices $A, B, C, \ldots$. At the top of the tree, we have the **OSVD** of A. At the next level we have the **OSVD** of AB and AB^{-t}. Note that we use B^{-T} rather than B^{-1}, so that in the general case of rectangular matrices, all dimensions will be compatible. The **OSVD** of AB leads to the **PSVD** of the matrix pair (A, B) as follows. Consider the **OSVD** of the product AB and insert a nonsingular matrix X and its inverse in order to obtain a combined factorization of the two matrices:

$$AB = USV^t = UI_nX^{-1}XSV^t \Longrightarrow \begin{array}{rcl} A & = & UI_nX^{-1} \\ B & = & XSV^t \end{array}$$

Our main Theorem states that such a common factor in the decompositions of A and B can always be found.
The **QSVD** of the matrix pair follows from the **OSVD** of AB^{-t} in a similar fashion as:

$$AB^{-t} = USV^t = UI_nX^{-1}XSV^t \Longrightarrow \begin{array}{rcl} A & = & UI_nX^{-1} \\ B & = & X^{-t}S^{-t}V^t \end{array}$$

where again, X is a nonsingular matrix. The factor X in the **PSVD** differs completely from the one in the **QSVD**, as do the algebraic, structural and geometric properties of both decompositions.
At the next level, we have the **OSVD**s of products and quotients for three matrices, ABC, ABC^{-t}, $AB^{-t}C^{-t}$ and $AB^{-t}C$. For instance, from the **OSVD** of $AB^{-t}C^{-t}$ we find, by inserting nonsingular matrices X and Y and their inverses, a decomposition of the matrix triplet (A, B, C) as:

$$AB^{-t}C^{-t} = USV^t = UX^{-1}XY^tY^{-t}SV^t \Longrightarrow \begin{array}{rcl} A & = & UI_nX^{-1} \\ B & = & X^{-t}I_nY^{-1} \\ C & = & YS^{-t}V^t \end{array} \qquad (1)$$

We will define a terminology such that a combination between two matrices F and G is of *type* P if they occur as FG or $F^{-t}G^{-t}$ and of *type* Q if they occur as FG^{-t} or $F^{-t}G$. Here P stands for 'product' (note that $F^{-t}G^{-t} = (FG)^{-t}$) and Q stands for 'quotient'. The decomposition of (1) will therefore be called a **QP-SVD**. The remaining factorizations for 3 matrices result then in a **PP-SVD** for ABC, a **PQ-SVD** for ABC^{-t} and a **QQ-SVD** for $AB^{-t}C$.
Note that the **QP-SVD** of (A, B, C) can be found from the **PQ-SVD** of (C^t, B^t, A^t). Hence, for 3 matrices, there are only three *theoretically distinct* generalized SVDs.
It is clear that we could find similar factorizations for $4, 5, etc\ldots$ real nonsingular matrices.

One contribution of this paper is to show how to obtain (at least theoretically) explicit factorizations of the *individual* matrices $A, B, C, \ldots$ corresponding to an **OSVD** of products and quotients of these matrices. Much of the complexity of this paper is introduced in order to handle fully general matrices of compatible dimensions which can be rectangular and possibly rank deficient. In this case, the inverse in a quotient is to be replaced with some generalized inverse. If the reader feels uncertain about our general derivation, it might be

instructive to return to the real nonsingular case.

This paper is organised as follows:

- The main Theorem is stated in section 2 and some nomenclature is proposed.
- A block factorization lemma is proved in section 3. It is one of the key results that will be needed.
- A constructive proof of the main Theorem is given in section 4.

Notation and abbreviations

Throughout the paper, matrices are denoted by capitals, vectors by lower case letters other than $i, j, k, l, m, n, p, q, r, s$, which are nonnegative integers. Scalars (complex) are denoted by Greek letters. The notation A $(m \times n)$ refers to a complex matrix with m rows and n columns. The conjugate of A is $\overline{A}$ while the complex conjugate transpose is $A^* = conj(A^t)$. A^{-*} is the inverse of A^* (nonsingular). I_k is the $k \times k$ identity matrix. a_i is the i-th column of the matrix A. We shall denote different matrices with subscripts as $A_1, A_2, A_3, \ldots$. Quantities related to the j-th matrix will have a subscript j. For instance, the rank of A_j will be denoted by r_j. We will adopt the following convention for block matrices: Any (possibly rectangular) block of zeros is denoted by 0, the precise dimensions being obvious from the context. The symbol I is an identity matrix of appropriate dimensions. Whenever a dimension indicating integer in a block matrix is negative or zero, the corresponding block row or block column should be omitted and all expressions and equations in which a block matrix of that block row or block column appears, can be discarded. An equivalent formulation would be that we allow $0 \times n$ or $n \times 0$ $(n \neq 0)$ to appear in matrices. This allows an elegant general treatment of several cases at once. The following abbreviations will be used throughout: **GSVD**: Generalized Singular Value Decomposition, **OSVD**: Ordinary Singular Value Decomposition, **PSVD**: Product Singular Value Decomposition, **QSVD**: Quotient Singular Value Decomposition, **RSVD**: Restricted Singular Value Decomposition.

2 The Main Theorem

Theorem 1
Generalized Singular Value Decompositions for k matrices.
Consider a set of k matrices with compatible dimensions: A_1 $(n_0 \times n_1), A_2$ $(n_1 \times n_2), \ldots,$ A_{k-1} $(n_{k-2} \times n_{k-1}), A_k$ $(n_{k-1} \times n_k)$. *Then there exist*

- *Unitary matrices* U_1 $(n_0 \times n_0)$ *and* V_k $(n_k \times n_k)$

- *Matrices* $D_j, j = 1,2,\ldots,(k-1)$ *of the form*

$$
\underset{n_{j-1}\times n_j}{D_j} = \begin{array}{c} \\ r_j^1 \\ r_{j-1}^1 - r_j^1 \\ r_j^2 \\ r_{j-1}^2 - r_j^2 \\ r_j^3 \\ \ldots \\ \ldots \\ r_j^j \\ n_{j-1} - r_{j-1} - r_j^j \end{array} \begin{array}{c} \begin{array}{ccccccc} r_j^1 & r_j^2 & r_j^3 & \ldots & \ldots & r_j^j & n_j - r_j \end{array} \\ \begin{pmatrix} I & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & I & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & \ldots & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & I & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \end{array} \tag{2}
$$

where

$$
r_0 = 0 \ , \ r_j = \sum_{i=1}^{j} r_j^i = rank(A_j) \tag{3}
$$

- *A matrix* S_k *of the form*

$$
\underset{n_{k-1}\times n_k}{S_k} = \begin{array}{c} \\ r_k^1 \\ r_{k-1}^1 - r_k^1 \\ r_k^2 \\ r_{k-1}^2 - r_k^2 \\ r_k^3 \\ \ldots \\ \ldots \\ r_k^k \\ n_{k-1} - r_{k-1} - r_k^k \end{array} \begin{array}{c} \begin{array}{ccccccc} r_k^1 & r_k^2 & r_k^3 & \ldots & \ldots & r_k^k & n_k - r_k \end{array} \\ \begin{pmatrix} S_k^1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & S_k^2 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & S_k^3 & \ldots & \ldots & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & S_k^k & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \end{array} \tag{4}
$$

where

$$
r_k = \sum_{i=1}^{k} r_k^i = rank(A_k) \tag{5}
$$

and the $r_k^i \times r_k^i$ *matrices* S_k^i *are diagonal with positive diagonal elements. The integers* r_j^i *are ranks of certain matrices as given in the constructive proof in section 4.*

- *Nonsingular matrices* X_j $(n_j \times n_j)$ *and* Z_j, $j = 1,2,\ldots,(k-1)$ *where* Z_j *is either* $Z_j = X_j^{-*}$ *or either* $Z_j = X_j$ *(i.e. both choices are always possible)*

such that the given matrices can be factorized as

$$
\begin{aligned}
A_1 &= U_1 D_1 X_1^{-1} \\
A_2 &= Z_1 D_2 X_2^{-1} \\
A_3 &= Z_2 D_3 X_3^{-1} \\
\ldots &= \ldots
\end{aligned}
$$

$$\begin{aligned} A_i &= Z_{i-1} D_i X_i^{-1} \\ \ldots &= \ldots \\ A_k &= Z_{k-1} S_k V_k^* \end{aligned}$$

Observe that the matrices D_j in (2) and S_k in (4) are in general not diagonal. Their only non-zero blocks however are diagonal block matrices. We propose to label them as *quasi-diagonal* matrices. The matrices $D_j, j = 1, \ldots, k-1$ are quasi-diagonal, their only nonzero blocks being identity matrices. The matrix S_k is quasi-diagonal and its nonzero blocks are diagonal matrices with positive diagonal elements. Observe that we always take the last factor in every factorization as the inverse of a nonsingular matrix, which is only a matter of convention (Another convention would result in a modified definition of the matrices Z_i). As to the name of a certain **GSVD**, we propose to adopt the following convention:

Definition 1
The nomenclature for GSVDs
If $k = 1$ in Theorem 1, then the corresponding factorization of the matrix A_1 will be called the ordinary singular value decomposition.
If for a matrix pair $A_i, A_{i+1}, 1 \leq i \leq k-1$ in Theorem 1, we have that

$$Z_i = X_i$$

then, the factorization of the pair will be said to be of P-type.
If on the other hand, for a matrix pair $A_i, A_{i+1}, 1 \leq i \leq k-1$ in Theorem 1, we have that

$$Z_i = X_i^{-*}$$

the factorization of the pair will be said to be of Q-type.
The name of a **GSVD** *of the matrices A_i, $i = 1, 2, \ldots k > 1$ as in Theorem 1, is then obtained by simply enumerating the different factorization types.*

Let us give some examples.

Example 1:
Consider two matrices A_1 $(n_0 \times n_1)$ and A_2 $(n_1 \times n_2)$. Then, we have two possible **GSVDs**:

	P-type	Q-type
A_1	$U_1 D_1 X_1^{-1}$	$U_1 D_1 X_1^{-1}$
A_2	$X_1 S_2 V_2^*$	$X_1^{-*} S_2 V_2^*$

The P-type factorization corresponds to the **PSVD** as in [7] (called ΠSVD there) and [5], while the Q-type factorization is nothing else than the **QSVD** in [8] [12] [14] (called generalized SVD there). This justifies the choice of names for the factorization of pairs: A P-type factorization is precisely the kind of transformation that occurs in the **PSVD**

while a Q-type factorization occurs in the **QSVD** [1].

Example 2
The **RSVD** for three matrices (A_1, A_2, A_3) as introduced and analysed in [3] [15] has the form:

$$\begin{aligned} A_1 &= U_1 S_1 X_1^{-1} \\ A_2 &= X_1^{-*} S_2 X_2^{-1} \\ A_3 &= X_2^{-*} S_3 V_3^* \end{aligned}$$

where S_1, S_2, S_3 are certain *quasi-diagonal* matrices. It can be verified that this **RSVD** can be rearranged into a **QQ-SVD** that is conform with the structure of Theorem 1.

Example 3
Let us write down the **PQQP-SVD** for 5 matrices, together with the structure of the matrices D_i, $i = 1, 2, 3, 4$, and S_5 (we have omitted the block dimensions).

$$A_1 = U_1 \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} X_1^{-1} \qquad A_2 = X_1 \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} X_2^{-1}$$

$$A_3 = X_2^{-*} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} X_3^{-1} \qquad A_4 = X_3^{-*} \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X_4^{-1}$$

$$A_5 = X_4 \begin{pmatrix} S_5^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_5^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S_5^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_5^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_5^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} V_5^*$$

[1] The interested reader may wish to consult [4], where we have proposed a standardized and mnemonic nomenclature for the several possible generalizations of the singular value decomposition

We also introduce the following notation, using powers, which symbolize a certain repetition of a letter or of a sequence of letters:

- $\mathbf{P^3Q^2}$**-SVD** = **PPPQQ-SVD**
- $\mathbf{(PQ)^2Q^3(PPQ)^2}$**-SVD** = **PQPQQQQPPQPPQ-SVD**

Despite the fact that there are 2^{k-1} different sequences of letters **P** and **Q** at level $k > 1$, not all of these sequences correspond to different **GSVDs**. The reason for this is that for instance the **QP-SVD** of (A^1, A^2, A^3) can be obtained from the **PQ-SVD** of $((A^3)^*, (A^2)^*, (A^1)^*)$. Similarly, the $\mathbf{P^2(QP)^3}$**-SVD** of $(A^1, \ldots, A^9)$ is essentially the same as the $\mathbf{(PQ)^3P^2}$**-SVD** of $((A^9)^*, \ldots, (A^1)^*)$. The following table gives the number of *different* factorizations for k matrices.

	k even	k odd
number of different **GSVDs**	$\frac{1}{2}(2^{k-1} + 2^{k/2})$	$\frac{1}{2}(2^{k-1} + 2^{(k-1)/2})$

A possible way to visualize Theorem 1 is to build a tree with all *different* factorizations for 1, 2, 3, etc ... matrices as follows:

$$
\begin{array}{lccccccccccccc}
1 & & & & & & & \mathbf{O} & & & & & & \\
2 & & & & & & \mathbf{P} & & \mathbf{Q} & & & & & \\
3 & & & & & \mathbf{P^2} & & \mathbf{PQ} & & \mathbf{Q^2} & & & & \\
4 & & \mathbf{P^3} & & \mathbf{P^2Q} & & \mathbf{PQP} & & \mathbf{PQ^2} & & \mathbf{QPQ} & & \mathbf{Q^3} & \\
\ldots & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}
$$

3 A block factorization lemma

In this section, we shall construct a factorization of a block matrix into three factors that have some special properties: The first factor is lower block triangular, the second one is quasi-diagonal and the third one is unitary. This result will be of key importance in our constructive proof of the **GSVD**, which will be developed in section 4.

Lemma 1
On the factorization of a block matrix
Consider a $p \times q$ matrix B with k block rows:

$$
B = \begin{array}{c} \\ p_1 \\ p_2 \\ \ldots \\ p_k \end{array}
\begin{array}{c} q \\ \begin{pmatrix} B_1 \\ B_2 \\ \ldots \\ B_k \end{pmatrix} \end{array}
$$

where $\sum_{i=1}^{k} p_i = p$. *The matrix* B *can be factorized as*

$$B = TSV^*$$

where

- T *is a* $p \times p$ *block lower triangular matrix*

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ \dots \\ p_k \end{array} \begin{array}{c} \begin{array}{cccccccccc} r_1 & p_1 - r_1 & r_2 & p_2 - r_2 & r_3 & p_3 - r_3 & \dots & \dots & r_k & p_k - r_k \end{array} \\ \begin{pmatrix} T_{11} & T_{12} & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ T_{21} & 0 & T_{23} & T_{24} & 0 & 0 & \dots & \dots & 0 & 0 \\ T_{31} & 0 & T_{32} & 0 & T_{35} & T_{36} & \dots & \dots & 0 & 0 \\ \dots & 0 & \dots & 0 & \dots & 0 & \dots & \dots & 0 & 0 \\ T_{k1} & 0 & T_{k2} & 0 & T_{k5} & 0 & \dots & \dots & T_{k\,2k-1} & T_{k\,2k} \end{pmatrix} \end{array}$$

and the $p_i \times p_i$ *matrices* $(T_{i\,(2i-1)} \;\; T_{i\,(2i)})$ *are unitary.*

- S *is a* $p \times q$ *quasi-diagonal matrix with the following block structure:*

$$S = \begin{array}{c} \\ r_1 \\ p_1 - r_1 \\ r_2 \\ p_2 - r_2 \\ r_3 \\ p_3 - r_3 \\ \dots \\ r_k \\ p_k - r_k \end{array} \begin{array}{c} \begin{array}{ccccccc} r_1 & r_2 & r_3 & \dots & \dots & r_k & q - \sum_{i=1}^{k} r_i \end{array} \\ \begin{pmatrix} S^1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & S^2 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & S^3 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & S^k & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix} \end{array}$$

and the $r_i \times r_i$ *matrices* S^i *are diagonal with positive diagonal elements.*

- V *is a* $q \times q$ *unitary matrix.*
- *The integers* r_i *are defined recursively as*

$$\begin{aligned} r_1 &= rank(B_1) \\ r_i &= rank \begin{pmatrix} B_1 \\ B_2 \\ \dots \\ B_i \end{pmatrix} - rank \begin{pmatrix} B_1 \\ \dots \\ B_{i-1} \end{pmatrix} \qquad i = 2, 3, \dots, k \end{aligned}$$

and satisfy

$$\sum_{i=1}^{k} r_i = rank(B)$$

Proof: Let the **OSVD** of B_1 be

$$B_1 = U_1 S_1 V_1^* = (\ U_1^1\ \ U_1^2\ ) \begin{pmatrix} S^1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (V_1^1)^* \\ (V_1^2)^* \end{pmatrix}$$

with S^1 a $r_1 \times r_1$ diagonal matrix with positive diagonal elements that are the singular values of B_1 and $r_1 = rank(B_1)$. The matrix $(U_1^1\ \ U_1^2)$ is $p_1 \times p_1$ unitary while $(V_1^1\ \ V_1^2)$ is $q \times q$ unitary. It is straightforward to obtain the following factorization of the matrix B:

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \\ B_k \end{pmatrix} = \begin{pmatrix} U_1^1 & U_1^2 & 0 & 0 & \dots & 0 \\ B_2 V_1^1 (S^1)^{-1} & 0 & I_{p_2} & 0 & \dots & 0 \\ B_3 V_1^1 (S^1)^{-1} & 0 & 0 & I_{p_3} & \dots & 0 \\ \dots & 0 & \dots & \dots & \dots & 0 \\ B_k V_1^1 (S^1)^{-1} & 0 & \dots & \dots & \dots & I_{p_k} \end{pmatrix} \begin{pmatrix} S^1 & 0 \\ 0 & 0 \\ 0 & B_2 V_1^2 \\ 0 & B_3 V_1^2 \\ \dots & \dots \\ 0 & B_k V_1^2 \end{pmatrix} \begin{pmatrix} (V_1^1)^* \\ (V_1^2)^* \end{pmatrix} \quad (6)$$

In case that $r_1 = q$, there is no matrix V_1^2. In that case, simply omit the corresponding blocks in the second and third factor. The obtained factorization is then the one of the Lemma and we may stop here. (Observe that it is also possible that there is no matrix U_1^2, namely if $r_1 = p_1$. In that case, we simply follow our convention regarding matrix blocks with possibly zero dimensions. In this case, the construction does not stop however).

Assume that $r_1 < q$. The proof now proceeds by considering the block matrix

$$\begin{pmatrix} B_2 V_1^2 \\ B_3 V_1^2 \\ \dots \\ B_k V_1^2 \end{pmatrix} \quad (7)$$

and applying a similar decomposition as in (6), using the **OSVD** of the matrix $B_2 V_1^2$:

$$B_2 V_1^2 = (U_2^1\ \ U_2^2) \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (V_2^1)^* \\ (V_2^2)^* \end{pmatrix} \quad (8)$$

with S^2 an $r_2 \times r_2$ diagonal matrix with positive diagonal elements and

$$r_2 = rank(B_2 V_1^2) = rank \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} - rank(B_1) \quad (9)$$

We then find that

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \\ B_k \end{pmatrix} = \begin{pmatrix} U_1^1 & U_1^2 & 0 & 0 & 0 & 0 & \dots & 0 \\ B_2 V_1^1 (S^1)^{-1} & 0 & U_2^1 & U_2^2 & 0 & 0 & \dots & 0 \\ B_3 V_1^1 (S^1)^{-1} & 0 & B_3 V_1^2 V_2^1 (S^2)^{-1} & 0 & I_{p_3} & 0 & \dots & 0 \\ B_4 V_1^1 (S^1)^{-1} & 0 & B_4 V_1^2 V_2^1 (S^2)^{-1} & 0 & 0 & I_{p_4} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ B_k V_1^1 (S^1)^{-1} & 0 & B_k V_1^2 V_2^1 (S^2)^{-1} & 0 & 0 & \dots & \dots & I_{p_k} \end{pmatrix}$$

$$\times \begin{pmatrix} S^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3V_1^2V_2^2 \\ 0 & 0 & B_4V_1^2V_2^2 \\ \dots & \dots & \dots \\ 0 & 0 & B_kV_1^2V_2^2 \end{pmatrix} \begin{pmatrix} (V_1^1)^* \\ (V_2^1)^*(V_1^2)^* \\ (V_2^2)^*(V_1^2)^* \end{pmatrix}$$

In case that $r_1 + r_2 = q$, there is no matrix V_2^2. The block rows and columns in the factorization that contain V_2^2 should then be omitted. In that case, we have obtained the decomposition as in the Lemma and we may stop here.
Assume that $r_1 + r_2 < q$. Next consider a similar factorization of

$$\begin{pmatrix} B_3V_1^2V_2^2 \\ B_4V_1^2V_2^2 \\ \dots \\ B_kV_1^2V_2^2 \end{pmatrix}$$

using the **OSVD** of $B_3V_1^2V_2^2$ and so on .... In general, the recursive construction will stop after $j \leq k$ steps, when we have:

$$\sum_{i=1}^{j} r_i = q$$

which completes the proof. □

Observe that in the extreme case where $j = k$, we obtain a block factorization of the matrix B as $B = TSV^*$ where

- The matrix T is given by

$$T = \begin{pmatrix} U_1^1 & U_1^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ B_2V_1^1(S^1)^{-1} & 0 & U_2^1 & U_2^2 & 0 & 0 & \dots & 0 & 0 \\ B_3V_1^1(S^1)^{-1} & 0 & B_3V_1^2V_2^1(S^2)^{-1} & 0 & U_3^1 & U_3^2 & \dots & 0 & 0 \\ B_4V_1^1(S^1)^{-1} & 0 & B_4V_1^2V_2^1(S^2)^{-1} & 0 & B_4V_1^2V_2^2V_3^1(S^3)^{-1} & 0 & \dots & 0 & 0 \\ \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & 0 \\ B_kV_1^1(S^1)^{-1} & 0 & B_kV_1^2V_2^1(S^2)^{-1} & 0 & B_kV_1^2V_2^2V_3^1(S^3)^{-1} & 0 & \dots & U_k^1 & U_k^2 \end{pmatrix} \quad (10)$$

  in which we have the **OSVDs**

$$B_iV_1^2V_2^2 \dots V_{i-1}^2 = (U_i^1 \;\; U_i^2) \begin{pmatrix} S^i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (V_i^1)^* \\ (V_i^2)^* \end{pmatrix} \qquad i = 2, \dots, k \qquad (11)$$

  and S^i is $r_i \times r_i$ diagonal with positive diagonal elements.

- The matrix S has the structure as stated in the lemma.

- The matrix V is then given by

$$V = (V_1^1 \;\; V_1^2 V_2^1 \;\; V_1^2 V_2^2 V_3^1 \;\; \ldots \;\; (V_1^2 V_2^2 \ldots V_{k-1}^2 V_k^1) \;\; (V_1^2 V_2^2 \ldots V_{k-1}^2 V_k^2)) \qquad (12)$$

and is unitary.

4 A constructive proof of the main Theorem

In this section, we derive a constructive proof of the main Theorem. The proof is based upon two ideas:

- An inductive argument: Any **GSVD** at level k for k matrices $A_1, A_2, \ldots, A_k$, can be constructed from a corresponding **GSVD** of the first $k-1$ matrices $A_1, A_2, \ldots, A_{k-1}$. We basically use one of the factors of the matrix A_{k-1} and the block factorization lemma of the previous section, to find a factorization of the matrix A_k.

- Once such a factorization of the matrix A_k has been obtained, the combined factorizations of the matrices $A_1, \ldots, A_{k-1}$ have to be modified, a procedure which will be described in terms of the so-called *ripple-through-phenomenon.*

The inductive argument will be described in section 4.1, while the *ripple-through-phenomenon* is the subject of section 4.2. The details of both basic ideas are developed in section 4.3. A summary of the algorithmic proof is provided in section 4.4.

4.1 The induction step

In order to construct a **GSVD** of the set of matrices $(A_1, A_2, \ldots, A_k)$, we assume that we have obtained the corresponding **GSVD** for the matrices $(A_1, A_2, \ldots, A_{k-1})$.

$$\begin{aligned} A_1 &= U_1 D_1 X_1^{-1} \\ A_2 &= Z_1 D_2 X_2^{-1} \\ \ldots &= \ldots \\ A_{k-1} &= Z_{k-2} S_{k-1} V_{k-1}^* \end{aligned} \qquad (13)$$

Recall that the matrices D_j, $j = 1, \ldots, k-2$ are quasi-diagonal with nonzero blocks equal to identity matrices. The matrix S_{k-1} is quasi-diagonal with as its nonzero blocks diagonal matrices with positive diagonal elements. Also recall that for a $P-type$ step, $Z_{k-1} = X_{k-1}$ while for a $Q-type$ step, $Z_{k-1} = X_{k-1}^{-*}$.
Our first step is to rewrite the last equation as follows: Partition the matrix V_{k-1} according to the block columns of S_{k-1} (4) as:

$$V_{k-1} = \begin{array}{c} \begin{matrix} r_{k-1}^1 & r_{k-1}^2 & \ldots & r_{k-1}^{k-1} & n_{k-1} - r_{k-1} \end{matrix} \\ \begin{pmatrix} V_{k-1}^1 & V_{k-1}^2 & \ldots & V_{k-1}^{k-1} & V_{k-1}^k \end{pmatrix} \end{array}$$

where

$$r_{k-1} = \sum_{i=1}^{k-1} r_{k-1}^i$$

Then, the matrix Y_{k-1} in

$$S_{k-1} V_{k-1}^* = D_{k-1} Y_{k-1} \tag{14}$$

can be chosen to be nonsingular. This matrix is obtained from a scaling of the rows of V_{k-1}^* with the diagonal blocks in S_{k-1}, and replacing these diagonal blocks with identity matrices in order to obtain the quasi-diagonal matrix D_{k-1} in (14). A matrix X_{k-1} is defined as:

$$X_{k-1} = Y_{k-1}^{-1} \tag{15}$$

so that we can rewrite the factorization for A_{k-1} as

$$A_{k-1} = Z_{k-2} S_{k-1} V_{k-1}^* = Z_{k-2} D_{k-1} X_{k-1}^{-1}$$

Consider now the matrix $Z_{k-1}^{-1} A_k$. It can be partitioned into k block rows B_i, $i = 1, \ldots, k$, having the same dimensions as the block columns of S_{k-1}:

$$Z_{k-1}^{-1} A_k = \begin{array}{c} \\ r_{k-1}^1 \\ r_{k-1}^2 \\ \ldots \\ r_{k-1}^{k-1} \\ n_{k-1} - r_{k-1} \end{array} \begin{array}{c} n_k \\ \left(\begin{array}{c} B_1 \\ B_2 \\ \ldots \\ B_{k-1} \\ B_k \end{array} \right) \end{array} \tag{16}$$

Assume now, for the time being, that we had a way of factorizing the matrix $Z_{k-1}^{-1} A_k$ as

$$Z_{k-1}^{-1} A_k = P_k S_k V_k^* \tag{17}$$

where P_k is nonsingular, S_k is quasi-diagonal with the structure as in (4) and V_k is an $n_k \times n_k$ unitary matrix. Then we would have a factorization of the matrix A_k as

$$A_k = (Z_{k-1} P_k) D_k X_k^{-1} \tag{18}$$

However, the set of factorizations of the matrices $A_1, \ldots, A_{k-1}$ (13) completed with this factorization for the matrix A_k (18), is not a **GSVD** as in the main Theorem, unless the matrix P_k has some special structure, as will now be explained via a so-called *ripple-through-phenomenon.*

The result of this discussion will be that P_k can be obtained by applying the block factorization lemma of section 3 to a certain permutation of the block rows of $Z_{k-1}^{-1} A_k$ that is determined by the sequence of letters P and Q in the **GSVD**-name.

4.2 The ripple-through-phenomenon

Assume that we have obtained a factorization of the matrix A_k as in (18). The combined factorizations of the matrices A_{k-1}, A_k are now:

$$A_{k-1} = Z_{k-2} D_{k-1} X_{k-1}^{-1} \quad (19)$$

$$A_k = Z_{k-1} P_k S_k V_k^* \quad (20)$$

where $Z_{k-1} = X_{k-1}$ for a $P - type$ and $Z_{k-1} = X_{k-1}^{-*}$ for a $Q - type$ factorization. Because of the presence of the factor P_k in (20) and its absence in (19), the combined factorization of A_{k-1}, A_k does **not** have the structure as required by the main Theorem. The idea is now to preserve the factorization of A_k and modify the factorization of A_{k-1} as follows:

- If the combined factorization is of $P - type$, the factorization of A_{k-1} is modified by introducing a nonsingular matrix P_{k-1} as

  $$A_{k-1} = Z_{k-2} P_{k-1} D_{k-1} P_k^{-1} X_{k-1}^{-1}$$

  where

  $$P_{k-1} D_{k-1} P_k^{-1} = D_{k-1}$$

- If the combined factorization is of $Q - type$, we modify the factorization of A_{k-1} by introducing a nonsingular matrix P_{k-1} as

  $$A_{k-1} = Z_{k-2} P_{k-1} D_{k-1} P_k^{*} X_{k-1}^{-1}$$

  where

  $$P_{k-1} D_{k-1} P_k^{*} = D_{k-1}$$

Assume that such a modification is always possible (and it is as, will be shown furtheron), then, in either case, we have changed the first factor of A_{k-1} from Z_{k-2} to $Z_{k-2} P_{k-1}$. But, in order to be conform with the structure of the **GSVD** as in the main Theorem, we need to change the factorization of A_{k-2} by introducing a nonsingular matrix P_{k-2} that satisfies:

- for a combined factorization of A_{k-2}, A_{k-1} of $P - type$:

  $$P_{k-2} D_{k-2} P_{k-1}^{-1} = D_{k-2}$$

- for a combined factorization of A_{k-2}, A_{k-1} of $Q - type$:

  $$P_{k-2} D_{k-2} P_{k-1}^{*} = D_{k-2}$$

However, we also have to change the factorization of A_{k-3} etc ..... This goes on, backwards through the sequence of factorizations of $A_{k-1}, A_{k-2}, A_{k-3} \ldots, \ldots$ till we arrive at the factorization of the matrix A_1, where it will be necessary to change the matrix U_1 into $U_1 P_1$, where P_1 must be unitary (because $U_1 P_1$ must be unitary from Theorem 1) and

- for a combined factorization of A_1, A_2 of $P-type$:

$$P_1 D_1 P_2^{-1} = D_2 \qquad P_1 \quad \text{unitary}$$

- for a combined factorization of A_1, A_2 of $Q-type$:

$$P_1 D_1 P_2^{*} = D_1 \qquad P_1 \quad \text{unitary}$$

The conclusion is that the inductive argument of section 4.1, where we assumed that we had a factorization of the matrix A_k as (18), leads to a necessary modification of the factorizations of the matrices $A_{j-1}, j = 2, \ldots, k$ by inserting nonsingular matrices $P_{j-1}, j = 2, \ldots, k$ such that

- For a combined factorization of A_{j-1}, A_j of $P-type$:

$$P_{j-1} D_{j-1} P_j^{-1} = D_{j-1} \tag{21}$$

- For a combined factorization of A_{j-1}, A_j of $Q-type$:

$$P_{j-1} D_{j-1} P_j^{*} = D_{j-1} \tag{22}$$

- P_1 is a unitary matrix.

This backward modification will be called the *ripple-through-phenomenon.*

4.3 A detailed analysis of the inductive argument and the ripple-through-phenomenon

Having established the two main ideas of the proof, we shall now show how the **GSVD** of $A_1, \ldots, A_k$ can be obtained from the **GSVD** of $A_1, \ldots, A_{k-1}$, essentially by using the block factorization lemma of section 3. We'll show that:

1. The matrix P_k in (18) has a certain zero structure which is determined by the specific sequence of Ps and Qs in the **GSVD**-name. This will be investigated in section 4.3.1.

2. The zero structure of P_k corresponds to a block row block column permutation of a lower block triangular matrix T, i.e. there exists a block permutation matrix N such that

$$P_k = N^t T N \tag{23}$$

   The block permutation N is determined by the specific sequence of Ps and Qs in the **GSVD**-name. This is demonstrated in section 4.3.2.

3. In section 4.3.3. we show that the block diagonal blocks of T must be unitary.

4. From (18) and (23) we find

$$NZ_{k-1}^{-1}A_k = TNS_kV_k^* \tag{24}$$

While the matrix NS_k is not a quasi-diagonal matrix, it will be shown that there always exists a block permutation matrix M such that NS_kM is quasi-diagonal. Then (24) becomes:

$$NZ_{k-1}^{-1}A_k = T(NS_kM)(M^tV_k^*) \tag{25}$$

The factorization in (25) could be obtained from a straightforward application of the block factorization lemma of section 3. The algorithm to find the block permutation matrix M is given in section 4.3.4.

It is now obvious that we have to demonstrate that P_k is a block row block column permutation of a lower block triangular matrix as in (23) and that we have to find the precise structure of the block permutation matrices N and M.

4.3.1 Fact 1: The matrix P has a certain zero-block structure

First, we will show that the equations (21) and (22) impose a certain block zero structure on the matrix P_k.
Consider the combined factorization of the matrices A_{j-1}, A_j. There are two possible cases: The combined factorization is of $Q-type$ or it is of $P-type$.

First assume that it is of $P-type$. Then, from (21) it follows that:

$$P_{j-1}D_{j-1} = D_{j-1}P_j \tag{26}$$

The idea is to find a nonsingular solution P_{j-1} when P_j and D_{j-1} are known, and to discuss the conditions for consistency of the matrix equation (26). Recall the structure of the matrix D_{j-1}:

$$\underset{n_{j-2}\times n_{j-1}}{D_{j-1}} = \begin{array}{l} r_{j-1}^1 \\ r_{j-2}^1 - r_{j-1}^1 \\ r_{j-1}^2 \\ r_{j-2}^2 - r_{j-1}^2 \\ \dots \\ r_{j-1}^{j-1} \\ n_{j-2} - r_{j-2} - r_{j-1}^{j-1} \end{array} \begin{array}{c} \begin{array}{ccccc} r_{j-1}^1 & r_{j-1}^2 & \dots & r_{j-1}^{j-1} & n_{j-1}-r_{j-1} \end{array} \\ \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & I & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \end{array} \tag{27}$$

We partition the matrix P_j according to the block columns of D_{j-1} as

$$\underset{n_{j-1}\times n_{j-1}}{P_j} = \begin{array}{l} r_{j-1}^1 \\ r_{j-1}^2 \\ \dots \\ n_{j-1} - r_{j-1} \end{array} \begin{array}{c} \begin{array}{ccccc} r_{j-1}^1 & r_{j-1}^2 & \dots & r_{j-1}^{j-1} & n_{j-1}-r_{j-1} \end{array} \\ \begin{pmatrix} P_j^{11} & P_j^{12} & \dots & P_j^{1(j-1)} & P_j^{1j} \\ P_j^{21} & P_j^{22} & \dots & P_j^{2(j-1)} & P_j^{2j} \\ \dots & \dots & \dots & \dots & \dots \\ P_j^{j1} & P_j^{j2} & \dots & P_j^{j(j-1)} & P_j^{jj} \end{pmatrix} \end{array} \tag{28}$$

Hence we find that

$$D_{j-1}P_j = \begin{array}{c} \\ r_{j-1}^1 \\ r_{j-2}^1 - r_{j-1}^1 \\ r_{j-1}^2 \\ r_{j-2}^2 - r_{j-1}^2 \\ \cdots \\ r_{j-1}^{j-1} \\ n_{j-2} - r_{j-2} - r_{j-1}^{j-1} \end{array} \begin{array}{c} \begin{array}{cccccc} r_{j-1}^1 & r_{j-1}^2 & \cdots & r_{j-1}^{j-1} & n_{j-1} - r_{j-1} \end{array} \\ \begin{pmatrix} P_j^{11} & P_j^{12} & \cdots & P_j^{1(j-1)} & P_j^{1j} \\ 0 & 0 & \cdots & 0 & 0 \\ P_j^{21} & P_2^{22} & \cdots & P_2^{2(j-1)} & P_2^{2j} \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_j^{(j-1)j} & P_j^{(j-1)2} & \cdots & P_j^{(j-1)(j-1)} & P_j^{(j-1)j} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array} \tag{29}$$

The following two observations are crucial:

- It follows from (26) and the fact that the last block column of D_{j-1} (27) is zero that the last block column in (29) must be zero. This implies that the matrix equation (26) is only consistent if

$$P_j^{1j} = 0 \quad P_j^{2j} = 0 \quad \cdots \quad P_j^{(j-1)j} = 0 \tag{30}$$

- It can be seen from (26), (28) and (29) that the last block row of P_j plays no role in the determination of P_{j-1}.

If the consistency condition (30) is satisfied, a nonsingular solution P_{j-1} of the matrix equation (26) is given by:

$$\underset{n_{j-2} \times n_{j-2}}{P_{j-1}} = \begin{array}{c} \\ r_{j-2}^1 \\ r_{j-2}^2 \\ \cdots \\ r_{j-2}^{j-2} \\ n_{j-2} - r_{j-2} \end{array} \begin{array}{c} \begin{array}{cccccc} r_{j-2}^1 & r_{j-2}^2 & \cdots & r_{j-2}^{j-2} & n_{j-2} - r_{j-2} \end{array} \\ \begin{pmatrix} \begin{pmatrix} P_j^{11} & 0 \\ 0 & I \end{pmatrix} & \begin{pmatrix} P_j^{12} & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} P_j^{1(j-2)} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} P_j^{1(j-1)} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} P_j^{21} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} P_j^{22} & 0 \\ 0 & I \end{pmatrix} & \cdots & \begin{pmatrix} P_j^{2(j-2)} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} P_j^{2(j-1)} & 0 \\ 0 & 0 \end{pmatrix} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \begin{pmatrix} P_j^{(j-2)1} & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \cdots & \begin{pmatrix} P_j^{(j-2)(j-2)} & 0 \\ 0 & I \end{pmatrix} & \begin{pmatrix} P_j^{(j-2)(j-1)} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} P_j^{(j-1)1} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} P_j^{(j-1)2} & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} P_j^{(j-1)(j-2)} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} P_j^{(j-1)(j-1)} & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \end{array}$$

Observe that P_{j-1} is nonsingular if all its block diagonal blocks are nonsingular. Also note that P_{j-1} is partitioned in $j-1$ block columns and block rows.

Next, assume that the combined factorization of A_{j-1}, A_j is of $Q-type$. Then from (22) we find:

$$D_{j-1}P_j^* = P_{j-1}^{-1}D_{j-1} \tag{31}$$

Exploiting the structure of D_{j-1} (27), we arrive at the following observations in a similar fashion as above:

- The matrix equation (31) is consistent only if

$$P_j^{j1} = 0 \quad P_j^{j2} = 0 \quad \cdots \quad P_j^{j(j-1)} = 0$$

- The last block column of P_j plays no role in the determination of P_{j-1}.

From (31) we can find a solution for P_{j-1}^{-1} as

$$
\underset{n_{j-2}\times n_{j-2}}{P_{j-1}^{-1}} = \begin{array}{c} \\ r_{j-2}^1 \\ r_{j-2}^2 \\ \cdots \\ r_{j-2}^{j-2} \\ n_{j-2}-r_{j-2} \end{array}
\begin{array}{c}
\begin{array}{cccc} r_{j-2}^1 & r_{j-2}^2 & \cdots & r_{j-2}^{j-2} \end{array} \\
\begin{pmatrix} (P_j^{11})^* & 0 \\ 0 & I \end{pmatrix} \quad \begin{pmatrix} (P_j^{21})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} (P_j^{(j-2)1})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} (P_j^{(j-1)1})^* & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} (P_j^{12})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} (P_j^{22})^* & 0 \\ 0 & I \end{pmatrix} \quad \cdots \quad \begin{pmatrix} (P_j^{(j-2)2})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} (P_j^{(j-1)2})^* & 0 \\ 0 & 0 \end{pmatrix} \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\begin{pmatrix} (P_j^{1(j-2)})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \cdots \quad \cdots \quad \begin{pmatrix} (P_j^{(j-2)(j-2)})^* & 0 \\ 0 & I \end{pmatrix} \quad \begin{pmatrix} (P_j^{(j-1)(j-2)})^* & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} (P_j^{1(j-1)})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} (P_j^{2(j-1)})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} (P_j^{(j-2)(j-1)})^* & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} (P_j^{(j-1)(j-1)})^* & 0 \\ 0 & I \end{pmatrix}
\end{array}
\tag{32}
$$

Observe that for this $Q-type$ factorization, we determine the inverse P_{j-1}^{-1} from D_{j-1} and P_j instead of the matrix P_{j-1} itself as with the $P-type$ factorization. It is not necessary to determine P_{j-1} explicitly because we can easily determine P_{j-2} or its inverse from either P_{j-1} or P_{j-1}^{-1} as can be seen as follows:
Again, consider the two cases of a $P-type$ and a $Q-type$ factorization of A_{j-2}, A_{j-1}.

- Assume that the combined factorization of A_{j-2}, A_{j-1} is of $P-type$, then $D_{j-2}P_{j-1}^{-1} = P_{j-2}D_{j-2}$ which is consistent only if the first $j-2$ blocks of the last block column of P_{j-1}^{-1} are zero. But from (32) it then follows that $P_j^{(j-1)1} = 0, P_j^{(j-1)2} = 0, \ldots, P_j^{(j-1)(j-2)} = 0$. The last block row of P_{j-1}^{-1} plays no role in the determination of P_{j-2}.

- Assume that the combined factorization of A_{j-2}, A_{j-1} is of $Q-type$, then $P_{j-1}^{-*}D_{j-2} = P_{j-2}D_{j-2}$ which is consistent only if the first $j-2$ blocks of the last block row of P_{j-1}^{-1} are zero: $P_j^{1(j-1)} = 0, P_j^{2(j-1)} = 0, \ldots, P_j^{(j-2)(j-1)} = 0$. The last block column of P_{j-1}^{-1} plays no role in the determination of P_{j-2}.

The conclusion is that there is no need for explicit inversion of the matrix P_{j-1} but that we can reach similar conclusions from the matrix P_{j-1}^{-1}.

Depending on the sequence in which the letters P and Q appear we now apply for $j = k, k-1, \ldots, 2$ either equation (21) or (22) to determine the matrices $P_{k-1}, P_{k-2}, \ldots, P_1$ or their inverses. The consistency of these equations implies that certain blocks of P_k must be zero, in a structured manner as will be explained furtheron. Let us first give an example.

Example 4:
In order to illustrate the *ripple-through-phenomenon*, we'll derive the **QPQ-SVD** of the matrices (A_1, A_2, A_3, A_4) from a **QP-SVD** of the matrices (A_1, A_2, A_3):

$$
\begin{aligned}
A_1 &= U_1 D_1 X_1^{-1} \\
A_2 &= X_1^{-*} D_2 X_2^{-1} \\
A_3 &= X_2 S_3 V_3^*
\end{aligned}
$$

First, the factorization for A_3 is rewritten as $A_3 = X_2 D_3 X_3^{-1}$ where D_3 is a quasi-diagonal matrix, the nonzero blocks of which are identity matrices. Next, assume that we have a factorization of $X_3^* A_4$ as $X_3^* A_4 = P_4 S_4 V_4^*$ where P_4 is nonsingular, S_4 is quasi-diagonal and V_4 is unitary. The *ripple-through-phenomenon* now consists of a modification of the factorizations of A_1, A_2, A_3 by introducing nonsingular matrices P_3, P_2, P_1 such that

$$\begin{array}{lcll} A_4 & = & X_3^{-*} P_4 S_4 V_4^* & \\ A_3 & = & X_2 P_3 D_3 P_4^* X_3^{-1} & \rightarrow \quad P_3 D_3 P_4^* = D_3 \\ A_2 & = & X_1^{-*} P_2 D_2 P_3^{-1} X_2^{-1} & \rightarrow \quad P_2 D_2 P_3^{-1} = D_2 \\ A_1 & = & U_1 P_1 D_1 P_2^* X_1^{-1} & \rightarrow \quad P_1 D_1 P_2^* = D_1 \end{array}$$

The equations on the right are only consistent if the matrix P_4 has a certain structure. Partition P_4 according to the block dimensions of S_4 as

$$P_4 = \begin{pmatrix} P_4^{11} & P_4^{12} & P_4^{13} & P_4^{14} \\ P_4^{21} & P_4^{22} & P_4^{23} & P_4^{24} \\ P_4^{31} & P_4^{32} & P_4^{33} & P_4^{34} \\ P_4^{41} & P_4^{42} & P_4^{43} & P_4^{44} \end{pmatrix}$$

Then, using the definitions of the matrices P_3, P_2, P_1 as above, we find that:

- From $P_3 D_3 P_4^* = D_3$, it follows that $P_4^{41} = 0, P_4^{42} = 0, P_4^{43} = 0$.
- From $P_2 D_2 P_3^{-1} = D_2$, it follows that $P_4^{31} = 0, P_4^{32} = 0$.
- From $P_1 D_1 P_2^* = D_1$, it follows that $P_4^{12} = 0$.

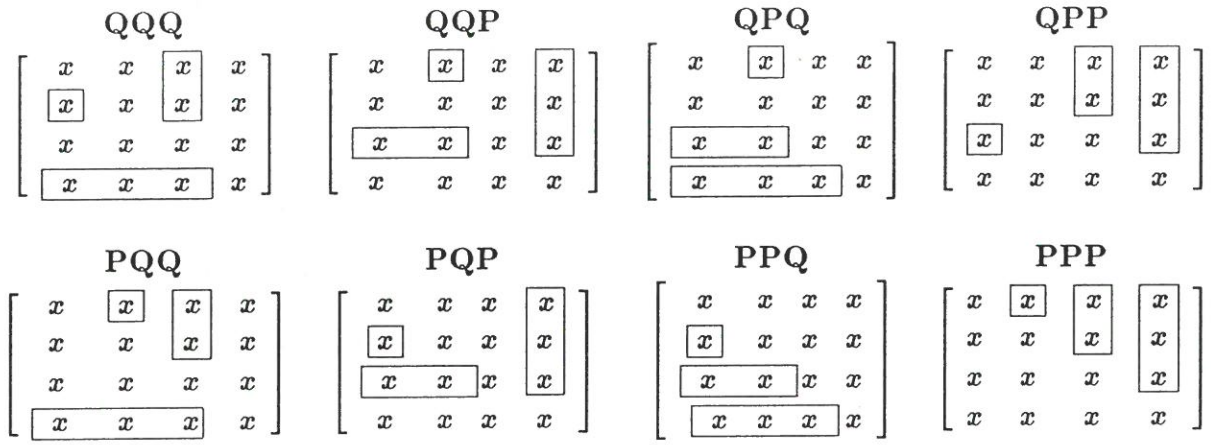
Hence

$$P_4 = \begin{pmatrix} P_4^{11} & 0 & P_4^{13} & P_4^{14} \\ P_4^{21} & P_4^{22} & P_4^{23} & P_4^{24} \\ 0 & 0 & P_4^{33} & P_4^{34} \\ 0 & 0 & 0 & P_4^{44} \end{pmatrix}$$

4.3.2 Fact 2: The matrix P is a block row - block column permutation of a lower block triangular matrix

The following observation follows immediately from the recursive definition of the matrices $P_j, j = 1, \ldots, k-1$ in section 4.3.1:
The net effect of the consistency condition (21) or (22) is that the first $j-1$ blocks of the j-th block row or column of P_k are zero, with $j = 1, \ldots, k$.
The sequence in which block rows or columns are zeroed is determined by the sequence in which the letters P and Q appear in the **GSVD**-name. Consider as an illustration all possible **GSVD**s for 4 matrices:

Example 5:
The reader may wish to verify that the zero structure of the matrix P_4 in every **GSVD** for 4 matrices is given by (The zero elements are enclosed by a box):



Observe that in each of the cases, there are $j-1$ zero blocks in either block row j *or* block column j, $j = 1,2,3,4$. An important consequence of this observation also holds for the general case:
The matrix P_k is always a block row block column permutation of a lower block triangular matrix T, i.e. there exists a $n_{k-1} \times n_{k-1}$ block permutation matrix N such that

$$P_k = N^t T N \tag{33}$$

A careful analysis reveals that in general the block permutation matrix N can be obtained from the following algorithm:

Algorithm for the block permutation matrix N:

- Denote by $N_{(i)}$ the block permutation matrix obtained in step i of the recursion that follows.
- Define the **block row reverse** of a partitioned identity matrix

$$I_{p_1+\ldots+p_q} = \begin{pmatrix} I_{p_1} & 0 & \ldots & \ldots & 0 \\ 0 & I_{p_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & I_{p_q} \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} 0 & \ldots & \ldots & 0 & I_{p_q} \\ 0 & \ldots & \ldots & I_{p_{q-1}} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I_{p_1} & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

  Obviously, the block row reverse of the second matrix is again the identity matrix.
- **Initialisation**
  Let $N_{(k)}$ be equal to the partitioned identity matrix

$$N_{(k)} = \begin{pmatrix} I_{r_{k-1}^1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & I_{r_{k-1}^2} & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & I_{r_{k-1}^{k-1}} & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & I_{n_{k-1}-r_{k-1}} \end{pmatrix} \tag{34}$$

in which the block dimensions correspond to the block dimensions of the partitioned matrix P_k.

- **For i=k-1, k-2, ..., 1:**

  - If the factorization of A_i, A_{i+1} is of
    - $P-type$: Then, $N_{(i)} = N_{(i+1)}$.
    - $Q-type$: Then, $N_{(i)}$ is obtained from $N_{(i+1)}$ as follows:
      - Take the submatrix of $N_{(i+1)}$ formed by its first $i+1$ block columns.
      - Reverse the block rows that have a nonzero block.

- The matrix $N_{(1)}$ is the desired block permutation matrix: $N = N_{(1)}$.

Example 6
As an example, the block permutation matrix for a **QPQ-SVD** is obtained as:

$$N_{(4)} = \begin{pmatrix} I_{r_3^1} & 0 & 0 & 0 \\ 0 & I_{r_3^2} & 0 & 0 \\ 0 & 0 & I_{r_3^3} & 0 \\ 0 & 0 & 0 & I_{n_3-r_3} \end{pmatrix} \rightarrow N_{(3)} = \begin{pmatrix} 0 & 0 & 0 & I_{n_3-r_3} \\ 0 & 0 & I_{r_3^3} & 0 \\ 0 & I_{r_3^2} & 0 & 0 \\ I_{r_3^1} & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow N_{(2)} = \begin{pmatrix} 0 & 0 & 0 & I_{n_3-r_3} \\ 0 & 0 & I_{r_3^3} & 0 \\ 0 & I_{r_3^2} & 0 & 0 \\ I_{r_3^1} & 0 & 0 & 0 \end{pmatrix} \rightarrow N_{(1)} = \begin{pmatrix} 0 & 0 & 0 & I_{n_3-r_3} \\ 0 & 0 & I_{r_3^3} & 0 \\ I_{r_3^1} & 0 & 0 & 0 \\ 0 & I_{r_3^2} & 0 & 0 \end{pmatrix}$$

4.3.3 Fact 3: The block diagonal blocks of T must be unitary

An important consequence of the fact that P_k is a block row block column permutation of a lower block triangular matrix T, is that the block diagonal blocks of P_k and T must be the same (but ordered differently). The matrix P_1 or its inverse at the end of the *ripple-through-phenomenon* in section 4.3.1 is of the form:

$$\begin{pmatrix} P_k^{11} & 0 \\ 0 & I \end{pmatrix}$$

where P_k^{11} is the first block diagonal block of P_k, which can be any block diagonal block of T, depending on the specific sequence of Ps and Qs in the **GSVD**-name. But because $U_1 P_1$ must be unitary, it follows that P_k^{11} must be unitary, hence, all block diagonal blocks of T must be unitary!

4.3.4 Fact 4: A block permutation such that NSM is quasi-diagonal

Let us first summarize what we have obtained so far:
Consider the **GSVD** of $A_1, \ldots, A_{k-1}$ as in (13). If a nonsingular matrix P_k could be obtained such that $A_k = Z_{k-1} P_k S_k V_k^*$, then P_k must be a block row block column permutation of a lower block triangular matrix T, i.e. there exists a block permutation matrix N such that $P_k = N^t T N$. An algorithm to obtain N was given in section 4.3.2. We then find that

$$N Z_{k-1}^{-1} A_k = T N S_k V_k^*$$

However, if S_k has the quasi-diagonal structure as in (4), then NS_k is a block row permutation of that quasi-diagonal structure. It is however always possible to find an $n_k \times n_k$ block permutation matrix M, such that NS_kM is quasi-diagonal. We then have:

$$\begin{aligned} N Z_{k-1}^{-1} A_k &= T N S_k M M^t V_k^* \\ &= T (N S_k M)(V_k M)^* \\ &= T \tilde{S} \tilde{V}^* \quad \text{say} \end{aligned}$$

where $\tilde{S}$ is quasi-diagonal and $\tilde{V}$ is unitary. Obviously, the matrices T, $\tilde{S}$ and $\tilde{V}$ follow immediately from the block factorization lemma of section 3 applied to the matrix $N Z_{k-1}^{-1} A_k$, in which the scalars p_i (which are the block row dimensions) follow from the dimensions of the block rows of N.
The block permutation matrix M can be obtained as follows:

Algorithm for block permutation matrix M:

- The upper $r_k \times r_k$ block of M^t (the transpose of M) has the same zero-identity block structure as the matrix N, the identity matrix in block column j of M^t being $I_{r_k^j}$.
- In block position $(k+1, k+1)$ of M^t (lower right hand corner), there is an $(n_k - r_k) \times (n_k - r_k)$ identity matrix $I_{n_k - r_k}$.

Example 7:
Let us proceed with the **QPQ-SVD** of example 6. From the block factorization lemma, one obtains a decomposition as:

$$N_{(1)} Z_3^{-1} A_1 = T \tilde{S} \tilde{V}^*$$

$$= T \begin{array}{c} \\ \tilde{r}_1 \\ n_3 - r_3 - \tilde{r}_1 \\ \tilde{r}_2 \\ r_3^3 - \tilde{r}_2 \\ \tilde{r}_3 \\ r_3^1 - \tilde{r}_3 \\ \tilde{r}_4 \\ r_3^2 - \tilde{r}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \tilde{r}_1 & \tilde{r}_2 & \tilde{r}_3 & \tilde{r}_4 & n_4 - r_4 \end{array} \\ \begin{pmatrix} S^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & S^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S^4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tilde{V}^*$$

where the integers $\tilde{r}_i, i = 1,2,3,4$ are the recursively defined integers of the block factorization lemma.
It is easily found that

$$N^t \tilde{S} = \begin{array}{c} \\ \tilde{r}_3 \\ n_3 - r_3 \\ \tilde{r}_4 \\ r_3^3 - \tilde{r}_4 \\ \tilde{r}_2 \\ r_3^2 - \tilde{r}_2 \\ \tilde{r}_1 \\ r_3^1 - \tilde{r}_1 \end{array} \begin{array}{c} \begin{array}{ccccc} \tilde{r}_1 & \tilde{r}_2 & \tilde{r}_3 & \tilde{r}_4 & n_4 - r_4 \end{array} \\ \begin{pmatrix} 0 & 0 & S^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S^4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & S^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ S^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The dimensions $r_4^j, j = 1,2,3,4$ are now defined as:

$$r_4^1 = \tilde{r}_3 \quad r_4^2 = \tilde{r}_4 \quad r_4^3 = \tilde{r}_2 \quad r_4^4 = \tilde{r}_1$$

The matrix M is constructed via

$$M^t = \begin{pmatrix} 0 & 0 & 0 & I_{r_4^4} & 0 \\ 0 & 0 & I_{r_4^3} & 0 & 0 \\ I_{r_4^1} & 0 & 0 & 0 & 0 \\ 0 & I_{r_4^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_4 - r_4} \end{pmatrix}$$

4.4 Summary of the constructive proof

Putting together all elements of section 4.3, we find the following algorithm, which at the same time is a constructive proof, to derive the **GSVD** of the matrices $A_1, \ldots, A_k$ from a corresponding one of $A_1, \ldots, A_{k-1}$ as in (13):

- Determine the block permutation matrices N and M from the algorithms in sections 4.3.2 and 4.3.4.

- Apply the block factorization lemma of section 3 to the partitioned matrix:

$$NZ_{k-1}^{-1}A_k = T\tilde{S}_k\tilde{V}_k^*$$

  where T is lower block triangular with unitary block diagonal matrices, $\tilde{S}_k$ is quasi-diagonal and $\tilde{V}_k$ is unitary.

- Determine the matrices $P_k = N^tTN$, $S_k = N^t\tilde{S}_kM^t$ and $V_k = \tilde{V}_kM^t$. The factorization of the matrix A_k becomes $A_k = (Z_{k-1}P_k)S_kV_k^*$.

- Determine the matrices $P_{k-1}, P_{k-2}, \ldots, P_1$ from the *ripple-through-recursion* of section 4.3.1. and update the factorizations of the matrices $A_{k-1}, \ldots, A_1$ accordingly.

- The obtained factorizations of $A_1, \ldots, A_k$ constitute a **GSVD**, the structure and properties of which are conform with the statement of the main Theorem.

5 Conclusions

In this paper, we have proposed a tree of generalizations of the singular value decomposition, with at the top the **OSVD** for one matrix and the **PSVD** and the **QSVD** for two matrices. Our proof is constructive, the main idea being an induction step which allows to find a **GSVD** at level k from a corresponding one at level $k-1$. This necessitates a detailed analysis of the backward modification of the chain of decompositions at level $k-1$, which we have called the *ripple-through-phenomenon.* A key role is played by a certain block factorization lemma.
We are convinced that this tree of generalized singular value decompositions will open new and exciting fields of research with respect to numerical algorithms, geometrical interpretations and applications. For instance, the uniqueness issues, the relation to generalized eigenvalue problems, genericity and sensitivity analysis and numerical and implementational aspects definitely deserve more attention. The authors have already found and collected several interesting applications of these generalizations, which will be reported in subsequent publications. We have also derived expressions for the scalars r_j^i, which are the dimensions of the blocks in the quasi-diagonal matrices D_j amd S_k in the main Theorem, in terms of ranks of the matrices $A_j, k = 1, \ldots, k$ and products and concatenations thereof.

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