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The Visual Computer

Original articles

Artistic design with fractal matrices

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A simple recursive procedure that grows integer matrices with a fractal nature is presented. Colorful translations of subsets of these matrices produce aesthetically appealing patterns that extend the known classes of fractal images. This is illustrated with several examples.

Key words: Fractals – Computer art – Patterns – Textures – Ornaments – Tilings – Carpets – Dynamical systems

Introduction

Fractal matrices is the name given to the class of unbounded matrices obtained by continued application of the recursion (Barbé 1992)

$$E_{m+1} = E_1 \oplus E_m. \tag{1}$$

E_1 is an $r \times s$ (rows × columns) matrix of *integers* $e_1(i,j)$, $0 \leqq i < r$, $0 \leqq j < s$. It is called the *generator matrix*. $A \oplus B$ denotes the so-called *bigsum* operation between two matrices A and B: it is the matrix obtained by replacing each element $a(i,j)$ in A by the matrix B to which first $a(i,j)$ has been added to all its elements. Equation (1) thus defines a two-dimensional symbolic-substitution dynamical system (Mozes 1989), where the symbols are integers and the substitution rule is arithmetical in nature.

The class of matrices obtained through Eq.(1) for different generator matrices is infinite. It emerged from a generalization of the fractal-state space associated with a particular dynamical system known as cellular automaton (Barbé 1990). These matrices also form an extension of a special subset in the class of iterated Kronecker products, as described by Shallit and Stolfi (1989).

From the definition in Eq.(1), it follows that E_m is an $r^m \times s^m$ matrix. m will be designated as the *logsize*, an abbreviation of "logarithmic size", of the matrix. Choosing $e_1(0,0) = 0$ makes E_{m+1} grow from E_m by just adding new elements to the right, to the bottom, and to the right bottom of E_m. As an example, consider the generator matrix

$$E_1 = \begin{bmatrix} 0 & -4 & 1 \\ 1 & 2 & 0 \end{bmatrix}. \tag{2}$$

It follows that

$$E_2 = \left[\begin{array}{ccc|ccc|ccc} 0 & -4 & 1 & -4 & -8 & -3 & 1 & -3 & 2 \\ 1 & 2 & 0 & -3 & -2 & -4 & 2 & 3 & 1 \\ \hline 1 & -3 & 2 & 2 & -2 & 3 & 0 & -4 & 1 \\ 2 & 3 & 1 & 3 & 4 & 2 & 1 & 2 & 0 \end{array}\right]. \tag{3}$$

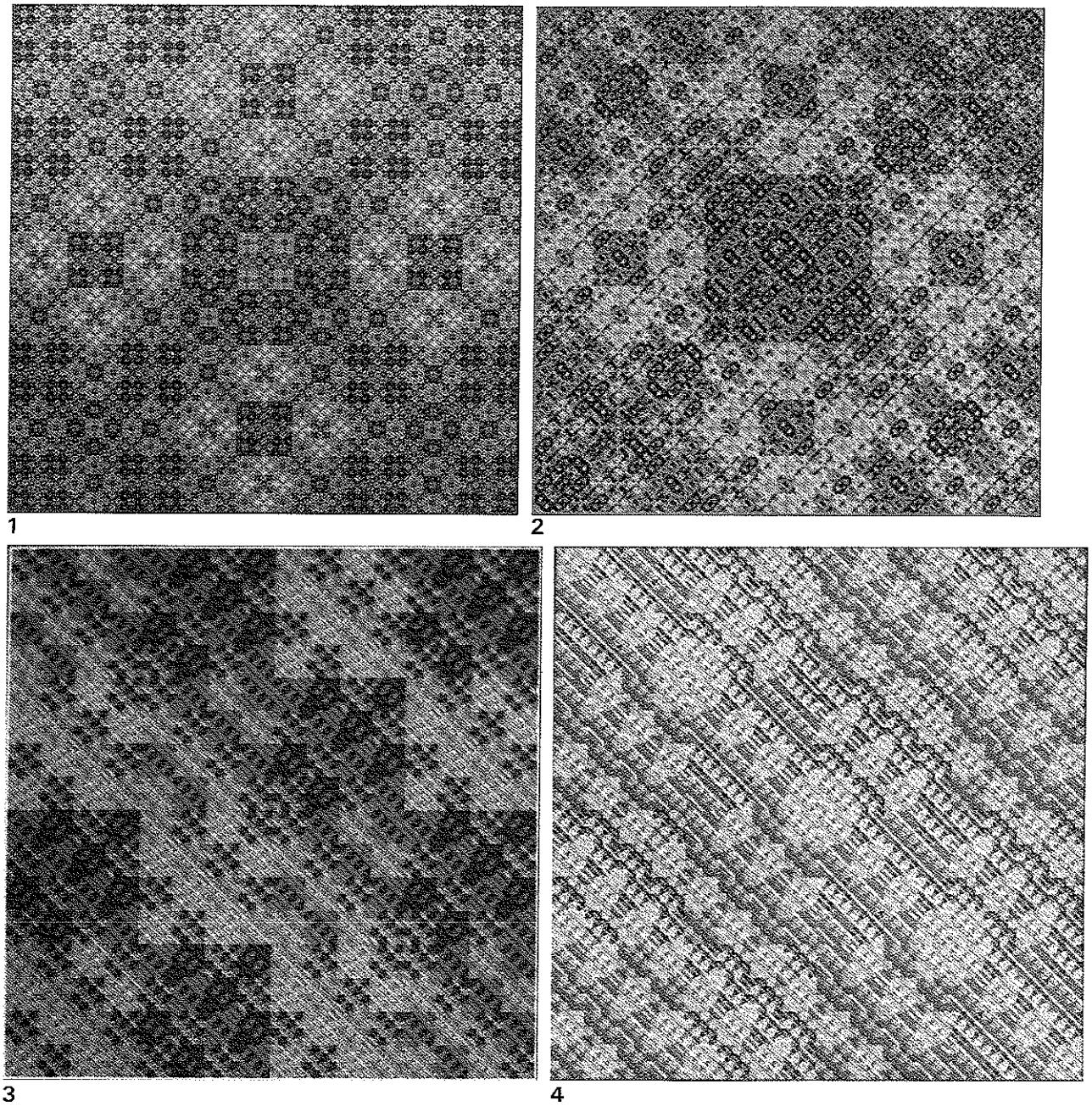
E_2 already shows how Eq.(1) produces an intricate distribution of integers over the growing matrix. It can be shown that

$$E_m = E_k \oplus E_{m-k} = E_{m-k} \oplus E_k \tag{4}$$

for all k: $0 < k < m$. This means that E_m can be grown in a way that doubles the logsize at each step according to the sequence $E_1, E_2, E_4, E_8 \ldots$. Equation (4) also reflects a particular *similarity* between E_m and some of its parts in the following sense: when E_m is partitioned in submatrices of the size of E_k (the so-called *logsize-k agglomerates* of E_m), the resulting matrix of logsize-k agglomer-

The Visual Computer (1993) 9:233–238
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ates is *structurally* (or symbolically) similar to E_{m-k}, while all the logsize-k agglomerates are similar to each other and to E_k. The similarity referred to concerns a structural equivalence under a 1-1 mapping: i.e., two equally sized matrices A and B whose elements are symbols (possibly of a different nature in A and B) are structurally similar when equal/different elements in A correspond to equal/different elements in B under elementwise mapping, and vice versa. Consider e.g. the matrices E_1 and E_2 above: the matrix, whose elements are the submatrices in the partitioning of E_2 as shown, is



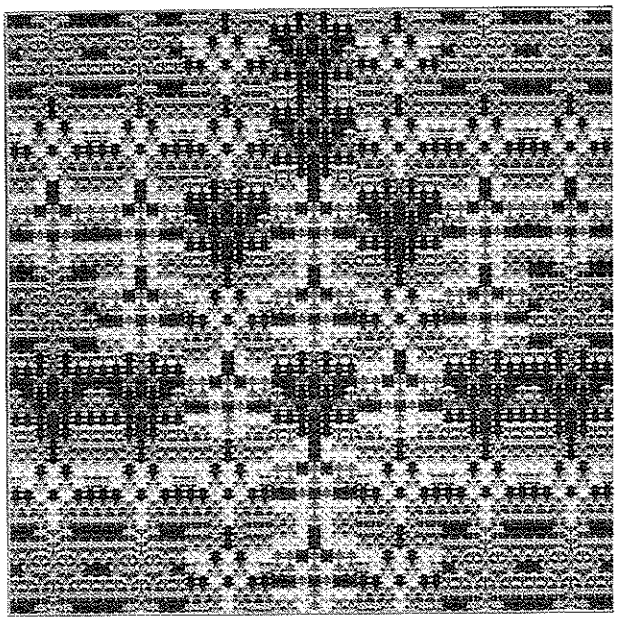
Figs. 1 and 2. $3^5 \times 3^5 (= 243 \times 243)$ E_5-matrices generated respectively from the matrices A and B

Figs. 3 and 4. $4^4 \times 4^4 (= 256 \times 256)$ E_4-matrices with the same generator matrix C, but with different palettes

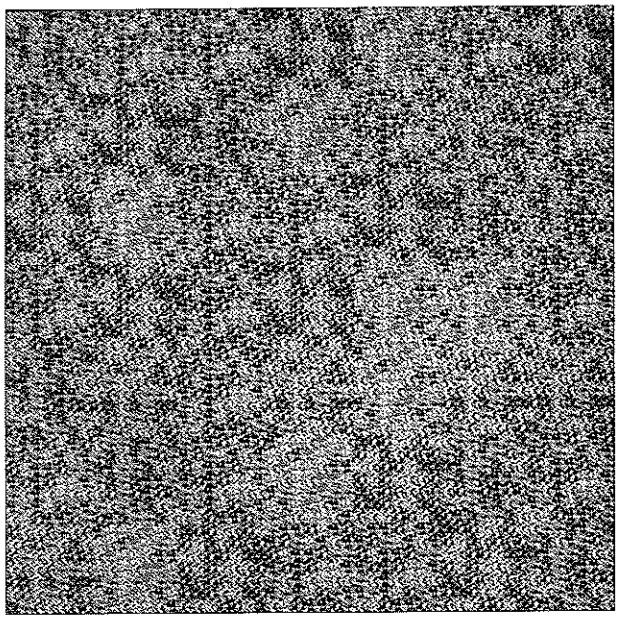
structurally similar to the generator E_1. Also, all these submatrices are similar to each other. It is this kind of similarity between the whole and its parts that allows the matrices to be classified as fractal objects (strictly speaking, this denomination should be reserved for infinite matrices only). An in-depth analysis of this, other self-similarities, and additional properties of these fractal matrices can be found in Barbé (1992).

Colorful translation

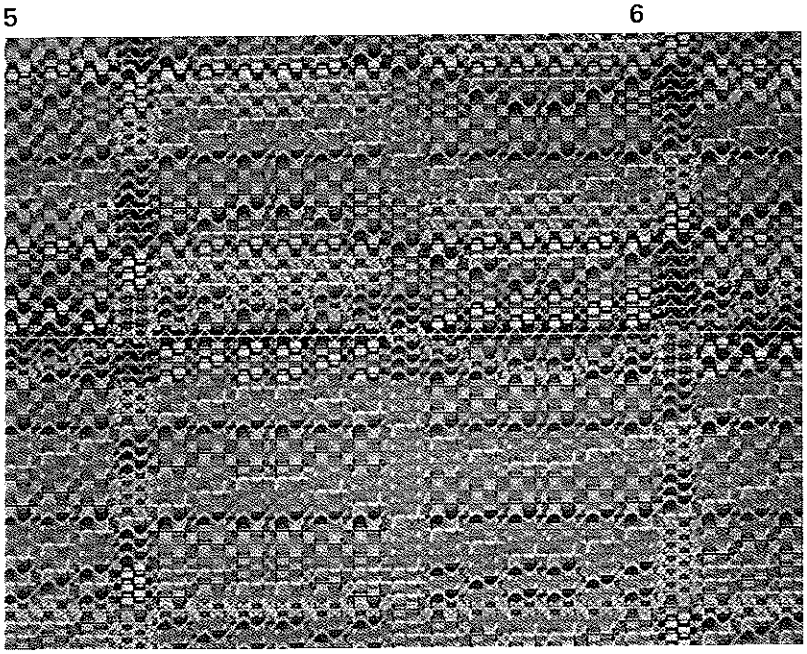
The perceptual nature of matrix E_m changes dramatically when it is displayed through a 1-1 substitution of its integer elements by colored pixels: nice patterns with much aesthetic appeal emerge. This is illustrated in Figs. 1 to 7. All figures were generated using a 16-color palette on a VGA-screen having 640×480 pixels. For each figure, a particular



5



6



7

Figs. 5 and 6. $7^3 \times 7^3 (= 343 \times 343)$ E_3-patterns obtained from generators D and F respectively

Fig. 7. $7^3 \times 21^2 (= 343 \times 441)$ submatrix of the $7^3 \times 21^3$ E_3-matrix obtained from generator G

selection was made from a randomly chosen mapping of the integers into the 16 colors. By limiting the range of the integers in the generator matrix and by also limiting the logsize of the matrices on display, each number can be assigned a different color. Otherwise, more colors are needed or different numbers should be attributed to the same color (which allows for an additional artistic degree of freedom). These patterns can be used as designs for tilings, carpets, and mosaics. The way in which they are obtained supplements the known classical ways, both those based on ordinary geometry [see e.g. the standard work of Grünbaum and Shephard (1987)] and those based on fractal geometry [e.g. Mandelbrot (1982) and Barnsley (1988)].

Figure 1 shows the E_5 matrix obtained from the generator matrix

$$A=\begin{bmatrix}0 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & 0\end{bmatrix}.$$

At the resolution of the picture, E_2, E_3, E_4, E_5 are easily distinguishable, as are all agglomerates of logsize 2 to logsize 5 (these correspond to the submatrices obtained by dividing E_5 respectively in $27\times 27{,}9\times 9{,}3\times 3{,}1\times 1$ equally sized parts). The structure of the generator matrix was chosen to show an extremely simple symmetry, so that the above mentioned structural similarities between all these agglomerates immediately catch the eye.

$$C=\begin{bmatrix}0 & 1 & -1 & 1\\ -1 & 0 & 1 & -1\\ 1 & -1 & 0 & 1\\ -1 & 1 & -1 & 0\end{bmatrix}.$$

Figure 4 represents the same matrix, but with another mapping from the integers into the colors. This illustrates how a change in palette may induce a dramatical change in the perception of the picture.

Figures 5 and 6 are E_3-matrixed obtained from the 7×7 generators D and F respectively:

$$D=\begin{bmatrix}0 & 0 & -1 & 2 & -1 & 0 & 0\\ -1 & -1 & -1 & 2 & -1 & -1 & -1\\ -2 & -2 & 2 & -2 & 2 & -2 & -2\\ 0 & -2 & -1 & -2 & -1 & -2 & 0\\ 2 & 2 & -2 & 2 & -2 & 2 & 2\\ -1 & -1 & -1 & -2 & -1 & -1 & -1\\ 0 & 0 & -1 & -2 & -1 & 0 & 0\end{bmatrix}$$

$$F=\begin{bmatrix}0 & 1 & 1 & -2 & 2 & 1 & -2\\ 1 & -2 & 0 & 2 & 1 & -1 & 1\\ -2 & 2 & 1 & 1 & -2 & 0 & 0\\ 1 & -2 & 1 & -2 & 2 & 2 & 1\\ 0 & -1 & 1 & -1 & 2 & -1 & -1\\ 1 & 0 & 1 & 2 & -2 & 0 & 1\\ -1 & 2 & -2 & 1 & 1 & -2 & 0\end{bmatrix}.$$

Figure 7 is an abstract pattern obtained from only a part of the E_3 matrix generated from the 7×21 matrix

$$G=\begin{bmatrix}0 & 2 & 0 & -2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 0 & 3 & 0 & 0 & 3 & 0 & -1 & 2 & 2 & 1\\ 2 & 0 & 2 & -2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 3 & 0 & 3 & 3 & 0 & 3 & -1 & 2 & 1 & 2\\ 0 & 2 & 0 & -2 & 1 & 3 & 3 & 3 & 3 & 1 & 2 & 0 & 3 & 0 & 0 & 3 & 0 & -1 & 1 & 2 & 2\\ -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1\\ 2 & 2 & 1 & -1 & 0 & 3 & 0 & 0 & 3 & 0 & 2 & 1 & 3 & 3 & 3 & 3 & 1 & -2 & 0 & 2 & 0\\ 2 & 1 & 2 & -1 & 3 & 0 & 3 & 3 & 0 & 3 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & -2 & 2 & 0 & 2\\ 1 & 2 & 2 & -1 & 0 & 3 & 0 & 0 & 3 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & -2 & 0 & 2 & 0\end{bmatrix}$$

Figure 2 is another E_5 matrix generated by the 3×3 matrix

$$B=\begin{bmatrix}0 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & 0\end{bmatrix}.$$

Again, notice that the whole picture and its agglomerates are similar to B.

Figure 3 is the E_4 matrix grown from the 4×4 generator

The horizontal partitioning of the picture in Fig. 7 in seven equally sized submatrices is structurally similar to the first column in G. All these submatrices have the same structure: they only differ through the use of a different palette of colors.

Extensions

The procedure presented above can be applied to any tiling that covers the plane in a self-similar

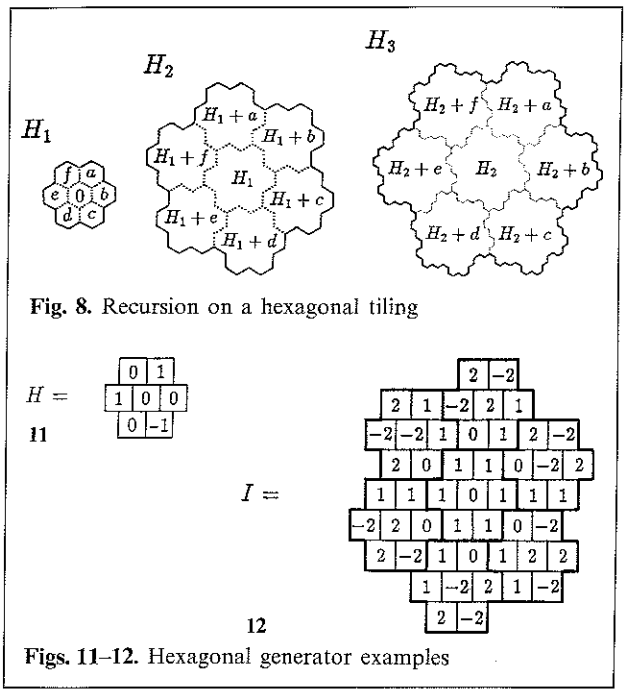


Fig. 8. Recursion on a hexagonal tiling

Figs. 11–12. Hexagonal generator examples

geometrical way. This will be illustrated for a honeycomb tiling. We start with a hexagonally shaped generator matrix H_1 consisting of seven elementary hexagons in which seven integers a to f are inscribed, the one in the center being 0 (see Fig. 8).

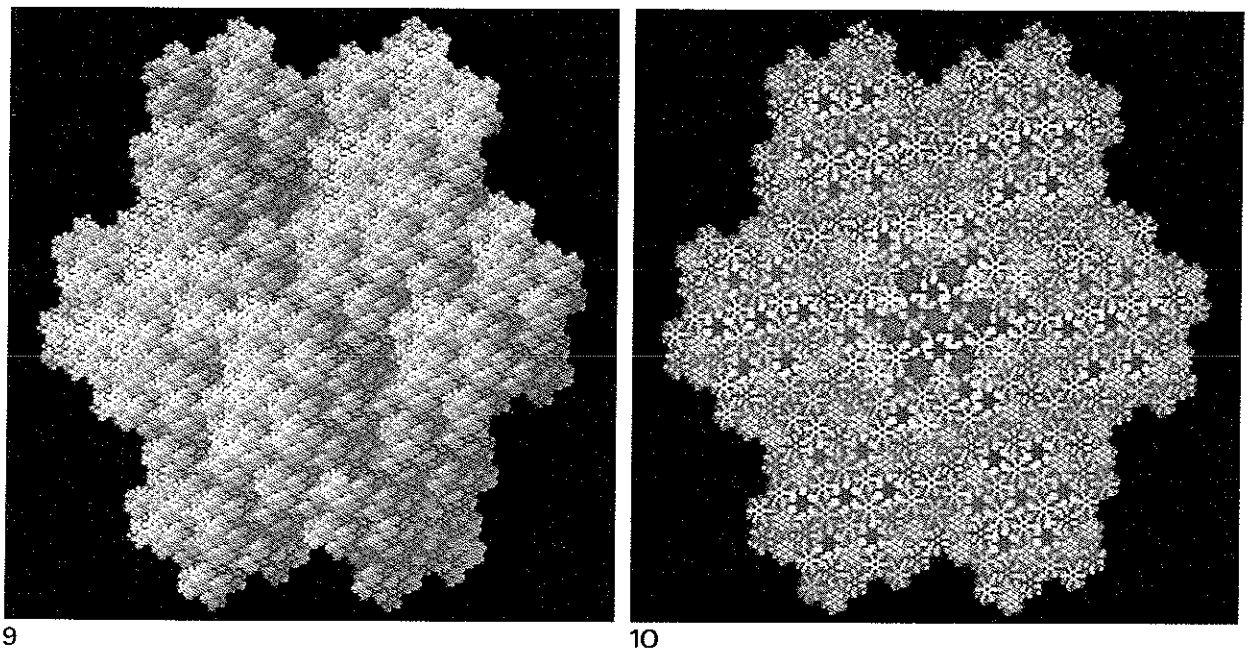
In a next step, this matrix is expanded into a hexagonal cluster H_2 of seven H_1-like shapes, where the 49 numbers in H_2 are obtained as indicated (H_1+a means that a is added to all elements of H_1). This process can be repeated endlessly in an obvious way.

Figure 9 shows the H_5 pattern obtained from a square-pixel based approximation of the hexagons, starting with the generator (see Fig. 11).

One could equally well start with H_2, H_3... as a generator matrix, filling it with 7^2, 7^3 ... elements, thereby increasing the number of interesting basic patterns in the generator itself. Figure 10 displays a pattern (obtained after two complete recursion steps $+1/7$ of the third step) produced by the 49-cell generator I (see Fig. 12).

Conclusion

An easily implementable recursive procedure for generating fascinating patterns has been presented. It can be used in all kinds of artistic ornamental design. The examples shown give only a partial impression about the multitude and diversity of figures that can be produced. Much of how the result obtained from a specific generator matrix looks like depends on the motif formed by the integers in this generator. But, even for a generator



9 10

Figs. 9 and 10. Hexagonal patterns ($7^5 = 49^2 \times 7 = 16807$ cells) obtained from the generators H and I respectively

with a given structural pattern, the perception of the result may dramatically depend on both the specific integers chosen to embody that pattern and on the palette through which the integers are mapped into colors. In this way, there are at least three degrees of freedom along which the designer may influence the result. Selecting the shape and size of the generator, allowing regular or random permutations or a change of integers in the generator matrix at each recursion step in Eq.(1) (by which strict structural self-similarity breaks down), may provide additional degrees of freedom in exploring the almost psychedelic universe of colorfully transfigurating number-filled matrices.

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