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A cellular automaton (binary difference field) under a strange conservation law.

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Abstract

Binary difference fields are isomorphic to the state-time evolution of the rule-102 binary cellular automaton. Several structural and dynamical properties of such fields exhibiting global rowwise constancy of change in value for a given change in the initial state are presented. An isomorphic structure involving a certain integer-valued fractal matrix is introduced.[1]

1 Introduction

Consider a field of cells arranged in a triangular lattice (Fig. 1). Each cell in the field can be *occupied* or *not* (this will be represented by the binary values 1/0 or by the symbols +/−, or by a cell being black or white in the figures). A group of three neighbouring cells in a elementary *top-down* triangular mesh will be called *elementary triangular neighbourhood.* The local evolution rule under consideration here requires that the value in each cell in an elementary triangular neighbourhood equals the modulo-2 sum of the two other neighbours.

The appearance of 1 in a cell also reflects the existence of a difference in value in the two other cells, hence the name : *binary difference field* (BDF). Each row can be considered as being the *difference sequence* of the row that precedes it from above. The rowwise downward evolution of the state of the cells starting from a certain initial row is isomorphic to the evolution of the classical 1-dimensional binary cellular automaton obeying rule 102 [1]. The BDFs will have potentially unbounded size, starting with a certain top row. Each finite segment of length N on the top row completely determines an equilateral top-down triangular subfield of the same size. The complete field is also determined by the state of a finite segment f_0 on the top row together with the state of all cells on the adjacent left-skew and right-skew columns f_L and f_R (Fig. 2). Instead of f_L and f_R, corresponding sequences h_L and h_R on the top row can be considered. So a field is completely determined by the triple (f_0, f_L, f_R)

[1] Part of this paper was presented at the workshop "*Cellular Automata*: Theory and experiment", September 1989, J.R. Oppenheimer Study Center, Los Alamos National Laboratory, New Mexico. An adapted version will appear in the journal Physica D under the title : "A CA ruled by an eccentric conservation law".

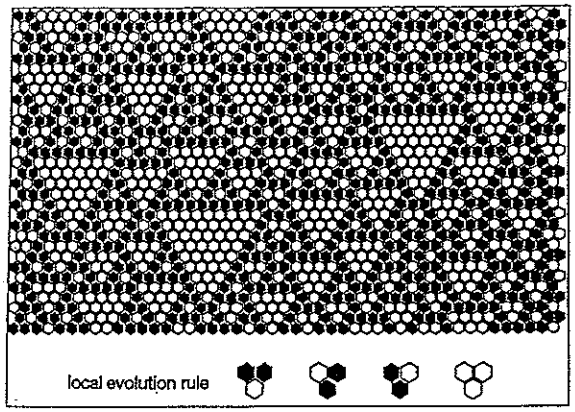


Figure 1: A binary difference field.

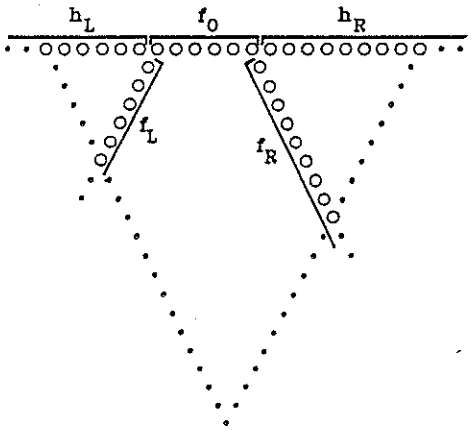


Figure 2: The triples (h_L, f_0, h_R) and (f_L, f_0, f_R) that completely determine a BDF.

or (f_0, h_L, h_R). In reference [2], a thorough analysis was given of the periodic and fractal behaviour of the overall field in case the cells along f_L and f_R were enforced to show periodic occupational patterns.

Also notice the 120° rotational symmetry in the local evolution rule. This means that left-skew and right-skew columns also evolve according to CA-rule 102. It is an *additive rule*, and allows for different BDF to be superposed modulo-2 (or EXORed), while the result still satisfies the local evolution rule.

If some cells on the top row change their value, and it is required that the overall field remains a difference field, each row has some cells that have to change their value too. The set of all these changing cells is called the *change propagation net* (CPN). In itself it is a new difference field in which all changing cells in the original field are now presented by occupied cells and the others by empty cells (Fig. 3).

Any change propagation net can also be obtained as the EXOR (or modulo-2) superposition of the simple change propagation nets evolving from each value-changing cell on the top row. Such simple propagation net is isomorphic to the set of odd coefficients in Pascal's triangle. The change propagation net has a *dual* interpretation : when it is turned upside down, it indicates rowwise invariant parity of the cells in a difference field that might be covered by it [2].

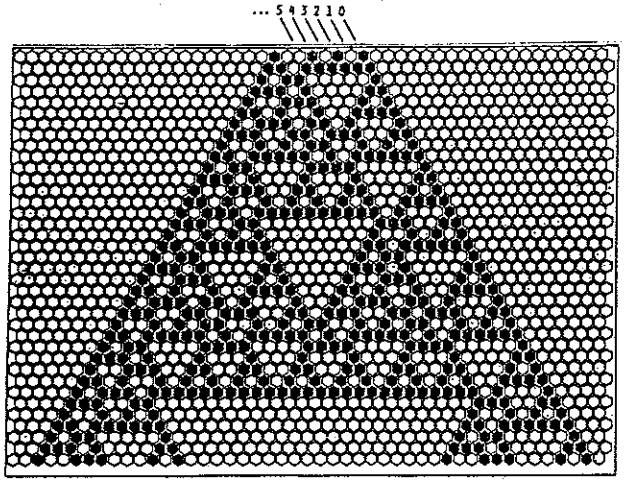


Figure 3: A change propagation net (the set of black cells).
The CPN-seed $g_o = 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1$.

Now consider a CPN over a BDF. On each row k, the number of occupied and non-occupied cells in the BDF that are covered by the CPN are represented by N_k^+ and N_k^-. The difference $e_k \stackrel{\Delta}{=} N_k^+ - N_k^-$ is the *occupational excess.* In what follows, properties of BDF for which e_k is *constant (over k) under a given* CPN will be presented. The problems under investigation here emerged from an attempt to characterize the randomness of a binary sequence according to the number of occupied cells in its consecutive difference sequences [3]. The invariance of these numbers under certain changes, which is equivalent to the condition $e_k = 0$, is a problem that then arises spontaneously.

It will be convenient to rank the rows and columns. Row ranking starts with zero at the top row. When a CPN over a BDF is considered, ranking of the right-skew columns starts with zero at the rightmost CPN-cell on the top row and increases when moving to the left (Fig. 3). On any row k, the CPN can be characterized by either

- a. the binary number g_k formed by placing a 1 at each cell that belongs to the CPN on that row, and a 0 at the other cells. The rightmost/leftmost digit is the 1 coinciding with the rightmost/leftmost cell of the CPN.

- b. the corresponding decimal number γ_k : it is always an odd integer as g_k has always a 1 as its rightmost digit.

- c. a polynomial $G_k(z)$ over the Galois-field $GF(2)$ (see [4]) : $G_k(z) = \Sigma\alpha_{k,i}z^i$. $\alpha_{k,i}$ is the value of the binary digit of rank i in g_k. In $zG(z)$, z can be considered as an operator that shifts the $G(z)$-configuration one place to the left. From the local evolution rule, it then follows that $G_k(z) = (1+z)G_{k-1}(z) = (1+z)^kG_0(z)$ (addition and multiplication in $GF(2)$).

Notice that the complete CPN is determined by its top-row configuration g_0. This will be called the *CPN-seed.*

Main problem statement : *Specify a CPN-seed g_0 (and so a complete CPN). Find all*

binary difference fields that have rowwise constant occupational excess e_k over all rows k under this CPN.

Any BDF that has a constant e_k will be called a *particular solution* to the problem, and will be represented by $P(f_0, f_l, f_R; g_0, e_k)$, where (f_0, f_L, f_R) is the BDF-representing triple, with f_0 the BDF-sequence confined within the range of the CPN-seed g_0 (i.e. the numbers of cells in g_0 and f_0 are the same). As yet, we have not been able to formulate a general answer to this problem. This is also the case for the *equivalent problem* of finding particular solutions in which (g_0, f_0) is given and (f_L, f_R) are to be found. The solution is then represented by $P(f_L, f_R; g_0, f_0, e_k)$. Nevertheless, numerous computer experiments have shown that the solutions to the problem have rather peculiar structural properties. Notice that f_0, and in general f_k when an arbitrary row k is considered, can be represented in the same ways as $g_0(g_k)$. Restricted-size parts (as to the number of rows and columns) of particular solutions to the given (g_0, f_0, e_k)-problem can be produced by the computer in the following way.

Procedure for finding partial solutions :

Start with a given $(g_0, f_0; e)$ at the top row. (g_0, f_0) completely determines the (inner)difference sequence at row 1 (Fig. 4). The occupational state of the two outer CPN-cells on this row have to be chosen properly. If the inner sequence already has the proper excess-value e, then we have one degree of freedom for the states of the outer cells : they have to be of opposite sign. The choice is made *randomly*, by making the leftmost cell occupied with a certain probability. If the excess of the difference sequence of (g_0, f_0) equals $e \pm 2$, then the choice is not free : the outer cells must both be assigned the same state 0 or 1, so as to make $e_1 = e$. In case the excess of the difference sequence of (g_0, f_0) is $e \pm k, (k > 2)$ then e_1 cannot be made equal to e, and there is no solution for the given $(g_0, f_0; e)$.

If there is a solution for row 1 (providing now a $(g_1, f_1; e)$), we can proceed in the same way to row 2, and so on. When it is impossible to obtain the proper excess value at a certain row, one should go back to the last row where there was a free choice, change the states of the edge-cells which now make the choice unfree, and then proceed further downwards until the next conflicting situation arises, whereby the retrograde procedure is then initiated again. So the solution is found by a *random search procedure.*

Observe that the whole evolution of the CPN and the underlying BDF as presented in Fig. 4 and as implemented in the search procedure can also be considered as the evolution of a *quaternary* cellular automaton. The local evolution rule is obtained by conjoining the evolution rule for the two independent BDF and CPN fields. The combined states (g_0, f_0) are then represented by one quaternary number q_0 or by a polynomial Q_0 over $GF(4)$, the rowwise evolution of which is also given by the expression $(1 + z)^k Q_0(z)$. The excess value e is then just the difference between the number of $\oplus$ cells and $\ominus$ cells on each row.

Fig. 5(a) shows the structure of the inner field generated by a given seed (g_0, f_0). The CPN-information is then removed, uncovering a part of the underlying BDF (Fig. 5(b)). This field is then completed to the left and to the right (Fig. 5(c)). The total size of the pictures shown is about 320 × 300 cells (width × depth).

2 Properties of the solutions $P(f_L, f_R; g_0, f_0, e)$

The computer experiments seem to suggest that there are 2 types of solution :

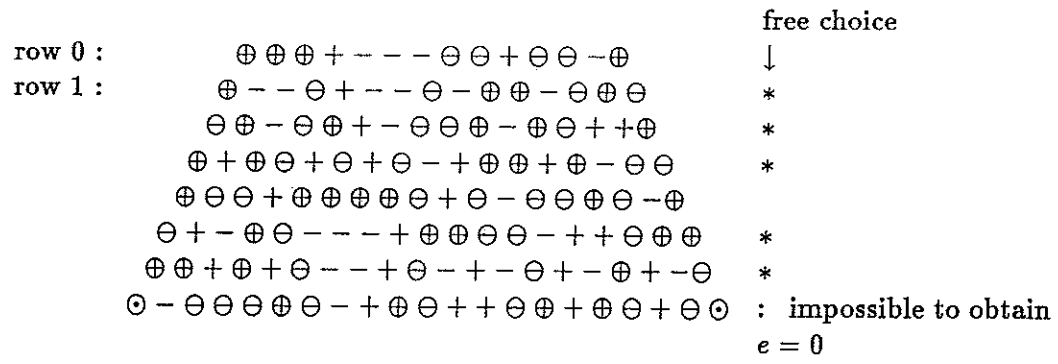
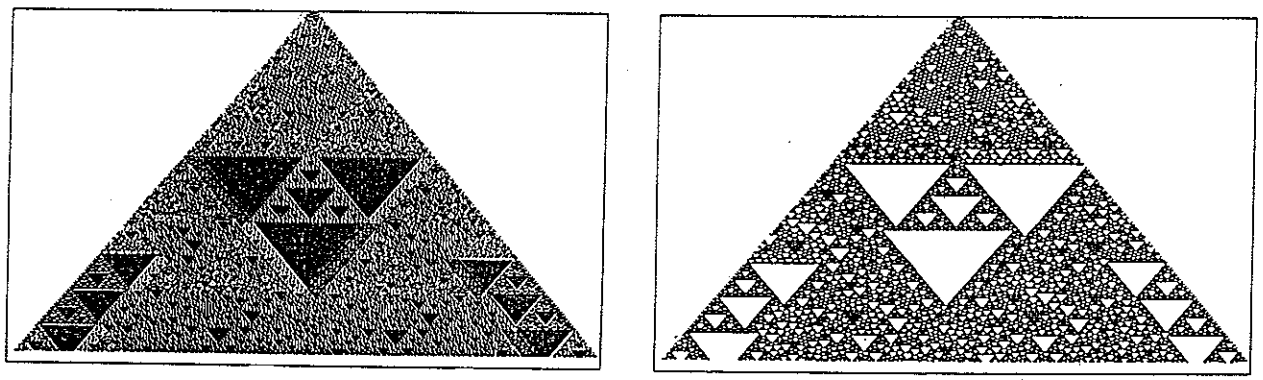
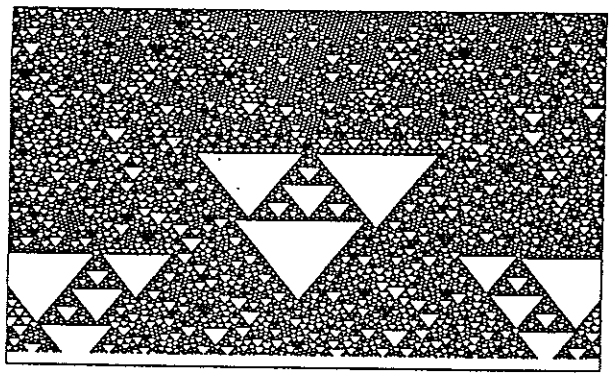


Figure 4: Evolutionary procedure for finding a partial solution to the (g_0, f_0, e)-problem. The CPN-cells are encircled.
+/- represent the occupational states of the BDF-cells.
$g_0 = 1\,1\,1\,0\,0\,0\,0\,1\,1\,0\,1\,1\,0\,1$, $f_0 = 1\,1\,1\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,1$, $e = 0$.



a. b.



c.

Figure 5: (a) : a constant excess quaternary field (confined within CPN-area), (b) : the corresponding binary field, (c) : the corresponding completed field. $[(g_0, f_0) = (10111, 01011)]$
Size : 320 x 300 cells (width x depth).

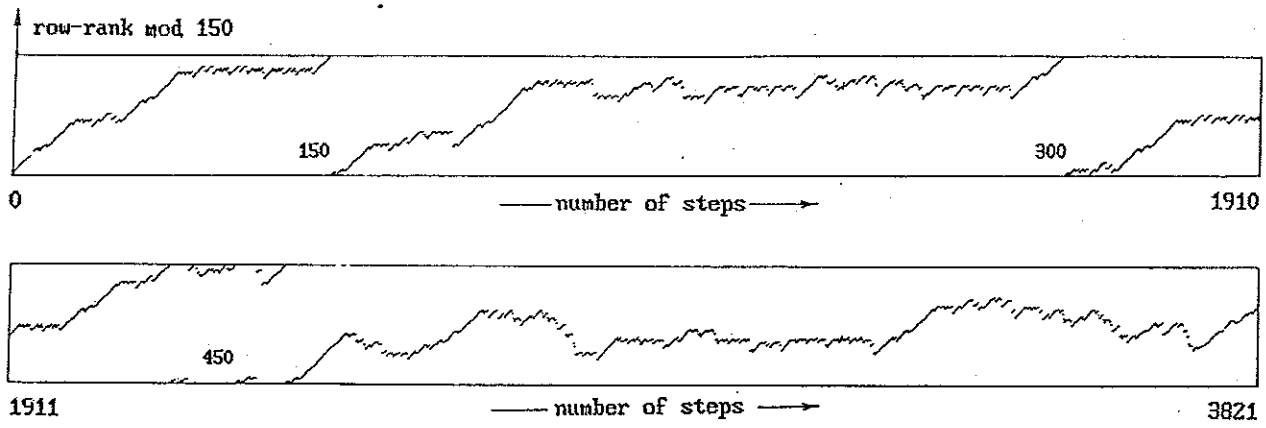


Figure 6: A typical search history : evolution of solution-size versus the number of steps.

1. *Finite-depth solutions*:
   for most $(g_0, f_0; e)$-problems, the search procedure returns quickly to the initial seed (g_0, f_0), mostly after excursions to a row-depth which is of the order of the seed size (number of digits in g_0), or even less. These are *nonproper* solutions to the problem and will not be considered further.

2. *Potentially unbounded solutions*:
   for particular $(g_0, f_0; e)$-configurations, a full-screen-size solution was found. In most cases, this is only obtained after a search time (i.e. the number of rows for which a decision about the state of the CPN-edge cells has to be made) that may fairly exceed the number of rows in the image. A typical search history is shown in Fig. 6. The search time needed to get a full-screen solution may vary widely, and depends on $(g_0, f_0; e)$ and on the particular values for seed and probability of the random generator. On one occasion, a search time as high as 100.000 was not sufficient to get further then the zone around rows 300-320.

Although we still have only a solution of limited size, we are tempted to conjecture that they show a part of a *potentially unbounded solution* (for some cases, this will be proven). This is corroborated by two facts :

1. For initial seeds of comparable size, there is a clear distinction between the finite-depth solutions and the full-screen-size solutions. The former are very limited in size as compared to the full-screen solutions. There seems to exist a boundary level for the size of the finite-depth solutions (which may however depend on the seed size). Once this level is crossed, a full-screen solution seems to emerge, albeit possibly with long times of indecisive wandering back and forth at certain zones before proceeding downward. Also, retrogression may become quite important on several occasions, but they seem to be overcome in the long run (Fig. 6).

2. Some of the potentially unbounded solutions show particular periodic properties, and these just suggest that the obtained partial solution can be extrapolated inductively.

It has been observed that there are g_0 for which there are no potentially unbounded solutions (using an exhaustive search for the proper f_0). When there are solutions for a particular g_0, the set of *costates* f_0 for which a solution exist is rather restricted.

Several properties of the solutions will now be presented in a more systematic way.

Property 1 *The excess e_k must be an even integer.*

This follows from the fact that the number of CPN cells on any row $k \neq 0$ is always even. As a consequence, because e_k must be constant for all k, the CPN-seed g_0 must also have an even number of cells.

Property 2 *If g_0^m is the mirrored version of a CPN-seed g_0 (i.e. left-right interchange), its solution is the mirrored version of the solution of (g_0, f_0, e).*

Property 3 *Each solution $P(f_L, f_R; g_0, f_0, e)$ corresponds to a solution $P(\bar{f}_l, \bar{f}_R, g_0, f_0', -e)$ for the opposite excess value. $\bar{f}_L, \bar{f}_R$ represent the complementary sequences of f_L, f_R. f_0' represents the sequence obtained from f_0 by changing the values of the cells covered by the CPN at row 0.*

This is a consequence of the definition of the CPN. The solution to $-e$ is actually obtained by changing the states of the CPN-covered cells in the solution for e.

Property 4 *Each solution $P(f_L, f_R; g_0, f_0, e)$ provides also a solution for the $(g_k, f_k; e)$-problem, where g_k and f_k are the g- and f-states in the k^{th} row of the field $P(f_L, f_R; g_0, f_0, e)$. Indeed, $P(f_L^k, f_R^k; g_k, f_k, e)$ is the part of the original solution starting at row k.*

Any CPN-seed $(g_0; e)$ for which there is no possible *preseed* (i.e. a CPN-configuration from which g_0 can be generated and that has an even number of cells not smaller than $|e|$, and so is able to cover an excess e configuration), will be called a *primitive CPN-seed.* It should be mentioned that property 4 has no converse in the sense that it would be sufficient to consider only solutions for the primitive seed in order to find the solutions for all other seeds generated by this primitive seed. This is illustrated in Fig. 7 by means of an example representing the preconfiguration of all three potentially unbounded solutions to the problem ($g_0 = 10010010010010010001, e = 8$). These solutions were found by exhaustive search on f_0. The only preseed that will generate the same set of solutions is g_{-1}. g_{-3} will only generate the first two solutions and g_{-5} only the second solution. So g_{-1} is the only *equivalent preseed* to g_0 (for $e = 8$). It is certainly not a primitive seed. It also follows from property 4 that none of the preseeds in the example, when filled to obtain excess $e = 8$ (but in a different configuration for those already showing $e = 8$), will lead to a potentially unbounded solution. Indeed, this would have been found otherwise from g_0. The full-screen partial solutions corresponding to the example in Fig. 7 are shown in Fig. 8(a,b,c). In order to stress the importance of this fact, we would like to state it as a

(Non)-property 5 *The search for solutions for primitive CPN-seeds is insufficient to provide the solutions for nonprimitive seeds.*

The jointly additive property of the combined CPN and BDF fields (or of the equivalent quaternary field) has already been mentioned. If we have two such fields $(CPN, BDF)_1$ and $(CPN, BDF)_2$ then the field resulting from addition (in GF(4), or modulo-2 addition separately in CPN and BDF) of these fields where one is shifted horizontally over an arbitrary number of cells, is also a (CPN,BDF) field. A way of representing this is : $(CPN, BDF)_1 + z^l(CPN, BDF)_2$ where z^l represents a shift over l cells to the left. Unfortunately, we have to state

(Non)-property 6 *Rowwise occupational excess is generally not additive, nor is it conserved under shift-addition of two rowwise constant excess solutions.*

$(g_i, f_i) : i =$		
-7	$\oplus\oplus-\oplus\oplus-\oplus\oplus\oplus-\oplus\oplus-\oplus\oplus$	
-6	$\oplus-\oplus\oplus-\oplus\oplus--\oplus\oplus-\oplus\oplus-\oplus$	excess 10
-5	$\oplus\oplus\oplus-\oplus\oplus-\oplus-\oplus-\oplus\oplus-\oplus\oplus\ominus$	10
-4	$\oplus--\oplus\oplus-\oplus\oplus\oplus\oplus\oplus\oplus-\oplus\oplus-+\oplus$	12
-3	$\oplus\oplus-\oplus-\oplus\oplus-----\oplus\oplus-\oplus+\ominus\oplus$	8
-2	$\oplus-\oplus\oplus\oplus\oplus-\oplus----\oplus-\oplus\oplus\ominus\oplus+\ominus$	8
-1	$\oplus\oplus\oplus---\oplus\oplus\oplus---\oplus\oplus\oplus-++\ominus\oplus\ominus$	8
0	$\oplus--\oplus--\oplus--\oplus--\oplus--\oplus+-\oplus++\oplus$	8

1st solution : $f_0 = 1001001001001001101111$

g_{-6} is the primitive seed corresponding to $(g_o, e_o = 8)$.

g_{-7} is not a primitive seed (it has an odd number of CPN-cells).

$i =$		
-6	$\oplus-\oplus\oplus-\oplus\oplus--\oplus\oplus-\oplus\oplus+\oplus$	excess 10
-5	$\oplus\oplus\oplus-\oplus\oplus-\oplus-\oplus-\oplus\oplus-\ominus\ominus\oplus$	8
-4	$\oplus--\oplus\oplus-\oplus\oplus\oplus\oplus\oplus\oplus-\oplus\ominus-+\ominus$	8
-3	$\oplus\oplus-\oplus-\oplus\oplus-----\oplus\oplus+\ominus+\oplus\oplus$	8
-2	$\oplus-\oplus\oplus\oplus\oplus-\oplus----\oplus-\ominus\oplus\oplus\ominus-\oplus$	8
-1	$\oplus\oplus\oplus---\oplus\oplus\oplus---\oplus\oplus\ominus+-+\ominus\oplus\oplus$	8
0	$\oplus--\oplus--\oplus--\oplus--\oplus-+\oplus++\oplus+-\oplus$	8

2nd solution : $f_0 = 1001001001001011111101$

$i =$		
-6	$\oplus-\oplus\oplus-\oplus\oplus--\oplus\oplus-\oplus\ominus+\ominus$	excess 6
-5	$\oplus\oplus\oplus-\oplus\oplus-\oplus-\oplus-\oplus\oplus+\oplus\oplus\oplus$	12
-4	$\oplus--\oplus\oplus-\oplus\oplus\oplus\oplus\oplus\oplus-\ominus\ominus--\ominus$	6
-3	$\oplus\oplus-\oplus-\oplus\oplus-----\oplus\ominus-\ominus-\ominus\oplus$	4
-2	$\oplus-\oplus\oplus\oplus\oplus-\oplus----\oplus+\ominus\ominus\ominus\ominus+\oplus$	4
-1	$\oplus\oplus\oplus---\oplus\oplus\oplus---\oplus\ominus\oplus---\oplus\ominus\oplus$	8
0	$\oplus--\oplus--\oplus--\oplus--\oplus++\oplus--\oplus+\oplus$	8

3rd solution : $f_0 = 1001001001001111001111$

Figure 7: Prestates of the three configurations (g_0, f_0) with potentially unbounded solutions for $g_0 = 1001001001001001001001$. The mirror versions are not considered as they are redundant.

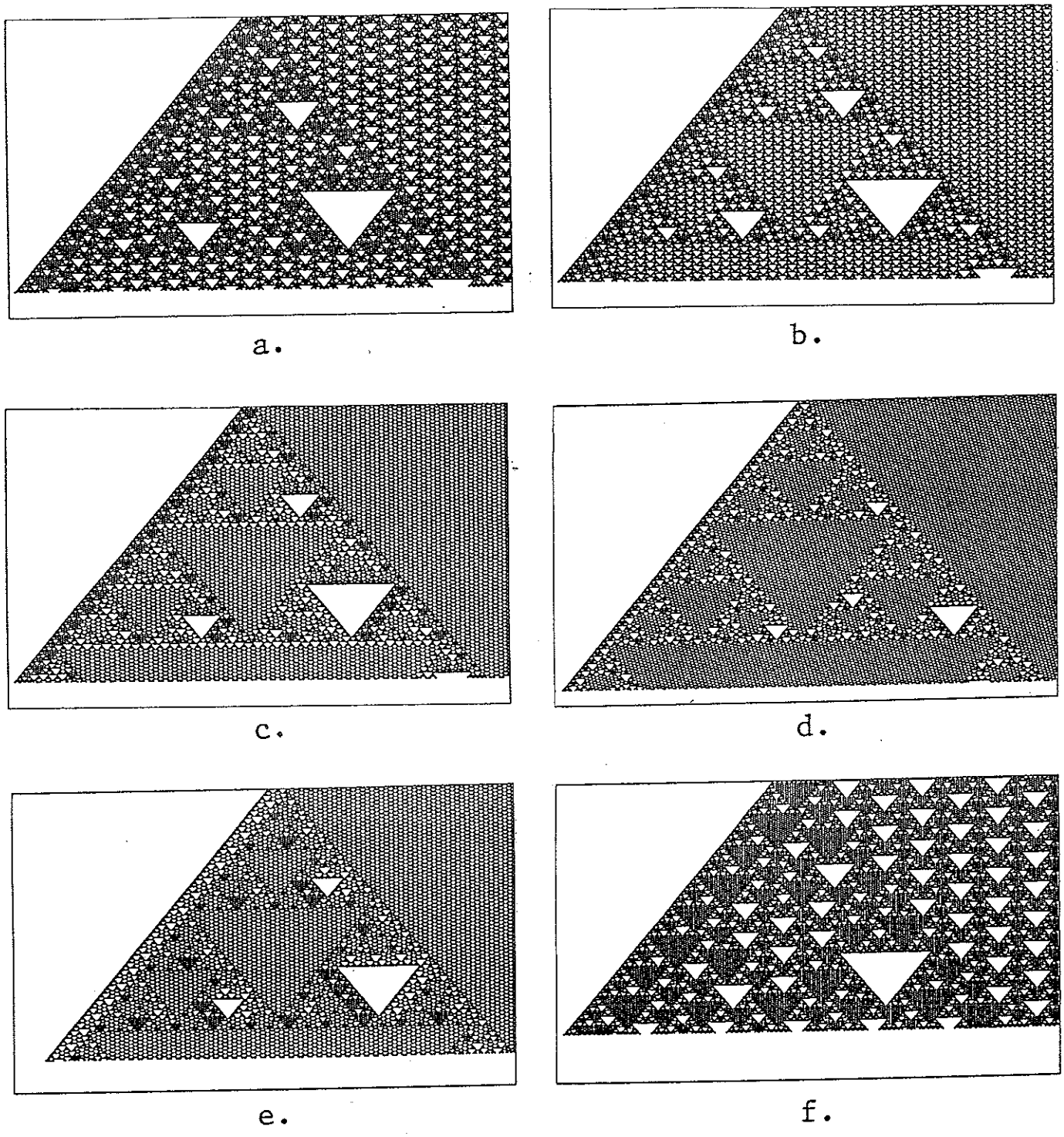


Figure 8: Partial solutions for excess $e \neq 0$.
a.b.c. the solutions for $g_0 = 1(001)^7$, $e = 8$
d. $(g_0, f_0, e) = (10011000111, 10011001110, 4)$
e. $(g_0, f_0, e) = (1(001)^9, (0)^{16}(110000)^2, -10)$
f. $(g_0, f_0, e) = (1(001)^{15}, (100)^{10}10110110(1)^8, 16)$
(exponents refer to repeated concatenation).

So, counting numbers of CPN-cells and excess-values is a *nonlinear* operation. There are provable exceptions to this non-additiveness for certain solutions. These will be mentioned later.

For the properties that now follow and that are indicated as being conjectural, there is no proof available as yet. They are based on a multitude of computer generated examples and on the nonavailability of counterexamples.

(Conjectural) Property 7 *For $e \neq 0$, any solution $P(f_L, f_R; g_0, f_0, e)$ has periodic f_L, f_R sequences. One of these sequences has always period length 1 and consists of a sequence of all occupied cells when $e > 0$, and of empty cells when $e < 0$. It also follows that any solution for a given (g_0, f_0) is* unique.

A number of solutions are presented in Fig. 8.

Property 8 *To any solution for a $(g_0; e \neq 0)$ problem, (in the condition of satisfying property 7), corresponds a solution for $(g_0; e = 0)$.*

This $(g_0, e = 0)$ solution shows the same period as the one whose period length differs from 1 in the $(g_0, e \neq 0)$ solution.

Proof: this is a consequence of the combined overall periodic structure of a given CPN and of a BDF that has a skew column with all cells occupied. Consider Fig. 9 which schematically represents the CPN- and BDF-structure of a solution with $e \neq 0$, up to row level 2^{k+1} (for an arbitrary k). From the analysis in reference [2], it follows that the parts (B,D) are complete copies of the parts (A,M) (for both the CPN-structure *and* the underlying BDF). On the other hand, C is a copy of A only for the CPN part. Consider now the rows 2^k to $2^{k+1} - l_s$ (l_s is the length of the CPN-seed on the top row). The global row excess there is e, but as the excess in part B already equals e, the excess of the cells of CPN-subset C must equal 0. The periodic structure on f_R extends over C too.

Moreover, the sequence segments XY and UV are complementary (see [2]). The local periodicity of the segment UV, induced by the periodicity of f_R, will be inherited by XY in its complementary form.

It is conjectured (based on observations) that for each $(g_0, e = 0)$-solution, there is at least one solution for the same g_0 and an excess-value $e \neq 0$.

Property 9 *If*

a) $e = 0$

b) g_0 *is a* symmetric *CPN-seed*

c) the CPN-cells in g_0 *show an* antisymmetric *occupation*

d) the non CPN-cells in g_0 *show a* symmetric *occupation,*

then there are an infinite number of completely arbitrary solutions in the sense that there is a free choice at each row. This means that f_L in the solution $P(f_L, f_R; g_0, f_0, 0)$ is completely arbitrary, while f_R is the complement of f_L. It follows that there is complete symmetry for the BDF-cells not covered by the CPN.

The symmetry conditions mentioned together under the points a, b, c, d above will be called the *cosymmetry* conditions on (g_0, f_0).

Proof : when (g_0, f_0) satisfies the cosymmetry conditions, then the resulting (g_1, f_1) on the next row is also cosymmetric. Besides, there is a free choice for the occupational states of the boundary CPN-cells. By induction, this holds then for every row. The following example clarifies the situation.

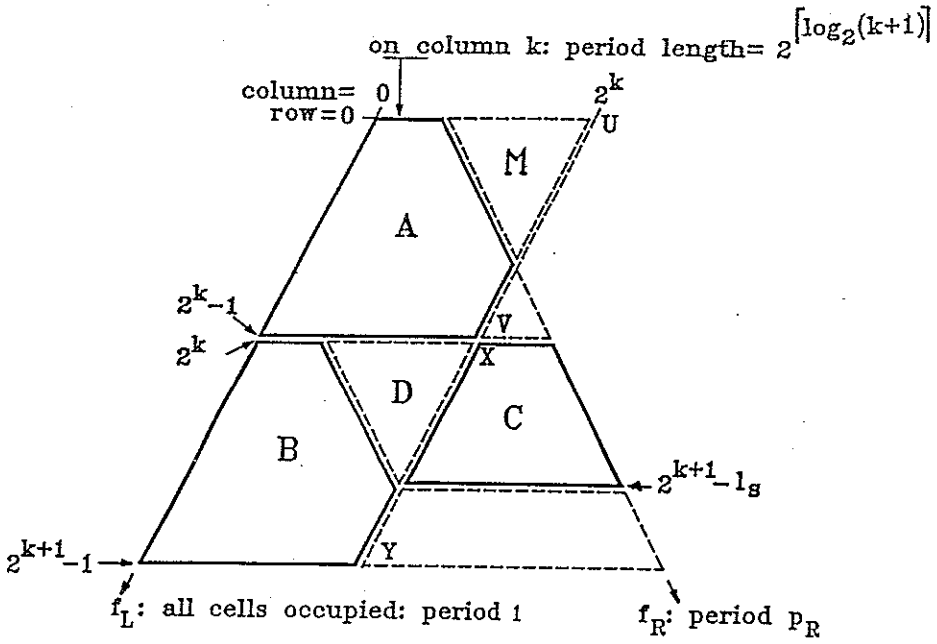


Figure 9: Structure of a (CPN,BDF)-field.

↓ symmetry axis

$$\begin{array}{c} \oplus\ominus-+\oplus-\ominus+-\oplus\ominus \\ \ominus+\ominus+\ominus\oplus\ominus\oplus+\oplus+\oplus \\ \oplus\oplus\oplus\oplus\oplus+-+\ominus\ominus\ominus\ominus\ominus \\ \oplus----\ominus++\oplus----\ominus \end{array}$$

A full-screen example of a solution with a *random* f_L is given in Fig. 11(a).

Property 10 *The existence of a completely arbitrary solution to* $(g_0, f_0, e = 0)$ *implies that* (g_k, f_k) *is cosymmetric for all* k.

Proof : by contradiction. First observe that the fact of a CPN-seed being symmetric or not is inherited by the whole CPN. Then consider a nonsymmetric g_0 and assume there exists a completely arbitrary solution to it. A *general change propagation net* (GPCN) is a net consisting of all cells in a BDF that must change value together while the overall field remains a BDF. It is always a net, the elementary meshes of which have the forms given in Fig. 10(a).

Now select some particular row r_p on which there are two cells on symmetric locations, one cell (A) belonging to the g_0-related CPN, the other (B) not. Then it is always possible to construct a symmetric GCPN that only covers the 2 selected cells on that row (see Fig. 10(b)). Because of the symmetry of this GCPN, the left and right-skew boundaries of the original CPN are both covered by this GCPN at the same row-levels. The assumed complete arbitrariness of the solution allows these particular boundary cells to change their value, while the row excess remains $e = 0$. However, this cannot be true for row r_p, as only one CPN-cell of this row also belongs to the GCPN (cell A). This means that complete arbitrariness of the solution is only possible for a symmetric CPN. It also follows that symmetrically located CPN-cells must

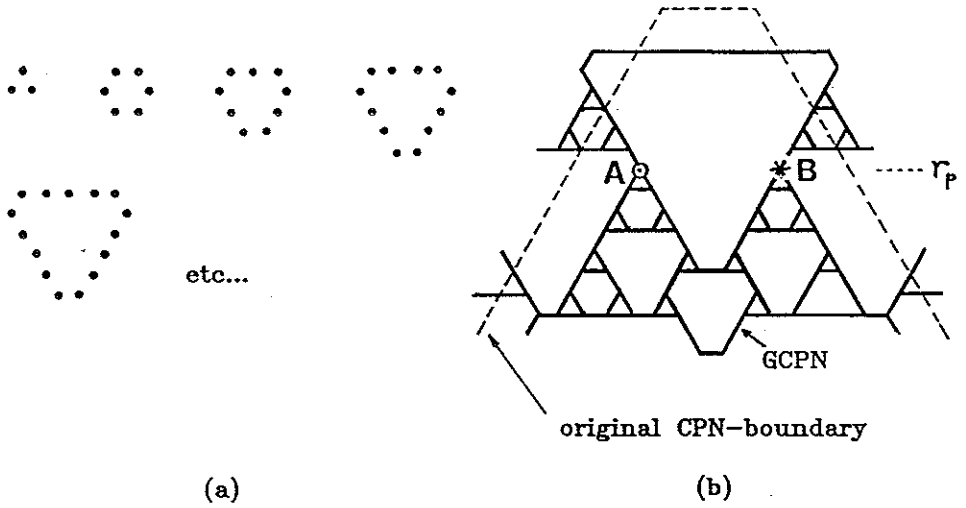
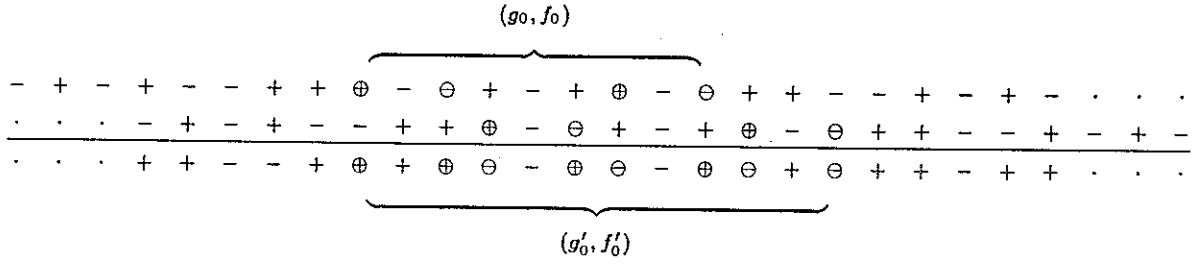


Figure 10: (a) : elementary meshes of a GCPN. (b) : a symmetric GCPN covering two particular cells

have complementary occupational states. This fact taken together with the local evolution rule implies that symmetrically located non CPN-cells must have the same state.

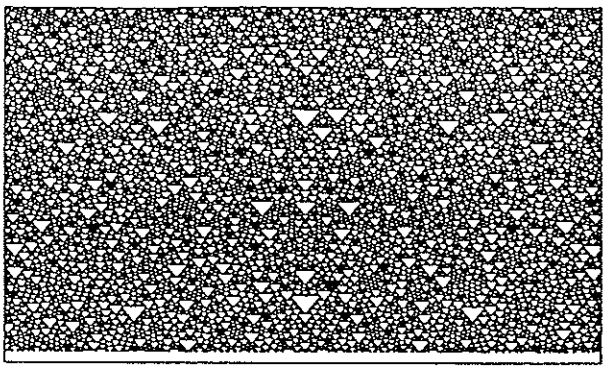
Property 11 *When (g_0, f_0) is a cosymmetric seed, and $P(h_L, h_R; g_0, f_0, 0)$ is an arbitrary solution to the constant excess = 0 problem, the shiftsum $(1+z^k)P(h_L, h_R; g_0, f_0, 0)$ for $k \neq 0$ is also an arbitrary solution to some new cosymmetric (g_0', f_0').*
Solutions with this general shift-additive property will be called perfect.

Proof: this is a consequence of the particular symmetries involved : h_L and h_R are completely symmetric (property 7). The following example clarifies this :

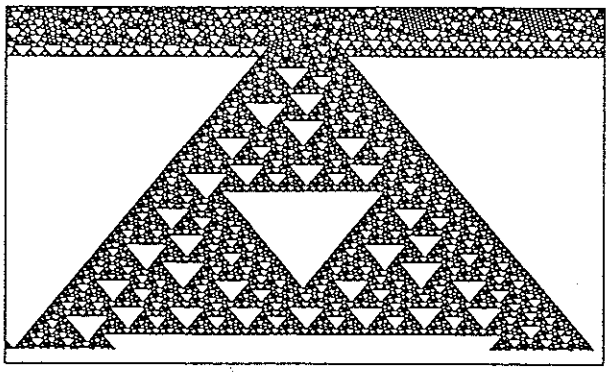


Observe that for solutions to non-cosymmetric (g_0, f_0), this shift-additivity also holds for shifts over distances $2^j (j = 0, 1, 2 \ldots)$. Such shifts do indeed generate the subfields of the original field starting at rowrank 2^j. Experimental results suggest that perfect solutions may also occur for non-cosymmetric seeds (e.g. a particular solution to $(g_0, f_0) = \ominus - + - \ominus + + \oplus - \oplus$ has been observed to be a candidate for perfectness).

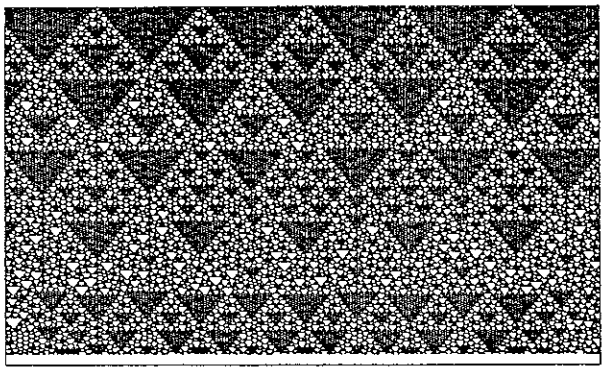
Fig. 11 shows different potentially unbounded solutions for different seeds $(g_0, f_0, e = 0)$. Fields with rowwise excess $e = 0$ under a given CPN, will be called *rowwise balanced.* Apart from the completely arbitrary random solution in Fig. 11(a), all other balanced solutions seem to be highly structured. Some of the solutions give rise to periodic patterns in the left- and right outer field, after perhaps an initial transient zone. In these cases, the whole



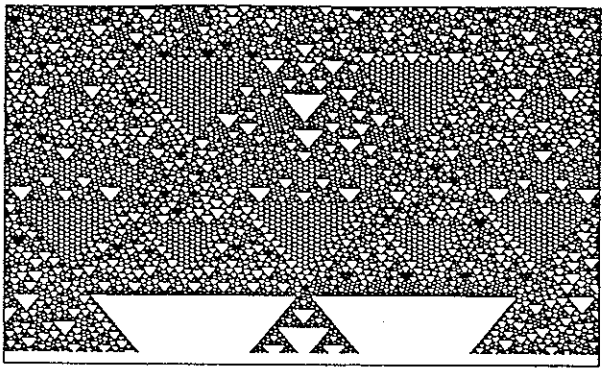
a. $(g_o, f_o) = (1001, 1000)$
random solution



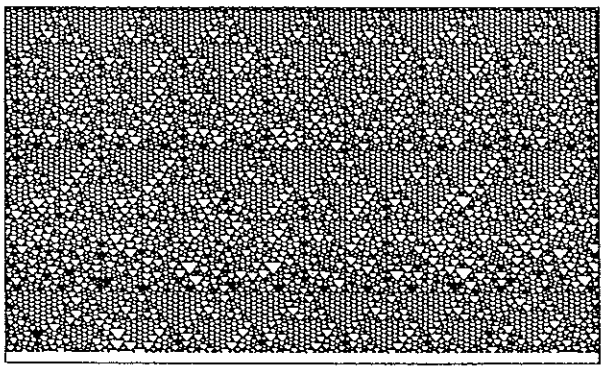
b. $(g_o, f_o) = (1001101, 0011110)$



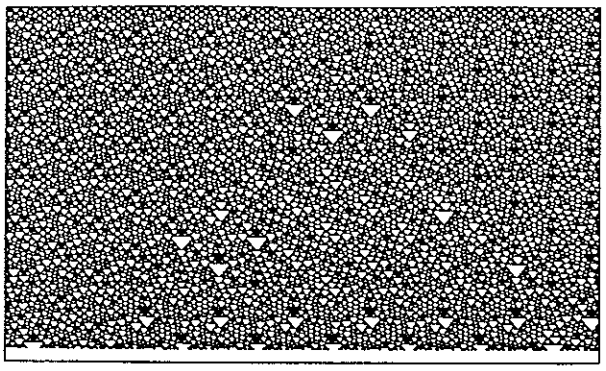
c. $(g_o, f_o) = (1001, 1010)$



d. $(g_o, f_o) = (10111, 01011)$



e. $(g_o, f_o) = (1001, 1010)$



f. $(g_o, f_o) = (100001, 100010)$

Figure 11: Examples of rowwise balanced fields.

Table 1: Auxiliary sequences

n	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$M(n)$	:	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	...
$T(n)$	:	2	4	6	2	6	4	2	4	6	4	2	6	2	4	6	2	...
$S(n)$	:	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	...
$I(n)$	:	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	...
$O(n)$	:	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	...
$\Gamma(n)$	:	3*	5	9*	15	17	23*	27	29*	33*	39	43*	45	51	53*	57	63*	...
$\Phi(n)$	:	0	2	6	4	12	14	10	8	24	26	30	28	20	22	18	16	...
$Q(n)$	:	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4	...

pattern seems to emerge from the collision of two periodic half-infinite sequences h_L and h_R on the top row. Moreover, the eventually induced periodic patterns along f_L, f_R seem to have the same period length. It may happen that left and right periodic patterns are the same, or symmetric versions of each other, or differ. For several of the examples (Fig. 11(d), Fig. 5(c)), no overall periodicity seems to emerge, at least not for the limited-size solutions available (some examples have been analyzed up to a depth of 800 rows). Nevertheless, these solutions too are highly structured, showing a strange interweaving between *locally periodic* (crystalline) and *locally amorphous* structures. One really wonders whether this feature is not manifest on ever increasing scales, and whether it has anything in common with self-similar and overall nonperiodic sequences such as the *Morse-Thue* sequence which will be considered more closely in the next section. Or are they just part of a long transient ?

As for the number of potentially unbounded solutions, the observations can be summarized in the following statements :

1. there are seeds for which there is no solution for a given e-value.
2. if there are solutions for g_0, the number of *costates* f_0 for which a solution exists is rather limited.
3. when $e \neq 0$, the solution for (g_0, f_0) is unique.
4. one particular (g_0, f_0)-pair may produce many different rowwise balanced fields. These solutions depend on the particular random sequences generated during the search procedure, and on the particular regression depth.

3 An isomorphism between the invariant-excess problem and a walk on a particular fractal dust-matrix.

The evolution of the solution of the constant excess problem starting with an initial (g_0, f_0) is given by (g_k, f_k). Remind that g_k reflects the CPN-state at row k, and f_k the state of the underlying BDF confined to the CPN-range on row k. From now on, the decimal equivalent representations (γ_k, φ_k) for (g_k, f_k) will be used. Of interest is the path that (γ_k, φ_k) may follow in a corresponding two-dimensional discrete phase-space.

A number of evolutionary rules will be stated without proofs, as these are rather cumbersome. It will be helpful to introduce the sequences as given in table 1. M(n) is the sequence

representing the parity of the set of digits in the binary representations of n (= modulo-2 sum of the digits). It is known in the literature as the Morse-Thue sequence [6], and was already mentioned in the previous section. Thue and Morse seem to have independently introduced this sequence in 1906 and 1921 respectively [6,7]. A multitude of references discussing properties of this sequence and its generalizations can be found in [8]. M(n) can be constructed recursively starting with 0. Each following step then appends the complement of the already existing part, thus doubling the sequence length.
S(n) can be reconstructed from M(n) by substituting each 0 in M(n) by 0001 and each 1 in M(n) by 0100. T(n) is derived from M(n) by replacing each 01,11,10,00 in M(n) respectively by 2,4,6,4. I(n) starts with 1, and in each size-doubling step, the first part is repeated alternatively with and without complementation.

3.1 The evolution of γ_k.

The evolution of g_k as a binary number, where k is the *row-index*, is easy to obtain :

$$g_{k+1} = 1 \circ Dg_k \circ 1, \tag{1}$$

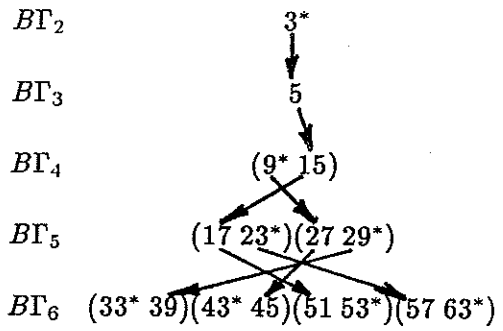
where $\circ$ denotes concatenation, and Dg_k is the binary difference sequence of g_k. The corresponding decimal representation γ_k is odd. As the CPN-states g_k also have an even number of ones, the set Γ of allowable CPN-states is given by those odd natural numbers n for which $M(n) = 0$. These are precisely the numbers n for which $S(n) = 1$. So $\Gamma = \{n : S(n) = 1\} = \{3, 5, 9, 15, \ldots\}$. The corresponding sequence is $\Gamma(n)$. Observe that $\Gamma(n+1) = 3 + \sum_{i=0}^{n} T(n)$ for $n > 0$, with $\Gamma(0) = 3$. CPN-states γ_k that have no prestates in Γ (i.e., the γ_k for which a possible prestate g_{k-1} has an odd number of ones) will be called *irreducible*. The irreducible states in $\Gamma(n)$ are found to be those for which $I(n) = 1$. These have been marked by an asterisk in $\Gamma(n)$.
It will be convenient to consider pairs of states in $\Gamma(n)$, as indicated below

$$\begin{aligned}\Gamma(n) \;=\; & (3^*\,5)(9^*\,15)(17\,23^*)(27\,29^*)(33^*\,39)\\ & (43^*\,45)(51\,53^*)(57\,63^*)\ldots\end{aligned}$$

Observe that each pair consists of just one reducible and one irreducible state, but in changing order. If the order (a^*, b) is denoted by 1, and (a, b^*) by 0, the orientations vary according to the sequence $O(n) \triangleq I(2n)$.
Define *block* B_j as the interval of integers from 2^{j-1} to $2^j - 1$, for $j = 1, 2, 3 \ldots$ $B_0 = [0]$, and j is the *block-rank*. The number of states $\#B_j = 2^{j-1}$. Define $B\Gamma_j$ as the interval of allowable states γ in block B_j. $\#B\Gamma_j = 2^{j-3}$.

We will now present an algorithm to find the successor γ_{k+1} of γ_k. First observe that if $\gamma_k \in B\Gamma_j$, then $\gamma_{k+1} \in B\Gamma_{j+1}$ (as a consequence of eq.(1)). From $B\Gamma_2$ up to $B\Gamma_6$, the

following transitions are found :



Now, each element in $B\Gamma_j$ will be mapped into the reducible state of a pair in $B\Gamma_{j+1}$ (so each state in $B\Gamma_j$ maps into a pair in $B\Gamma_{j+1}$). This mapping is obtained by a rather complex shuffle involving $\log_2(\#B\Gamma_j) = (j-3)$ steps.

All operations to be introduced will work upon sequences with a cardinality equal to an integer power of 2, so that they can be divided into 2, 4, 8,... subsequences of equal length. Consider such a sequence S. *A splitting operator of rank r*, $D_r(S)$ will divide S into 2^r equally-sized parts. Define an *ordering sequence* Ω_r of rank r. $\Omega_0 = L, \Omega_1 = LR, \Omega_2 = LRRL\ldots$ etc. In general, Ω_r corresponds to the first 2^r elements of the Thue-Morse sequence M(n) where 0 has been replaced by L and 1 by R. L and R operate on sequences. L means that the elements of the sequence on which it operates should be rearranged to assure an increase of value in its elements to the left. R will rearrange the elements in increasing order to the right.

A *shuffle of rank r* on a sequence S , is the combined operation $\Omega_r(D_r(S))$, whereby the 2^r elements of Ω_r operate in their order of appearance on the 2^r subsequences in $D_r(S)$, also in their proper order. Such a shuffle will be denoted by $Sh_r(S)$.

The procedure for mapping $B\Gamma_j$ into $B\Gamma_{j+1}$ then goes as follows :

- Step 1 : Consider the sequence S_j of pairs in $B\Gamma_j$. This sequence shows its natural order.
  Apply $Sh_0(S_j)$, i.e. reverse the order of the pairs (the order of the γ-values inside the pairs should remain unchanged). Call the resulting sequence S_j^0.

- Step s : ($s = 2$ to $j-4$)
  Obtain $S_j^{s-1} = Sh_{s-1}(S_j^{s-2})$ (in-pair orientation unchanged)

- Step $j-3$:
  Reorient the elements inside the pairs by performing $Sh_{j-4}(S_j^{j-5})$ to obtain $\hat{S}_j (= S_j^{j-4})$.

$\hat{S}_j$ now represents a permutation of the elements in $B\Gamma_j$. They map in their order of appearance into the pairs in $B\Gamma_{j+1}$ in their natural order.
The whole procedure is illustrated in Fig. 12 for the map $B\Gamma_7 \rightarrow B\Gamma_8$.

Following properties are related to this algorithm :

1. Neighbouring elements, pairs, 2-pairs ... 2^l-pairs in S_j (where an m-pair is an aggregate of m neighbouring pairs as possibly covered by the $D_r(S_j)$ subsequences), remain neighbours in $\hat{S}_j$.

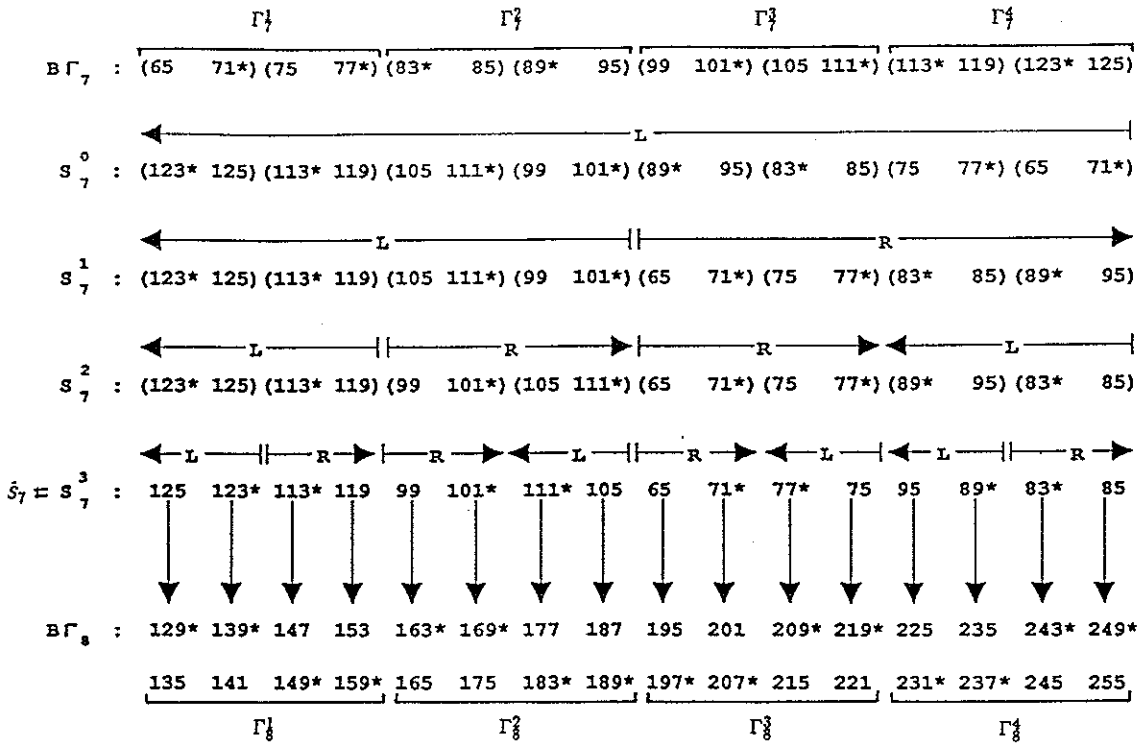


Figure 12: The map from $B\Gamma_7$ into $B\Gamma_8$. The pairs in $B\Gamma_8$ are represented vertically. The actual mapping is into the reducible states (without *).

2. Splitting up $B\Gamma_j$ and $B\Gamma_{j+1}$ in 4 equally-sized subsequences indexed $\Gamma_j^1, \Gamma_j^2, \Gamma_j^3, \Gamma_j^4$ and $\Gamma_{j+1}^1, \Gamma_{j+1}^2, \Gamma_{j+1}^3, \Gamma_{j+1}^4$, will show the mapping

$$\begin{array}{cccc} \Gamma_j^1 & \Gamma_j^2 & \Gamma_j^3 & \Gamma_j^4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \Gamma_{j+1}^3 & \Gamma_{j+1}^4 & \Gamma_{j+1}^2 & \Gamma_{j+1}^1 \end{array}$$

This reflects a constant permutation for every shuffle $j \rightarrow j+1$. Such constancy does not exist for the permutations of smaller subsequences in a shuffle. We learn from this that there is a large variation in position of the state inside the blocks when it moves through the consecutive steps (i.e. when the CPN evolves along the rows). The state moves from a larger (smaller) value in $B\Gamma_j$ to one of the smaller (larger) values in $B\Gamma_{j+1}$. This will be referred to further as the *agitation* property.

3.2 The evolution of φ_k.

Recall that g_k is a possible CPN-state at row k, with $l(g_k) = \lfloor \log_2 \gamma_k \rfloor + 1$ binary digits. f_k is the binary number representing the state of the underlying BDF-cells at row k confined to the g_k-range. f_k also has $l(g_k)$-digits. But, as opposed to g_k, f_k may have zeroes everywhere. Where γ_k (dec. equiv. of g_k) $\in [2^{l(g_k)-1}+1, 2^{l(g_k)}-1]$ and belongs to block $B_{l(g_k)}$, φ_k (dec. equiv. of f_k) may range from 0 to $2^{l(g_k)}-1$. φ_k belongs to block B_{ρ_k}, with $\rho_k = \lfloor \log_2 \varphi_k \rfloor + 1$. When $\rho_k = l(g_k)$, the costates (γ_k, φ_k) will be called *syntopic*, as they belong to blocks of the same rank. Otherwise $\rho_k < l(g_k)$, and the costates will be called *non-syntopic*. Syntopicness of (γ_k, φ_k) requires a 1 as leftmost digit in f_k.

As opposed to γ_k, φ_k may be any natural number. Its evolution is given by

$$f_{k+1} = [0|1] \circ Df_k \circ [0|1] \tag{2}$$

where [0|1] stands for a selection between 1 and 0. The particular selection should assure the proper excess-value on row $k+1$. For φ_k, this will once again translate into a more complex

scheme. First define the $\Phi(n)$-sequence (cf. table 1). It can be recursively constructed in a size-doubling way :

- step 0 : start with 0
- step j : take the previous partial sequence, add 2^j to its elements and append it in reverse order to the previous partial sequence.

The transition from φ_k to φ_{k+1} is then obtained as follows :

- If φ_k is not syntopic with γ_k, then

$$\varphi_{k+1} = [0|2^{l(g_k)}] + \Phi(\varphi_k) + [0|1], \tag{3}$$

where the particular choice must correspond to the one in eq.(2). If $2^{l(g_k)}$ is selected in the first term, φ_{k+1} will become syntopic with γ_{k+1}. If 0 is selected, $(\varphi_{k+1}, \gamma_{k+1})$ will be nonsyntopic. In the latter case, $\rho_{k+1} = \rho_k + 1 < l(g_{k+1}) = l(g_k) + 1$.

- If (φ_k, γ_k) are syntopic, i.e. $\rho_k = \lfloor \log_2 \varphi_k \rfloor + 1 = l(g_k)$:

$$\varphi_{k+1} = [0|2^{l(g_k)}] + \Phi(2^{\rho_k} - \varphi_k - 1) + [0|1] \tag{4}$$

The selection $2^{l(g_k)}$ will again produce syntopicness of $(\varphi_{k+1}, \gamma_{k+1})$.

If 0 is selected, φ_{k+1} may fall into a block of rank which may be much smaller than the blockrank of φ_k : $0 \leq \rho_{k+1} \leq \rho_k$. The higher the φ_k-value in its block, the smaller the blockrank ρ_{k+1}.

As a consequence, costates (γ_k, φ_k) only get *syntopicized* when the leftmost CPN-cell on row k becomes occupied.

We are now in a position that enables us to trace the path of (γ_k, φ_k), starting from an initial seed. A typical movement over different blocks for increasing k is shown in Fig. 13. All blocks are shown at logarithmic scale (consecutive blocks actually double in size). The particular jump-sizes depend on the position of φ_k in the φ-blocks. The motion is only possible in the right upper triangular part.

Among all possible paths that may evolve from a given initial seed, we want to find those that guarantee constant excess.

3.3 The excess-matrix covering the (γ_k, φ_k) phase-plane.

Consider (g_k, f_k) as a pair of binary sequences of equal length. An occupied cell in the BDF under the CPN at row k is found at the locations where both g_k and f_k show a 1. The number of occupied cells in (g_k, f_k) can be found by counting the number of 1's in $g_k \otimes f_k$, where $\otimes$ represents the bitwise AND-ing operation of g_k and f_k. In the same way, the number of nonoccupied cells in (g_k, f_k) is found by counting the number of 1's in $g_k \otimes \bar{f}_k$, where $\bar{f}_k$ is the complemented sequence of f_k.

Moving to the equivalent decimal number representations of binary sequences, the result of the AND-operation for arbitrary sequences (g, f) can be found in a matrix C with the particular structure shown in Fig. 14(a).

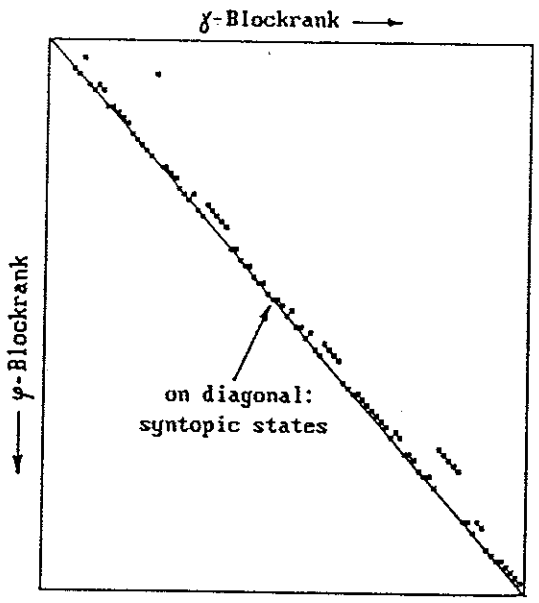


Figure 13: Possible motion over the (γ, φ)-blocks in the state space.

$\varphi\backslash\gamma$	0	1	2	3	4	5	6	7	...
0	0	0	0	0	0	0	0	0	...
1	0	1	0	1	0	1	0	1	...
2	0	0	2	2	0	0	2	2	...
3	0	1	2	3	0	1	2	3	...
4	0	0	0	0	4	4	4	4	...
5	0	1	0	1	4	5	4	5	...
6	0	0	2	2	4	4	6	6	...
7	0	1	2	3	4	5	6	7	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

(a)

$\varphi\backslash\gamma$	0	1	2	3	4	5	6	7	...
0	0	0	0	0	0	0	0	0	...
1	0	1	0	1	0	1	0	1	...
2	0	0	1	1	0	0	1	1	...
3	0	1	1	2	0	1	1	2	...
4	0	0	0	0	1	1	1	1	...
5	0	1	0	1	1	2	1	2	...
6	0	0	1	1	1	1	2	2	...
7	0	1	1	2	1	2	2	3	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

(b)

Figure 14: a) AND-ing two binary sequences in decimal form. b) The number of 1's in the sequence resulting from an AND-operation.

This matrix C can be constructed in a size-doubling recursive way :

$$C_{m+1} = \begin{bmatrix} C_m & C_m \\ C_m & C_m + 2^m \end{bmatrix} \quad \text{with } C_0 = [0] \tag{5}$$

$C_m + 2^m$ means that 2^m must be added to *all* elements in C_m. The size of C_m equals 2^m.

The number of 1's in the binary representation of a number n can be found as the corresponding element in the sequence $Q(n)$ (cf. table 1).

$Q(n)$ can also be obtained in a recursive way. Start with $Q_0 = 0$, and at each next stage, append the previous sequence augmented by 1. If all odd numbers in $Q(n)$ are replaced by 1 and the even by 0, the Morse-Thue sequence appears once more.

Performing the substitution $n \rightarrow Q(n)$ in the AND-matrix C, results in the C^+-matrix, which can also be derived using the recursion

$$C^+_{m+1} = \begin{bmatrix} C^+_m & C^+_m \\ C^+_m & C^+_m + 1 \end{bmatrix} \quad \text{with } C^+_0 = [0] \tag{6}$$

C^+ is shown in Fig. 14(b).

The corresponding matrices for the $g \otimes \bar{f}$ operation are found by turning the matrices upside down, as $\bar{f}$ takes the complementary value of f. This results in

$$C^-_{m+1} = \begin{bmatrix} C^-_m & C^-_m + 1 \\ C^-_m & C^-_m \end{bmatrix} \quad \text{with } C^-_0 = [0] \tag{7}$$

The excess value of occupied cells over nonoccupied cells corresponding to a (g, f)-pair can then be found in the excess-matrix $E \triangleq C^+ - C^-$. Its recursive form is :

$$\begin{aligned} E_{m+1} &\triangleq C^+_{m+1} - C^-_{m+1} \\ &= \begin{bmatrix} E_m & E_m - 1 \\ E_m & E_m + 1 \end{bmatrix} \quad \text{with } E_0 = [0] \end{aligned}$$

Part of the excess-matrix is shown in Fig. 15. E_m displays all integers from $-m$ to m, and these are distributed over the matrix in a very intricate way. Fig. 16 shows the distributions for different excess values in E_9 (matrix dimension 512×512). The *selfsimilarity* is apparent, and these distributions remain quite similar for growing matrix dimensions. In the unbounded limit, these subsets form a *fractal dust* [9] with fractal dimension 2 (!). The overall matrix exhibits several interesting scaling and recursive properties. Notice the symmetry between upper half and lower half for opposite excess-values. Values in one column are either all odd or all even.

It is clear now that solutions to the invariant excess problem are given by those excursions allowed by the (γ_k, φ_k)-state, which remain on the particular fractal dust associated with the given excess-value. Recall that the allowable γ_k-states belong only to a particular subset Γ of the γ along the matrix-columns.

Consider a *positive excess*-value. When asked for a solution, it would be rather plausible to suggest that it should move as close as possible along the diagonal blocks, as the dust concentration is clearly highest there. This would maximize the chance of hitting a dust-particle. The agitation property of γ_k even seems to strengthen this urge. So, the solutions

$\varphi\backslash\gamma$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	−1	−1	−2	−1	−2	−2	−3	−1	−2	−2	−3	−2	−3	−3	−4
1	0	1	−1	0	−1	0	−2	−1	−1	0	−2	−1	−2	−1	−3	−2
2	0	−1	1	0	−1	−2	0	−1	−1	−2	0	−1	−2	−3	−1	−2
3	0	1	1	2	−1	0	0	1	−1	0	0	1	−2	−1	−1	0
4	0	−1	−1	−2	1	0	0	−1	−1	−2	−2	−3	0	−1	−1	−2
5	0	1	−1	0	1	2	0	1	−1	0	−2	−1	0	1	−1	0
6	0	−1	1	0	1	0	2	1	−1	−2	0	−1	0	−1	1	0
7	0	1	1	2	1	2	2	3	−1	0	0	1	0	1	1	2
8	0	−1	−1	−2	−1	−2	−2	−3	1	0	0	−1	0	−1	−1	−2
9	0	1	−1	0	−1	0	−2	−1	1	2	0	1	0	1	−1	0
10	0	−1	1	0	−1	−2	0	−1	1	0	2	1	0	−1	1	0
11	0	1	1	2	−1	0	0	1	1	2	2	3	0	1	1	2
12	0	−1	−1	−2	1	0	0	−1	1	0	0	−1	2	1	1	0
13	0	1	−1	0	1	2	0	1	1	2	0	1	2	3	1	2
14	0	−1	1	0	1	0	2	1	1	0	2	1	2	1	3	2
15	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4

Figure 15: The excess-matrix E_4.

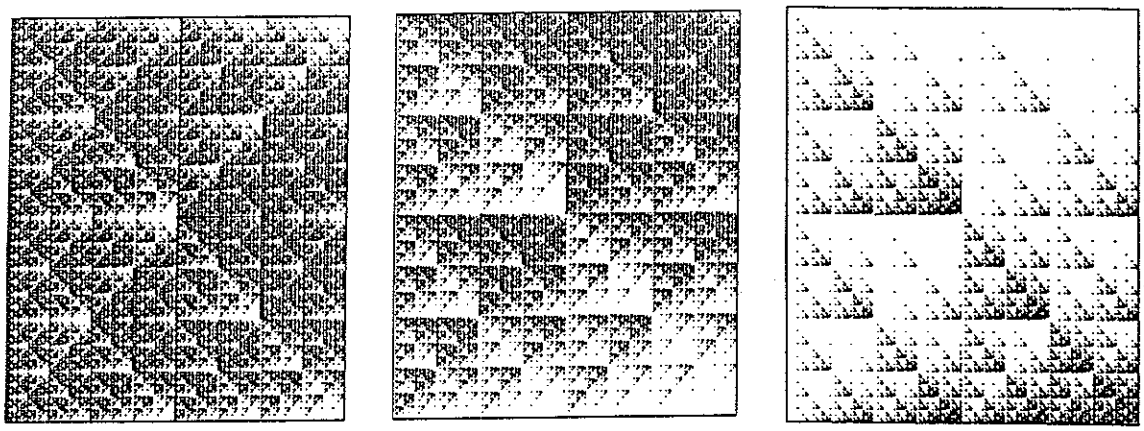


Figure 16: Distribution for different excess values : $e = 0, e = -2, e = 4 \ldots$

would have overall syntopic costates (γ_k, φ_k), meaning that all f_k would have a 1 as leftmost digit. Although this provides no proof, it gives an intuitive understanding of property 7. This stated that for positive excess-values, one of the boundary sequences f_L or f_R in the solution must be full of 1's. f_L can always be made this boundary by left-right mirroring of the (g, f)-states if necessary. And this corresponds to the states being syntopic everywhere. But syntopicness is not sufficient, as there are initial states (γ_0, φ_0) for which there are no solutions. On the other hand, property 7 also implies that for the equivalent mirrored version of the problem (where f_R is full of 1's), the corresponding traject in the phase-plane is not fully syntopic, but is periodically syntopicized. For *negative excess*-values, the dust is more concentrated in the upper right triangular part of the E-matrix, which makes it plausible that the (γ_k, φ_k) are never syntopic for these solutions. This is again confirmed in property 7. For an *excess* $e = 0$, there is a more regular spread of the dust in all regions of the matrix. This makes it less surprising that the solutions are not exclusively syntopical or nonsyntopical, whatever the particular left/right orientation of the (g, f)-states. Nevertheless, the excursions from the diagonal blocks seem to remain rather small, (Fig. 13). In this context, cosymmetric evolutions have the particular property that they may become syntopicized or unsyntopicized in a completely arbitrary way : they will always remain on the o-dust.

4 Concluding remarks.

The nature of the evolutionary patterns of a very simple binary cellular automaton that also satisfies the more global condition of rowwise excess conservation has been discussed.

It has been shown that solutions exist for particular initial seeds, and that in the most general case (noncosymmetry), the emerging fields are complex, but highly structured. This is what makes them particularly interesting.

A number of properties could only be presented in a conjectural status, as they were based on observations of partial computer results. One should keep in mind that the solutions shown have negligible size as compared to the conjectured unboundedness. It is presently not known if it will actually be possible to extend the solutions beyond any limit, nor how the partial solutions already obtained will change eventually while continuing to search for an extended solution. The possibilities for thorough mathematical analysis have to be investigated further.

The fact that excess-values on a given row starting from an arbitrary initial (g, f)-seed are not readily expressible, and that solutions are generally non-additive (property 6), seems to make the general solutions to the main problem hard to find. An attempt for a new attack on the problem was given by the isomorphism with an evolution on a fractal dust, which gave a better intuitive understanding of some results.

So, apart from possible generalizations to other automata-rules, finding more complete answers to the excess-invariant problem remains a challenging task. Meantime, obtaining views of the overall-structure of partial solutions much beyond the size available yet, especially for solutions which do not seem strictly periodic, remains a fascinating undertaking, which is however limited by the currently available computing resources.

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