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ON A CLASS OF FRACTAL MATRICES (I) EXCESS-MATRICES AND THEIR SELF-SIMILAR PROPERTIES

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An infinite class of integer valued matrices of unbounded size and of arbitrary dimensions is presented here as a generalization of the so-called excess-matrix associated with a particular cellular automaton. These matrices are constructed through a recursive arithmetical procedure, and show an intricate fractal-like distribution of integers. The elements in these matrices have a number theoretical interpretation which is directly related to a proper number-base representation of the coordinates of the element. These excess-matrices exhibit numerical and symbolical self-similarities under agglomerating and decimating coarse-graining operations, and under so-called disguising transformations.

1. Introduction

In Barbé [1990], it was shown that the state space of a particular cellular automaton that combined a local evolution rule with a certain constraint on the automaton's state, is isomorphic to a fractal-like subset of an integer matrix E. This matrix E can be grown from the unlimited application of the recursive procedure

$$E_{m+1} = \begin{bmatrix} E_m & E_m - 1 \\ E_m & E_m + 1 \end{bmatrix}, \text{ with } E_1 = \begin{bmatrix} 0 & -1 \\ 0 & +1 \end{bmatrix}. \tag{1}$$

In this and forthcoming expressions, $E_m + z$ (or $z + E_m$), where z is a scalar, means that z is added to all elements of E_m. At each recursion step, the size of the matrix doubles, the size of E_m being $2^m \times 2^m \cdot E$ shows an intricate distribution of the integers over its elements (this is shown for E_4 in Fig. 1, while Fig. 2 shows a colorful translation of E_7). E has been called an *excess-matrix*, because the element $e(r, c)$ at row r and column c (row and column ranking starts at 0) equals the excess of 1's over 0's in the binary representation of r at all places where the binary representation of c shows a 1. For example, $e(15, 25) = 1$ as can be seen from

binary representation of $c = 25$: $\boxed{1}$ $\boxed{1}$ 0 0 $\boxed{1}$
binary representation of $r = 15$: $\boxed{0}$ $\boxed{1}$ 1 1 $\boxed{1}$ (2)

excess-count = number of 1's − number of 0's in the full square boxes = 1.

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	−1	−1	−2	−1	−2	−2	−3	−1	−2	−2	−3	−2	−3	−3	−4
1	0	1	−1	0	−1	0	−2	−1	−1	0	−2	−1	−2	−1	−3	−2
2	0	−1	1	0	−1	−2	0	−1	−1	−2	0	−1	−2	−3	−1	−2
3	0	1	1	2	−1	0	0	1	−1	0	0	1	−2	−1	−1	0
4	0	−1	−1	−2	1	0	0	−1	−1	−2	−2	−3	0	−1	−1	−2
5	0	1	−1	0	1	2	0	1	−1	0	−2	−1	0	1	−1	0
6	0	−1	1	0	1	0	2	1	−1	−2	0	−1	0	−1	1	0
7	0	1	1	2	1	2	2	3	−1	0	0	1	0	1	1	2
8	0	−1	−1	−2	−1	−2	−2	−3	1	0	0	−1	0	−1	−1	−2
9	0	1	−1	0	−1	0	−2	−1	1	2	0	1	0	1	−1	0
10	0	−1	1	0	−1	−2	0	−1	1	0	2	1	0	−1	1	0
11	0	1	1	2	−1	0	0	1	1	2	2	3	0	1	1	2
12	0	−1	−1	−2	1	0	0	−1	1	0	0	−1	2	1	1	0
13	0	1	−1	0	1	2	0	1	1	2	0	1	2	3	1	2
14	0	−1	1	0	1	0	2	1	1	0	2	1	2	1	3	2
15	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4

Fig. 1. Subset E_4 of the generic excess-matrix E obtained from recursion Eq. (1).

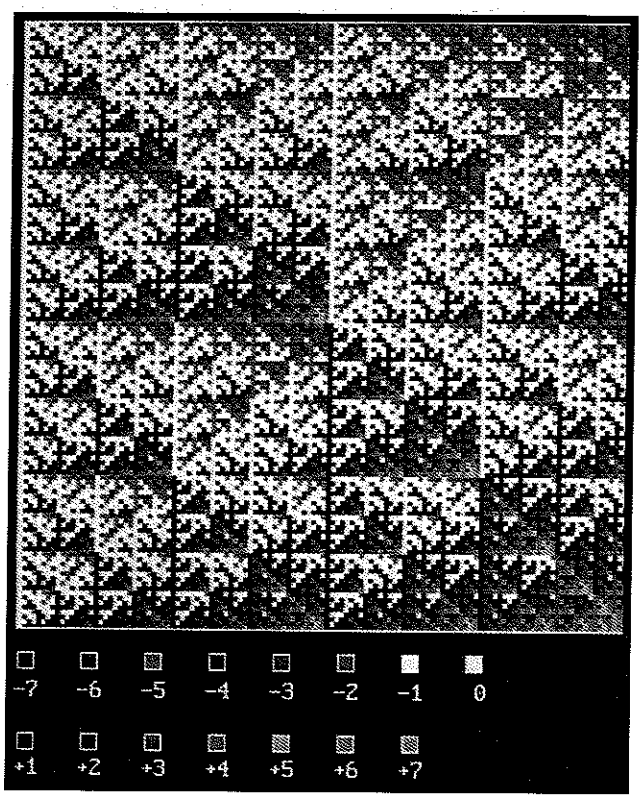


Fig. 2. A colorful representation of E_7.

In Sec. 2, Eq. (1) will be generalized to a recursion

$$E_{m+1} = E_1 \oplus E_m \,, \tag{3}$$

starting with an integer-valued *generator matrix* E_1 of arbitrary size and dimension. $\oplus$ denotes the operation that is similar to the Kronecker product[1] of two matrices, but with the product replaced by an addition. Recursion (3) produces an infinite class of infinitely sized *excess-matrices.*[2] For the two-dimensional case, this recursion produces a subclass of the matrices generated by the iterated products introduced by Shallit & Stolfi [1989]. These authors also showed the equivalence with what they called iterated matrix valued homomorphisms, also known as dynamical substitution systems from other publications (See Queffélec [1987] for one-dimensional substitutions, Peyriére [1987] and Mozes [1989] for two-dimensional substitutions, all over a finite alphabet. The alphabet in E is an infinite subset of $\mathbb{Z}$, the integer set). So, recursion (3) also defines a dynamical substitution system, where the symbolic substitution is based on an arithmetical rule.

It will be shown in Sec. 2.2 how the value of the element $e(\boldsymbol{\kappa})$ at location $\boldsymbol{\kappa}$ in an excess-matrix ($\boldsymbol{\kappa}$ is a vector), is related to the representation of $\boldsymbol{\kappa}$ in a proper number base depending on the size of the generator matrix E_1, in a way that generalizes the interpretation in scheme (2). This is actually an extension of similar properties concerning fractal structures associated with Pascal's Triangle, and attributed to the nineteenth century mathematician E. Lucas (see Chaitin [1988, 1990] referring to Lucas, and Stewart [1989] referring to Chaitin, and extending Lucas).[3]

Section 3 is devoted to self-similarity properties of excess-matrices under different coarse-graining operations. The basis for this lies in the fact that the following fundamental semigroup relationship holds:

[1] $\oplus$ will be called the "bigsum," it is *not* the operation known as Kronecker sum. See, for example, Graham [1981].

[2] This name is chosen in reference to the generic example above, although the "excess-count" will have a different interpretation in the general case.

[3] In Peitgen *et al.* [1992], reference is made to the nineteenth century mathematician E. Kummer in this context.

$$\forall k, l \geq 0 : E_{l+k} = E_l \oplus E_k = E_k \oplus E_l .$$

Section 3.1 looks at the structure obtained by certain *regular partitionings* of an excess-matrix into equally sized parts (called agglomerates), while considering each agglomerate as one single symbol. This structure is structurally similar to E itself. Also, all agglomerates are structurally similar to each other.

Section 3.2 considers *decimating operations* on E. Hereby, specific rows and columns are deleted. Joining the remaining parts in their proper order produces a connected part (agglomerate) of E. This part can be found from a renormalization equation associated with the decimating operation. E itself emerges as an invariant under this renormalization.

Section 3.3 investigates what happens to the structure of E under a so-called "disguising" transformation. Such a transformation produces a matrix $\tilde{E}$ by a many-to-one substitution of the values occurring in E. $\tilde{E}$ itself is also fractal-like, but can never be an excess-matrix itself. However, such a disguised version $\tilde{E}$ is not able to hide the fine structure of E when the disguising function satisfies some rather weak condition.

In Sec. 4, the own structure of lower-dimensional-plane subsets of high dimensional excess-matrices is analyzed. This leads to rules for growing such lower-dimensional planes, without having to construct the whole higher-dimensional matrix. This leads to hybrid versions of the basic recursion scheme, and is particularly useful for displaying two-dimensional subsets of higher-dimensional matrices.

This paper is also preparatory for a second part that will deal with the distribution and fractal dimension of a specific selection of integers over the excess-matrices and their disguised versions. The development in that part will rely on the integer character of the excess-matrices and on the additive form of the recursion. This is the reason why we will almost exclusively use this type of integer/additive formalism in the development here, instead of a more general substitution dynamics formalism.

2. The Class of Excess-Matrices

2.1. *Generalized excess-matrices*

Recursions (1) and (3) will be generalized by considering generator matrices of *arbitrary size, arbitrary dimension*, and with *arbitrary integers* distributed over the elements. First, it will be convenient to introduce some notations and definitions to facilitate working with matrices of arbitrary dimension and size.

Notations 2.1. Let A be a *p-dimensional matrix*. The vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p)$, where σ_i is the size of the matrix in the ith dimension (or direction), is the associated *size-vector*. **Vectors** will always be written as **bold** characters.

$a(\boldsymbol{\alpha})$ represents the element of A at location with coordinate vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)$. As ranking of coordinates starts at 0, coordinates for the elements in A must satisfy $0 \leq \alpha_i < \sigma_i$, for all i.

Logical, arithmetical or combinatorial conditions or operations involving vectors, should be interpreted *componentwise*. For example, $\mathbf{0} \leq \boldsymbol{\alpha} < \boldsymbol{\sigma}$ is equivalent to $\forall i,\ 1 \leq i \leq p$: $0 \leq \alpha_i < \sigma_i$; $\boldsymbol{\sigma}^l \triangleq (\sigma_1^l, \sigma_2^l, \ldots, \sigma_p^l)$; $\boldsymbol{\alpha} \bmod \boldsymbol{\sigma}^l \triangleq (\alpha_1 \bmod \sigma_1^l, \alpha_2 \bmod \sigma_2^l, \ldots, \alpha_p \bmod \sigma_p^l)$; $\lfloor \boldsymbol{\kappa}/\boldsymbol{\sigma} \rfloor \triangleq (\lfloor \kappa_1/\sigma_1 \rfloor, \lfloor \kappa_2/\sigma_2 \rfloor \cdots \lfloor \kappa_p/\sigma_p \rfloor)$; etc...

Definition 2.1. *The bigsum of two matrices.* Let A and B be two arbitrary matrices of the same dimension p, with size vectors $\boldsymbol{\sigma}^{(A)} = (\sigma_1^{(A)}, \sigma_2^{(A)}, \ldots, \sigma_p^{(A)})$ and $\boldsymbol{\sigma}^{(B)} = (\sigma_1^{(B)}, \sigma_2^{(B)}, \ldots, \sigma_p^{(B)})$ respectively. The bigsum $A \oplus B$ is then defined as the matrix resulting from substituting each element $e(\boldsymbol{\alpha})$ in A by the matrix $e(\boldsymbol{\alpha}) + B$. This is shown below for a two-dimensional example involving the matrices A and B of sizes (p, q) and (r, s) respectively.

$$C = A \oplus B \triangleq \begin{bmatrix} a(0,0) + B_{r,s} & a(0,1) + B_{r,s} & \ldots & a(0,q-1) + B_{r,s} \\ a(1,0) + B_{r,s} & a(1,1) + B_{r,s} & \ldots & a(1,q-1) + B_{r,s} \\ \vdots & \vdots & \ddots & \vdots \\ a(p-1,0) + B_{r,s} & a(p-1,1) + B_{r,s} & \ldots & a(p-1,q-1) + B_{r,s} \end{bmatrix} \tag{4}$$

The following properties of the bigsum are easy to derive:

(a) $C = A \oplus B$ is also p-dimensional, and has size $\sigma_i^{(A)}\sigma_i^{(B)}$ in direction i.
(b) Element $c(\boldsymbol{\kappa})$ is given by

$$c(\boldsymbol{\kappa}) = a\left(\left\lfloor \frac{\boldsymbol{\kappa}}{\boldsymbol{\sigma}^{(B)}} \right\rfloor\right) + b(\boldsymbol{\kappa} \bmod \boldsymbol{\sigma}^{(B)}) . \quad (5)$$

(c) The bigsum is associative, but generally *non-commutative.*

Definition 2.2. *The excess-matrix.* Let E_1 be a p-dimensional *integer-valued matrix* with size-vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p)$, then the associated excess-matrix E is the p-dimensional matrix of unbounded size obtained by an unlimited application of the following recursion:

$$E_{m+1} = E_1 \oplus E_m . \quad (6)$$

E_1 will be called the *generator matrix.*

The matrix E_m has linear size $\boldsymbol{\sigma}^m$, (i.e., σ_i^m in direction i). m is called the *logsize*, it is the σ_i-base logarithm of the linear size σ_i^m in direction i, and that for any direction. The total number of elements in E_m equals $(\sigma_1\sigma_2 \ldots \sigma_p)^m$. Each additional step in Eq. (6) multiplies the linear size in direction i by σ_i (or: the logsize is increased by 1).

Example 2.1. Consider the two-dimensional generator matrix $E_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \end{bmatrix}$. Then $\boldsymbol{\sigma} = (2, 4)$, and

$$E_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & 2 & 1 & -2 & -2 & 1 \\ 1 & 2 & 2 & 1 & -2 & -1 & -1 & -2 & -2 & -1 & -1 & -2 & 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & 2 & -1 & -4 & -4 & -1 & -1 & -4 & -4 & -1 & 2 & -1 & -1 & 2 \end{bmatrix} .$$

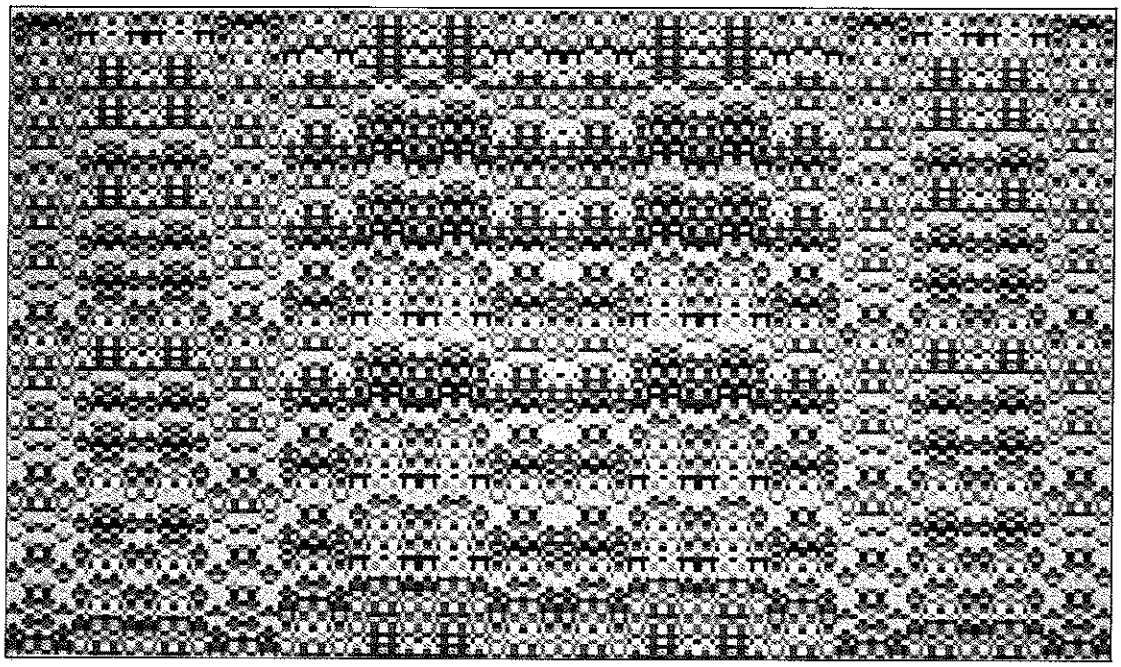


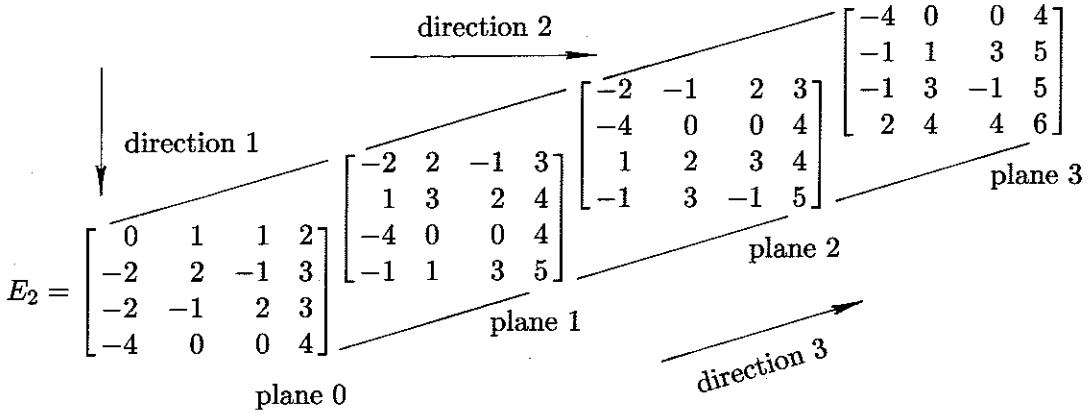
Fig. 3. The 128×256 left submatrix of the 128×16384 E_7-matrix generated from $E_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \end{bmatrix}$.

Figure 3 shows a colorful equivalent of the top left submatrix of the corresponding E of linear size 128×256 (= first 256 columns of E_7, as the complete E_7 should show $2^7 = 128$ rows and $4^7 = 16384$ columns).

Example 2.2. Consider the following three-dimensional generator matrix of size $\boldsymbol{\sigma} = (2, 2, 2)$

$$E_1 = \left[\begin{matrix} 0 & 1 \\ -2 & 2 \end{matrix}\middle|\begin{matrix} -2 & 2 \\ 1 & 3 \end{matrix}\right].$$

Then:



$$E_2 = \left[\begin{matrix} 0 & 1 & 1 & 2 \\ -2 & 2 & -1 & 3 \\ -2 & -1 & 2 & 3 \\ -4 & 0 & 0 & 4 \end{matrix}\middle|\begin{matrix} -2 & 2 & -1 & 3 \\ 1 & 3 & 2 & 4 \\ -4 & 0 & 0 & 4 \\ -1 & 1 & 3 & 5 \end{matrix}\middle|\begin{matrix} -2 & -1 & 2 & 3 \\ -4 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & 3 & -1 & 5 \end{matrix}\middle|\begin{matrix} -4 & 0 & 0 & 4 \\ -1 & 1 & 3 & 5 \\ -1 & 3 & -1 & 5 \\ 2 & 4 & 4 & 6 \end{matrix}\right]$$

Definition (6) and the associativity of the bigsum lead to the following property:

Property 2.1. *E_m satisfies the semigroup property:*

$$\forall m,\ k \geq 1:\ E_{m+k} = E_m \oplus E_k = E_k \oplus E_m\,. \qquad (7)$$

The next property allows the element $e_m(\boldsymbol{\kappa})$ to be decomposed into the sum of elements of remarkably lower rank. It is a direct consequence of Eqs. (5) and (7).

Property 2.2. *Let $e_m(\boldsymbol{\kappa})$ denote the element at location $\boldsymbol{\kappa}$ in E_m. Then*

$$\forall \boldsymbol{l} > 0:\ e_m(\boldsymbol{\kappa}) = e_{m-l}\left(\left\lfloor \frac{\boldsymbol{\kappa}}{\boldsymbol{\sigma}^l} \right\rfloor\right) + e_l(\boldsymbol{\kappa} \bmod \boldsymbol{\sigma}^l) \qquad (8)$$

or, equivalently

$$\forall \boldsymbol{\beta} < \boldsymbol{\sigma}^l:\ e_m(\boldsymbol{\sigma}^l \cdot \boldsymbol{\alpha} + \boldsymbol{\beta}) = e_{m-l}(\boldsymbol{\alpha}) + e_l(\boldsymbol{\beta})\,. \qquad (9)$$

2.2. *A number theoretic interpretation of $e_m(\boldsymbol{\kappa})$*

Definition 2.3. Let's symbolize the size-n, base-σ_i representation of component κ_i in the coordinate vector $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_p)$ as a string of length n:

$$\kappa_{i,n-1} \ldots \kappa_{i,2} \kappa_{i,1} \kappa_{i,0} (0 \leq \kappa_{i,j} < \sigma_i \text{ for all } j)\,. \qquad (10)$$

The minimal number of digits in Eq. (10) should equal $\lceil \log_{\sigma_i}(\kappa_i + 1) \rceil$. However, an arbitrary number of 0's may be added to the left of the minimum-length representation, thus increasing n.

When $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$ are p-vectors, the *size-n, base-$\boldsymbol{\sigma}$ representation of $\boldsymbol{\kappa}$* is the $p \times n$ symbol matrix where the ith row is the base-σ_i representation of κ_i:

$$\boldsymbol{\kappa}_{\boldsymbol{\sigma}}^{(n)} \triangleq \begin{bmatrix} \kappa_{1,n-1} & \kappa_{1,n-2} & \cdots & \kappa_{1,0} \\ \kappa_{2,n-1} & \kappa_{2,n-2} & \cdots & \kappa_{2,0} \\ & \vdots & \vdots & \\ \kappa_{p,n-1} & \kappa_{p,n-2} & \cdots & \kappa_{p,0} \end{bmatrix}, \qquad (11)$$

$$n \geq \max_i \lceil \log_{\sigma_i}(\kappa_i + 1) \rceil\,.$$

The *column* at rank j from the right will be called the *jth conjunction*, with associated *conjunction vector* $\boldsymbol{c}_j(\boldsymbol{\kappa}) = (\kappa_{1,j}, \kappa_{2,j}, \ldots, \kappa_{p,j})$.

Observe that $\boldsymbol{c}_j(\boldsymbol{\kappa})$ defines a location in the generator matrix E_1. When $e_1(\boldsymbol{c}_j(\boldsymbol{\kappa}))$ is considered as a "weight" associated with $\boldsymbol{c}_j(\boldsymbol{\kappa})$, and so E_1 as a "weight"-table for all possible conjunctions, the next property shows that $e_m(\boldsymbol{\kappa})$ equals the sum total of the weights of all conjunction vectors in $\boldsymbol{\kappa}_{\boldsymbol{\sigma}}^{(m)}$. This is the extension of a similar dependency on locations in the modulo representations of Pascal's Triangle as mentioned in the introduction.

Property 2.3. *Fundamental decomposition property. Element $e_m(\kappa)$ in E_m equals the sum of the elements in E_1 at those locations that are given by the conjunction vectors of the base-σ representation of κ:*

$$e_m(\kappa) = \sum_{j=0}^{m} e_1(\kappa_{1,j}, \kappa_{2,j} \ldots \kappa_{p,j}), \qquad m \geq \max_i \lceil \log_{\sigma_i}(\kappa_i + 1) \rceil . \tag{12}$$

Proof. Apply Eq. (8) repeatedly with $l = 1$. This gives

$$\begin{aligned} e_m(\kappa) &= e_{m-1}\left(\left\lfloor \frac{\kappa}{\sigma} \right\rfloor\right) + e_1(\kappa \bmod \sigma) \\ &= e_{m-2}\left(\left\lfloor \frac{1}{\sigma} \left\lfloor \frac{\kappa}{\sigma} \right\rfloor \right\rfloor\right) + e_1\left(\left\lfloor \frac{\kappa}{\sigma} \right\rfloor \bmod \sigma\right) \\ &\quad + e_1(\kappa \bmod \sigma) \\ \ldots &= \sum_{j=0}^{m-1} e_1\left(\left\lfloor \frac{\kappa}{\sigma} \right\rfloor_j \bmod \sigma\right). \end{aligned}$$

where $\lfloor \frac{\kappa}{\sigma} \rfloor_j \triangleq \lfloor \frac{1}{\sigma} \lfloor \frac{\kappa}{\sigma} \rfloor_{j-1} \rfloor$, to start with $\lfloor \frac{\kappa}{\sigma} \rfloor_0 = \kappa$.

Now, $\lfloor \frac{\kappa_i}{\sigma_i} \rfloor_j \bmod \sigma_i$ is nothing other than the digit of rank j in the base-σ_i representation of κ_i. Hence $\lfloor \frac{\kappa}{\sigma} \rfloor_j \bmod \sigma = c_j(\kappa)$. ■

Remark. Equation (12) is a translation, in this paper's specific context, of expressions that appear in the context of substitution dynamical systems as related to automata: proper base representations of κ act as input sequences on finite state automata calculating $e(\kappa)$. See, for example, Allouche [1987] for the one-dimensional case, Salon [1987] and also Shallit & Stolfi [1989] for the two-dimensional square case.

Example 2.3. Consider the following three-dimensional example:

$$E_1 = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}_{\kappa_3=0} \quad \begin{bmatrix} -2 & 1 & -3 \\ 0 & 4 & -2 \end{bmatrix}_{\kappa_3=1},$$

so

$$\sigma = (2, 3, 2).$$

What would be the value of $e_{10}(125, 88, 54)$ in the corresponding E_{10}? The base-(2, 3, 2) representation of (125, 88, 54), 10 digits long, is given by

base-2 representation of 125:	0	0	0	1	1	1	1	1	0	1
base-3 representation of 88:	0	0	0	0	0	1	0	0	2	1
base-2 representation of 54:	0	0	0	0	1	1	0	1	1	0
weight of the conjunctions:	2	2	2	−1	0	4	−1	0	−3	3

The conjunctions are found at the columns in the array above, and the corresponding weights are found in E_1. The sum of all these weights equals 8. So $e_{10}(125, 88, 54) = 8$. The element at the same location in E_7, i.e., $e_7(125, 88, 54)$ equals 2, as the leftmost three (0, 0, 0) conjunctions in the array would disappear. Observe that when $e_1(\mathbf{0}) = 0$, the weight for the (0, 0, ... 0)-conjunctions would be 0, and $e_m(\kappa)$ would be independent of the size m of the matrix E_m. Only the position of the element would then define its value.

Definition 2.4. A matrix A whose **0**th element $a(\mathbf{0}) = 0$, will be called *unbiased*. When in addition, the greatest common divisor of all elements of A equals 1, A will be called *reduced*.

Let A' be an arbitrary integer valued matrix. Then there always exists a unique reduced matrix A such that $A' = aA + b$, where $b = a'(\mathbf{0})$ and $a = g.c.d$ of the elements of $A' - b$.

The next property follows easily.

Property 2.4. *If E_1 is a reduced generator matrix, then:*

(a) $e_m(\kappa)$ is independent of m, so it can be further denoted by $e(\kappa)$
(b) all E_m are reduced and vice versa.

Remarks. When E_1 is unbiased, the sequence E_1, E_2, E_3 ... represents a "growing matrix": E_{m+k} is obtained from E_m by just adding new elements to E_m.

When E_1 is the reduced version of $E_1' = aE_1 + b$, then one shows easily that $E_m' = aE_m + mb$. So, if suffices to consider reduced excess-matrices only, without loss of generality. **From here on, reducedness of the excess-matrices will be implicitly assumed.**

Corollary 2.1. *For reduced excess-matrices, the expressions* (8), (9) *and* (12) *become*

$$\forall l > 0 : e(\boldsymbol{\kappa}) = e\left(\left\lfloor \frac{\boldsymbol{\kappa}}{\boldsymbol{\sigma}^l} \right\rfloor\right) + e(\boldsymbol{\kappa} \bmod \boldsymbol{\sigma}^l), \quad (13)$$

$$\forall \boldsymbol{\beta} < \boldsymbol{\sigma}^l : e(\boldsymbol{\sigma}^l \cdot \boldsymbol{\alpha} + \boldsymbol{\beta}) = e(\boldsymbol{\alpha}) + e(\boldsymbol{\beta}), \quad (14)$$

$$e(\boldsymbol{\kappa}) = \sum_{j=0}^{n} e(\boldsymbol{c}_j(\boldsymbol{\kappa})) \text{ with } n \geq \max\lceil \log_{\sigma_i}(\kappa_i + 1)\rceil . \quad (15)$$

Remark. Equation (14) states that the function $e(\cdot)$ is $\boldsymbol{\sigma}$-additive, in a multi-dimensional sense. See Delange [1972] referring to the originator of this concept [Gelfond, 1968]. Use of this notion in spectral analysis of sequences generated by one-dimensional substitution systems can be found in Coquet *et al.* [1977] and Queffélec [1987].

2.3. *The fractal nature of excess-matrices*

The examples presented until now show that excess-matrices are objects with a very complex structure. Especially the colorful "translations" bring associations with fractal objects into mind. However, by squeezing E_m onto an equally spaced lattice in a (hyper) cube of unit size, and calling this transfiguration $\mathcal{L}(E_m)$, it appears that $\mathcal{L}(E_m)$ might converge to a genuine fractal $\mathcal{L}(E)$. $\mathcal{L}(E_m)$, for m sufficiently large, and even $\mathcal{L}(E)$ would then correspond to what Mandelbrot [1989, p. 328] has called latticed fractals. Actually, they are a kind of "latticed" versions of multifractals, as they represent a mapping of the lattice points into a symbol set (for example, colors; or just $\mathbb{Z}$, replacing the "measure set" usually encountered in the definition of multifractals, see, for example, Takayasu [1990] and Falconer [1990]). In this context, it is worthwhile to mention that the so-called binomial multifractal [Takayasu, 1990], is completely isomorphic with the noninteger, nonreduced matrix obtained from applying the basic recursion to the generator matrix $E_1 = [\log p \ \log q]$, $(p + q = 1)$. The elements of E_m thus obtained, represent the logarithms of the measures obtained in the mth step. Also, the two-dimensional version presented by Barnsley [1988, example 6.3] is in the same sense isomorphic to the excess-matrix obtained with the generator

$$E_1 = \begin{bmatrix} \log p_1 & \log p_3 \\ \log p_2 & \log p_4 \end{bmatrix}, \quad \left(\sum p_i = 1\right).$$

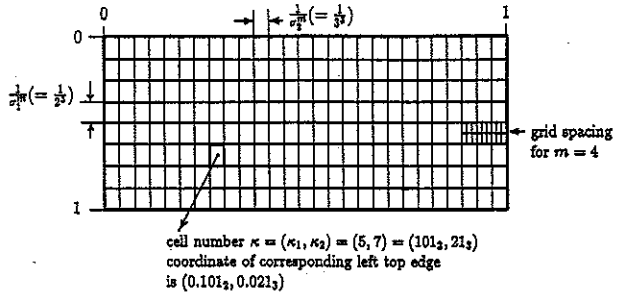


Fig. 4. The grid for the lattice $\mathcal{L}(E_3)$ associated with a 2×3 generator matrix E_1.

Actually, there is no strict need for the elements of the excess-matrices to be integers (rational matrices can be readily transformed into numerically similar integer ones (see Def. 3.1), while this is not always possible for real matrices). The restriction to integers just makes working with mapping sequences easier. These sequences will be defined in Sec. 3, and will play an important role in a forthcoming Part II of this paper, where fractal dimensions of excess-matrices will be discussed.

Let's have a closer look at the lattice form. The grid spacing for $\mathcal{L}(E_m)$ in direction σ_i is σ_i^{-m}. $e(\boldsymbol{\kappa})$ can be associated with the point with coordinates $\boldsymbol{\kappa}/\boldsymbol{\sigma}^m$, or with the cell that has this point as left-top edge. Figure 4 illustrates this for a two-dimensional example. Observe that the base-$\boldsymbol{\sigma}$ representation of this "lattice-address" corresponds to a shift of the base-$\boldsymbol{\sigma}$ representation of $\boldsymbol{\kappa}_{\boldsymbol{\sigma}}^{(m)}$, m places to the right of the "decimal" point. This will be denoted by $\mathbf{0} \cdot \boldsymbol{\kappa}_{\boldsymbol{\sigma}}^{(m)}$.

E_m and $\mathcal{L}(E_m)$ are in a 1–1 correspondence. What about E and $\mathcal{L}(E)$? E could be considered as having an infinite but countable number of elements. On the other hand, the points in $\mathcal{L}(E)$ seem to fill the continuum inside the unit cube $[0, 1)^p$, as each point in $\mathcal{L}(E)$ has an address with components $0 \cdot \beta_{\sigma_i}^{\infty}$, where $\beta_{\sigma_i}^{\infty}$ is an infinite sequence of digits in the range $[0, \sigma_i - 1]$. But then, the 1–1 correspondence between E and $\mathcal{L}(E)$ must have disappeared somewhere, as E has a countably infinite and $\mathcal{L}(E)$ an uncountably infinite number of elements. It is felt that this paradox can only be solved by invoking transfinite and infinitesimal numbers, thus

transgressing the potential infinity towards the actual one. E and $\mathcal{L}(E)$ should be considered as being E_ω and $\mathcal{L}(E_\omega)$, whereby ω is the first limit ordinal. Then both E_ω and $\mathcal{L}(E_\omega)$ remain in 1-1 correspondence and have $2^\omega = \aleph_1$ elements.[4] The points in $\mathcal{L}(E_\omega)$ are then the points in $[0, 1)^p$, but considered as a set which extends the set of reals, in the sense that the numbers whose difference is infinitesimal should be considered as different. For example, in the binary case, points with addresses $0.01\bar{0}\ldots$ and $0.00\bar{1}$ (the overbar denotes unlimited repetition of the underlying digit) should be considered as different (as should be $0.013\bar{0}$ and $0.012\bar{4}$ in the 5-ary case). Notice the difference in "graphical" representation of E_ω and $\mathcal{L}(E_\omega)$. E_ω can only be partially visualized through the potentially infinite E, whose corresponding part in $\mathcal{L}(E_\omega)$ always starts at the left-top edge in $\mathcal{L}(E_\omega)$. Also, any finite subset of E_ω is an infinitesimally sized subset in $\mathcal{L}(E_\omega)$, while any noninfinitesimal subset of $\mathcal{L}(E_\omega)$ is an infinite subset of E_ω. As to the values associated with points in $\mathcal{L}(E_\omega)$: whenever the base-σ representation of an address does not end with an infinite sequence of conjunctions whose weight is 0, the value is ω. Otherwise it takes some value in $\mathbb{Z}$. In what follows, we will almost exclusively deal with E and its finite subsets E_m [or the finite $\mathcal{L}(E_m)$].

2.4. *The range of the integers in E_m*

This section will deal with the question of how the set of integers that appear in E_m evolves when m moves from 1 to ∞. Some aspects of this will be needed in later sections.

Definition 2.5. The set of integers appearing in E_m will be called the *range of* E_m, and will be denoted by $R(E_m)$.

It is a direct consequence of the relationship $E_{l+k} = E_l \oplus E_k$, that $R(E_{l+k})$ can be obtained from $R(E_l)$, as follows:

$$R(E_{l+k}) = \bigcup_{\lambda \in R(E_k)} R(E_l + \lambda)\,. \tag{16}$$

$R(E_l+\lambda) = R(E_l)+\lambda$ is the set that is obtained from $R(E_l)$ by adding λ to all elements in $R(E_l)$. With $e_{1\,\min}$ and $e_{1\,\max}$ the extremal values in E_1, we find

$$\min R(E_l) = l \cdot e_{1\,\min} \text{ and } \max R(E_l) = l \cdot e_{1\,\max} \tag{17}$$

[4]Under the continuum hypothesis. See any book on transfinite ordinal & cardinal numbers, for example, Henle [1986].

Example 2.4. Let $E_1 = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 0 & 6 \end{bmatrix}$, then $R(E_1) = \{-2, 0, 1, 3, 6\}$. Applying Eq. (16) with $k = 1$ results in $R(E_2) = \{-4, -2, -1, 0, 1, 2, 3, 4, 6, 7, 9, 12\}$, and $R(E_3) = \{-6, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 18\}$. For arbitrary l: $R(E_l) = [-2l, 6l]\backslash\{-2l+1, 6l-1, 6l-2, 6l-4, 6l-7\}$, and $R(E) = \lim_{l\to\infty} R(E_l) = \mathbb{Z}$ = the set of integers.

Observe how $R(E_l)$ tends to be filled with consecutive integers near its "centre", at least when E_1 is reduced. The condition $e_1(\mathbf{0}) = 0$ implies that $R(E_1) \subset R(E_m)$ when $l < m$. The matrix E_1 from the example is of *mixed type*, i.e., it contains integers of both signs. All E_l and $R(E_l)$ are then mixed. In the same way, nonnegativity (or nonpositivity) of E_1 is inherited by the subsequent E_l and the associated $R(E_l)$. When $R(E_1)$ is *dense*, i.e. $R(E_1)$ is a set of consecutive integers, $R(E_l) = l \cdot R(E_1) = [le_{1\,\min}, le_{1\,\max}]$ is also dense, and $R(E) = \mathbb{Z}$ (or $\mathbb{Z}^+$, or $\mathbb{Z}^-$) in case E_1 is of mixed type (or nonnegative, or nonpositive). When $R(E_1)$ is not dense, all $R(E_l)$ are also not dense. In this case, nonpositive and nonnegative $R(E)$ will possibly differ from $\mathbb{Z}^-$ or $\mathbb{Z}^+$, but only at some finite region around 0.

Example 2.5

(a) $R(E_1) = \{0, 3, 5\}$, then $R(E_l) = [0, 5l]\backslash\{1, 2, 4, 7, 5l-3, 5l-1\}$ and $R(E) = \mathbb{Z}^+\backslash\{1, 2, 4, 7\}$

(b) $R(E_1) = \{-5, -1, 0\}$, then $R(E_l) = [-5l, 0]\backslash\{-5l+1, -5l+2, -5l+3, 5l+6, -5l+7, -5l+11\}$, and $R(E) = \mathbb{Z}^-$.

Observe that neither the dimension nor the size of the underlying matrix does play any role in the evolution of $R(E_l)$.

3. Self-Similarity and Renormalization of E under Different Coarse-Graining Transformations

In this section we will consider certain *coarse-graining* transformations of the matrix E, and it will be shown that the results are similar to E itself, in the sense of the following definitions.

Definition 3.1. Two matrices A and B will be called *numerically similar* when $A = c_2B + c_1$, with $c_2 \neq 0$ and c_1 constants. c_1 and c_2 will respectively be called the *additive* and *multiplicative bias.*

Definition 3.2. Two matrices A and B of equal size and with respective elements $a(\boldsymbol{\kappa})$ and $b(\boldsymbol{\kappa})$ are *symbolically similar* (SS) when $a(\boldsymbol{\alpha}) = a(\boldsymbol{\beta})$ if and only if $b(\boldsymbol{\alpha}) = b(\boldsymbol{\beta})$ for all pairs of coordinates

(α, β): i.e., when equal elements in A correspond to equal elements in B and vice-versa. This will be denoted by $A \overset{SS}{\leftrightarrow} B$. Notice that the elements of both matrices A and B may differ in nature.

As a direct consequence of these definitions, we obtain

Property 3.1. *Numerical similarity of matrices implies symbolic similarity.*

The converse of this property is generally not true.

3.1. *Self-similarity under agglomerating coarse-graining*

In what follows, we will look at partitions of the matrix E in blocks of logsize l (linear size σ^l). $B_l(\kappa)$ will represent the block of rank κ, i.e., that submatrix of E whose elements have coordinates μ that satisfy:

$$\kappa \cdot \sigma^l \leq \mu < \kappa \cdot \sigma^{l+1}. \tag{18}$$

Figure 1 illustrates this for a two-dimensional case. It shows the square blocks $B_2(i, j)$ $(i = 0, \ldots, 3,\ j = 0, \ldots, 3)$ in the E_4-matrix slightly separated.

The matrix of l-blocks of E_m will be denoted by $\mathcal{B}_l^m$, for E this will be $\mathcal{B}_l$. The matrix $\mathcal{B}_l$ whose elements are the $\mathcal{B}_l(\kappa)$-blocks, represents a coarse-graining view of E at a rather abstract level. Each element $\mathcal{B}_l(\kappa)$ is hereby considered as a single entity itself: an *agglomerate* of elements of E.

Property 3.2

(a) *All logsize-l agglomerates $B_l(\kappa)$ are numerically similar to E_l with additive bias $e(\kappa)$:*

$$B_l(\kappa) = E_l + e(\kappa)\ (for\ all\ l \geq 0) \tag{19}$$

and $B_l(0) = E_l$.

(b) *All logsize-l agglomerates $B_l(\kappa)$ are symbolically similar.*

Proof

(a) follows directly from $E_{k+l} = E_k \oplus E_l$, which substitutes $e(\kappa) \in E_k$ by $E_l + e(\kappa)$, and this is nothing else than the block $B_l(\kappa)$.

(b) is a direct consequence of (a) in view of the fact that numerical similarity implies symbolical similarity. ■

Equation (19) implies that there is a 1–1 correspondence between $B_l(\kappa)$ and $e(\kappa)$: all elements $e(\kappa)$ in E_k that show the same value, correspond to elements $B_l(\kappa) = E_l + e(\kappa)$ in $\mathcal{B}_l^{k+l}$ that are also equal and vice-versa. Hence, we can state

Property 3.3. *$\mathcal{B}_l^m$ is symbolically simlar to E_{m-l}, (for all nonnegative l, m; $m \geq l$). $\mathcal{B}_l$ is symbolically similar to E.*

This property reveals a first kind of self-similarity or invariance of E: any agglomerating coarse-graining operation on E creates a new matrix which has a structure that is identical to the structure of E (on a symbolical level).

Example 3.1. Figure 5 shows E_5 obtained from the generator matrix $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. E_1 was chosen to have an eye-catching symmetry, so that similarities on different scales are readily perceivable. $\mathcal{B}_4^5$, $\mathcal{B}_3^5$ are easily obtained by dividing E_5 in 9 and 81 squares respectively (by dividing the sides in 3 and 9 equal parts respectively). The corresponding agglomerates of logsize 4 (linear size $3^4 = 81$) and logsize 3 (linear size 27) are easily detectable. First observe that all agglomerates of the same size are symbolically similar: when they differ, they do so only by a 1–1 substitution of colors. Also, observe that $\mathcal{B}_4^5 \overset{SS}{\leftrightarrow} E_1$, $\mathcal{B}_3^5 \overset{SS}{\leftrightarrow} E_2$. Through further refinement, one can even check that $\mathcal{B}_2^5 \overset{SS}{\leftrightarrow} E_3$ and $\mathcal{B}_1^5 \overset{SS}{\leftrightarrow} E_4$, while of course $\mathcal{B}_0^5 \equiv E_5$.

The next property extends the symbolic similarity between E and $\mathcal{B}_l$ towards an isomorphic version of Eq. (14).

Property 3.4

$$\forall \alpha, \beta, m, n, \sigma^l > \beta:$$

$$B_{m+n}(\alpha \cdot \sigma^l + \beta) = B_m(\alpha) \oplus B_n(\beta) \tag{20}$$

Proof. This follows directly from Eqs. (19) and (14). ■

3.2. *Self-similarity of E under decimating coarse-graining*

Decimating operations on E involve the deletion of some rows and columns of E. The remaining elements form a new matrix that gives a certain coarse-graining view on the original matrix. It will be shown that such a view is identical to the view on a specific agglomerate of E.

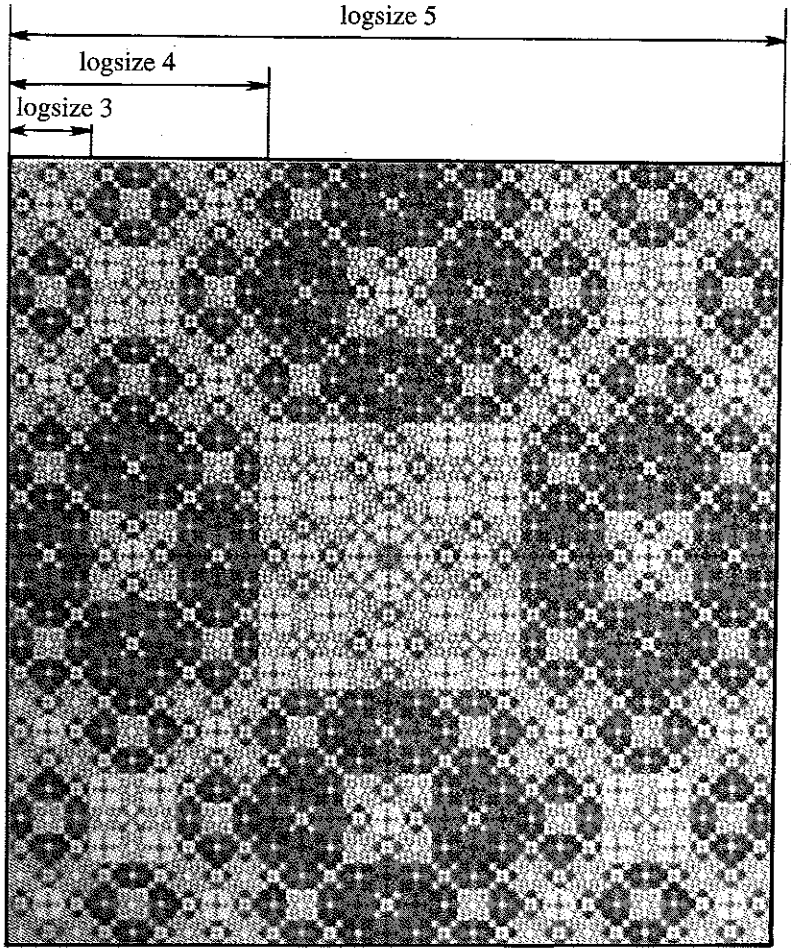


Fig. 5. Illustration of the symbolic self-similarity of an excess-matrix. $\left(\text{Generator } E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)$.

Notation 3.1. $\bar{\kappa}^{(l)}$ will denote the equivalence class of all places in E_l with the same values as $e(\kappa)$. $\bar{\kappa}_a^{(l)}$ denotes the class of all addresses κ in E_l for which $e(\kappa) = a$. $\bar{\kappa}$ and $\bar{\kappa}_a$ will do so with respect to E.

Property 3.5. *Consider $\mathcal{B}_l^m$, i.e., E_m partitioned in logsize-l agglomerates. In each agglomerate, pick out the element of rank κ ($0 \le \kappa < \sigma^l$), and delete all other elements. This results in a new matrix symbolized by $f_\kappa^{l,0}(E_m)$, the 0 in the superscript referring to the logsize of one element. It satisfies*

$$f_\kappa^{l,0}(E_m) = E_{m-l} + e(\kappa) \tag{21}$$

$$= B_{m-l}(\bar{\kappa}^{(l)}). \tag{22}$$

Also,

$$f_\kappa^{l,0}(E_m) = E + e(\kappa). \tag{23}$$

Proof. The αth element in $f_\kappa^{l,0}(E_m)$ is also the κth element in agglomerate $B_l(\alpha)$. It is given by

$$e(\sigma^l \cdot \alpha + \kappa) \overset{(14)}{=} e(\alpha) + e(\kappa).$$

As the logsize of $f_\kappa^{l,0}(E_m)$ equals $m - l$, this gives Eq. (21). Equation (22) follows by invoking Property 3.2 and from the fact that all $B_{m-l}(\kappa_j)$ with $\kappa_j \in \bar{\kappa}^{(l)}$ are equal. ■

Remarks. Equation (22) shows how elements in E_m which lie at distances σ_i^l apart along direction i, when joined together in their proper order, make up a well-defined part of E_m: the agglomerate $B_{m-l}(\kappa)$.

Equation (23) uncovers a second type of self-similarity or invariance of an excess matrix E: *E is numerically self-similar with additive bias under the decimating operation $f_\kappa^{l,0}$* (for any κ and l).

Observe that Eq. (22) has no meaning when the whole matrix E is considered, as $B_{m-l}(\boldsymbol{\kappa})$ would then equal $B_\infty(\boldsymbol{\kappa})$, which is not meaningful. However, when the lattice $\mathcal{L}(E_m)$ is considered, Eq. (22) makes sense for all m, even when $m \to \infty$. In this lattice, $B_{m-l}(\boldsymbol{\kappa})$ always represents that agglomerate of $\mathcal{L}(E_m)$ that lies in the p-dimensional hypercube $\{\boldsymbol{\alpha} \in \mathbb{R}^p : \frac{\boldsymbol{\kappa}}{\boldsymbol{\sigma}^l} \le \boldsymbol{\alpha} < \frac{\boldsymbol{\kappa}+1}{\boldsymbol{\sigma}^l}\}$, and this is independent of m. When combined with Eq. (23), this means that the $B_{m-l}(\boldsymbol{\kappa})$-associated hypercube in $\mathcal{L}(E_w)$ is numerically and symbolically similar to $\mathcal{L}(E_\omega)$ itself.

The next property is a first step towards a general decimating property.

Property 3.6. *Consider the partitioning of E_m in logsize-l agglomerates B_l, and then partition these B_l-blocks themselves into logsize-k agglomerates (so $k < l < m$). Then pick out the $\boldsymbol{\kappa}$th logsize-k agglomerate in each B_l-block ($0 \le \boldsymbol{\kappa} < \boldsymbol{\sigma}^{l-k}$), and delete all the other k-blocks. This results in a new matrix $f_{\boldsymbol{\kappa}}^{l,k}(E_m)$, and*

$$f_{\boldsymbol{\kappa}}^{l,k}(E_m) = E_{m-l+k} + e(\boldsymbol{\kappa}) \tag{24}$$

$$= B_{m-l+k}(\bar{\boldsymbol{\kappa}}^{(l-k)}) . \tag{25}$$

Also,

$$f_{\boldsymbol{\kappa}}^{l,k}(E) = E + e(\boldsymbol{\kappa}) . \tag{26}$$

Proof. Consider once again the relationship $E_m = E_{m-k} \oplus E_k$ and observe that

$$f_{\boldsymbol{\kappa}}^{l,k}(E_m) = f_{\boldsymbol{\kappa}}^{l-k,0}(E_{m-k}) \oplus E_k . \tag{27}$$

Now Eqs. (21) and (22) can be used to obtain Eqs. (24) and (25). ■

Example 3.2. Figure 6 shows E_5 for the generator matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and its partitioning $\mathcal{B}_4^5$ ($l = 4$ in Property 3.6). In each B_4-block, the cell of logsize $k = 2$ (linear size 9) at location (2, 6) is selected. When joined together, they form the agglomerates $B_{5-4+2}(\overline{2, 6})^{(2)} = B_3(\overline{2, 6})^{(2)}$, where $(\overline{2, 6})^{(2)}$ is the set of locations in E_2 that show the same value as $e(2, 6)$ and hence $B_3(\overline{2, 6})^{(2)}$ represents all agglomerates in E_5 that are identical to $B_3(2, 6)$.

Definition 3.3. $f_{\boldsymbol{\kappa}}^{l,k}$ will be called a *simple k-block decimating operation of (logarithmic) stepsize l.* Successive application of simple k-block decimating operations, each with possibly different size k and stepsize l forms a *composite decimating operation* and is symbolized by

$$f_{\boldsymbol{\kappa}_n,\boldsymbol{\kappa}_{n-1},\ldots\boldsymbol{\kappa}_1}^{l_n,k_n;l_{n-1},k_{n-1};\ldots l_1,k_1}(E_m)$$

$$\triangleq f_{\boldsymbol{\kappa}_n}^{l_n,k_n}(\ldots f_{\boldsymbol{\kappa}_2}^{l_2,k_2}(f_{\boldsymbol{\kappa}_1}^{l_1,k_1}(E_m))\ldots) . \tag{28}$$

Equation (28) is only defined for

$$l_1 + l_2 \ldots + l_n \le m \text{ and } 0 \le \boldsymbol{\kappa}_i \le \boldsymbol{\sigma}^{l_i} - 1 . \tag{29}$$

Applying Eq. (24) n times produces the decimating property in its most general form.

Property 3.7. *The general decimating property.*

$$f_{\boldsymbol{\kappa}_n,\boldsymbol{\kappa}_{n-1},\ldots\boldsymbol{\kappa}_1}^{l_n,k_n;l_{n-1},k_{n-1};\ldots l_1,k_1}(E_m)$$

$$= E_{m-(l_1+l_2\ldots+l_n)+(k_1+k_2\ldots+k_n)} + \sum_{i=1}^{n} e(\boldsymbol{\kappa}_i) \tag{30}$$

$$= B_{m-\Sigma(l_i-k_i)}\left(\bar{\boldsymbol{\kappa}}_{\Sigma e(\boldsymbol{\kappa}_i)}^{(\Sigma l_i - \Sigma k_i)}\right) \tag{31}$$

and

$$f_{\boldsymbol{\kappa}_n,\boldsymbol{\kappa}_{n-1},\ldots\boldsymbol{\kappa}_1}^{l_n,k_n;l_{n-1},k_{n-1};\ldots l_1,k_1}(E) = E + \sum_{i=1}^{n} e(\boldsymbol{\kappa}_i) \tag{32}$$

Recall that the $\bar{\boldsymbol{\kappa}}$-class in Eq. (31) is nothing other than the equivalence class of locations in $E_{(\Sigma l_i - \Sigma k_i)}$ where the value $\sum_{i=1}^{n} e(\boldsymbol{\kappa}_i)$ is found.

Remarks. Equations (30) and (32) show that decimating operations are commutative, as the terms in $\sum e(\boldsymbol{\kappa}_i)$ are commutable. In fact, the set of all decimating operations

$$F \triangleq \{f_{\boldsymbol{\kappa}}^{l,k};\ l > 0,\ k \le l,\ 0 \le \boldsymbol{\kappa} \le \boldsymbol{\sigma}^l - 1\} , \tag{33}$$

together with the concatenation $\circ$ of simple decimating operations defined by

$$f_{\boldsymbol{\kappa}_2}^{l_2,k_2} \circ f_{\boldsymbol{\kappa}_1}^{l_1,k_1} \triangleq f_{\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_1}^{l_2,k_2;l_1,k_1} , \tag{34}$$

forms an infinite *commutative semigroup* $\langle F, \circ \rangle$. This follows from the fact that $f_{\boldsymbol{\kappa}_2}^{l_2,k_2} \circ f_{\boldsymbol{\kappa}_1}^{l_1,k_1} = f_{\boldsymbol{\kappa}_3}^{l_2+l_1,k_1+k_2} \in F$, where $\boldsymbol{\kappa}_3 =$ location such that $e(\boldsymbol{\kappa}_3) = e(\boldsymbol{\kappa}_1) + e(\boldsymbol{\kappa}_2)$. This equality is a consequence of Eqs. (34), (30) and (24).

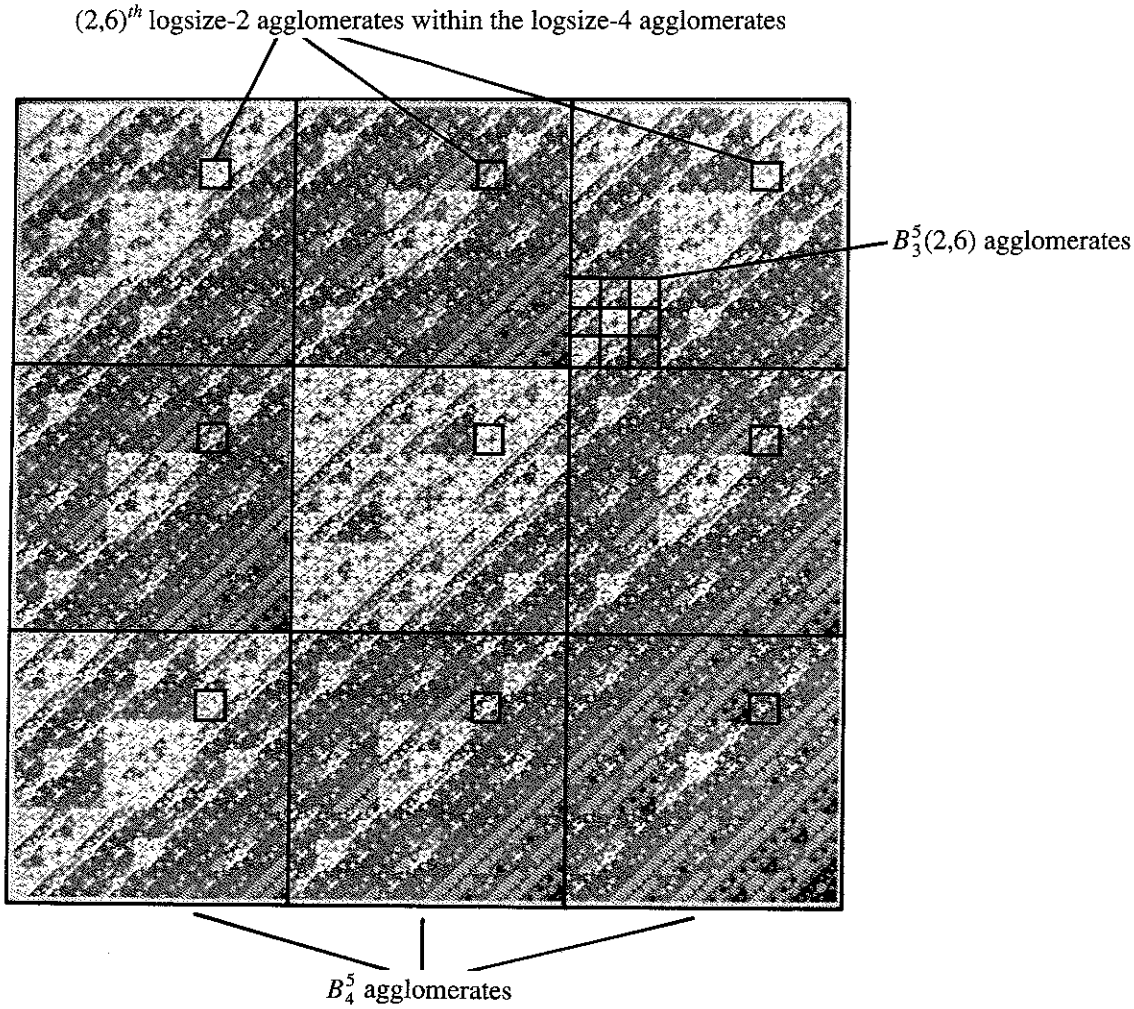


Fig. 6. Decimating an excess-matrix reproduces one of its agglomerates: joining the logsize-2 agglomerates at location (2, 6) in the logsize-4 agglomerates of E_5 produces the logsize-3 agglomerate of E_5 located at (2, 6).

All this is strongly reminiscent of *renormalization group* concepts [Takayasu, 1990; Willson, 1979]: Equations (30) and (32) can somehow be considered as the renormalization equations of the "quantities" E_m and E under the decimating coarse-graining operations $f_\kappa^{l,k}$. Equation (32) shows that the *excess-matrix E emerges as an invariant under this renormalization* (up to an additive constant).[5]

3.3. *Self-similarity under disguise*

We will now try to hide the fine-structure of an excess-matrix E by substituting different values over E by equal symbols, what results in a "disguised" version $\tilde{E}$ of E. As this section will show, many such disguises still reflect the fine structure of E on the level of their agglomerate-matrix $\tilde{\mathcal{B}}_l$ of some minimal size l.

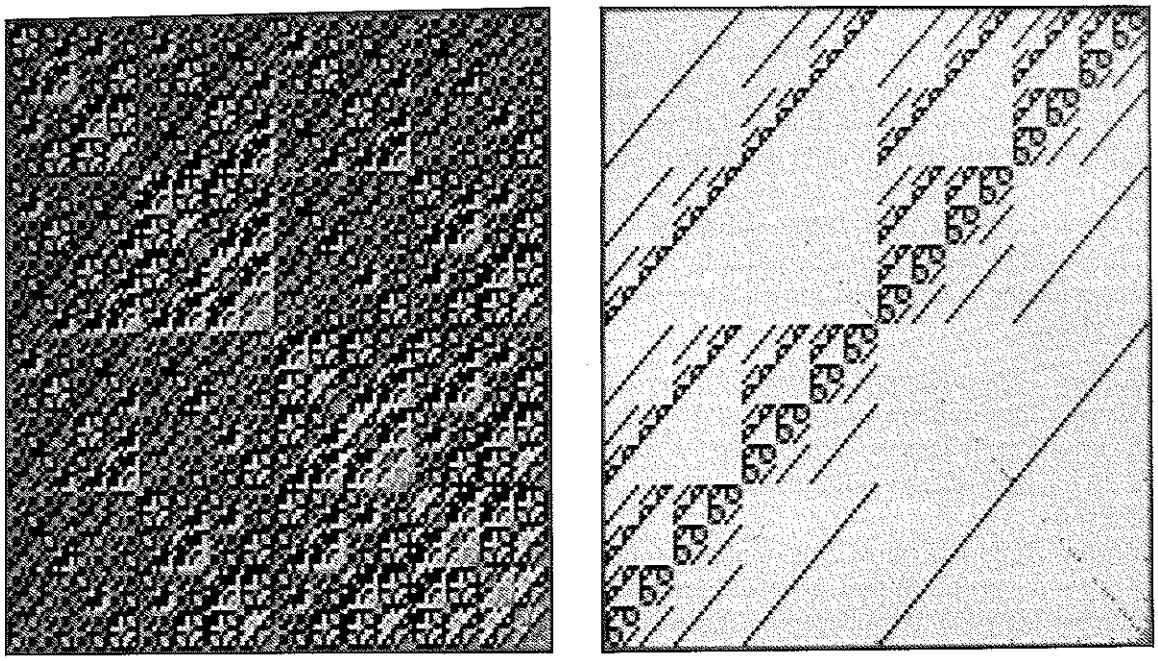
3.3.1. *Similarity of E and $\tilde{\mathcal{B}}_l$*

Definition 3.4. A matrix B is a *disguised version* (DV) of the matrix A, when $a(\boldsymbol{\alpha}) = a(\boldsymbol{\beta})$ implies that $b(\boldsymbol{\alpha}) = b(\boldsymbol{\beta})$ for all corresponding pairs $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in A and B, *and* when at least two different values in A are mapped into the same value in B. This will be denoted by $A \overset{DV}{\rightarrow} B$.

Any function $\varphi(e)$, $e \in \mathbb{Z}$, that is applied elementwise on an excess-matrix E, produces a version $\tilde{E} = \varphi(E)$ that is either symbolically similar to, or a disguised version of E. When φ is meant to produce a disguised version, it will be called a *disguising function.*

Example 3.3. Figure 7 shows E_7 obtained from $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ and the corresponding $\tilde{E}_7$ under the

[5]For discussions about the connection between renormalization group concepts and fractals, see, for example, Mandelbrot [1982], Takayasu [1990], Schroeder [1991], Peitgen *et al.* [1992].



(a) (b)

Fig. 7. (a) The excess-matrix from Example 3.3 and (b) its disguised version.

disguising function $\varphi(e) = 0$ for $|e| \leq 4$, while $\varphi(e) = |e|$ for $|e| > 4$. Observe how $\tilde{E}_7$ differs from E_7, but the matrix $\tilde{\mathcal{B}}_4^7$ of logsize-4 agglomerates of $\tilde{E}_7$ is symbolically similar to E_3.

In the next property, recall from Sec. 2.4 that $R(E_l)$ represents the range of E_l, $R(E) = \mathbb{Z}$ for mixed type excess-matrices, $R(E) =$ "almost" $\mathbb{Z}^+$ or $\mathbb{Z}^-$ for nonnegative or nonpositive excess-matrices.

Property 3.8. *Let $\tilde{E} = \varphi(E)$ be a disguised version of E. Then $\mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l$ (for some fixed $l \geq 0$) if and only if there exist two different integers a, $b \in R(E)$ such that*

$$\forall k \in R(E_l) : \varphi(a+k) = \varphi(b+k)\,. \tag{35}$$

When restricted to E_m, the property holds under replacement of $\mathcal{B}_l$ and $\tilde{\mathcal{B}}_l$ by $\mathcal{B}_l^m$ and $\tilde{\mathcal{B}}_l^m$, and under the condition a, $b \in R(E_{m-l})$.

Proof. $\mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l$ if and only if there are at least two different l-blocks in $\mathcal{B}_l$ whose images in $\tilde{\mathcal{B}}_l$ are identical, i.e., there exists some $\boldsymbol{\kappa}_a$ and $\boldsymbol{\kappa}_b$ for which

$$B_l(\boldsymbol{\kappa}_a) \neq B_l(\boldsymbol{\kappa}_b) \tag{36}$$

and

$$\tilde{B}_l(\boldsymbol{\kappa}_a) = \tilde{B}_l(\boldsymbol{\kappa}_b) \tag{37}$$

Observe that, in view of the fact that $B_l(\boldsymbol{\kappa}) = E_l + e(\boldsymbol{\kappa})$, $e(\boldsymbol{\kappa}_a) = a$ and $e(\boldsymbol{\kappa}_b) = b$, Eqs. (36) and (37) are equivalent to

$$a \neq b\,, \tag{38}$$

$$\varphi(E_l + a) = \varphi(E_l + b)\,. \tag{39}$$

Considering the last condition elementwise over E_l, produces Eq. (35). ■

Corollary 3.1

(a) If $\mathcal{B}_{l_0} \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_{l_0}$ for some l_0, then $\mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l$ for all $l \leq l_0$.

(b) If $\mathcal{B}_{l_0} \overset{SS}{\rightarrow} \tilde{\mathcal{B}}_{l_0}$ for some l_0, then $\mathcal{B}_l \overset{SS}{\rightarrow} \tilde{\mathcal{B}}_l$ for all $l \geq l_0$.

Proof

(a) If Eq. (35) holds for some l, then it holds for any $l \leq l_0$, as $R(E_l) \subset R(E_{l_0})$ (unbiasedness of E, i.e., $e(\mathbf{0}) = 0$, is essential).

(b) by contradiction: suppose $l_m > l_0$ and $\mathcal{B}_{l_m} \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_{l_m}$, then according to (a): $\mathcal{B}_{l_0} \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_{l_0}$. ■

The strength of Property 3.8 can best be appreciated in its "negative" version: if $E \overset{DV}{\rightarrow} \tilde{E}$, then $\mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l$ for $l \geq l_0$ if and only if *no* two different integers a, $b \subset R(E)$ can be found for which

$$\forall \boldsymbol{k} \in R(E_{l_0}) : \varphi(a+k) = \varphi(b+k)\,. \tag{40}$$

This would also mean that $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} E$, as $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{SS}{\leftrightarrow} E$ and $\overset{SS}{\leftrightarrow}$ is obviously transitive. So, even if $\tilde{E} = \varphi(E)$ is a disguised version of E, the structure of E remains present in $\tilde{E}$ as the structure of the agglomerate matrix $\tilde{\mathcal{B}}_l$, when condition (40) is satisfied.

The function φ maps $R(E)$ into a subset of the integers. Recall that $R(E)$ is not necessarily $\mathbb{Z}$, but it will be convenient to associate φ with a sequence (or string) along $\mathbb{Z}$.

Definition 3.5. The *mapping sequence* $s(e)$, $e \in \mathbb{Z}$ associated with the function φ applied to E is the two-sided infinite sequence for which $s(e) = \varphi(e)$ when $e \in R(E)$, and $s(e) = *$ (don't care symbol) when $e \notin R(E)$.

Recall that $R(E) = \mathbb{Z}$ when E is of mixed type, and then $s(e) \equiv \varphi(e)$. When E is of nonnegative or nonpositive type, it suffices to consider the corresponding half-infinite sequences; several $*$ may then appear.

Now consider the disguised agglomerate $\tilde{B}_l(\kappa)$. As it equals $\varphi(B_l(\kappa)) = \varphi(E_l + e(\kappa))$, it follows that it is completely determined by that part of $s(e)$ for which $e \in R(E_l) + e(\kappa)$. This part of $s(e)$ is a substring that will be denoted by $s_l(e(\kappa))$: the *$R(E_l)$-substring of $s(e)$ located at $e(\kappa)$*. It can also be said that $s_l(a)$ completely defines the content of all $B_l(\kappa)$ for which $e(\kappa) = a$, i.e., of all $\tilde{B}_l(\bar{\kappa}_a)$. When $s(a) \neq *$, $s_l(a)$ is called a *proper* substring, otherwise it is an *improper* substring which has no real importance.

Example 3.4. Consider some excess-matrix for which $R(E_1) = \{0, 3, 5\}$ (see also Example 2.5.a.). Then, $R(E_2) = \{0, 3, 5, 6, 8, 10\}$ and $R(E) = \mathbb{Z}^+ \backslash \{1, 2, 4, 7\}$. Use the following mapping sequence:

e:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$s(e)$:	1	*	*	**2**	*	2	**4**	*	**4**	**0**	4	**0**	−2	**0**	−5	0	3	...

The range of the agglomerates $B_2(\bar{\kappa}_0) = R(E_2) + 0$, so:

$R(B_2(\kappa_0)) = R(E_2) + 0$	:	0	3	5	6	8	10
and we find $s_2(0)$	:	1	2	2	4	4	4

All $B_2(\bar{\kappa}_0)$ are identical and contain the integers $(0, 3, 5, 6, 8, 10) \cdot \tilde{B}_2(\bar{\kappa}_0)$ is obtained from $B_2(\bar{\kappa}_0)$ by the mapping sequence restricted to $s_2(0)$. So $\tilde{B}_2(\bar{\kappa}_0)$ contains the integers $(1, 2, 4)$.

In the same way we find that

$R(B_2(\bar{\kappa}_3)) = R(E_2) + 3$	:	3	6	8	9	11	13
$s_2(3)$	:	2	4	4	0	0	0

and

$R(B_2(\bar{\kappa}_5)) = R(E_2) + 5$	:	5	8	10	11	13	15
$s_2(5)$	:	2	4	4	0	0	0

Observe that the strings $s_2(3)$ and $s_2(5)$ are identical. Although $B_2(\bar{\kappa}_3) \neq B_2(\bar{\kappa}_5)$, their disguised counterparts are equal: $\tilde{B}_2(\bar{\kappa}_3) = \tilde{B}_2(\bar{\kappa}_5)$. The string $s_l(k)$ can be obtained practically by putting a mask that has openings at locations specified by $R(E_l)$ over the mapping sequence. The opening corresponding to location 0 on the mask, should be put at $s(k)$. For example, the openings in the $R(E_2)$-mask located at $s(3)$ is represented above by the square windows. $s_2(3)$ is readily perceivable. By shifting the mask along $s(e)$, all l-strings $s_l(k)$ can easily be obtained. All $s_l(j)$, $j = 1, 2, 4, 7$ are improper substrings.

With these new concepts of mapping sequence and associated $R(E_l)$-substrings, Property 3.9 and Corollary 3.1 can be reformulated in a more practical way as given by the next property.

Property 3.9. *When $E \overset{DV}{\rightarrow} \tilde{E}$:*

(a) *$\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \mathcal{B}_l (\overset{SS}{\leftrightarrow} E)$ if and only if there are no two proper $R(E_l)$-substrings $s_l(a)$ and $s_l(b)$ in the mapping sequence that are identical (for $a \neq b$).*

(b) *Let l^* denote the maximal value of l for which there exists a pair of proper identical $R(E_l)$-substrings in $s(e)$. l^* will be called the* maximal extent *of matching substrings in $s(e)$. Then:*

$$\text{for all } l \leq l^* : \mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l ,$$

$$\text{and for all } l > l^* : \mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l .$$

Remark. When, for practical purposes, E is restricted to E_m so that only $\tilde{\mathcal{B}}_l^m$ can be considered, one should only search for identical proper $R(E_l)$-strings in the substring $s_m(0)$ in order to verify whether $\tilde{\mathcal{B}}_l^m$ is DV or SS to $\mathcal{B}_l^m$.

Example 3.5. For p a given positive integer, let $\varphi(e) = 0$ for $|e| \leq p$, $\varphi(e) = |e|$ for $|e| > p$. Take $R(E_1) = \{-1, 0, 1\}$, so $R(E_l) =$ integer interval $[-l, +l]$ and $s(e) = \varphi(e)$ for all $e \in \mathbb{Z}$. This corresponds to the situation in Example 3.3 and Fig. 7, where $p = 4$. For $l \leq p-1$, it is always possible to find two identical $R(E_l)$-substrings (all full of 0's) at different places. This is no longer the case when $l \geq p$. So $l^* = p - 1$ and we conclude that

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l \quad \text{for} \quad l \leq p-1$$

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l \quad \text{for} \quad l \geq p\,.$$

This is clearly visible in Fig. 7, for the restricted version, where $E_{7-l} \overset{SS}{\leftrightarrow} \mathcal{B}_l^7 \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l^7$ for $l \leq 3$, and $E_{7-l} \overset{SS}{\leftrightarrow} \mathcal{B}_l^7$ for $4 \leq l < 7$.

Now, let $R(E_1) = \{0, 3, 5\}$, and $p = 4$ while still considering the same disguising function. The corresponding mapping sequence is (starting at $e = 0$):

$$s(e) = 0\ *\ *\ 0\ *\ 5\ 6\ *\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ \ldots$$

The proper $R(E_1)$-substrings are $s_1(0) = 0, 0, 5$; $s_1(3) = 0, 6, 8$; $s_1(5) = 5, 8, 10, \ldots$, and it is clear that none of these are identical. It follows that for excess-matrices with this $R(E_1)$, $E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l$ for all $l \geq 1$. On the other hand, for $p \geq 8$, we find that $E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l$ for $l \leq 1$, and $E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l$ for $l > 1$. This can be generalized for arbitrary p into

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l \quad \text{for} \quad l \leq \left\lfloor \frac{p-3}{5} \right\rfloor,$$

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l \quad \text{for} \quad l > \left\lfloor \frac{p-3}{5} \right\rfloor.$$

The same disguising function may thus lead to different mapping sequences, and different correspondences between $\mathcal{B}_l$ and $\tilde{\mathcal{B}}_l$, depending on the range of the generator matrix.

Example 3.6. Let $s(e)$ be a periodic sequence with period P. Then obviously l^* is infinite, and $\mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l$ for all l. It also follows that $\tilde{B}_l(\bar{\kappa}_a) = \tilde{B}_l(\bar{\kappa}_{a+qP})$ for all integers a, q.

3.3.2. *Similarity of $\tilde{\mathcal{B}}_l$ and $\tilde{E}$*

It is already known from Property 3.2 that $E \overset{SS}{\leftrightarrow} \mathcal{B}_l$. Properties 3.8 and 3.9 on the other hand, dealt with the symbolic similarity between $\mathcal{B}_l$ and $\tilde{\mathcal{B}}_l$ when $E \overset{DV}{\rightarrow} \tilde{E}$. What about the relationship between $\tilde{E}$ and $\tilde{\mathcal{B}}_l$? Could it be that $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$, even when $E \overset{DV}{\rightarrow} \tilde{E}$? The following properties will throw a light on this.

Definition 3.6. $l^*_{a,b}$ denotes the maximal value of l for which $\{s_l(a), s_l(b)\}$ is a pair of identical proper strings (for given a, b). $(a, b; l^*_{a,b})$ will refer to that pair.

Property 3.10. *If there is any proper identical string pair $(a, b; l^*_{a,b})$ with finite maximal extent $l^*_{a,b}$ in $s(e)$, then*

$$\tilde{\mathcal{B}}_l \overset{DV}{\rightarrow} \tilde{E} \text{ for all } l \geq 1\,.$$

Proof. When $(a, b; l^*_{a,b})$ is a pair of identical strings, then $(a+\max R(E_{l^*_{a,b}}), b+\max R(E_{l^*_{a,b}}); 0)$ is also a matching pair of substrings containing one element. [One should consider the pair $(a + \min R(E_{l^*_{a,b}}), b + \min R(E_{l^*_{a,b}}); 0)$ when E is nonpositive]. As $\max R(E_{l^*_{a,b}}) = l^*_{a,b} \cdot e_{1\,\max}$, it then follows from Property 3.8 that

$$\tilde{B}_l(\bar{\kappa}_{a+l^*_{a,b}\cdot e_{1\,\max}}) \neq \tilde{B}_l(\bar{\kappa}_{b+l^*_{a,b}\cdot e_{1\,\max}}) \text{ for all } l \geq 1$$

while $$\tilde{e}(\bar{\kappa}_{a+l^*_{a,b}\cdot e_{1\,\max}}) = \tilde{e}(\bar{\kappa}_{b+l^*_{a,b}\cdot e_{1\,\max}}).$$

On the other hand, $\tilde{B}_l(\bar{\kappa}_s) = \tilde{B}_l(\bar{\kappa}_t)$ always implies $\tilde{e}(\bar{\kappa}_s) = \tilde{e}(\bar{\kappa}_t)$. Indeed, Property 3.8 requires that $\forall k \in R(E_l) : \varphi(e(\bar{\kappa}_s) + k) = \varphi(e(\bar{\kappa}_t) + k)$, and as $0 \in R(E_l)$, this means $\varphi(e(\bar{\kappa}_s)) = \varphi(e(\bar{\kappa}_t))$ or $\tilde{e}(\bar{\kappa}_s) = \tilde{e}(\bar{\kappa}_t)$.

By definition, all this together means that $\tilde{\mathcal{B}}_l \overset{DV}{\rightarrow} \tilde{E}$. ■

Property 3.11. *Let E be an excess-matrix of mixed type and $\tilde{E} = \varphi(E)$ a disguised version. Then $\tilde{\mathcal{B}}_{l_0} \overset{SS}{\leftrightarrow} \tilde{E}$ for some $l_0 > 0$, if and only if $s(e)$ is periodic and one-to-one within the minimal period. ($1 - 1$ within the minimal period means that, if $P_0[s]$ represents a substring over one minimal period-length P_0, none of its symbols appears more than once).*

Corollary 3.2. *If $\tilde{\mathcal{B}}_{l_0} \overset{SS}{\leftrightarrow} \tilde{E}$ for some $l_0 > 0$, then $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$ for all l.*

Proof

(a) *Sufficiency:*
Recall that, as E is of mixed type, $R(E) = Z$ and $s(e) \equiv \varphi(e)$. Suppose $s(e)$ is periodic and one-to-one within $P_0[s]$. Then

$$\varphi(a+l) = \varphi(a+kP_0+l)\,, \tag{41}$$

for all a, k and all $l \geq 0$. On the other hand,

$$\varphi(a+l) \neq \varphi(b+l) \quad \text{if } b \neq a+kP_0\,, \text{ for any } l \geq 0\,. \tag{42}$$

According to Property 3.8, Eqs. (41) and (42) are equivalent to

$$\tilde{B}_l(\kappa_a) = \tilde{B}_l(\kappa_a + kP_0) \text{ for all } l \geq 0\,, \tag{43}$$

and

$$\tilde{B}_l(\kappa_a) \neq \tilde{B}_l(\kappa_b) \quad \text{if } b \neq a+kP_0\,, \text{ also for any } l \geq 0\,. \tag{44}$$

Equations (43)–(44) together imply that $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$, for all $l \geq 0$.

(b) *Necessity:*
Suppose $\tilde{\mathcal{B}}_{l_0} \overset{SS}{\leftrightarrow} \tilde{E}$ for some fixed $l_0 > 0$, then we must have

$$\varphi(a) = \varphi(b) \Leftrightarrow \tilde{B}_{l_0}(\bar{\kappa}_a) = \tilde{B}_{l_0}(\bar{\kappa}_b) \tag{45}$$

Now, let (a, b) be two values for which $\varphi(a) = \varphi(b)$ and for which there is no $a < \zeta < b$ for which $\varphi(\zeta) = \varphi(a) = \varphi(b)$. Then the right-hand side of Eq. (45) implies that (cf. Property 3.8)

$$\varphi(a+k) = \varphi(b+k) \text{ for all } k \in R(E_{l_0})\,. \tag{46}$$

But according to $\Rightarrow$ in Eq. (45), this would imply that

$$\forall k \in R(E_{l_0}) : \tilde{B}_{l_0}(\bar{\kappa}_{a+k}) = \tilde{B}_{l_0}(\bar{\kappa}_{b+k})\,. \tag{47}$$

This however again implies through $\Leftarrow$ in Eq. (45), that the conditions in Eq. (46) must extend to

$$\varphi(a+k) = \varphi(b+k) \text{ for all } k \in R(E_{2l_0})\,. \tag{48}$$

Continuing this go-between among left-hand side and right-hand side of the equivalence in Eq. (45) in an endless manner eventually leads to the conclusion that $\tilde{\mathcal{B}}_{l_0} \overset{SS}{\leftrightarrow} \tilde{E}$ implies

$$\forall k \in R(E) : \varphi(a+k) = \varphi(b+k)\,. \tag{49}$$

From Sec. 2.4 we know that $R(E)$ equals the set of all integers when E is of mixed type. So Eq. (49) means that the sequences $s(a+k)$ and $s(b+k)$ are identical, or equivalently that $s(k)$ and $s(b-a+k)$ are identical. Or, the whole sequence $s(k)$ must be identical with itself when shifted over a distance $b-a$. This is only possible if $s(k)$ is periodic with period length $b-a$ (not necessarily the minimal period length P_0, $b-a$ may be a multiple of P_0).

Within the minimal period, $s(k)$ has to be one-to-one. To show this, suppose there are two elements a', b' within such minimal period, i.e. $|a'-b'| < P_0$, for which $\varphi(a') = \varphi(b')$. This would imply that $|a'-b'|$ had to be an integer multiple of P_0, by the same argument that brought us from Eq. (49) to the fact that $(b-a)$ had to be an integer multiple of P_0. But as $|a'-b'| < P_0$, $|a'-b'|$ cannot be an integer multiple of P_0. Hence, within a minimal period sequence $P_0[s]$, any symbol can occur only once.

So, the proposition $\tilde{\mathcal{B}}_{l_0} \overset{SS}{\leftrightarrow} \tilde{E}$ for some $l_0 \geq 0$, requires that the mapping-sequence $s(e)$ be periodic and one-to-one, but according to part (a) of the proof, this implies that $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$ for all l. This also proves the corollary. ■

Remarks

(a) When E is not of mixed type, Property 3.11 has to be modified. For example, when E is nonnegative, $s(e)$ is a half-infinite sequence only existing for $e \geq 0$. Recall from Sec. 2 that in this case $R(E) = \mathbb{Z}^+ \backslash T$, where T is a finite set of finite integers. Let $\lambda = \max T + 1$. Then it can be shown that if a is the smallest value for which there exists a value b such that $\varphi(a) = \varphi(b)$, $s(e)$ has to be 1-1 periodic over all $e \geq a + \lambda$, in order for $\tilde{\mathcal{B}}_l$ to be symbolically similar with $\tilde{E}$. For values $e < a + \lambda$, less strict conditions are needed, although they seem to be compatible with an extrapolation of the periodic behavior for $e \leq a + \lambda$. A similar modified version of Property 3.11 can be formulated for nonpositive E. *1-1 periodicity of $\varphi(e)$ is always a sufficient condition for $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}_l$.*

(b) When a finite matrix E_m is considered, it only makes sense to look for symbolic similarity between $\tilde{\mathcal{B}}^m_{l_0}$ and $\tilde{E}_{m-l_0}$ for some $l_0 (1 < l_0 < m)$. It would be sufficient that $s(e)$, for $e \in [m \cdot e_{1\,\min}, m \cdot e_{1\,\max}]$ (cf. Eq. (17)), can be considered as a segment of an overall 1-1 periodic sequence. Strict necessary conditions are much more involved, but can be obtained along the lines of the necessity part of the proof of Property 3.11.

Example 3.7. Let $E_1 = \begin{bmatrix} 0 & 3 \\ 3 & 5 \end{bmatrix}$ and consider a disguising function φ that forces the excess-values 6 and 10 in E_7 to be mapped into the same number. So $\varphi(6) = \varphi(10)$. If, in addition, it is required that $\tilde{\mathcal{B}}^7_l \overset{SS}{\leftrightarrow} \tilde{E}_{7-l}$, it can be shown that the mapping sequence must satisfy (the sequence starts at $e = 0$): $s(e) = a, *, *, b, *, c, 0, *, d, 1, 0, 2, 3, 1, (5, 2, 3, 6)^4, 5, 2, *, 6, *, 2$ or any other 1-1 symbolic substitution of this. $*$ corresponds to e-values that do not occur in E_7, and so their mapping $\varphi(e)$ need not be defined. $(5, 2, 3, 6)^4$ means that the substring $(5, 2, 3, 6)$ has to be repeated 4 times. Observe that $R(E) = \mathbb{Z}^+ \backslash \{1, 2, 4, 7\}$, so $\lambda = 8$ (cf. Remark (a) above), and as the smallest value of e that maps into the same symbol as some other e-value under φ equals 6 ($\varphi(6) = \varphi(10) = 0$), $s(e)$ starts to be (locally) periodic at value $e = 8 + 6 = 14$. When it is required that $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$, $s(e)$ must be globally periodic from $e = 14$ on. Notice that $\varphi(e) = (3, 6, 5, 2)^\infty$, with corresponding $s(e) = 3, *, *, 2, *, 6, 5, *, (3, 6, 5, 2)^\infty$ would be sufficient for obtaining an $\tilde{E}$ for which $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$. Figure 8 shows E_7 and $\tilde{E}_7$ under the mapping associated with the latter sequence.

Example 3.8. Consider the one-dimensional matrix Q obtained from $Q_1 = [0 \quad 1]$:

$$Q = [0\ 1\ 1\ 2\ 1\ 2\ 2\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 3\ 4 \ldots].$$

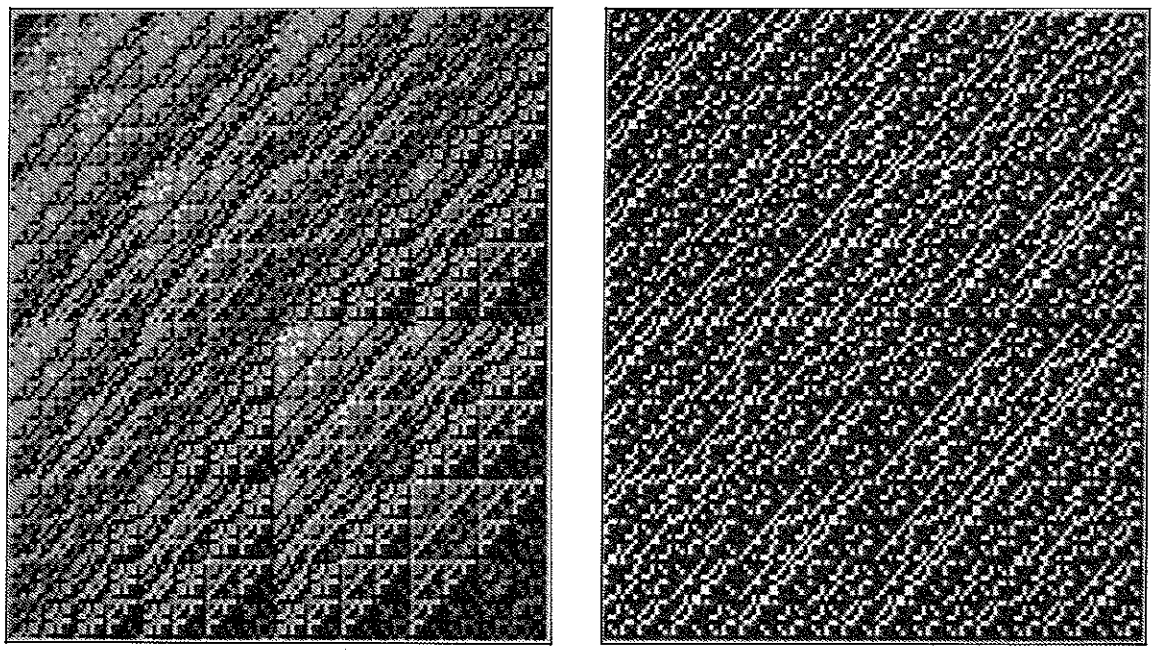
Consider the map

$$\begin{aligned} \varphi(e) &= 0 \quad \text{if} \quad e = \text{even} \\ &= 1 \quad \text{if} \quad e = \text{odd}. \end{aligned}$$

The associated mapping sequence is periodic and one-to-one. The disguised version of Q under the mapping φ is the sequence

$$M = [0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0 \ldots].$$

This sequence is the well-known Morse–Thue sequence of the parity in the binary representations of the integers [Schroeder, 1991; Queffélec, 1987].



(a) (b)

Fig. 8. (a) The excess-matrix E_7 obtained from $E_1 = \begin{bmatrix} 0 & 3 \\ 3 & 5 \end{bmatrix}$. (b) Its disguised version under the periodic mapping-sequence $s(e) = (3, 6, 5, 2)^\infty$.

Any logsize-l partitioning of M is symbolically similar with M (for Q, there is numerical similarity). With $\varphi(e) = e \bmod p$, Q would be transformed in a p-ary sequence M_p, the logsize-l partitioning of which would be symbolically similar with M_p itself.

In summary, we have found that when $E \overset{DV}{\rightarrow} \tilde{E}$, and l^* is the maximal extent of matching proper substrings in $s(e)$, either

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l \overset{DV}{\rightarrow} \tilde{E}, \quad \text{for} \quad l \leq l^*$$

and

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{SS}{\leftrightarrow} \tilde{\mathcal{B}}_l \overset{DV}{\rightarrow} \tilde{E}, \quad \text{for} \quad l > l^*,$$

or

$$E \overset{SS}{\leftrightarrow} \mathcal{B}_l \overset{DV}{\rightarrow} \tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}, \quad \text{for all } l \text{ when } \varphi \text{ is 1-1 periodic.}$$

(1-1 periodicity is also necessary when E is of mixed type, quasi-necessary otherwise).

Could the fact that $\tilde{\mathcal{B}}_l \overset{SS}{\leftrightarrow} \tilde{E}$ for some disguised version $\tilde{E}$ of E imply that $\tilde{E}$ itself is an excess-matrix? In other words, does $\tilde{E}$ satisfy the relationship $\tilde{E}_{m+k} = \tilde{E}_m \oplus \tilde{E}_k$. The answer is negative, as this would require $R(\tilde{E}_l)$ to expand with increasing l. And this can never be the case in view of the (quasi)-periodic requirements for $s(e)$ that are necessary for symbolic similarity between $\tilde{\mathcal{B}}_l$ and $\tilde{E}$. It can be concluded that *proper disguised versions of an excess matrix E are never excess-matrices themselves.* For any disguised version $\tilde{E}$, the only relationship that still holds under the decimating operations from Sec. 3.2, is the nonnumerical one (cf. Eq. (31), with E_m and B replaced by $\tilde{E}_m$ and $\tilde{B}$). The reason is that the decimating operation and the disguising operation do commute.

4. Displaying One- and Two-Dimensional Subsets in Higher-Dimensional E

Displaying E as a whole when the dimension p exceeds 2 is not feasible. The best thing that can be done is to show subsets of E along some two-dimensional plane intersections of E. The most obvious sections are the ones that keep all but two coordinates in $e(\kappa_1, \kappa_2, \ldots, \kappa_p)$ constant. Can such a plane be generated on its own in an efficient way without having to expand E in all directions? Recall that the expansion of E itself can be done in a way that squares the linear size of the matrix at each recursion step. Can a plane subset of E be obtained in a comparably efficient way? The answer will be affirmative. For convenience of representation, the problem will first be analyzed in its simplest two-dimensional form, after which the solution for the general case will be presented in an inductive way.

4.1. *Generating rows and columns separately in a two-dimensional excess-matrix*

Consider an arbitrary *two-dimensional* reduced excess-matrix E_l obtained from a generator matrix with size vector (σ_1, σ_2). It has σ_1^l rows (ranked from 0 to $\sigma_1^l - 1$) of σ_2^l-elements (ranked from 0 to $\sigma_2^l - 1$); see Fig. 9. Its row of rank r extends along the direction or dimension 2, and will be represented by $\Pi_l^2(r)$. Its column of rank c will, in analogy, be represented by $\Pi_l^1(c)$. $\Pi_l^2(0)$ and $\Pi_l^1(0)$ thus denote the first row and first column of E_l. Observe that the first *row of $B_l(0, j)$-blocks* in E has the structure

$$E_l + e(0, 0) \vdots E_l + e(0, 1) \vdots E_l + e(0, 2) \vdots \ldots = \Pi^2(0) \oplus E_l, \tag{50}$$

where $\Pi^2(0)$ represents the first row in the infinite E-matrix.

For any row $\Pi^2(r)$ in E, with $r < \sigma_1^l$, Eq. (50) translates into

$$\Pi^2(r) = \Pi^2(0) \oplus \Pi_l^2(r). \tag{51}$$

For *finite* extensions of $\Pi_l^2(r)$, we actually have

$$\forall k, l : \Pi_{k+l}^2(r) = \Pi_k^2(0) \oplus \Pi_l^2(r) \quad (\neq \Pi_l^2(r) \oplus \Pi_k^2(0)). \tag{52}$$

Notice that the *noncommutativity* of $\oplus$ applies here, as explicitly mentioned in the above formula. For the first row, Eq. (52) becomes

$$\forall k, l : \Pi_{k+l}^2(0) = \Pi_k^2(0) \oplus \Pi_l^2(0) \quad (= \Pi_l^2(0) \oplus \Pi_k^2(0)). \tag{53}$$

Definition 4.1. If S_k, $A_k (k = 1, 2, 3 \ldots)$ are matrices for which

$$\forall k, l : S_{k+l} = S_k \oplus S_l, \ \textit{initial condition } S_1 \tag{54}$$

$$A_{k+l} = S_k \oplus A_l, \ \textit{initial condition } A_1 \tag{55}$$

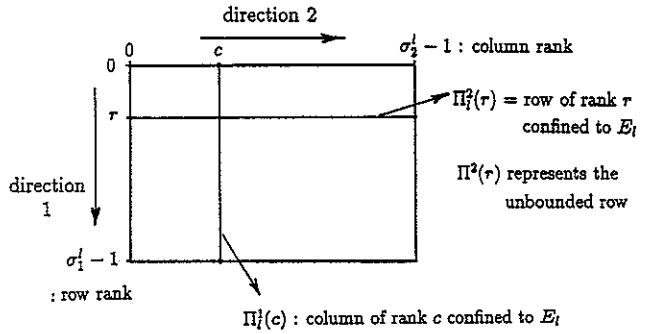


Fig. 9. Denoting one-dimensional subsets of a two-dimensional matrix.

then it will be said that the sequence of A_k's is obtained by applying the evolutionary rule of the S_k-sequence to A_1.

The slowest way to grow both sequences, is the one that uses $S_{k+1} = S_k \oplus S_1$, $A_{k+1} = S_k \oplus A_1$. The fastest way squares the linear size at each step:

$$\begin{aligned} S_{2l} &= S_l \oplus S_l \\ A_{2l} &= S_l \oplus A_l \,. \end{aligned} \tag{56}$$

Relationship (52) now provides the strategy for constructing row r in E:

Step 1. Let $m_0 = \lceil \log_{\sigma_1}(r+1) \rceil$ (= smallest integer m for which $r < \sigma_1^m$). Construct $\Pi^2_{m_0}(r)$, i.e., the elements on row r confined to E_{m_0}. This can be done most efficiently by reducing these elements to a sum of elements in E_1 by means of consecutive "square root" reductions in the linear size of E_{m_0}.

Step 2. Expand $\Pi_l^2(r)$ by applying the evolutionary rule of the first row to it, using Eq. (56):

$$\begin{aligned} \Pi^2_{2l}(0) &= \Pi_l^2(0) \oplus \Pi_l^2(0) \,, \\ \Pi^2_{2l}(r) &= \Pi_l^2(0) \oplus \Pi_l^2(r) \,. \end{aligned} \tag{57}$$

Remarks

(a) Observe that no row in E can be constructed by applying the evolutionary rule of some other row than the first one.

(b) The only row that is *self-generating* is the first one, the others are all generated by applying the evolutionary rule of the first row to some "strange" initial sequence: they are *hybrids*.

(c) The same strategy can be used to construct any column $\Pi^1(c)$ in E: the first column is self-generating, the others are hybrids developed from applying the first-column evolutionary rule to $\Pi^1_{m_0}(c)$ $(m_0 = \lceil \log_{\sigma_2}(c+1) \rceil)$.

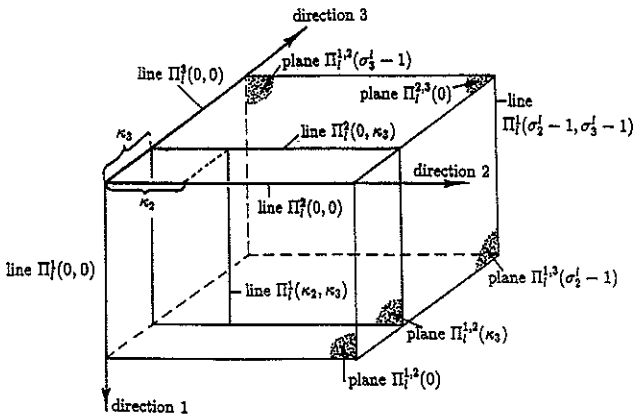


Fig. 10. Notation of planes in logsize-l "cube".

4.2. *Generalization to higher dimensional excess-matrices*

Although a bit more involved, the generalization to higher dimensional matrices is now straightforward.

Notations 4.1. Let p be the dimension of some general excess-matrix E. The separate "directions" in space are indexed (1, 2, 3,..., p). Let D_ν be a specific selection of ν indices from this set, and $\bar{D}_\nu$ the complementary set. $\boldsymbol{\kappa}_{\bar{D}_\nu}$ is a specific set of coordinates in the directions of $\bar{D}_\nu$ i.e., $\boldsymbol{\kappa}_{\bar{D}_\nu} = (\kappa_{i_1}, \kappa_{i_2}, \ldots, \kappa_{i_{p-\nu}})$, $i_j \in \bar{D}_\nu$. $\Pi_l^{D_\nu}(\boldsymbol{\kappa}_{\bar{D}_\nu})$ then represents the plane subset of E_l parallel to the directions in D_ν and localized at $\boldsymbol{\kappa}_{\bar{D}_\nu}$. $\Pi_l^{D_\nu}(\mathbf{0})$ represents the first plane (of rank $\mathbf{0}$) parallel to the D_ν-directions.

Figure 10 illustrates this for a three-dimensional example.

Property 4.1. $\Pi^{D_\nu}(\boldsymbol{\kappa}_{\bar{D}_\nu})$ *can be obtained in the following way:*

Step 1. Let $m_0 = \max_{i \in \bar{D}_\nu} \lceil \log_{\sigma_i}(\kappa_i + 1) \rceil$. Construct $\Pi^{D_\nu}_{m_0}(\boldsymbol{\kappa}_{\bar{D}_\nu})$ as a plane subset of E_{m_0}.

Step 2. Expand $\Pi^{D_\nu}(\boldsymbol{\kappa}_{\bar{D}_\nu})$ by applying the evolutionary rule of the first plane $\Pi^{D_\nu}(\mathbf{0})$ to the initial block $\Pi^{D_\nu}_{m_0}(\boldsymbol{\kappa}_{\bar{D}_\nu})$, i.e., apply

$$\Pi^{D_\nu}_{2l}(\boldsymbol{\kappa}_{\bar{D}_\nu}) = \Pi_l^{D_\nu}(\mathbf{0}) \oplus \Pi_l^{D_\nu}(\boldsymbol{\kappa}_{\bar{D}_\nu}) \,. \tag{58}$$

Again, only the first planes of each direction (rank-$\mathbf{0}$ planes) are self-generating. The others are

hybrid. Equation (58) becomes particularly interesting for displaying two-dimensional plane subsets of E: D_ν then selects two directions to which the generated planes will be parallel.

Example 4.1. The logsize-4 subset of plane 2 in Example 2.2 can be obtained by: "logsize-2 plane 0" $\oplus$ "logsize-2 plane 2" (the logsize-2 planes are the ones that are shown there), and is the 16×16 matrix produced by replacing each element α in plane 0 by $\alpha+$ the 4×4-matrix displayed in plane 2. The same holds for the logsize-4 subset in planes 1 and 3, but for plane 4 (5, 6, 7) one initially needs to know the logsize-3 ($=8\times8$) matrix in plane 4 (5, 6, 7) before it can be extended.

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