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A subspace identification algorithm for descriptor systems

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Abstract: We present a subspace algorithm for identifying state space models for descriptor systems, directly from input/output data. The subspace algorithm avoids the use of Markov parameters which are difficult to estimate in practice. It also applies to unstable systems without difficulty.

Keywords: Linear systems; descriptor systems; systems identification; state space models, subspace algorithms.

1. Introduction

Consider the linear, multivariable, time invariant system with state space representation

$$Ex_{k+1} = Ax_k + Bu_k, \tag{1a}$$

$$y_k = Cx_k + Du_k, \tag{1b}$$

where $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^l$ and $x_k \in \mathbb{R}^\nu$ denote, respectively, the input, output, and state vector at time k. For identification purposes, A, B, C, D, and E are unknown system matrices of appropriate dimensions, to be identified by means of recorded I/O-sequences $\{u_1, u_2, \ldots, u_N\}$ and $\{y_1, y_2, \ldots, y_N\}$. Furthermore, matrix E is assumed to be square but possibly rank deficient. A realization of the above form is known as a descriptor form when E is singular; otherwise it has a regular form. We should also point out that the system matrices are unique modulo nonsingular left and right transformations; in other words, an equivalent model is obtained as $\{PEQ, PAQ, PB, CQ, D\}$.

Descriptor systems arise in many real world applications. For instance, Leontieff systems in multisector economy [5], interconnected systems, and systems of partial differential equations [6], can be modeled with the above class of systems. The literature on descriptor systems is quite extensive and we refer the readers to the recent survey paper [7].

The identification problem for singular systems is well defined from a Markov parameter point of view [1,3]. That is, the system Hankel matrix is used in a form similar to Kung's algorithm [4]. The main

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limitation with these algorithms is the computation of the Markov parameters, which is basically the input to the algorithm. Here, we take a different and more meaningful route by working with the input/output data directly.

For the time being, we assume that the pencil $\lambda E - A$ is given in the Kronecker canonical form [2], which means that both A and E are block diagonal matrices. This is a rather standard assumption in the literature and bears no loss of generality. The system equations then read as follows:

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \bar{E}_S \end{array}\right]\left[\begin{array}{c} x_{k+1}^R \\ \hline \bar{x}_{k+1}^S \end{array}\right] = \left[\begin{array}{c|c} A_R & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c} x_k^R \\ \hline \bar{x}_k^S \end{array}\right] + \left[\begin{array}{c} B_R \\ \hline \bar{B}_S \end{array}\right] u_k, \tag{2a}$$

$$y_k = \left[C_R \mid \bar{C}_S\right]\left[\begin{array}{c} x_k^R \\ \hline \bar{x}_k^S \end{array}\right] + Du_k. \tag{2b}$$

Here, I denotes an identity matrix, and $\bar{E}_S$ is nilpotent (zero eigenvalues) with $\bar{E}_S^k = 0$ for $k \geqslant \mu$, where μ is the nilpotency index. The R-indices refer to the *strictly proper* (*regular*) *part* of the system transfer function, while the S-indices refer to the *polynomial* (*singular*) *part*. For our purpose, A_R and $\bar{E}_S$ need not be given in Jordan form, only the separation of the two blocks is important. Note that the complete reduction to Jordan form may introduce complex arithmetic.

In the sequel, we make use of a slightly different form of the state space model. By substituting $\bar{x}_{k+1}^S = \bar{x}_k^S$,

$$\left[\begin{array}{c} x_{k+1}^R \\ \hline \bar{x}_{k-1}^S \end{array}\right] = \left[\begin{array}{c|c} A_R & 0 \\ \hline 0 & \bar{E}_S \end{array}\right]\left[\begin{array}{c} x_k^R \\ \hline \bar{x}_k^S \end{array}\right] + \left[\begin{array}{c} B_R \\ \hline -\bar{B}_S \end{array}\right] u_k,$$

$$y_k = \left[C_R \mid \bar{C}_S\bar{E}_S\right]\left[\begin{array}{c} x_k^R \\ \hline \bar{x}_k^S \end{array}\right] + \left(D - \bar{C}_S\bar{B}_S\right)u_k.$$

We should point out that the singular part is not fully observable [1]. Naturally, the model that can be identified from the I/O data corresponds to part of the observable subsystem, which is excited (by either the input or the initial state). Hence, the model which is effectively identified will be a reduced model as follows:

$$\left[\begin{array}{c} x_{k+1}^R \\ \hline x_{k-1}^S \end{array}\right] = \left[\begin{array}{c|c} A_R & 0 \\ \hline 0 & E_S \end{array}\right]\left[\begin{array}{c} x_k^R \\ \hline x_k^S \end{array}\right] + \left[\begin{array}{c} B_R \\ \hline B_S \end{array}\right] u_k, \tag{3a}$$

$$y_k = \left[C_R \mid C_S\right]\left[\begin{array}{c} x_k^R \\ \hline x_k^S \end{array}\right] + D_{RS}u_k. \tag{3b}$$

Obviously E_S still has zero eigenvalues, but it has smaller dimensions than $\bar{E}_S$. The order of this reduced system is referred to as n, while n_R and n_S denote the order of the regular and singular part respectively [2].

In the next section, we introduce a subspace identification algorithm for descriptor systems which corresponds to a modified version of the algorithm of [8].

[1] As an example, for $\bar{E}_S = [0]$ the minimal system order for the singular part is zero instead of one, as $\bar{C}_S\bar{E}_S = 0$.

[2] $n = n_R + n_S$.

2. Subspace identification algorithm

Let the input/output matrices $H_{1|i}$ and $H_{i+1|2i}$ be concatenated block Hankel matrices as follows:

$$H_{1|i} = \left[\frac{U_{1|i}}{Y_{1|i}}\right] = \left[\begin{array}{cccc} u_1 & u_2 & \cdots & u_j \\ u_2 & u_3 & \cdots & u_{j+1} \\ \vdots & \vdots & & \vdots \\ u_i & u_{i+1} & \cdots & u_{i+j-1} \\ \hline y_1 & y_2 & \cdots & y_j \\ y_2 & y_3 & \cdots & y_{j+1} \\ \vdots & \vdots & & \vdots \\ y_i & y_{i+1} & \cdots & y_{i+j-1} \end{array}\right],$$

$$H_{i+1|2i} = \left[\frac{U_{i+1|2i}}{Y_{i+1|2i}}\right] = \left[\begin{array}{cccc} u_{i+1} & u_{i+2} & \cdots & u_{i+j} \\ u_{i+2} & u_{i+3} & \cdots & u_{i+j+1} \\ \vdots & \vdots & & \vdots \\ u_{2i} & u_{2i+1} & \cdots & u_{2i+j-1} \\ \hline y_{i+1} & y_{i+2} & \cdots & y_{i+j} \\ y_{i+2} & y_{i+3} & \cdots & y_{i+j+1} \\ \vdots & \vdots & & \vdots \\ y_{2i} & y_{2i+1} & \cdots & y_{2i+j-1} \end{array}\right],$$

where $j \gg i$, and also i should be 'large enough', see [8]. Let the state vector sequences X_k^{R} and X_k^{S} be defined as

$$X_k^{\mathrm{R}} = \left[x_k^{\mathrm{R}} \mid x_{k+1}^{\mathrm{R}} \mid \cdots \mid x_{k+j-1}^{\mathrm{R}}\right], \qquad X_k^{\mathrm{S}} = \left[x_k^{\mathrm{S}} \mid x_{k+1}^{\mathrm{S}} \mid \cdots \mid x_{k+j-1}^{\mathrm{S}}\right].$$

Then, it is readily shown that the following I/O equation holds:

$$Y_{1|i} = \Gamma\left[\frac{X_1^{\mathrm{R}}}{X_i^{\mathrm{S}}}\right] + H_{\mathrm{T}} U_{1|i} \tag{4}$$

where

$$\Gamma = \left[\Gamma_{\mathrm{R}} \mid \Gamma_{\mathrm{S}}\right] = \left[\begin{array}{c|c} C_{\mathrm{R}} & C_{\mathrm{S}} E_{\mathrm{S}}^{i-1} \\ C_{\mathrm{R}} A_{\mathrm{R}} & \vdots \\ \vdots & C_{\mathrm{S}} E_{\mathrm{S}} \\ C_{\mathrm{R}} A_{\mathrm{R}}^{i-1} & C_{\mathrm{S}} \end{array}\right]$$

and

$$H_{\mathrm{T}} = \left[\begin{array}{ccccc} D_{\mathrm{RS}} & C_{\mathrm{S}} B_{\mathrm{S}} & C_{\mathrm{S}} E_{\mathrm{S}} B_{\mathrm{S}} & \cdots & C_{\mathrm{S}} E_{\mathrm{S}}^{i-2} B_{\mathrm{S}} \\ C_{\mathrm{R}} B_{\mathrm{R}} & D_{\mathrm{RS}} & C_{\mathrm{S}} B_{\mathrm{S}} & \cdots & C_{\mathrm{S}} E_{\mathrm{S}}^{i-3} B_{\mathrm{S}} \\ C_{\mathrm{R}} A_{\mathrm{R}} B_{\mathrm{R}} & C_{\mathrm{R}} B_{\mathrm{R}} & D_{\mathrm{RS}} & \cdots & C_{\mathrm{S}} E_{\mathrm{S}}^{i-4} B_{\mathrm{S}} \\ \vdots & & & & \vdots \\ C_{\mathrm{R}} A_{\mathrm{R}}^{i-2} B_{\mathrm{R}} & C_{\mathrm{R}} A_{\mathrm{R}}^{i-3} B_{\mathrm{R}} & C_{\mathrm{R}} A_{\mathrm{R}}^{i-4} B_{\mathrm{R}} & \cdots & D_{\mathrm{RS}} \end{array}\right],$$

and similarly

$$Y_{i+1|2i} = \Gamma \begin{bmatrix} X_{i+1}^{R} \\ \hline X_{2i}^{S} \end{bmatrix} + H_T U_{i+1|2i}. \tag{5}$$

Under mild conditions – see below – we have:

Theorem. *The following properties hold*:
(P1) $\text{rank}\{H_{1|i}\} = mi + n$,
(P2) $\text{rank}\{H_{i+1|2i}\} = mi + n$,
(P3) $\text{rank}\{H_{1|i}/H_{i+1|2i}\} = 2mi + n$,
(P4) $\text{span}_{\text{row}}\{X_{i+1}^{R}/X_i^{S}\} = \text{span}_{\text{row}}\{H_{1|i}\} \cap \text{span}_{\text{row}}\{H_{i+1|2i}\}$.

Proof. [3] From equation (4), it is seen that the row space of $Y_{1|i}$ is a combination of the row spaces of

$$U_{1|i} \quad \text{and} \quad \begin{bmatrix} X_1^{R} \\ \hline X_i^{S} \end{bmatrix}.$$

A common assumption is that $\text{rank}\{U_{1|i}\} = mi$ (persistent excitation) and that the state vector sequences do not lie in $\text{span}_{\text{row}}\{U_{1|i}\}$ (generic property). Property (P1) then readily follows. (P2) and (P3) are similarly proven.

From (P1)–(P3) via Grassmann's dimension theorem, it then follows that

$$\dim\{\text{span}_{\text{row}}\{H_{1|i}\} \cap \text{span}_{\text{row}}\{H_{i+1|2i}\}\} n.$$

For (P4) to hold true, both X_{i+1}^{R} and X_i^{S} have to lie in the row spaces of $H_{1|i}$ and $H_{i+1|2i}$, which is shown as follows. From

$$\begin{bmatrix} X_1^{R} \\ \hline X_i^{S} \end{bmatrix} = \Gamma^{\dagger} Y_{1|i} - \Gamma^{\dagger} H_T U_{1|i}, \qquad \begin{bmatrix} X_{i+1}^{R} \\ \hline X_{2i}^{S} \end{bmatrix} = \Gamma^{\dagger} Y_{i+1|2i} - \Gamma^{\dagger} H_T U_{i+1|2i}$$

where † denotes the Moore–Penrose pseudo-inverse, it already follows that

$$\text{span}_{\text{row}}\{X_i^{S}\} \subset \text{span}_{\text{row}}\{H_{1|i}\}, \qquad \text{span}_{\text{row}}\{X_{i+1}^{R}\} \subset \text{span}_{\text{row}}\{H_{i+1|2i}\}.$$

From

$$X_{i+1}^{R} = A_R^i X_1^{R} + \left[A_R^{i-1} B_R \mid \cdots \mid A_R B_R \mid B_R \right] U_{1|i} =: A_R^i X_1^{R} + \Delta_R U_{1|i}$$

and $\text{span}_{\text{row}}\{X_1^{R}\} \subset \text{span}_{\text{row}}\{H_{1|i}\}$, it follows that

$$\text{span}_{\text{row}}\{X_{i+1}^{R}\} \subset \text{span}_{\text{row}}\{H_{1|i}\}.$$

Similarly, from

$$X_i^{S} = E_S^i X_{2i}^{S} + \left[B_S \mid E_S B_S \mid \cdots \mid E_S^{i-1} B_S \right] U_{i+1|2i} =: E_S^i X_{2i}^{S} + \Delta_S U_{i+1|2i}$$

and $\text{span}_{\text{row}}\{X_{2i}^{S}\} \subset \text{span}_{\text{row}}\{H_{i+1|2i}\}$, it follows that

$$\text{span}_{\text{row}}\{X_i^{S}\} \subset \text{span}_{\text{row}}\{H_{i+1|2i}\},$$

which completes the proof. □

[3] The proof is similar to the proof for regular systems (see [8]) adopted here for the sake of self-containedness.

A basis for the intersection of the row spaces of $H_{1|i}$ and $H_{i+1|2i}$ is easily computed with a singular value decomposition (SVD) of the compound matrix:

$$\left[\begin{array}{c} H_{1|i} \\ \hline H_{i+1|2i} \end{array}\right] = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \left[\begin{array}{c|c} S_{11} & 0 \\ \hline 0 & 0 \end{array}\right] V^{\mathrm{T}}.$$

$$\underbrace{}_{2mi+n} \quad \underbrace{}_{2li-n} \quad \underbrace{}_{2mi+n} \quad \underbrace{}_{2li-n}$$

From

$$U_{12}^{\mathrm{T}} H_{1|i} = -U_{22}^{\mathrm{T}} H_{i+1|2i}$$

it follows that the row space of $U_{12}^{\mathrm{T}} H_{1|i}$ equals the required intersection. Let us assume that $X'_{i|i+1}$ generates a basis for this intersection (n rows), then, from the above properties, it follows that

$$X'_{i|i+1} = Q \left[\begin{array}{c} X_{i+1}^{\mathrm{R}} \\ \hline X_i^{\mathrm{S}} \end{array}\right]$$

where Q is an unknown nonsingular transformation. Furthermore, since $X'_{i|i+1}$ is computed from the above defined intersection, we know that

$$X'_{i|i+1} = T H_{i+1|2i} \tag{6}$$

for some T, from which it also follows that

$$X'_{i-1|i} = T H_{i|2i-1}. \tag{7}$$

The system matrices are then computed as follows:

First, a matrix $[E' \mid A' \mid B']$ is computed, with n independent rows, such that

$$[E' \mid A' \mid B'] \left[\begin{array}{c} X'_{i|i+1} \\ \hline X'_{i-1|i} \\ \hline U_{i|i} \end{array}\right] = 0. \tag{8}$$

This can be done with e.g. an SVD of the right-hand matrix. From

$$\left[\begin{array}{cc|cc|c} -I & 0 & A_{\mathrm{R}} & 0 & B_{\mathrm{R}} \\ 0 & E_{\mathrm{S}} & 0 & -I & B_{\mathrm{S}} \end{array}\right] \left[\begin{array}{c} X_{i+1}^{\mathrm{R}} \\ X_i^{\mathrm{S}} \\ \hline X_i^{\mathrm{R}} \\ X_{i-1}^{\mathrm{S}} \\ \hline U_{i|i} \end{array}\right] = 0 \tag{9}$$

it follows that

$$[E' \mid A' \mid B'] = P \left[\begin{array}{cc} -I & 0 \\ 0 & E_{\mathrm{S}} \end{array}\right] Q^{-1} \left| \left[\begin{array}{cc} A_{\mathrm{R}} & 0 \\ 0 & -I \end{array}\right] Q^{-1} \right| \left[\begin{array}{c} B_{\mathrm{R}} \\ B_{\mathrm{S}} \end{array}\right] \tag{10}$$

for some nonsingular matrix P. The above formula reveals the Kronecker canonical form of the pencil $\lambda E' - A'$. Conversely, P and Q as well as A_{R} and E_{S} follow from the Kronecker canonical form of the

pencil. B_R and B_S then follow from $P^{-1}B'$. Finally the C and D matrices follow from a set of linear equations

$$[Y_{i|i}] = [C_R | C_S | D_{RS}] \left[\begin{array}{c} X_i^R \\ \hline X_i^S \\ \hline U_{i|i} \end{array}\right] \tag{11}$$

where

$$X_i^R = [I|0]Q^{-1}X'_{i-1|i}, \qquad X_i^S = [0|I]Q^{-1}X'_{i|i+1}.$$

This then completes the identification procedure. In conclusion, all the system matrices can be computed, once the intersection of the input/output matrices is known.

Finally, note that in the above algorithm the firstly computed intersection $X'_{i|i+1}$, contains X_{i+1}^R and X_i^S in a 'mixed' form. The separation is done afterwards, by computing the canonical form of a matrix pencil. Alternatively, the separation could also be done up front, combined with a few additional steps such that the obtained model will be in balanced coordinates. The procedure is then similar to the one of [9], but definitely more cumbersome and therefore omitted here. The interested reader is referred to [10].

3. Conclusion

A subspace algorithm has been presented for identifying state space models for descriptor systems, directly from input/output data. The subspace algorithm avoids the use of Markov parameters which are difficult to estimate in practice. It also applies to unstable systems without difficulty.

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References

[1] M.A. Christodoulou and B.G. Mertzios, Realization of singular systems via Markov parameters, *Internat. J. Control* **42** (6) (1985) 1433–1441.
[2] F.R. Gantmacher, *Theory of Matrices*, Vols. I–II (Chelsea, New York, 1959).
[3] W.S. Gray, E.I. Verriest and F.L. Lewis, A Hankel approach to singular system realization theory, *Proc. 29th CDC*, Honolulu, Hawaii (1990) 73–78.
[4] S.Y. Kung, A new identification and model reduction algorithm via singular value decomposition, *Proc. 12th Asilomar Conf. on Circuits, Systems and Computers*, Pacific Grove, CA (1978) 705–714.
[5] W. Leontieff, *Essays in Economics* (M.E. Sharp, New York, 1977).
[6] F.L. Lewis, W. Marszalek and B.G. Mertzios, Walsh function analysis of 2-D generalized continuous systems, *IEEE Trans. Automat. Control* **35** (10) (1990) 1140–1144.
[7] F.L. Lewis, A survey of linear singular systems, *Circuits Systems Signal Process.* **5** (1) (1986) 3–36.
[8] M. Moonen, B. De Moor, L. Vandenberghe and J. Vandewalle, On- and offline identification of linear state space models, *Internat. J. Control* **49** (1) (1989) 219–232.
[9] M. Moonen and J. Ramos, A subspace algorithm for balanced state space system identification, ESAT/SISTA Report 1991-07, Kath. Universiteit Leuven, Dept. E.E. Submitted.
[10] M. Moonen, B. De Moor, J. Ramos and S. Tan, A subspace identification algorithm for descriptor systems, ESAT/SISTA Report 1991-20, Kath. Universiteit Leuven, Dept. E.E.