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STEPPED SINE SYSTEM IDENTIFICATION, ERRORS-IN-VARIABLES AND THE QUOTIENT SINGULAR VALUE DECOMPOSITION

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We develop a parameter estimation algorithm for the stepped sine identification of the transfer function of a linear system when all measurements are corrupted by noise. It consists of three steps: (1) An overdetermined set of linear equations containing stepped-sine measurements is solved in a linear least squares sense. (2) The resulting error covariance matrix is used to construct the noise covariance matrix for a second set of linear equations. (3) This second set of equations, in which all data are noisy, is solved using the quotient singular value decomposition, which is a generalisation of the singular value decomposition. Simulation results show that this new method remains accurate even at extremely bad signal-to-noise ratios. It is also possible to take into account offset and drift of the measurement sensors. Finally, all algorithmic steps involved led themselves to recursive updating.

1. INTRODUCTION

In many disciplines, such as modal analysis of mechanical systems, frequency domain identification techniques are preferred to time domain approaches. Using measurements of frequency response functions (FRFs) that relate inputs to outputs in a certain frequency range, parameters of some *a priori* postulated models are estimated. A popular approach to scan a certain frequency range is broad-banded excitation such as swept sine, white noise, hammer impact, fast swept sine, chirp etc. (see e.g. [1]). These excitation signals and the measured responses are transformed to the frequency domain using the Fast Fourier Transform (FFT). The main drawbacks are limited frequency resolution and dynamic range. Another frequency response method is stepped sine excitation: The FRF is evaluated over a number of discrete frequency points within the range of interest. As is well known, the steady state frequency response for linear systems at a particular frequency is determined by the amplitude and phase relations between a sinusoidal input and resulting output. The most important advantages of the stepped sine approach are its high accuracy (typically because the effective signal-to-noise ratio is high) and the possibility of using frequency points that are unequally spaced so that resonance and anti-resonance frequency ranges are measured with a higher resolution.

For mechanical systems, most of these methods identify a modal model, describing the structure in terms of its modal parameters: natural frequencies, damping ratios and mode shapes. These modal parameters are very useful in structural analysis and design. The frequency domain identification method of this paper is developed with a control application in mind. For such applications, state space models [2] and input/output models [3-5] are more appropriate. Besides, the model parameters can easily be derived from these models.

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In this paper, we describe a new input/output model identification technique, which combines ideas of linear least squares estimation, errors-in-variables and numerical linear algebra. Section 2 describes a least squares method to estimate the FRF for a pure sinusoidal excitation. It is shown how to eliminate an offset and a trend from the measured response. In section 3, we introduce the quotient singular value decomposition (QSVD) as a numerical robust tool to solve errors-in-variables problems. In section 4, we combine the insights of sections 2 and 3 to develop a QSVD-based parameter estimation algorithm, both for the discrete time and continuous time case. The simulation results in section 5 show that the QSVD-based method remains accurate even at extremely bad signal-to-noise ratios, outperforming least squares and SVD based methods.

2. FREQUENCY RESPONSE FUNCTION ESTIMATION BASED ON STEPPED SINE EXCITATION

Let us consider a linear single input single output system with input signal $u(t)$ and output signal $y(t)$. The steady state response of a linear system to a sinusoidal input is a sinusoid with the same frequency but a different amplitude and phase. Let $u_{w_i}(t)$ be such a sinusoidal input at a particular pulsation ω_i:

$$w_{\omega_i}(t) = \alpha_i \sin(\omega_i t + \phi_i).$$

The output $y_{\omega_i}(t)$ is also a sinusoid of the same frequency but with a different amplitude and phase:

$$y_{\omega_i}(t) = \beta_i \sin(\omega_i t + \psi_i).$$

A Fourier analysis of these signals would result in two non-zero components for each signal: a component at ω_i, and a component at $-\omega_i$, which is not relevant for our purpose. The component at ω_i is a complex number which describes the amplitude and phase of the sinusoid. For the input $u_{\omega_i}(t)$ it is represented by $a^u_{\omega_i} + jb^u_{\omega_i}$ while for the output $y_{\omega_i}(t)$ it is $a^y_{\omega_i} + jb^y_{\omega_i}$. The complex value of the FRF at that particular pulsation ω_i is the ratio of these two complex numbers. At every particular frequency the complex numbers $a^u_{\omega_i} + jb^u_{\omega_i}$ and $a^y_{\omega_i} + jb^y_{\omega_i}$ can be calculated from the measured time domain input/output data by decomposing the measurements into a sine and a cosine. This decomposition is exact if both signals are pure sinusoidal:

$$u_{\omega_i}(k) = 2a^u_{\omega_i} \cos(\omega_i k T_s) - 2b^u_{\omega_i} \sin(\omega_i k T_s) \tag{1}$$

$$y_{\omega_i}(k) = 2a^y_{\omega_i} \cos(\omega_i k T_s) - 2b^y_{\omega_i} \sin(\omega_i k T_s) \tag{2}$$

where $u_{\omega_i}(k)$ and $y_{\omega_i}(k)$ are the sampled input and output signals and $k = 1, \ldots, r$ is the time index; The number of samples is r and T_s is the sampling period. One finds that

$$\begin{aligned} u_{\omega_i}(k) &= 2a^u_{\omega_i} \cos(\omega_i k T_s) - 2b^u_{\omega_i} \sin(\omega_i k T_s) \\ &= (a^u_{\omega_i} + jb^u_{\omega_i})\, e^{j\omega_i k T_s} + (a^u_{\omega_i} - jb^u_{\omega_i})\, e^{-j\omega_i k T_s}. \end{aligned}$$

The right-hand side of this last equation is the Fourier transform of $u_{\omega_i}(k)$. From equation (1) one finds immediately that $a^u_{\omega_i} + jb^u_{\omega_i}$ can be calculated from r observations of $u_{\omega_i}(k)$, $k = 1, 2, \ldots, r$, by solving the set of equations:

$$\begin{pmatrix} \cos(\omega_i T_s) & -\sin(\omega_i T_s) \\ \cos(2\omega_i T_s) & -\sin(2\omega_i T_s) \\ \cdots & \cdots \\ \cos(r\omega_i T_s) & -\sin(r\omega_i T_s) \end{pmatrix} \begin{pmatrix} a^u_{\omega_i} \\ b^u_{\omega_i} \end{pmatrix} = \begin{pmatrix} u_{\omega_i}(1) \\ u_{\omega_i}(2) \\ \vdots \\ u_{\omega_i}(r) \end{pmatrix} \tag{3}$$

which will be denoted as

$$A^u_{\omega_i}(z^u_{\omega_i}) = (u). \tag{4}$$

Here $A^u_{\omega_i}$ is an $r \times 2$ matrix. The exact solution is

$$(z^u_{\omega_i}) = (A^{ut}_{\omega_i} A^u_{\omega_i})^{-1} A^{yt}_{\omega_i}(u). \tag{5}$$

A similar set of equations with output measurements gives the complex numbers $a^y_{\omega_i} + jb^y_{\omega_i}$. In reality however, the steady state response of a system to a sinusoidal input is not purely sinusoidal. In control system applications, such as robotics, the observed steady state response is often a combination of a sinusoid, an offset, a trend and possibly higher harmonics excited by static non-linearities. The extra terms caused by offsets and linear trends are modeled as:

$$y_{\omega_i}(k) = a^y_{\omega_i} \cos(\omega_i k T_s) - b^y_{\omega_i} \sin(\omega_i k T_s) + c^y_{\omega_i} + d^y_{\omega_i} k. \tag{6}$$

For a set of r observations $y_{\omega_i}(k)$ at a particular frequency ω_i, the linear equations become:

$$\begin{pmatrix} \cos(\omega_i T_s) & -\sin(\omega_i T_s) & 1 & 0 \\ \cos(2\omega_i T_s) & -\sin(2\omega_i T_s) & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cos(r\omega_i T_s) & -\sin(r\omega_i T_s) & 1 & r-1 \end{pmatrix} \begin{pmatrix} a^y_{\omega_i} \\ b^y_{\omega_i} \\ c^y_{\omega_i} \\ d^y_{\omega_i} \end{pmatrix} = \begin{pmatrix} y_{\omega_i}(1) \\ y_{\omega_i}(2) \\ \vdots \\ y_{\omega_i}(r) \end{pmatrix} \tag{7}$$

which will be denoted by

$$A^y_{\omega_i}(z^y_{\omega_i}) = (y). \tag{8}$$

Here $A^y_{\omega_i}$ is an $r \times 4$ matrix. The exact solution of (8) is:

$$(z^y_{\omega_i}) = (A^{yt}_{\omega_i} A^y_{\omega_i})^{-1} A^{yt}_{\omega_i}(y). \tag{9}$$

The factor 2 that normally appears in the Fourier transform of the input and the output signal, is dropped in (3) and (7) since it has no influence on the value of the transfer function which is the ratio of $a^y_{\omega_i} + jb^y_{\omega_i}$ and $a_{\omega_i} + jb^u_{\omega_i}$.

Non-linearities, such as Coulomb friction, often excite higher harmonics. These higher harmonics are estimated by introducing $\cos(2\omega_i k T_s)$, $\sin(2\omega_i k T_s)$, $\cos(3\omega_i k T_s)$, etc., in equation (6), which can be useful for the analysis of the non-linearities. These harmonics do not influence the estimation of the fundamental if these harmonic terms are omitted from equation (6), if the measurement period is equal to an integer number of periods of the fundamental. They will not be taken into account in what follows although it should be kept in mind that such an extension is straightforward.

Another important deviation of ideal circumstances occurs when measurements are corrupted by additive noise, caused by the finite resolution of the measurement equipment. These errors will be represented by $\delta u_{\omega_i}(k)$ and $\delta y_{\omega_i}(k)$. It often occurs that the input signal is not measured but equal to the signal generated by the measurement equipment. The input signal is noiseless in that case, which is incorporated in the identification procedure by putting δ_{ω_i} equal to zero. Of course, in that case, more efficient algorithms could be devised, taking into account this valuable *a priori* information [7]. However, in some cases the input is measured and therefore disturbed with noise: e.g. when a mechanical structure is excited with a shaker and the input signal is measured with a load cell. Because of the central limit theorem (see e.g. [6]), which acts favourably in most engineering environments, we can assume that these noise sequences are identically and independently normally distributed with mean zero and variances $\sigma^2_{u_{\omega^i}}$ and $\sigma^2_{y_{\omega^i}}$. These measurement noises cause errors on the right hand side of equations (4) and (8).

Solving these equations in a least squares sense (which will give the maximum likelihood solution because of the *a priori* assumption of normality), the obtained solutions will differ from the exact (noise-free) solutions $z^u_{\omega_i}$ and $z^y_{\omega_i}$:

$$z^u_{\omega_i} + \delta z^u_{\omega_i} = (A^{ut}_{\omega_i} A^u_{\omega_i})^{-1} A^{ut}_{\omega_i}(u + \delta u) \tag{10}$$

$$z^y_{\omega_i} + \delta z^y_{\omega_i} = (A^{yt}_{\omega_i} A^y_{\omega_i})^{-1} A^{yt}_{\omega_i}(y + \delta y) \tag{11}$$

where δu and δy are the vectors that contain the error sequences $\delta u_{\omega_i}(k)$ and $\delta y_{\omega_i}(k)$. It follows from the zero mean assumption that

$$E(\delta z^u_{\omega_i}) = (A^{ut}_{\omega_i} A^u_{\omega_i})^{-1} A^{ut}_{\omega_i} E(\delta u) = 0$$

$$E(\delta z^y_{\omega_i}) = (A^{yt}_{\omega_i} A^y_{\omega_i})^{-1} A^{yt}_{\omega_i} E(\delta y) = 0.$$

Hence, the least squares estimates of the unknown coefficients are unbiased with error covariance matrices given by

$$E(\delta z^u_{\omega_i} \delta z^{ut}_{\omega_i}) = (A^{ut}_{\omega_i} A^u_{\omega_i})^{-1} \sigma^2_{u_{\omega^i}} \tag{12}$$

$$E(\delta z^u_{\omega_i} \delta z^{yt}_{\omega_i}) = 0 \tag{13}$$

$$E(\delta z^y_{\omega_i} \delta z^{ut}_{\omega_i}) = 0 \tag{14}$$

$$E(\delta z^y_{\omega_i} \delta z^{yt}_{\omega_i}) = (A^{yt}_{\omega_i} A^y_{\omega_i})^{-1} \sigma^2_{y_{\omega^i}}. \tag{15}$$

We are only interested in the correlations between $\delta a^u_{\omega_i}$, $\delta b^u_{\omega_i}$, $\delta a^y_{\omega_i}$, and $\delta b^y_{\omega_i}$ (and not in those involving $\delta c^y_{\omega_i}$ and $\delta d^y_{\omega_i}$). These correlations depend only on $(A^{ut}_{\omega_i} A^u_{\omega_i})^{-1}$ and on the leading principal 2×2 submatrix of $A^{yt}_{\omega_i}$. These two matrices are identical and will be denoted by Σ_{ω_i}. Its (k, l)-th element is denoted by $\Sigma^{(k,l)}_{\omega_i}$. For instance the correlation between $\delta a^u_{\omega_i}$ and $\delta b^u_{\omega_i}$ is $E(\delta a^u_{\omega_i} \delta b^u_{\omega_i}) = \sigma^2_{u_{\omega^i}} \Sigma^{(1,2)}_{\omega_i}$. In the next section, we show how the knowledge of these noise covariance matrices can be exploited in an errors-in-variables scheme to estimate the parameters of a transfer function.

3. ERRORS-IN-VARIABLES AND THE QUOTIENT SINGULAR VALUE DECOMPOSITION

In this section we first present a most elementary derivation of the idea of errors-in-variables. Its main feature is that it always leads to a (generalised) eigenvalue problem, which can be solved by the quotient singular value decomposition.

3.1. ERRORS-IN-VARIABLES

Let A be an $m \times n$ $(m > n)$ matrix of "exact" data, which is rank deficient:

$$\text{rank}\,(A) = n - 1 < n. \tag{16}$$

Let the vector x describe the linear relation between the columns of A:

$$Ax = 0. \tag{17}$$

Let N be the perturbation matrix (the matrix of errors, N stands for "noise") and assume that

$$E(N) = 0 \tag{18}$$

which of course implies that $E(A^tN)=0$ because A is a deterministic matrix. Let C be the $m\times n$ matrix which is the sum of A and N:

$$C=A+N.$$

Then it follows that

$$E(C^tC)=A^tA+E(A^tN)+E(N^tA)+E(N^tN)=A^tA+E(N^tN). \tag{19}$$

Define the matrices

$$\Sigma_C=E(C^tC)$$
$$\Sigma_N=E(N^tN)$$

and assume that we know Σ_N up to an unknown constant of proportionality λ. If we postmultiply (19), with x, we find the generalised eigenvalue problem

$$\Sigma_C x=\Sigma_N x\lambda. \tag{20}$$

When $\Sigma_N=I$, (20) reduces to an ordinary eigenvalue problem. The conclusions are the following:

Under the standing statistical assumptions (16)-(19), the identification of a linear relation always leads to a generalised eigenvalue problem.

The linear relation x is the eigenvector corresponding to the smallest eigenvalue λ (because $\Sigma_C-\Sigma_N\lambda$ must be non-negative definite since it is a model for the "covariance" matrix of the noise-free data).

A consistent estimate of the noise variance can be obtained from the smallest eigenvalue because $E\|N\|_F^2=\text{trace}\,(\lambda\Sigma_N)$. In particular, when $\Sigma_N=I$ and each element of N has a variance σ^2, we find $\lambda\approx m\sigma^2$.

Observe that we have derived the solution without minimising any norm, as is the case in the derivation of total linear least squares [8] and its generalisations [9]. In essence, these techniques also lead to an eigenvalue problem, but their formal derivation only "works" for unstructured matrices. In our application however, which will be developed in section 4, the data matrices have a special structure. Despite their structure, the solution as a generalised eigenvalue problem will simply follow from the standing assumptions (16)-(19)!

In practice the expected values of the matrices are replaced by their sample estimates. A discussion at length of the corresponding finite sample properties of both the covariance estimates and the generalised eigenvalue problems as well as the consistency properties as $m\to\infty$, can be found in [10].

3.2. THE QUOTIENT SINGULAR VALUE DECOMPOSITION

The relation between the singular value decomposition and the symmetric eigenvalue problem is well known (see e.g. [8]). Let the SVD of an $m\times n$ matrix A be $A=USV^t$. Then

$$AA^tU=U(SS^t)$$
$$A^tAV=V(S^tS).$$

Hence, the left singular vectors of A are the eigenvectors of AA^t and the right singular vectors are the eigenvectors of A^tA. The corresponding eigenvalues are the squares of the singular values: It is well known that the explicit and direct decomposition of the SVD is to be preferred over its computation as a symmetric eigenvalue problem. Indeed, the "squaring" of A in A^tA and AA^t causes a loss of half of the numerical relative accuracy and hence is to be avoided.

As an example, consider the matrix

$$A = \begin{pmatrix} -2{\cdot}822e+01 & -1{\cdot}6545e+00 & -3{\cdot}8510e+01 & 2{\cdot}0614e+01 \\ 3{\cdot}7042e+01 & 2{\cdot}9372e+00 & 5{\cdot}0627e+01 & -2{\cdot}7723e+01 \\ 1{\cdot}6885e+01 & 1{\cdot}9344e+00 & 2{\cdot}3147e+01 & -1{\cdot}3158e+01 \\ 2{\cdot}5953e+01 & 1{\cdot}7302e+00 & 3{\cdot}5433e+01 & -1{\cdot}9137e+01 \\ 1{\cdot}2205e+01 & 8{\cdot}4267e-01 & 1{\cdot}6666e+01 & -9{\cdot}0248e+00 \\ -2{\cdot}3699e+01 & -2{\cdot}0953e+00 & -3{\cdot}2416e+01 & 1{\cdot}7926e+01 \\ -2{\cdot}2129e+01 & -2{\cdot}0470e+00 & -3{\cdot}0279e+01 & 1{\cdot}6817e+01 \\ -1{\cdot}6855e+01 & -1{\cdot}9139e+00 & -2{\cdot}3104e+01 & 1{\cdot}3119e+01 \\ -3{\cdot}4587e+01 & -2{\cdot}6487e+00 & -4{\cdot}7262e+01 & 2{\cdot}5804e+01 \\ -3{\cdot}7946e+00 & -2{\cdot}5719e-01 & -5{\cdot}1812e+00 & 2{\cdot}8017e+00 \end{pmatrix}.$$

Observe that at first sight, there is nothing wrong with this matrix; in particular, the data seem to be balanced in their orders of magnitude.

The singular values of A† are 1·4142e+02, 1·4142e+00, 1·0000e−07 and 9·9999e−10. The numerical rank (defined as the index of the smallest singular value σ_r satisfying $\sigma_r > \sigma_1 \varepsilon_m$) is clearly 4. Obviously, the right singular vector corresponding to the smallest singular value is well determined. The computed eigenvalues of fl $(A'A)$‡ however are 2·0000e+04, 2·0000e+00, 2·6495e−12 and −9·8821e−13. The numerical rank is 2. The elements of the eigenvector corresponding to the smallest eigenvalue do not have even a vague resemblance to the corresponding singular vector of A. As a matter of fact, this eigenvector is not uniquely defined within the limited numerical precision. We want to emphasise that this phenomenon is *not* a consequence of the numerical algorithms used to compute the SVD and eigenvalue decompositions. The only cause of this catastrophic loss of accuracy is the explicit squaring in $A'A$. This is clarified if we reveal how we have generated our example. The matrix A is unitarily equivalent to the matrix

$$B = \begin{pmatrix} 1{\cdot}0000e+00 & 1{\cdot}0000e+00 & 0 & 0 \\ 1{\cdot}0000e-09 & 0 & 0 & 0 \\ 0 & 1{\cdot}0000e-09 & 0 & 0 \\ 0 & 0 & 1{\cdot}0000e+02 & 1{\cdot}0000e+02 \\ 0 & 0 & 1{\cdot}0000e-07 & 0 \\ 0 & 0 & 0 & 1{\cdot}0000e-07 \end{pmatrix}.$$

The singular vector corresponding to the smallest singular value is $[-1/\sqrt{2}, 1/\sqrt{2}, 0, 0]$ and is well-determined. The finite precision computation with $\varepsilon_m \approx 1{\cdot}00e-016$ of $B'B$ gives

$$\text{fl}\,(B'B) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 10\,000 & 10\,000 \\ 0 & 0 & 10\,000 & 10\,000 \end{pmatrix}$$

which has eigenvalues 20 000, 2, 0 and 0. Obviously, there is no *unique* eigenvector corresponding to the smallest eigenvalue (only the corresponding two-dimensional subspace is well-determined).

In the context of identification with noisy data, this example illustrates the importance of using the SVD instead of the corresponding symmetric eigenvalue problem. Imagine that the data in A are noisy measurements and that the covariance matrix of the noise

† Computed with MATLAB, numerical precision $\varepsilon_m \approx 1e-016$.

‡ fl$(\cdot)$ denotes the floating point result of a computation.

is an unknown multiple of the identity matrix. It follows from section 3.1, that the right singular vector of A corresponding to the smallest singular vector is an *unbiased* estimator of the "exact" (noise-free) linear relation. While we can find a good estimate of this linear relation, using the matrix A, this is clearly impossible if we first compute A^tA.

In a similar way one has to avoid the explicit formation of the matrices A^tA and B^tB in the symmetric generalised eigenvalue problem. This can be avoided by using the QSVD:

Lemma 1: *The quotient singular value decomposition*†

Every pair of matrices $A \in \mathcal{R}^{m\times n}$ *and* $B \in \mathcal{R}^{p\times n}$ *can be factorised as*

$$A = U_a S_a X^{-1}$$
$$B = U_b S_b X^{-1}$$

where U_a *and* U_b *are orthonormal. The matrices* S_a *and* S_b *have the following structure*:

$$S_a = \begin{array}{c} \\ r_{ab}-r_b \\ r_a+r_b-r_{ab} \\ m-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{ab}-r_b & r_a+r_b & r_{ab}-r_a & n-r_{ab} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & D_a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$S_b = \begin{array}{c} \\ p-r_b \\ r_a+r_b-r_{ab} \\ r_{ab}-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{ab}-r_b & r_a+r_b-r_{ab} & r_{ab}-r_a & n-r_{ab} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_b & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \end{array}$$

where $r_{ab} = \operatorname{rank}\binom{A}{B}$ *and* $D_a^2 + D_b^2 = I_{r_a+r_b-r_{ab}}$. *The matrix* $X \in \mathbb{R}^{n\times n}$ *is a non-singular matrix and is a common factor in the decompositions of A and B.*

The relation of the QSVD with the generalised eigenvalue problem is an immediate consequence of the orthonormality of U_a and U_b. It is easy to show that

$$A^tAXS_b^tS_b = B^tBXS_a^tS_a$$

and when B^tB is non-singular:

$$A^tAX = B^tBX(S_a^tS_a(S_b^tS_b)^{-1}).$$

Hence, the columns of X are the eigenvectors of the generalised eigenvalue problem while the diagonal elements of $(S_a^tS_a(S_b^tS_b)^{-1})$ are the corresponding generalised eigenvalues. The smallest eigenvector of the generalised eigenvalue problem corresponds to the smallest diagonal element of $(S_a^tS_a(S_b^tS_b)^{-1})$. Observe however that we have avoided completely the squaring of A and B.

Algorithms for the computation of the QSVD are described in [8] and the references therein. Observe that a QR-decomposition for both A and B might greatly reduce the computational complexity. This follows from

$$A = Q_aR_a = U_aS_aX^{-1}.$$
$$B = Q_bR_b = U_bS_bX^{-1}.$$

This implies that

$$R_a = (Q_a^tU_a)S_aX^{-1}$$
$$R_b = (Q_b^tU_b)S_bX^{-1}.$$

is a QSVD of the triangular factors of the QR-decomposition of A and B with other orthonormal factors but with the same quasi-diagonal factors S_a and S_b and non-singular factor X as the QSVD of (A, B). When a new row is appended to both A and B, it is

† This factorisation for two matrices is called the generalised SVD in [8]. However recently several other generalisations have been discovered [11] and the name QSVD was proposed in [12].

relatively easy to update the QR-decomposition of each using $n-1$ Givens rotations (see e.g. [8]). Observe that in most applications we do not need the orthogonal factors. Only the triangular factors need to be updated (for more details about updating the QSVD and related questions, we refer to [13]).

4. TRANSFER FUNCTION PARAMETER ESTIMATION

In this section, we combine the ideas of sections 2 and 3 to develop an identification technique that estimates the parameters of a transfer function, starting from noise-corrupted stepped sine measurements. In particular, it will be demonstrated how we can use the error covariance matrices of the least squares estimation step of section 2 to construct "square roots" M of the error covariance matrix Σ_N of the errors-in-variables method described in section 3. A QSVD of the matrix pair (C, M) where C is a certain structured matrix containing the results of the least squares step, will then deliver the parameters of the transfer function. Both discrete time and continuous time input/output models are considered.

A continuous time input/output model relates the Laplace transform of the system input $U(s)$ to the Laplace transform of the system output $Y(s)$.

$$\frac{Y(s)}{U(s)} = H_c(s). \tag{21}$$

$H_c(s)$ is the Laplace transform of the impulse response of the system, and can be modelled as a ratio of two polynomials in s

$$H_c(s) = \frac{\beta_0 + \beta_1 s + \cdots + \beta_q s^q}{\alpha_0 + \alpha_1 s + \cdots + \alpha_p s^p} \tag{22}$$

where q is the order of the numerator and p is the order of the denominator.

A discrete time input/output model relates the z-transform of the system input $U(z^{-1})$ to the z-transform of the system output $Y(z^{-1})$.

$$\frac{Y(z^{-1})}{U(z^{-1})} = H_d(z^{-1}). \tag{23}$$

$H_d(z^{-1})$ is the z-transform of the unit step response of the system, and can be modeled as a ratio of two polynomials in z^{-1}

$$H_d(z^{-1}) = \frac{\acute{\beta}_0 + \acute{\beta}_1 z^{-1} + \cdots + \acute{\beta}_q z^{-q}}{\acute{\alpha}_0 + \acute{\alpha}_1 z^{-1} + \cdots + \acute{\alpha}_p z^{-p}} \tag{24}$$

where q is the order of the numerator and p is the order of the denominator.

The orders of these continuous time and discrete time domain models are often estimated by inspection of the magnitude and phase of the measured FRF and we assume that they are known.

To identify the continuous time domain model, the transfer function $H_c(s)$ when evaluated at $s = j\omega_i$, is equated with the value of the FRF at that particular pulsation ω_i:

$$H_c(j\omega_i) = \frac{(a^y_{\omega_i} + \delta a^y_{\omega_i}) + j(b^y_{\omega_i} + \delta b^y_{\omega_i})}{(a^u_{\omega_i} + \delta a^u_{\omega_i}) + j(b^u_{\omega_i} + \delta b^u_{\omega_i})}. \tag{25}$$

The values of $(a^u_{\omega_i} + \delta a^u_{\omega_i}) + j(b^u_{\omega_i} + \delta b^u_{\omega_i})$ and $(a^y_{\omega_i} + \delta a^y_{\omega_i}) + j(b^y_{\omega_i} + \delta b^y_{\omega_i})$ result from equations (10) and (11), and will be denoted as $\hat{a}^u_{\omega_i} + j\hat{b}^u_{\omega_i}$ and $\hat{a}^y_{\omega_i} + j\hat{b}^y_{\omega_i}$. Combining (22) and (25) results in the following complex linear relation:

$$\begin{aligned} 0 = {} & (\alpha_0 + \alpha_1(j\omega_i) + \cdots + \alpha_p(j\omega_i)^p)(\hat{a}^y_{\omega_i} + j\hat{b}^y_{\omega_i}) \\ & - (\beta_0 + \beta_1(j\omega_i) + \cdots + \beta_q(j\omega_i)^q)(\hat{\alpha}^u_{\omega_i} + j\hat{b}^u_{\omega_i}). \end{aligned} \tag{26}$$

Both the real part and the imaginary part of this equation have to be zero. This results in the following two linear equations:

$$\begin{pmatrix} \hat{a}^y_{\omega_i} & -\hat{b}^y_{\omega_i}\omega_i & -\hat{a}^y_{\omega_i}\omega_i^2 & \cdots & -\hat{a}^u_{\omega_i} & \hat{b}^u_{\omega_i}\omega_i & \hat{a}^u_{\omega_i}\omega_i^2 & \cdots \\ \hat{b}^y_{\omega_i} & \hat{a}^y_{\omega_i}\omega_i & -\hat{b}^y_{\omega_i}\omega_i & \cdots & -\hat{b}^u_{\omega_i} & -\hat{a}^u_{\omega_i}\omega_i & \hat{b}^u_{\omega_i}\omega_i^2 & \cdots \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_p \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{pmatrix} = 0. \tag{27}$$

In order to identify the discrete tome domain model, the transfer function $H_d(z^{-1})$ when evaluated at $z = e^{j\omega_i T_s}$, is equated with the value of the frequency response function at that particular pulsation ω_i:

$$H_d(e^{-j\omega_i T_s}) = \frac{\hat{a}^y_{\omega_i} + j\hat{b}^y_{\omega_i}}{\hat{a}^u_{\omega_i} + k\hat{b}^u_{\omega_i}}. \tag{28}$$

T_s is the sampling period for which the discrete model has to be identified. Combining (24) and (28) results in the following complex linear relation:

$$\begin{aligned} 0 = &(\acute{\alpha}_0 + \acute{\alpha}_1(e^{-j\omega_i T_s}) + \cdots + \acute{\alpha}_p(e^{-j\omega_i pT_s})(\hat{a}^y_{\omega_i} + j\hat{b}^y_{\omega_i}) \\ &-(\acute{\beta}_0 + \acute{\beta}_1(e^{-j\omega_i T_s}) + \cdots + \acute{\beta}_q(e^{-j\omega_i qT_s})(\hat{a}^u_{\omega_i} + j\hat{b}^u_{\omega_i}). \end{aligned} \tag{29}$$

Both the real part and the imaginary part of this equation have to be zero. This results in the following two linear equations:

$$\begin{pmatrix} \hat{a}^y_{\omega_i}\mathcal{R}((\hat{a}^y_{\omega_i} + j\hat{b}^y_{\omega_i})e^{-j\omega_i T_s}) & \cdots & -\hat{a}^u_{\omega_i} & \mathcal{R}(-(\hat{a}^u_{\omega_i} + j\hat{b}^u_{\omega_i})e^{-j\omega_i T_s}) & \cdots \\ \hat{b}^y_{\omega_i}\mathcal{F}((\hat{a}^y_{\omega_i} + j\hat{b}^y_{\omega_i})e^{-j\omega_i T_s}) & \cdots & -\hat{b}^u_{\omega_i} & \mathcal{F}(-(\hat{a}^u_{\omega_i} + j\hat{b}^u_{\omega_i})e^{-j\omega_i T_s}) & \cdots \end{pmatrix} \begin{pmatrix} \acute{\alpha}_0 \\ \acute{\alpha}_1 \\ \vdots \\ \acute{\alpha}_p \\ \acute{\beta}_0 \\ \acute{\beta}_1 \\ \vdots \\ \acute{\beta}_q \end{pmatrix} = 0. \tag{30}$$

$\mathcal{R}(\cdot)$ denotes the real part of a complex number and $\mathcal{F}(\cdot)$ denotes the imaginary part of a complex number.

If the FRF is measured at m different frequencies ω_i, $i = 1, 2, \ldots, m$, (27) and (30) result in a set of linear equations of the following form

$$Cx = 0 \tag{31}$$

where A is a $(2m) \times (p+q+2)$ matrix and

$$x = (\alpha_0 \quad \alpha_1 \quad \cdots \quad \alpha_p \quad \beta_0 \quad \beta_1 \quad \cdots \quad \beta_q)'$$

in the continuous time case and,

$$x = (\acute{\alpha}_0 \quad \acute{\alpha}_1 \quad \cdots \quad \acute{\alpha}_p \quad \acute{\beta}_0 \quad \acute{\beta}_1 \quad \cdots \quad \acute{\beta}_q)'$$

in the discrete case.

The matrix C of equation (31) contains data which is corrupted with additive noise:

$$C = A + N.$$

The matrix A contains the exact data but is not available. The matrix N is the noise matrix and consists of two submatrices N_y and N_u:

$$N = (N_y \quad N_u).$$

N_u contains the errors on $a^u_{\omega_i}$ and $b^u_{\omega_i}$ while N_y contains the errors on $a^y_{\omega_i}$ and $b^y_{\omega_i}$. In the continuous time domain, both matrices have the following "graded" structure:

$$N_y = \begin{pmatrix} \delta a^y_{\omega_1} & -\omega_1 \delta b^y_{\omega_1} & -(\omega_1)^2 \delta a^y_{\omega_1} & (\omega_1)^3 \delta b^y_{\omega_1} & \cdots \\ \delta b^y_{\omega_1} & \omega_1 \delta a^y_{\omega_1} & -(\omega_1)^2 \delta b^y_{\omega_1} & -(\omega_1)^2 \delta a^y_{\omega_1} & \cdots \\ \delta a^y_{\omega_2} & -\omega_2 \delta b^y_{\omega_2} & -(\omega_2)^2 \delta a^y_{\omega_2} & (\omega_2)^3 \delta b^y_{\omega_2} & \cdots \\ \delta b^y_{\omega_2} & \omega_2 \delta a^y_{\omega_2} & -(\omega_2)^2 \delta b^y_{\omega_2} & -(\omega_2)^3 \delta a^y_{\omega_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta a^y_{\omega_m} & -\omega_m \delta b^y_{\omega_m} & -(\omega_m)^2 \delta a^y_{\omega_m} & (\omega_m)^3 \delta b^y_{\omega_m} & \cdots \\ \delta b^y_{\omega_m} & \omega_m \delta a^y_{\omega_m} & -(\omega_m)^2 \delta b^y_{\omega_m} & -(\omega_m)^3 \delta a^y_{\omega_m} & \cdots \end{pmatrix}.$$

and

$$N_u = \begin{pmatrix} -\delta a^u_{\omega_1} & \omega_1 \delta b^u_{\omega_1} & (\omega_1)^2 \delta a^u_{\omega_1} & -(\omega_1)^3 \delta b^u_{\omega_1} & \cdots \\ -\delta b^u_{\omega_1} & -\omega_1 \delta a^u_{\omega_1} & (\omega_1)^2 \delta b^u_{\omega_1} & (\omega_1)^3 \delta a^u_{\omega_1} & \cdots \\ -\delta a^u_{\omega_2} & \omega_2 \delta b^u_{\omega_2} & (\omega_1)^2 \delta a^u_{\omega_2} & -(\omega_2)^3 \delta b^u_{\omega_2} & \cdots \\ -\delta b^u_{\omega_2} & -\omega_2 \delta a^u_{\omega_2} & (\omega_2)^2 \delta b^u_{\omega_2} & (\omega_1)^2 \delta a^u_{\omega_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\delta a^u_{\omega_m} & \omega_m \delta b^u_{\omega_m} & (\omega_m)^2 \delta a^u_{\omega_m} & -(\omega_m)^3 \delta b^u_{\omega_m} & \cdots \\ -\delta b^u_{\omega_m} & -\omega_m \delta a^u_{\omega_m} & (\omega_m)^2 \delta b^u_{\omega_m} & (\omega_m)^3 \delta a^u_{\omega_m} & \cdots \end{pmatrix}.$$

This follows directly from (27).

The structure of submatrices N_y and N_u in the discrete time domain scheme is:

$$N_y = \begin{pmatrix} \delta a^y_{\omega_1} & \mathcal{R}((d\alpha^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-j\omega_1 T_s}) & \mathcal{R}((\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-2j\omega_1 T_s}) & \cdots \\ \delta b^y_{\omega_1} & \mathcal{F}((\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-j\omega_1 T_s}) & \mathcal{F}((\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-2j\omega_1 T_s}) & \cdots \\ \delta a^y_{\omega_2} & \mathcal{R}((\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-j\omega_2 T_s}) & \mathcal{R}((\delta a^y_{\omega_2} + jb\delta^y_{\omega_2})\, e^{-2j\beta_2 T_s}) & \cdots \\ \delta b^y_{\omega_2} & \mathcal{F}((\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-j\omega_2 T_s}) & \mathcal{F}((\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-2j\omega_2 T_s}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \delta a^y_{\omega_m} & \mathcal{R}((\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-j\omega_m T_s}) & \mathcal{R}((\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-2j\omega_m T_s}) & \cdots \\ \delta b^y_{\omega_m} & \mathcal{F}((\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-j\omega_m T_s}) & \mathcal{F}((\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-2j\omega_m T_s}) & \cdots \end{pmatrix} \tag{32}$$

and

$$N_u = \begin{pmatrix} -\delta a^y_{\omega_1} & \mathcal{R}(-(\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-j\omega_1 T_s}) & \mathcal{R}(-(\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-2j\omega_1 T_s}) & \cdots \\ -\delta b^y_{\omega_1} & \mathcal{F}(-(\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-j\omega_1 T_s}) & \mathcal{F}(-(\delta a^y_{\omega_1} + j\delta b^y_{\omega_1})\, e^{-2j\omega_1 T_s}) & \cdots \\ -\delta a^y_{\omega_2} & \mathcal{R}(-(\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-j\omega_2 T_s}) & \mathcal{R}(-(\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-2j\omega_2 T_s}) & \cdots \\ -\delta b^y_{\omega_2} & \mathcal{F}(-(\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-j\omega_2 T_s}) & \mathcal{F}(-(\delta a^y_{\omega_2} + j\delta b^y_{\omega_2})\, e^{-2j\omega_2 T_s}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ -\delta a^y_{\omega_m} & \mathcal{R}(-(\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-j\omega_m T_s}) & \mathcal{R}(-(\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-2j\omega_m T_s}) & \cdots \\ -\delta b^y_{\omega_m} & \mathcal{F}(-(\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-j\omega_m T_s}) & \mathcal{F}(-(\delta a^y_{\omega_m} + j\delta b^y_{\omega_m})\, e^{-2j\omega_m T_s}) & \cdots \end{pmatrix}. \tag{33}$$

The covariance matrix Σ_N of the noise can be estimated because of the knowledge of the correlations between the noise variables $\delta a^u_{\omega_i}$, $\delta b^u_{\omega_i}$, $\delta a^y_{\omega_i}$, $\delta b^y_{\omega_i}$ from the least squares estimation in section 2.

$$\Sigma_N = E(N^t N) = \begin{pmatrix} E(N_y^t N_y) & 0 \\ 0 & E(N_u^t N_u) \end{pmatrix}.$$

The expression for $E(N_y^t N_y)$ in the continuous time domain scheme is:

$$E(N_y^t N_y) = \begin{pmatrix} \sum_{i=1}^{m} \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & 0 & -\sum_{i=1}^{m} (\omega_i)^2 \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \cdots \\ \cdot & \sum_{i=1}^{m} (\omega_i)^2 \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & 0 & \cdots \\ \cdot & \cdot & \sum_{i=1}^{m} (\omega_i)^4 \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{34}$$

with Λ_{ω_i} equal to $\Sigma_{\omega_i}^{(1,1)} + \Sigma_{\omega_i}^{(2,2)}$. The expression for $E(N_y^t N_y)$ in the discrete time domain scheme is:

$$E(N_y^t N_y) = \begin{pmatrix} \sum_{i=1}^{m} \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \sum_{i=1}^{m} \mathcal{R}(e^{-j\omega_i T_s}) \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \sum_{i=1}^{m} \mathcal{R}(e^{-2j\omega_i T_s}) \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \cdots \\ \cdot & \sum_{i=1}^{m} \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \sum_{i=1}^{m} \mathcal{R}(e^{-j\omega_i T_s}) \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \cdots \\ \cdot & \cdot & \sum_{i=1}^{m} \sigma_{y_{\omega_i}}^2 \Lambda_{\omega_i} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{35}$$

The expressions for the corresponding covariance matrix $E(N_u^t N_u)$ are exactly the same, except that $\sigma_{y_{\omega_i}}^2$ is replaced by $\sigma_{u_{\omega_i}}^2$. Observe that the noise covariance matrix in (20) has to be known only up to within a non-negative real scalar. This implies that we do not need to know the values of $\sigma_{u_{\omega_i}}$ and $\sigma_{y_{\omega_i}}$ explicitly, but that only their ratio should be known (as is often the case in practice).

It is straightforward to find a "square root" M_y of the matrix $E(N_y^t N_y)$ in (34), i.e. a matrix M_y of size $(2m) \times (p+q+2)$ such that $E(N_y^t N_y) = M_y^t M_y$. Indeed, define

$$\varepsilon_i = \sqrt{\Sigma_{\omega_i}^{(1,1)}}$$

$$\eta_i = \sqrt{\Sigma_{\omega_i}^{(2,2)}}.$$

Then it is easily verified that

$$M_y = \begin{pmatrix} \varepsilon_1 & -\omega_1 \eta_1 & -(\omega_1)^2 \varepsilon_1 & (\omega_1)^3 \eta_1 & \cdots \\ \eta_1 & \omega_1 \varepsilon_1 & +(\omega_1)^2 \eta_1 & -(\omega)^3 \varepsilon_1 & \cdots \\ \varepsilon_2 & -\omega_2 \eta_2 & -(\omega_2)^2 \varepsilon_2 & (\omega_2)^3 \eta_2 & \cdots \\ \eta_2 & \omega_2 \varepsilon_2 & -(\omega_2)^2 \eta_2 & -(\omega_2)^3 \varepsilon_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon_m & -\omega_m \eta_m & -(\omega_m)^2 \varepsilon_m & (\omega_m)^3 \eta_m & \cdots \\ \eta_m & \omega_m \varepsilon_m & -(\omega_m)^2 \eta_m & -(\omega_m)^3 \varepsilon_m & \cdots \end{pmatrix} \tag{36}$$

Square roots of the matrices $E(N_y^t N_y)$ for the discrete time case and for the matrices $E(N_u^t N_u)$ can be obtained in a similar way.

Summarizing, we have the following algorithm to estimate the parameters of a transfer

function:

1. Obtain estimates of the values of the FRF from stepped sine measurements, together with estimates of the error covariances. Therefore solve the sets of linear equations (3) and (7) in a least squares sense. The least squares solutions are given by (10) and (11). The error covariances are given by (12)-(15).
2. Using the results from step 1, construct matrix C according to equation (27) if a continuous time domain model has to identified, or according to equation (30) if a discrete time domain model has to be identified. Construct the square roots M of the error covariance matrices, (34) and (35), according to equation (36).
3. Compute the QSVD of the matrix pair (C, M) as outlined in section 3.2: $C = U_c S_c X^{-1}$ and $M = U_m S_m X^{-1}$. The column of matrix X which corresponds to the smallest diagonal element of matrix $S_c^t S_c (S_m^t S_m)^{-1}$ is the identification result, i.e. parameter vector x of equation (31).

5. SIMULATION RESULTS

In this section, we illustrate the accuracy of our QSVD based identification method. Both the continuous time domain and the discrete time domain schemes are tested.

The test case is a fourth order continuous time model and its discrete time equivalent (for a sampling period of 10 msec). The model describes a system with resonance frequencies at 2·5 and 8·8 Hz, and an anti-resonance frequency at 5·5 Hz. The damping ratio of these modes is 0·01. The model was treated as a black box and submitted to a stepped sine excitation with a frequency step of 0·1 Hz in a frequency range going from 0·2 to 15 Hz. Additive measurement noise is simulated by adding normally distributed random sequences to the input sinusoids and corresponding model outputs. An offset and a linear trend were added to the outputs.

Figure 1 shows the amplitude and phase diagrams of the identified continuous time domain models. The standard deviation of the noise sequences on the input and output sinusoids is 20% of the amplitude of the respective signals. The dotted line shows the FRF that resulted from the corrupted time domain input/output data by solving equations (3) and (7). The dashed line shows the transfer function obtained with the QSVD algorithm outlined in section 4. The solid line shows the result when the covariance weighting is not taken into account (i.e. when it is taken to be equal to the identity matrix, in which

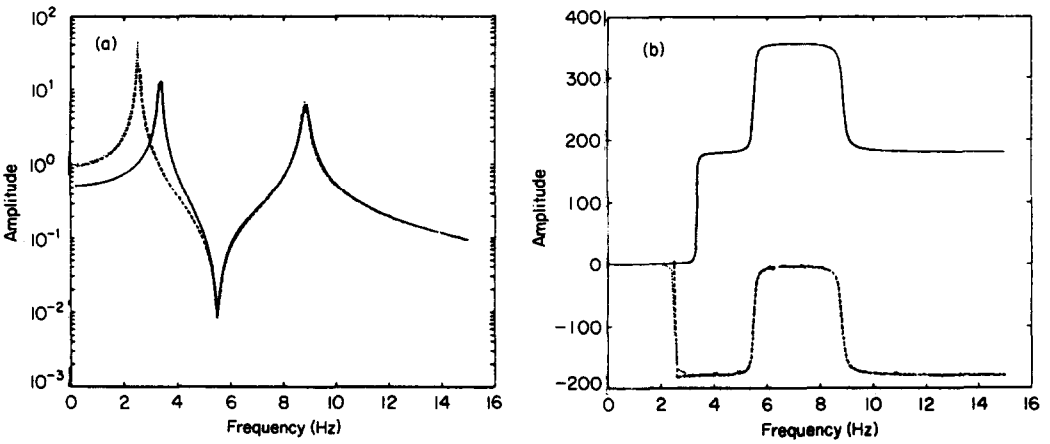


Figure 1. Identification result of the continuous time domain model. · · · ·, Frequency response function obtained from a stepped sine analysis of the model; - - -, QSVD identification result; ——, SVD result (= QSVD with identity covariance weighting).

case the QSVD reduces to an SVD of the data matrix C). This last result shows the importance of the appropriate covariance weighting. For this example, if the appropriate covariances are not taken into account, the estimation of the first mode is bad, resulting in an erroneous eigenfrequency and even a negative damping. The QSVD method finds a model that is much closer to the "exact" model.

Figure 2 shows the amplitude and phase diagrams of the identified discrete time domain models. The standard deviation of the noise sequences on the input and output sinusoids is 40% of the amplitude of the respective signal. The dotted line shows the frequency response estimates that resulted from the corrupted time domain input/output data by solving equations (3) and (7). The dashed line shows QSVD results. The solid line shows the solution when the noise covariance is set equal to the identity.

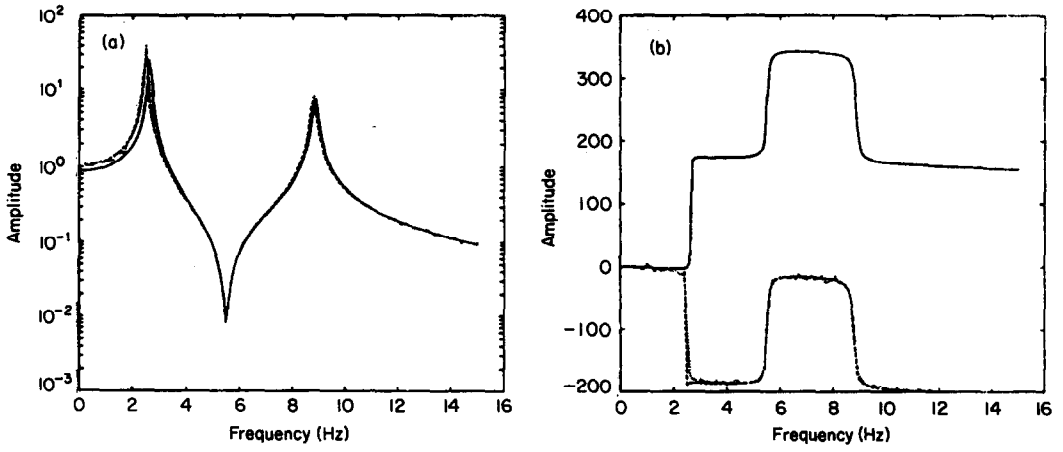


Figure 2. Identification result of the discrete time domain model. · · · ·, Frequency response function obtained from a stepped sine analysis of the model; - - -, QSVD identification result; ——, SVD result (= QSVD with identity covariance weighting).

One might observe that the identification results for the discrete time domain model are more accurate than the continuous time results. In the continuous time case, the elements of matrix C range over several orders of magnitude as they are proportional to powers of the pulsations. This leads to a bad condition number of the matrix which affects the sensitivity of the singular vector corresponding to the smallest generalised singular value. Unfortunately, there is no appropriate scaling to remedy this ill-conditioning. In the discrete time case, the elements of the data matrix C are all of the same order of magnitude leading to a much better balanced matrix and a smaller sensitivity of the solution vector.

6. CONCLUSIONS

We have presented a new identification method to estimate the parameters of the transfer function of a single input-single output linear system.

Starting from noisy stepped sine input-output measurements, the frequency response is estimated using linear least squares. This also gives us error covariance matrices. Offsets and trends can be taken into account as well as non-linearities that generate higher harmonics.

The results of the least squares are then used to estimate the parameters of the transfer function. Taking into account the structure of the data matrix and the noise covariances, this leads to a generalised eigenvalue problem.

Finally, we have shown how to solve the "square root" version of the generalised eigenvalue problem in a numerically robust way, using the quotient singular value decomposition for two matrices. We have also emphasised the feasibility of recursively updating all calculations.

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