QPQ-SVD based system identification*

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Abstract

The state of a system can be seen as an interaction between the past and the future of a process. Within this framework, a QPQ-SVD approach offers a straightforward method in order to estimate the system order, to compute a state vector sequence and to obtain a state space model. A senseful system order estimation is always possible after applying a so-called Orthogonality Theorem. The QPQ-SVD approach has also advantages from a numerical viewpoint.

Keywords: system identification, state space model, QPQ-SVD, canonical angles

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1 Introduction

A state space identification approach will be proposed here. The idea is based on input-output matrix equations for the past and the future of the process, derived from the state space model. It is proven then that a state vector sequence can be obtained as a transformation of a past or future input-output matrix.
An optimization problem is stated in order to evaluate dependencies between past and future of the process and to estimate the system order. The solution of this problem can be formulated in an elegant way by means of a QPQ-SVD. The system order is then estimated from an interaction matrix after applying a so-called Orthogonality Theorem. Finally, the system matrices are obtained from an overdetermined set of linear equations, depending on the known inputs and outputs and the computed state vector sequence.

Notations:
A^t: *transpose of matrix* A
A^+: *matrix related to the future*
A^-: *matrix related to the past*
A^{-t}: *inverse of* A^t
$(A)^+$: *pseudo-inverse of* A
$A(i:j,k:l)$: *submatrix of* A *consisting of rows* i *to* j *and columns* k *to* l *of* A
$span_{col(row)}(A)$: *space generated by the columns (rows) of* A
I_i: *identity matrix* $(i \times i)$
$\underline{x}$: *columnvector*
$\underline{x}_k$: *columnvector at time* k

$x_k(i)$: i-th component of $\underline{x}_k$
$\{.\}$: set of .
GSVD: generalized singular value decomposition
OSVD: ordinary singular value decomposition
PSVD: product SVD
QSVD: quotient SVD
QPQ-SVD: qoutient-product-quotient SVD

2 From state space model to matrix I/O-equations

Let us consider a process which can be modelled by the following state space formalism:

$$\begin{aligned} \underline{x}_{k+1} &= A.\underline{x}_k + B.\underline{u}_k \\ \underline{y}_k &= C.\underline{x}_k + D.\underline{u}_k \end{aligned} \tag{1}$$

This MIMO process is supposed to be time-discrete, time-invariant, linear and finite-dimensional. Here $\underline{u}_k(m \times 1)$ are the inputs, $\underline{y}_k(l \times 1)$ are the outputs and $\underline{x}_k(n \times 1)$ is the state of the system. Only for technical reasons, it is assumed that $\underline{u}_k$ and $\underline{y}_k$ are noiseless. In practice noise to a linear system masks the real system and makes a perfect identification impossible. The state space form (1) can be expressed more algebraically by matrix I/O-equations as stated in the next theorem.

Theorem 1:
The state space form (1) can also be written as:

$$Y^+ = \Gamma^+ X^+ + T^+ U^+ \tag{2}$$
$$Y^- = \Gamma^- X^- + T^- U^- \tag{3}$$

Here the I/O-equation (2) is related to the future of the process and (3) to the past of the process. The matrices Y^+ and U^+ are block Hankel, Y^- and U^- are block Toeplitz, Γ^+ and Γ^- can be seen as extended observability matrices and T^+ and T^- contain the Markov parameters of the process. The matrices of the I/O-equations are written out here:

$$Y^+(li \times j) = \begin{bmatrix} \underline{y}_k & \underline{y}_{k+1} & \cdots & \underline{y}_{k+j-1} \\ \underline{y}_{k+1} & \underline{y}_{k+2} & \cdots & \underline{y}_{k+j} \\ \vdots & \vdots & \cdots & \vdots \\ \underline{y}_{k+i-1} & \underline{y}_{k+i} & \cdots & \underline{y}_{k+i+j-2} \end{bmatrix}$$

$$Y^-(li \times j) = \begin{bmatrix} \underline{y}_{k-1} & \underline{y}_k & \cdots & \underline{y}_{k+j-2} \\ \underline{y}_{k-2} & \underline{y}_{k-1} & \cdots & \underline{y}_{k+j-3} \\ \vdots & \vdots & \cdots & \vdots \\ \underline{y}_{k-i} & \underline{y}_{k-i+1} & \cdots & \underline{y}_{k-i+j-1} \end{bmatrix}$$

For the definition of $U^+(mi \times j)$ and $U^-(mi \times j)$ just replace the symbol y by u.

$$X^+(n \times j) = [\underline{x}_k \; \underline{x}_{k+1} \cdots \underline{x}_{k+j-1}]$$
$$X^-(n \times j) = [\underline{x}_{k-i} \; \underline{x}_{k-i+1} \cdots \underline{x}_{k-i+j-1}]$$

$$\Gamma^+(li \times n) = \begin{bmatrix} C \\ C.A \\ C.A^2 \\ \vdots \\ C.A^{i-1} \end{bmatrix}, \; \Gamma^-(li \times n) = \begin{bmatrix} C.A^{i-1} \\ C.A^{i-2} \\ \vdots \\ C.A \\ C \end{bmatrix}$$

$$T^+(li \times mi) = \begin{bmatrix} D & O & O & \ldots & O \\ CB & D & O & \ldots & O \\ CAB & CB & D & \ldots & O \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ CA^{i-2}B & CA^{i-3}B & CA^{i-4}B & \ldots & D \end{bmatrix}$$

$$T^-(li \times mi) = \begin{bmatrix} D & \ldots & CA^{i-4}B & CA^{i-3}B & CA^{i-2}B \\ O & \ldots & \vdots & \vdots & \vdots \\ O & \ldots & D & CB & CAB \\ O & \ldots & O & D & CB \\ O & \ldots & O & O & D \end{bmatrix}$$

Proof:
straightforward repeated substitution of (1). □

The purpose of system identification is now to find matrices A, B, C, D of the state space description from the knowledge of $\{\underline{u}_k\}$ and $\{\underline{y}_k\}$. Our strategy to obtain these matrices consists mainly of 3 steps:

1. estimation of the system order
2. obtaining a state vector sequence X^+
3. computation of A, B, C, D from an overdetermined set of linear equations

A suggestion for obtaining X^+ is given in the next theorem.

Theorem 2:
There exist matrices F and G such that:

$$X^+ = F^t H^+ \tag{4}$$
$$X^- = G^t H^- \tag{5}$$

where H^+ and H^- are defined as

$$H^+ = \begin{bmatrix} Y^+ \\ U^+ \end{bmatrix}$$
$$H^- = \begin{bmatrix} Y^- \\ U^- \end{bmatrix}$$

Proof:
Fill expression (4) and (5) in into respectively (2) and (3):

$$Y^+ = \Gamma^+ F^t H^+ + T^+ U^+$$
$$Y^- = \Gamma^- G^t H^- + T^- U^-$$

Now split the matrices $\Gamma^+ F^t$ and $\Gamma^- G^t$ as follows:

$$\Gamma^+ F^t = [F_1 \; F_2]$$
$$\Gamma^- G^t = [G_1 \; G_2]$$

such that

$$Y^+ = F_1 Y^+ + F_2 U^+ + T^+ U^+$$
$$Y^- = G_1 Y^- + G_2 U^- + T^- U^-$$

We see that $F_1 = I$, $F_2 = -T^+$, $G_1 = I$ and $G_2 = -T^-$ is always a possible solution. Hence there exist matrices F and G such that $X^+ = F^t H^+$ and $X^- = G^t H^-$. □

3 Obtaining a state vector sequence and identification of the system matrices

3.1 Dependencies between past and future of the process

In order to evaluate relations between $L^t H^+$ and $M^t H^-$ (L and M are at this moment arbitrary square matrices), estimate the system order and obtain X^+, we will take a look at the following optimization problem:

$$extr_{\underline{l},\underline{m}} \{\underline{l}^t H^+ . (H^-)^t \underline{m}\} \tag{6}$$

under constraints:

$$L^t H^+ (H^+)^t L = I_{mi+li}$$
$$M^t H^- (H^-)^t M = I_{mi+li}$$

where

- $H^+ = [(Y^+)^t \; (U^+)^t]^t$, $H^- = [(Y^-)^t \; (U^-)^t]^t$
- $L((mi+li) \times (mi+li))$ and $M((mi+li) \times (mi+li))$ contain the solution vectors $\underline{l}$ and $\underline{m}$ of the optimization problem

The Lagrangian function has the form:

$$\mathcal{L}(\underline{l},\underline{m},\lambda,\gamma) = \underline{l}^t H^+(H^-)^t\underline{m} - \lambda.(\underline{l}^t H^+(H^+)^t\underline{l} - 1) - \gamma.(\underline{m}^t H^-(H^-)^t\underline{m} - 1) \qquad (7)$$

One can check that the constraints $\underline{l}_k^t H^+(H^+)^t\underline{l}_s = 0$ and $\underline{m}_k^t H^-(H^-)^t\underline{m}_s = 0$ $(s \neq k)$ are irrelevant to the problem. The extremal solutions are given by:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\underline{l}} = 0 \;&:\; H^+(H^-)^t\underline{m} = 2\lambda.H^+(H^+)^t\underline{l} \\
\frac{\partial\mathcal{L}}{\partial\underline{m}} = 0 \;&:\; H^-(H^+)^t\underline{l} = 2\gamma.H^-(H^-)^t\underline{m} \\
\frac{\partial\mathcal{L}}{\partial\lambda} = 0 \;&:\; \underline{l}^t H^+(H^+)^t\underline{l} = 1 \\
\frac{\partial\mathcal{L}}{\partial\gamma} = 0 \;&:\; \underline{m}^t H^-(H^-)^t\underline{m} = 1
\end{aligned}$$

Clearly these equations have several solutions for $\underline{l}$ and $\underline{m}$, λ and γ. After some manipulations we get the following generalized eigenvalue problems:

$$\begin{aligned}
H^+(H^-)^t[H^-(H^-)^t]^{-1}H^-(H^+)^t.L &= H^+(H^+)^t.L.\Lambda \\
H^-(H^+)^t[H^+(H^+)^t]^{-1}H^+(H^-)^t.M &= H^-(H^-)^t.M.\Gamma
\end{aligned} \qquad (8)$$

where Λ and Γ are diagonal matrices depending on the value of λ and γ.
Let us now take a **QPQ-SVD of the matrix set** $\{(H^+)^t, H^+, (H^-)^t, H^-\}$ (see [1][2]):

$$\begin{aligned}
(H^+)^t &= U_1 D_1 X_1^{-1} \\
H^+ &= X_1^{-t} D_2 X_2^{-1} \\
(H^-)^t &= X_2 D_3 X_3^{-1} \\
H^- &= X_3^{-t} D_4 V_4^t
\end{aligned} \qquad (9)$$

with U_1 and V_4 square orthonormal matrices; X_1, X_2 and X_3 (square) nonsingular matrices and D_1, D_2, D_3, D_4 quasi-diagonal matrices (D_1, D_2 and D_3 only contain elements 0 and 1,while D_4 not) with the same dimensions as respectively $(H^+)^t, H^+, (H^-)^t$ and H^-. The numerical advantage of the GSVD is expressed in (9): no products of H^+ with H^- are made, but an explicit factorization is obtained. Hence there will be no loss of accuracy.
Because of the orthonormality properties of X_1, X_2, X_3, we get:

$$\begin{aligned}
(D_2D_3).(D_4D_4^t)^{-1}.(D_2D_3)^t.[X_1^{-1}L] &= (D_1^tD_1).[X_1^{-1}L].\Lambda \\
(D_2D_3).(D_1^tD_1)^{-1}.(D_2D_3).[X_3^{-1}M] &= (D_4D_4^t).[X_3^{-1}M].\Gamma
\end{aligned} \qquad (10)$$

The matrices $D_1^tD_1$, D_2D_3 and $D_4D_4^t$ are diagonal matrices. Normally the matrices $D_1^tD_1$ and D_2D_3 are equal to the identity matrix. Thus:

$$\begin{aligned}
(D_4D_4^t)^{-1}.[X_1^{-1}L] &= [X_1^{-1}L].\Lambda \\
(D_4D_4^t)^{-1}.[X_3^{-1}M] &= [X_3^{-1}M].\Gamma
\end{aligned} \qquad (11)$$

In order to obtain a solution of (11), which satisfies the constraints, we apply a Cholesky factorization of $D_4D_4^t$:

$$D_4D_4^t = R_M^tR_M \qquad (12)$$

such that:

$$\begin{aligned} (D_4 D_4^t)^{-1}.[X_1^{-1}L] &= [X_1^{-1}L].\Lambda \\ R_M(D_4 D_4^t)^{-1}R_M^{-1}.[R_M X_3^{-1}M] &= [R_M X_3^{-1}M].\Gamma \end{aligned} \tag{13}$$

Because of the diagonality of R_M and $(D_4 D_4^t)^{-1}$ this equals:

$$\begin{aligned} (D_4 D_4^t)^{-1}.[X_1^{-1}L] &= [X_1^{-1}L].\Lambda \\ (D_4 D_4^t)^{-1}.[R_M X_3^{-1}M] &= [R_M X_3^{-1}M].\Gamma \end{aligned} \tag{14}$$

The matrix $(D_4 D_4^t)^{-1}$ is diagonal and hence there exist always the following solution:

$$\begin{aligned} \Lambda = \Gamma &= (D_4 D_4^t)^{-1} \\ X_1^{-1}L &= I_{mi+li} \\ R_M X_3^{-1}M &= I_{mi+li} \end{aligned} \tag{15}$$

Now the constraints are satisfied and the solution matrices L and M are then:

$$\begin{aligned} L &= X_1 \\ M &= X_3.(D_4 D_4^t)^{-1/2} \end{aligned} \tag{16}$$

The solution of the optimization problem can thus be written in function of the QPQ-SVD factorization of the matrix set $\{(H^+)^t, H^+, (H^-)^t, H^-\}$.
Let us now evaluate $L^t H^+ (H^-)^t M$ for the solution matrices of the optimization problem:

$$\begin{aligned} L^t H^+ (H^-)^t M &= X_1^t.X_1^{-t}D_2 X_2^{-1}.X_2 D_3 X_3^{-1}.X_3(D_4 D_4^t)^{-1/2} \\ &= (D_4 D_4^t)^{-1/2} = \Lambda^{1/2} = \Gamma^{1/2} = S = diag(s_1, \ \ldots \ , s_{mi+li}) \end{aligned} \tag{17}$$

The solution matrices L and M transform the matrix

$$\begin{bmatrix} Y^+(Y^-)^t & Y^+(U^-)^t \\ U^+(Y^-)^t & U^+(U^-)^t \end{bmatrix}$$

into S. Because of the orthonormality of $L^t H^+$ and $M^t H^-$, the elements of S can be interpreted as the cosines of the canonical angles θ_i between the spaces $span_{row}(L^t H^+)$ and $span_{row}(M^t H^-)$ ($s_i = cos\theta_i$), e.g. all elements $s_i = 0$ means that the spaces $span_{row}(L^t H^+)$ and $span_{row}(M^t H^-)$ are orthogonal to each other and all elements $s_i = 1$ means that $span_{row}(L^t H^+)$ and $span_{row}(M^t H^-)$ generate the same space.

3.2 The Orthogonality Theorem and the interaction matrix

Theorem 3:
Consider a fixed subspace $\mathcal{S}^r$ in the j dimensional vector space of j-tuples $\mathcal{R}^j$ ($j \gg 1$). Assume that a random vector $\underline{v}$ is generated with equal probability for all directions in $\mathcal{R}^j$. Then the directional probability density function $p(\alpha, r, j)$ of the angle α between $\underline{v}$ and the fixed subspace $\mathcal{S}^r$ is given by:

$$p(\alpha, r, j) = K(r, j).(sin\alpha)^{j-r-1}.(cos\alpha)^{r-1} \tag{18}$$

where $K(r, j) = 2.\frac{C_r C_{j-r}}{C_j}$ and the C_k can be obtained recursively as:

- $C_1 = 1$
- $C_2 = \pi$
- $C_k = \frac{2\pi}{k-2}.C_{k-2},\ \ k > 2$

Remark: the angle α is measured between $\underline{v}$ and the orthogonal projection of $\underline{v}$ on $\mathcal{S}^r$.
Proof:
see [3]. □

In figure 1 an example of this Orthogonality Theorem is given for $r = 2$. Let us now apply this theorem to our identification problem. Let $\mathcal{S}^r$ $(r = mi + li)$ be equal to $span_{row}(M^tH^-)$; the vectors $\underline{l}^tH^+$ play then the role of $\underline{v}$ in the case that they do not contribute to the model. However the angle α does not correspond to the angle between $\underline{l}^tH^+$ and $\underline{m}^tH^-$, we may conclude that for large values of j, orthogonal vectors $\underline{l}^tH^+$ and $\underline{m}^tH^-$ in (17) do not contribute to the model. From this Orthogonality Theorem we construct the matrix F in $X^+ = F^tH^+$ which contains the vectors $\underline{l}^tH^+$ that make a contribution to the linear model. The matrices X^+ and X^- are then equal to $X^+ = F^tH^+$ and $X^- = G^tH^-$ with $F^t = L^t(1:n,:)$ and $G^t = M^t(1:n,:)$ (suppose that the elements of S are ordered as in an OSVD). An $n \times n$ diagonal matrix S_x is then obtained as $S_x = X^+(X^-)^t$.
For a large j, we can state then:

Orthogonality is absence of linear dependencies

and

The linear model explains all linear dependencies between X^+ and X^-

thus

Orthogonal vectors $\underline{l}^tH^+$ and $\underline{m}^tH^-$ do not contribute to the model

Definition 1:
The matrix S in:

$$L^tH^+(H^-)^tM = S = (D_4D_4^t)^{-1/2}$$

is called the *interaction* matrix.

This definition arises from the fact that S measures the interaction between the past and the future of the process [8]. So much canonical angles between $span_{row}(L^tH^+)$ and $span_{row}(M^tH^-)$ near to 90^o refers to a *low* process order and a low interaction level between past and future of the process and less near to 90^o refers to a *high* process order and a high interaction level.

3.3 Obtaining a state vector sequence X^+ and identification of the system matrices

Only $X^+ = F^tH^+$ can serve as a valid state vector sequence while $X^- = G^tH^-$ not [1]. An explanation for this can be found in [7]. In that paper it is proven that $span_{row}(X^+) = span_{row}(H^-) \cap span_{row}(H^+)$ (in the case that the I/O-data are noiseless). But it cannot be

[1] We would like to thank Marc Moonen for remarking this.

proven that $span_{row}(X^-) = span_{row}(H^-) \cap span_{row}(H^+)$.
Once X^+ is found, we can calculate A, B, C, D from an overdetermined set of linear equations, which can be solved in a least squares sense:

$$\begin{bmatrix} \underline{x}_{k+1} & \cdots & \underline{x}_{k+j-1} \\ \underline{y}_k & \cdots & \underline{y}_{k+j-2} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_k & \cdots & \underline{x}_{k+j-2} \\ \underline{u}_k & \cdots & \underline{u}_{k+j-2} \end{bmatrix} \tag{19}$$

4 Example

A difference has to be made between artificially constructed and noiseless linear systems and real-life systems. In the case of artificially constructed linear systems we can find an intersection space between $span_{row}(L^t H^+)$ and $span_{row}(M^t H^-)$ with dimension equal to the system order n. For real-life systems we get no intersection and we have to apply the Orthogonality Theorem on the interaction matrix S in order to estimate the process order n, to obtain X^+ together with the matrices A, B, C, D. In figure 2 a real-life example is given of a glassfurnace. This system has 3 inputs: a gas input of a first feeder section, a cooling air input and a gas input of a second feeder section. The 6 outputs are temperatures of the glass at 6 points in a cross section of the feeder close to the spout. In figure 2 the canonical angles between the spaces $span_{row}(L^t H^+)$ and $span_{row}(M^t H^-)$ are computed as a function of j for values of i equal to 3 and 4. Remark that *independent* of the value of i, this system order seems to be equal to 10.
In figure 3 a simulation and prediction is given for one of the outputs of the glassfurnace process. Behind the vertical line one can see how well the model acts as a predictor. The design parameters were chosen as follows: $i = 3$, $k = 4$, $m = 3$, $l = 6$, $j = 200$, $n = 10$.

5 System identification algorithm

As a summary we propose the following QPQ-SVD algorithm for system identification :

1. given $\{\underline{u}_k\}$ and $\{\underline{y}_k\}$, construct the past and future I/O-matrices H^- and H^+:

$$H^- = \begin{bmatrix} Y^- \\ U^- \end{bmatrix}, \; H^+ = \begin{bmatrix} Y^+ \\ U^+ \end{bmatrix}$$

   with Y^- and U^- block Toeplitz matrices and Y^+ and U^+ block Hankel matrices.

2. compute a QPQ-SVD of the matrix set $\{(H^+)^t, H^+, (H^-)^t, H^-\}$:

$$\begin{aligned} (H^+)^t &= U_1 D_1 X_1^{-1} \\ H^+ &= X_1^{-t} D_2 X_2^{-1} \\ (H^-)^t &= X_2 D_3 X_3^{-1} \\ H^- &= X_3^{-t} D_4 V_4^t \end{aligned}$$

3. estimate the process order from the interaction matrix $S = (D_4 D_4^t)^{-1/2}$ and select vectors $\underline{l}^t H^+$ by applying the Orthogonality Theorem in order to construct X^+.

$$\begin{aligned} X^+ &= F^t H^+ \\ X^- &= G^t H^- \end{aligned}$$

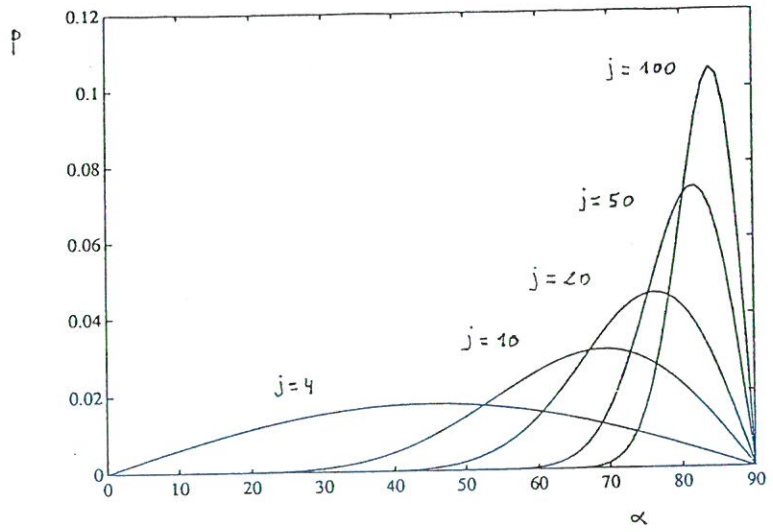
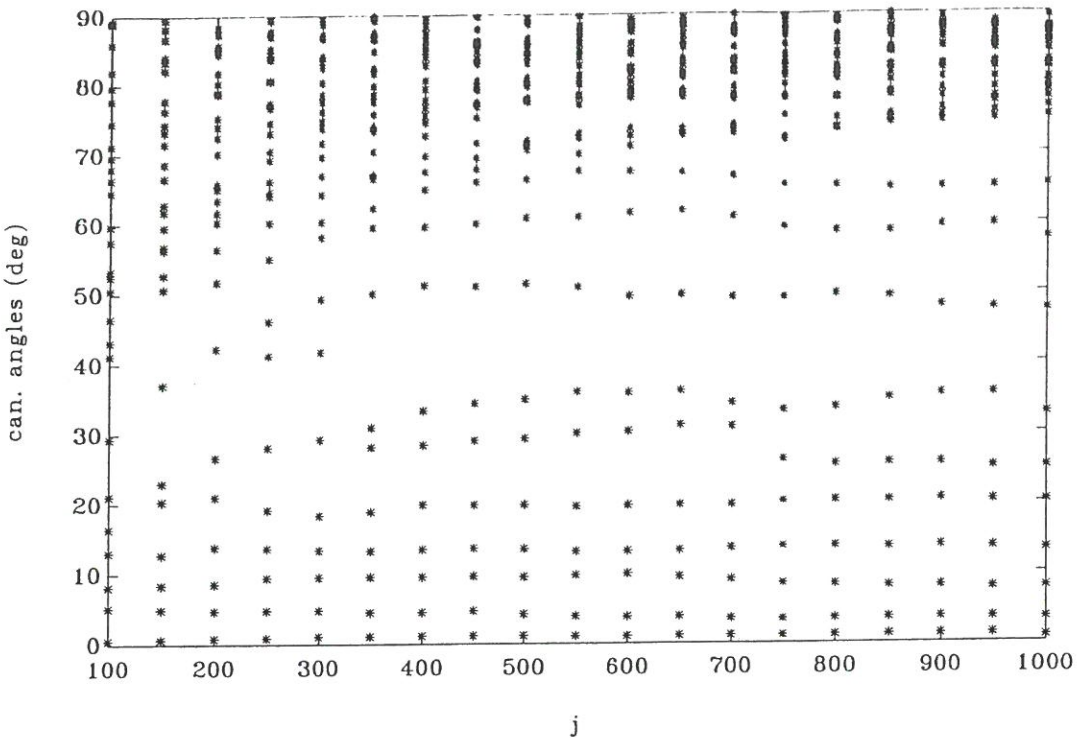


Figure 1: *Probability density function of angle between random vector and fixed subspace S^2 for different values of j.*

a.



b.

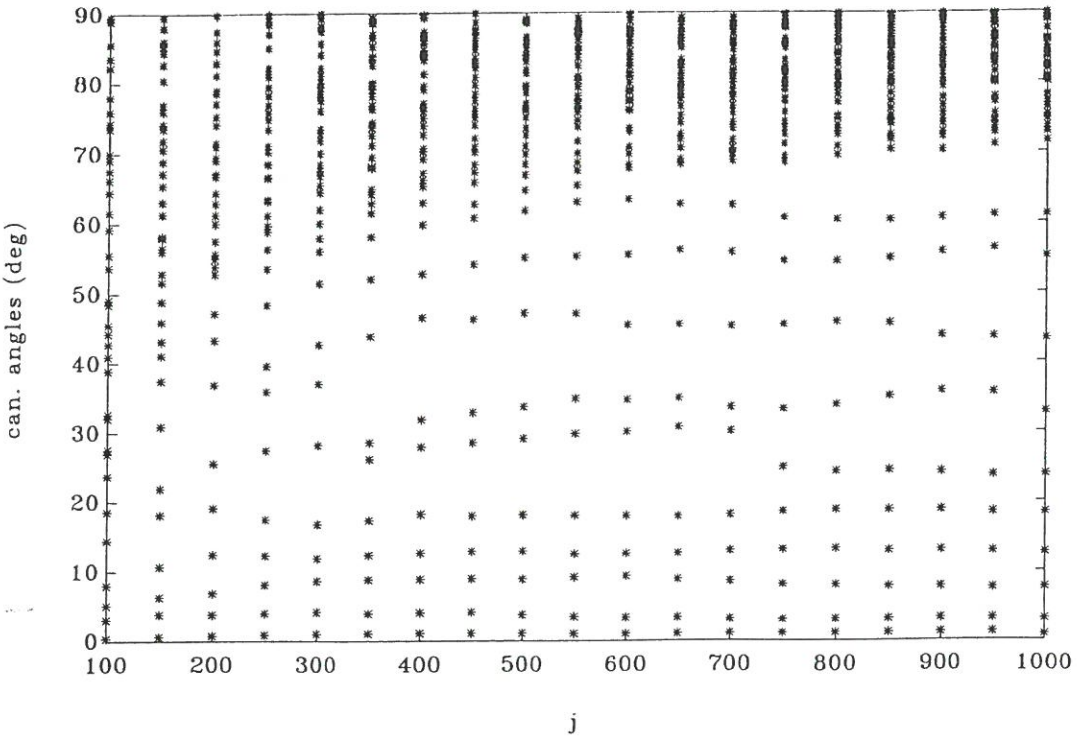


Figure 2: *System order estimation from the interaction matrix for a glassfurnace process, fig.a.* $i = 3$, *fig.b.* $i = 4$

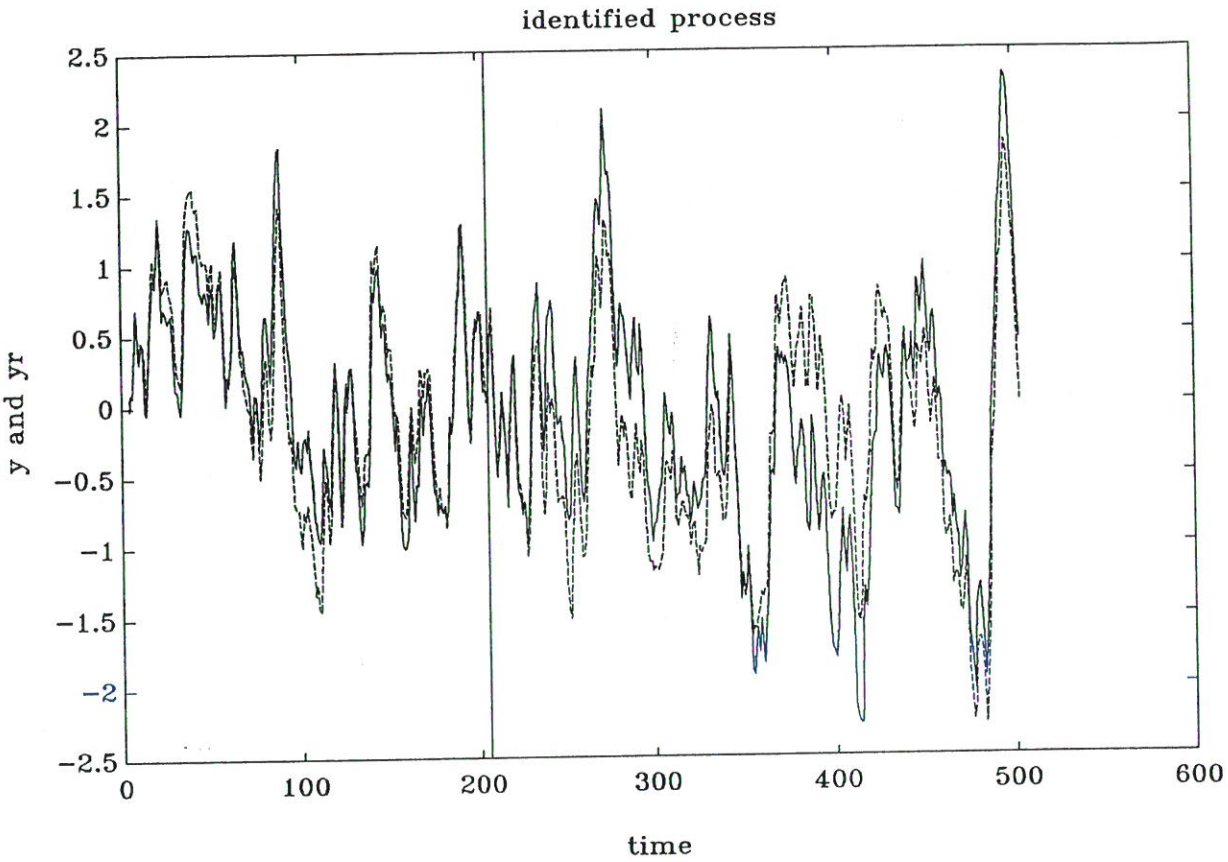


Figure 3: *Simulation and prediction of one of the outputs of a glassfurnace process with* $i = 3$, $k = 4$, $m = 3$, $l = 6$, $j = 200$, $n = 10$ *(full line = original measurements).*

with $L = X_1$, $M = X_3.S$ and $F^t = L^t(1:n,:)$, $G^t = M^t(1:n,:)$ (suppose that the elements of S are ordered as in an OSVD).

4. solve the following overdetermined set of linear equations in least squares sense in order to obtain the matrices A, B, C, D:

$$\begin{bmatrix} \underline{x}_{k+1} & \cdots & \underline{x}_{k+j-1} \\ \underline{y}_k & \cdots & \underline{y}_{k+j-2} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_k & \cdots & \underline{x}_{k+j-2} \\ \underline{u}_k & \cdots & \underline{u}_{k+j-2} \end{bmatrix}$$

Remark:
Instead of taking a QPQ-SVD in step 2 of this algorithm, we can get the same result with 3 OSVD's:

$$\begin{aligned} (H^+)^t &= U_1 \Sigma_1 (V_1)^t \\ (H^-)^t &= U_2 \Sigma_2 (V_2)^t \\ U_1^t U_2 &= U S V^t \end{aligned}$$

where S has the same meaning as in (17). The matrices X^+ and X^- are then:

$$\begin{aligned} X^+ &= U^{-1}(1:n,:) U_1^t \\ X^- &= V^{-1}(1:n,:) U_2^t \end{aligned}$$

+ (opm. QSVD methode en methode met 3 OSVD's zijn beide geprogrammeerd in MATLAB)

6 Conclusions

It is shown that a QPQ-SVD approach is very successful in order to evaluate relations between past and future of a process and hence useful for estimating the system order and obtaining a state vector sequence, all this in a numerical reliable way. Making use of an Orthogonality Theorem, it is always possible to make a senseful system order estimation.
This Orthogonality Theorem has to be applied to an interaction matrix, which measures the interaction between the past and the future of the process and it can be expressed in an elegant way by means of the QPQ-SVD of past and future I/O-measurements.

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