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The quotient SVD and transfer matrix norms *

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Abstract

Directional aspects of two transfer matrices relative to one another are studied. It is shown how the relative directional gains can be visualized using signal-to-signal ratio plots. The numerical tool is the quotient singular value decomposition (**QSVD**). Using the **QSVD** it is also possible to define norms between two transfer matrices: the H_{Q-2} and the $H_{Q-\infty}$-norm. It is also shown how these norms can be calculated.

Keywords: Oriented energy and signal-to-signal ratio, field of values, H_2-norm, H_∞-norm.

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1 Introduction

In this paper we will analyze some "directional" aspects of two multivariable systems *relative* to one another. The term "directionality" refers to the fact that multivariable systems possess properties that vary spatially as well as with frequency. For one multiple input-multiple output system (MIMO), this concept of directionality can be made precise by the *singular value decomposition* (**SVD**) (see e.g. Golub and Van Loan (1989), as was shown in e.g. Callier and Desoer (1982)).

As an example, consider the determination of 2 bounded input signals of a stable 2×2 MIMO system, such that the energy of the output is as large as possible. This will occur at any frequency ω_0 where the H_∞-norm of the transfer function is attained. The H_∞-norm is defined as

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{\max}[H(j\omega)]$$

where $\sigma_{\max}[.]$ denotes the largest singular value of the matrix between brackets.
Let the **SVD** of $H(j\omega_0)$ be given by

$$H(j\omega_0) = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix}$$

where $v_1 = \begin{pmatrix} \alpha + j\beta \\ \gamma + j\delta \end{pmatrix}$ with $v_1^* v_1 = 1$ or $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.
Then the input signal that will be amplified most is given by:

$$u(t) = \begin{pmatrix} \sqrt{\alpha^2 + \beta^2}\sin(\omega_0 t + \arctan\frac{\beta}{\alpha}) \\ \sqrt{\gamma^2 + \delta^2}\sin(\omega_0 t + \arctan\frac{\delta}{\gamma}) \end{pmatrix}$$

with $u_1 = \begin{pmatrix} \epsilon + j\phi \\ \kappa + j\lambda \end{pmatrix}$ and $u_1^* u_1 = 1$, the corresponding output vector will be given by

$$y(t) = \sigma_1 \begin{pmatrix} \sqrt{\epsilon^2 + \phi^2}\sin(\omega_0 t + \arctan\frac{\phi}{\epsilon}) \\ \sqrt{\kappa^2 + \lambda^2}\sin(\omega_0 t + \arctan\frac{\lambda}{\kappa}) \end{pmatrix}$$

In this paper, we will show how a generalization of the **SVD** for two matrices, called the *quotient SVD*, can be used to study directional aspects of 2 transfer functions, relative to one another. In section 2, we will summarize the main results on the **QSVD**. In section 3, we briefly discuss the well-know indicator ellipsoid, which is a singular value related plot in the *output* space. We also introduce the idea of an oriented energy plot, which is the corresponding plot in the input space. In our view, the oriented energy plot allows for a clearer visualization of the directional gain of a MIMO system. Next, we introduce the oriented signal-to-signal ratio of 2 systems and show that the **QSVD** is the appropriate tool to characterize it numerically. In section 4, we define the H_{Q-2}-norm of a system, relative

to another one. It is a measure of the energy of the output of one system relative to the output of a second system. In section 5, the corresponding $H_{Q-\infty}$-norm is considered, which can be interpreted as a weighted H_∞-norm for one system. We also derive an algorithm to compute it.

2 The QSVD

In this section the main results on the **QSVD** are summarized. While the **SVD** was already discovered more than 100 years ago (see De Moor and Zha (1991) for a collection of historical references), the **QSVD** is much more recent and is only one member of an infinite set of generalizations of the **SVD** for any number of matrices (De Moor and Zha 1991). A standardized nomenclature for these generalizations was proposed in De Moor and Golub (1989).

Theorem 1 *Any pair of complex matrices, $A \in \mathbb{C}^{l\times m}$ and $B \in \mathbb{C}^{p\times m}$ with the same number of columns, can be jointly factorized as*

$$\begin{aligned} A &= U_a S_a X^{-1} \\ B &= U_b S_b X^{-1} \end{aligned}$$

where $U_a \in \mathbb{C}^{l\times l}$ and $U_b \in \mathbb{C}^{p\times p}$ are unitary, $X \in \mathbb{C}^{m\times m}$ is a nonsingular square matrix and $S_a \in \mathbb{R}^{l\times m}$ and $S_b \in \mathbb{R}^{p\times m}$ are matrices with the following structure:

$$S_a = \begin{array}{c} \\ r_{a|b}-r_b \\ r_a+r_b-r_{a|b} \\ l-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{a|b}-r_b & r_a+r_b-r_{a|b} & r_{a|b}-r_a & m-r_{a|b} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & D_a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$S_b = \begin{array}{c} \\ p-r_b \\ r_a+r_b-r_{a|b} \\ r_{a|b}-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{a|b}-r_b & r_a+r_b-r_{a|b} & r_{a|b}-r_a & m-r_{a|b} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_b & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \end{array}$$

where

$$r_a = rank(A),\ r_b = rank(b)\ and\ r_{a|b} = rank\begin{pmatrix} A \\ B \end{pmatrix}$$

This decomposition is called the quotient singular value decomposition *(***QSVD***).*

Proof:
Our proof of the **QSVD** is constructive and uses three **SVD**s: First consider the **SVD** of the concatenation of the matrices A and B.

$$\begin{pmatrix} A \\ B \end{pmatrix} = U \begin{array}{c} \\ r_{a|b} \\ n-r_{a|b} \end{array} \begin{array}{c} \begin{array}{cc} r_{a|b} & n-r_{a|b} \end{array} \\ \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \end{array} V^* = \begin{array}{c} \\ l \\ p \end{array} \begin{array}{c} \begin{array}{cc} r_{a|b} & n-r_{a|b} \end{array} \\ \begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cc} r_{a|b} & n-r_{a|b} \end{array} \\ \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \end{array} V^*$$

Next consider the **SVD**s of the matrices U_1 and U_2:

$$\begin{aligned} U_1 &= U_a S_1 V_1^* \\ U_2 &= U_b S_2 V_2^* \end{aligned}$$

It follows from the orthonormality of U that $U_1^*U_1 + U_2^*U_2 = I_{r_{a|b}}$. This can be used to show that

$$\begin{aligned} S_1^t S_1 + S_2^t S_2 &= I_{r_{a|b}} \\ V_1 &= V_2 \end{aligned}$$

so that

$$\begin{aligned} A &= U_a(\ S_1\ 0\ )X^{-1} \\ B &= U_b(\ S_2\ 0\ )X^{-1} \end{aligned}$$

where

$$X^{-1} = \begin{pmatrix} V_1^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} V^*$$

which basically proves the Theorem. The structure of S_a and S_b follows from the ranks of A, B and $\begin{pmatrix} A \\ B \end{pmatrix}$. □

If A and B are real matrices, all the **SVD**s in the constructive proof are real and this automatically results in a real version of the **QSVD**.

The original formulation of the **QSVD** (which used to be called *the generalized SVD*, but see also De Moor et al. (1988)) is due to Paige (Paige and Saunders 1981) and Van Loan (Van Loan 1976).

D_a and D_b are diagonal matrices with size $r_q = r_a + r_b - r_{a|b}$. The diagonal elements are strictly positive and smaller than 1. Call the diagonal elements of D_a, $\alpha_1, \ldots, \alpha_{r_q}$ and those of D_b, $\beta_1, \ldots, \beta_{r_q}$. The pairs of real numbers (α_i, β_i), $i = 1, \ldots, r_q$ are called the quotient singular value pairs and by convention, we put

$$\alpha_i^2 + \beta_i^2 = 1$$

and order the pairs so that

$$\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \geq \cdots$$

These ratios will be called the *quotient singular values* (**QSV**). We will use an extended definition of the **QSV**: Take as α_i the diagonal elements of S_a. Call the elements on the diagonal of S_b that contains the diagonal elements of D_b, β_i. i is now an index from $1, \ldots, r_{a|b}$. With this definition the **QSV**s may be 0, when A is singular, and they may be infinite, when B is singular. The **QSV**s will be denoted as σ_{qi}, where σ_{qi} is the ith **QSV**.

Note that the block dimension $r_a + r_b - r_{a|b}$ gives the dimension of the intersection of the row spaces of A and B, by Grassmann's Dimension Theorem.

If B is square invertible, the **QSVD** of the matrix pair (A, B) delivers the **SVD** of AB^{-1}, since

$$AB^{-1} = U_a S_a X^{-1} X S_b^{-1} U_b^* = U_a (S_a S_b^{-1}) U_b^*$$

Observe that the **QSVD** is however more general because it still exists if B is not square or singular. Another interpretation goes via the generalized eigenvalue problem. Note that

$$\begin{array}{cccccccc} A^* & A & = & X^{-*} & S_a^t & S_a & X^{-1} \\ m \times l & l \times m & & m \times m & m \times l & l \times m & m \times m \\ B^* & B & = & X^{-*} & S_b^t & S_b & X^{-1} \\ m \times p & p \times m & & m \times m & m \times p & p \times m & m \times m \end{array}$$

so that

$$A^* A X S_b^t S_b = B^* B X S_a^t S_a$$

If $S_b^t S_b$ is invertible, then

$$A^* A X = B^* B X \Lambda$$

with

$$\Lambda = (S_a^t S_a)(S_b^t S_b)^{-1}$$

Although there is a close relationship between the **QSVD** of a matrix pair (A, B) and the **SVD** of AB^{-1} if B is invertible, or between the **QSVD** and the corresponding generalized eigenvalue problem $A^* A x = B^* B x \lambda$, from a numerical point of view, any exploitation of these connections is a bad idea, as is witnessed by the following example:
Suppose

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & \mu \\ \mu & 0 & 0 \\ 0 & -\mu & 0 \end{pmatrix}$$

where ϵ and μ are larger than the machine precision, but ϵ^2 and μ^2 smaller. $A^t A$ and $B^t B$ are then

$$A^t A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1+\epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B^t B = \begin{pmatrix} 1+\mu^2 & 1 & \mu \\ 1 & 1+\mu^2 & \mu \\ \mu & \mu & \mu^2 \end{pmatrix}$$

However, due to the finite machine precision this will be represented in the computer as:

$$fl(A^t A) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad fl(B^t B) = \begin{pmatrix} 1 & 1 & \mu \\ 1 & 1 & \mu \\ \mu & \mu & \mu^2 \end{pmatrix}$$

where $fl(.)$ represents the floating point result of the computation.
On the computer the products $A^t A$ and $B^t B$ are singular, while the exact product is not.

Therefore, it is not a good idea of performing the matrix products to find the **QSVD**. Taking the inverse of B is not possible either. The condition number of B is μ^2. Thus B is numerically singular. A direct **QSVD** should be used.

It seems that the best existing algorithm for the numerically robust computation of the **QSVD** is one by Bai and Demmel (1991).

We will also need the following lemma:

Lemma 1
Let u, v and x be three vectors satisfying

$$\begin{aligned} Ax &= u\alpha \\ Bx &= v\beta \\ A^* u\beta &= B^* v\alpha \end{aligned}$$

where α and β are 2 real numbers satisfying $\alpha^2 + \beta^2 = 1$. Then, u, v and x are vectors of the **QSVD** *of (A, B) as described in Theorem 1.*

The proof is straightforward.

3 Oriented energy and signal-to-signal ratios

In this section the definitions of the indicator ellipsoid, the oriented energy and the signal-to-signal ratio are stated. They are first illustrated on matrices and subsequently extended to transfer matrices.

3.1 The indicator ellipsoid

A complete picture of the action of a matrix $A \in \mathbb{C}^{l \times m}$ is provided by the image under A of the unit sphere of $\mathbb{C}^m$:

Theorem 2 **Indicator ellipsoid of a matrix $A \in \mathbb{C}^{l \times m}$** *(Callier and Desoer 1982)*
Let $A \in \mathbb{C}^{l \times m}$ be of rank r and consider the linear transformations $y = Ax$. The domain and codomain transformations induced by its **SVD** *$A = U \Sigma V^*$, where $\Sigma = diag(\sigma_i)$ contains the singular values σ_i, are $x = V\xi$ and $y = U\eta$. Let $S_m = \{ x \in \mathbb{C}^m \mid \|x\| = 1 \}$ denote the unit sphere of $\mathbb{C}^m$. Let $A[S_m]$ denote the image under A of S_m. Then the following properties hold:*

If $r = l = m$: $A[S_m] = \{ y \in \mathbb{C}^l \mid y = U\eta \, , \, \sum_{i=1}^r (\eta_i/\sigma_i)^2 = 1 \}$

If $r = l < m$: $A[S_m] = \{ y \in \mathbb{C}^l \mid y = U\eta \, , \, \sum_{i=1}^r (\eta_i/\sigma_i)^2 \leq 1 \}$

If $r = m < l$: $A[S_m] = \{ y \in \mathbb{C}^l \mid y = U\eta \, , \, \sum_{i=1}^r (\eta_i/\sigma_i)^2 = 1, \eta_{r+1} = \ldots = \eta_l = 0 \}$

If $r < min(l,m)$: $A[S_m] = \{\, y \in \mathrm{C}^l | y = U\eta, \sum_{i=1}^{r}(\eta_i/\sigma_i)^2 \leq 1, \eta_{r+1} = \ldots = \eta_l = 0 \,\}$

The matrix A maps the unit sphere S_m of C^m onto an r-dimensional ellipsoid in $\mathbf{R}(A)$ with as principal axes the left singular vectors u_i of U that correspond to nonzero singular values $\sigma_i > 0$. If A is not of full column rank, the points in $\mathbf{R}(A)$ interior to the ellipsoid have to be included. If A has full row rank, the ellipsoid has l principal axes with positive length. Note that the four cases of the theorem are mutually exclusive.

Let $A \in \mathrm{C}^{l \times m}$, $x \in \mathrm{C}^m$ and $y = Ax$. Then

$$\begin{aligned} max_{\|x\|=1} \|y\| &= \sigma_1 \\ min_{\|x\|=1} \|y\| &= \begin{cases} \sigma_m & \text{if } l \geq m \\ 0 & \text{if } l < m \end{cases} \end{aligned}$$

For the case $l \geq m$, the following minimax properties hold:

$$\begin{aligned} \max_{Q^r \subset \mathrm{C}^m} \{ \min_{x \in Q^r, \|x\|=1} \|y\| \} &= \sigma_r \\ \min_{Q^r \subset \mathrm{C}^m} \{ \max_{x \in Q^r, \|x\|=1} \|y\| \} &= \sigma_{m-r+1} \end{aligned}$$

where 'max' and 'min' denote operators, maximizing or minimizing over all r-dimensional subspaces Q^r of the ambient range space C. These are the classical (Courant-Fisher) minimax characterizations of the singular values (see e.g. Golub and Van Loan (1989)).

The Frobenius norm of A is defined as $\|A\|_F^2 = \sum_{i=1}^{l}\sum_{j=1}^{m}|a_{ij}|^2$. It is easy to show that $\|A\|_F^2 = trace(A^*A) = trace(AA^*)$. Using the fact that $\|UA\|_F = \|AV\|_F = \|A\|_F$ for arbitrary unitary matrices U and V (i.e. the Frobenius norm is *unitarily invariant*), we find from the **SVD** of A that

$$\|A\|_F^2 = \|U\Sigma V^*\|_F^2 = \|U^*U\Sigma V^*V\|_F^2 = \|\Sigma\|_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2$$

Summarizing, we have $\|A\|_F^2 = \sum_{i=1}^{r}\sigma_i^2 = trace(A^*A) = trace(AA^*)$. We have also the following statistical result, which puts the indicator ellipsoid in a statistical framework. For the time being, we will restrict ourselves to the real case.

Let $x \in \mathsf{R}^m$ be a random vector, which is zero mean and Gaussian distributed with covariance I_m. This implies that the directional distribution of x, which is the distribution of the projections of x onto the unit sphere in R^m, is uniform: All directions in R^m are equally probable [1]. We find for the average length of x:

$$\mathrm{E}\|x\|^2 = m$$

Next let $y = Ax$ with $A = U\Sigma V^t$ the singular value decomposition of A. Then $y \in \mathsf{R}^l$ will be normally distributed too by linearity and:

$$\mathrm{E}\|y\|^2 = \mathrm{E}(x^tA^tAx) = \mathrm{E}(z^t\Sigma^t\Sigma z) = \sigma_1^2\zeta_1^2 + \ldots + \sigma_r^2\zeta_r^2$$

[1] The reverse assertion is less trivial: a uniform directional distribution of vectors x is generated by a zero mean and Gaussian distribution with covariance I_m, if $m > 3$, see e.g. Letac (1981).

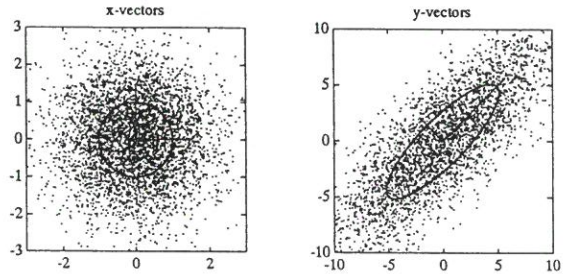

Figure 1: 1000 random points (normal distribution, zero mean, covariance I_2) are shown in the left plot. The unit circle is generated by normalizing these random points to have norm one. For each random point x, we compute $y = Ax$ and show the obtained points y in the right plot. The ellipsoid is obtained by plotting the image under A of the points on the unit circle. Observe that its principal axes lie in the direction of the left singular vectors, with half lengths equal to the singular values.

where $z = V^tx = (\,\zeta_1\,, \ldots\,, \zeta_m\,)$ and $r = rank(A)$. But from the unitarily invariance of the 2-norm, we find that

$$\mathrm{E}\|y\|^2 = \sigma_1^2 + \ldots + \sigma_r^2$$

Hence, we have proved:

Theorem 3
Let $x \in \mathsf{R}^m$ be a random vector, which is independently and identically normally distributed with mean zero and covariance matrix I_m. Then $y = Ax$ is normally distributed with mean zero and covariance matrix AA^t and $\mathrm{E}\|y\|^2 = \sigma_1^2 + \ldots + \sigma_r^2$.

Example:
The statistical interpretation of the SVD is illustrated in figures 1 and 2 for the matrix:

$$A = \begin{pmatrix} 4.9497e+00 & -1.4142e+00 \\ 4.9497e+00 & 1.4142e+00 \end{pmatrix} = \begin{pmatrix} 7.0711e-01 & -7.0711e-01 \\ 7.0711e-01 & 7.0711e-01 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.2 Oriented energy of a matrices

The geometrical interpretation of the static transfer performed by a matrix A can also be shown in another way. Let x be a unit vector in R^m and $y = Ax \in \mathsf{R}^l$. An *oriented energy*

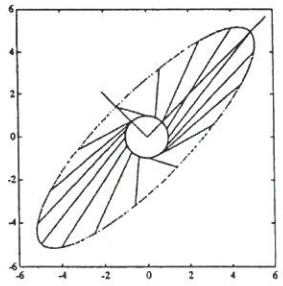

Figure 2: The small circle is generated by 1000 random points on the unit circle, representing input vectors. Also shown are the 1000 corresponding points on the indicator ellipsoid, some of which are connected by a line with their corresponding point on the unit circle. The right singular vectors are in the directions $(1,0)$ for $\sigma_1 = 7$ and $(0,1)$ for $\sigma_2 = 2$. The directions of the left singular vectors are given by the long lines. It can be observed that on the average points are relatively more 'attracted' to the left singular vector corresponding to the largest singular value, especially in the neighborhood of the right singular vector corresponding to the smallest singular value.

plot can be obtained by plotting the vector

$$x\sqrt{x^tA^tAx}$$

for each unit direction vector x. Such a plot tells us how much a vector x will get amplified when the matrix A is applied to it. The quantity $\|Ax\|$ will be called the oriented energy of the matrix A in the direction of the vector x. In general, there is one maximum (in the direction of the first right singular vector) and one minimum (in the direction of the m-th right singular vector). The other extrema, which are saddle points, occur at the remaining right singular vectors. An illustration can be found in figure 3. There is an important difference between the *indicator ellipsoid* and an *oriented energy plot*: The indicator ellipsoid is a 'picture' in the output space R^l of the vectors y while the oriented energy plot is a 'picture' in the domain R^m of the input space.
The set of numbers x^tA^tAx, where x goes over all vectors with $\|x\| = 1$, is called the *field of values* of the symmetric matrix A^tA.
The indicator ellipsoid and the oriented energy plot give complementary information about the action of a linear transformation, represented by a matrix A, on an input vector x resulting in an output vector y. The indicator ellipsoid shows the geometric extent of the set of vectors y that can be reached by applying a unit input vector x. If however we pick one point on the indicator ellipsoid, it does not tell us in which direction the corresponding input vector x could be situated. This is revealed by an oriented energy plot, which shows, in the input space, the effect of a certain input direction, on the energy of the

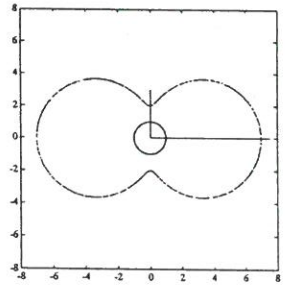

Figure 3: Shown is the unit circle obtained by normalizing 1000 random points (normal distribution, zero mean, covariance I_2) to have norm 1. For each normalized point x, the corresponding point $x\|Ax\|$ is also plotted, generating the oriented energy plot. The minimum and maximum occur at the directions of the *right* singular vectors, which are indicated by the lines.

output. Clearly, such information has its value in the sense that the input causally affects the output. If certain effects are to be avoided, for instance by designing an appropriate controller, one should operate in the input space. Hence the importance of an oriented energy plot.

3.3 Signal-to-signal ratio of two matrices

Until now we have been looking at the properties of only 1 matrix. In this section, we look at the properties of one matrix *relative* to another one. Hereto, we introduce:

Definition 1 *The signal-to-signal ratio* SSR *of 2 real matrices A and B in the direction x is the ratio of the oriented energies of the matrices A and B in the direction x:*

$$R_x[A,B] = \frac{\|Ax\|}{\|Bx\|}$$

Consider two matrices A and B with an equal number of columns and let their QSVD be:

$$\begin{cases} A = U_a S_a X^{-1} \\ B = U_b S_b X^{-1} \end{cases}$$

Let us now demonstrate how the QSVD provides a canonical decomposition for the signal-to-signal ratios just as the SVD does for the oriented energy. Hereto, consider the QSVD of $[A,B]$. Then,

$$R_x^2[A,B] = \frac{x^tA^tAx}{x^tB^tBx} = \frac{(x^tX^{-t})S_a^tS_a(X^{-1}x)}{(x^tX^{-t})S_b^tS_b(X^{-1}x)}$$

Obviously, when $x = x_i$, the ith column of X, then

$$R_{x_i}[A,B] = \frac{\alpha_i}{\beta_i}$$

Let $A \in \mathbb{C}^{l\times m}$ and $B \in \mathbb{C}^{p\times m}$, with $l \geq m$ and $p \geq m$, then

$$\begin{aligned} \max_{x,\|x\|=1} R_x[A,B] &= \frac{\alpha_1}{\beta_1} \\ \min_{x,\|x\|=1} R_x[A,B] &= \frac{\alpha_m}{\beta_m} \end{aligned}$$

The directions of maximal and minimal **SSR** are given by column x_1 and x_m of X. These extremal directions are, in general, *not* orthogonal to one another, because X is not an orthogonal matrix. Other relative minimax properties are:

$$\begin{aligned} MmR[A,B,r] \equiv \max_{Q^r\subset\mathbb{C}^m}\{\min_{x\in Q^r} R_x[A,B]\} &= \frac{\alpha_r}{\beta_r} \\ mMR[A,B,r] \equiv \min_{Q^r\subset\mathbb{C}^m}\{\max_{x\in Q^r} R_x[A,B]\} &= \frac{\alpha_{m-r+1}}{\beta_{m-r+1}} \end{aligned}$$

In words: Consider two matrices A and B and a certain r-dimensional subspace Q^r. Then there is a direction $x \in Q^r$ for which the signal-to-signal ratio is minimal. If we now manipulate also Q^r, we can find the maximum of all these minima, which is exactly $\frac{\alpha_r}{\beta_r}$. This ratio will be called the maximal minimal signal-to-signal ratio over r-dimensional subspaces: $MmR[A,B,r]$. The corresponding subspace is the range of the matrix $[x_1,\ldots,x_r]$ where x_i, $i = 1,\ldots,r$, are the first r vectors of X. A dual argument applies for the minimal maximal signal-to-signal ratio $mMR[A,B,r]$.

A signal-to-signal ratio plot is obtained by plotting the vector $x\frac{\|Ax\|}{\|Bx\|}$ for each unit direction x. Again, a **SSR**-plot is a picture in the input space. The extremal directions, which are given by the column vectors of X, are not orthogonal.

Example:
The signal-to-signal ratio is plotted for 2 pairs of matrices: $[A,B_1]$ and $[A,B_2]$

$$A = \begin{pmatrix} .25 & 0 \\ 0 & 2 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & -1.5 \\ 0 & 1.5 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

The directions of maximal and minimal **SSR** are shown and also the circle with radius $\frac{\alpha_1}{\beta_1}$ and $\frac{\alpha_2}{\beta_2}$. The maximal **SSR** is given by $\frac{\alpha_1}{\beta_1}$ and the minimal by $\frac{\alpha_2}{\beta_2}$. The directions of maximal and minimal **SSR** are given by the column vectors of the matrix X in the **QSVD**. These directions are *not* orthogonal as in the case of the oriented energy. The direction and size of the maximal **SSR** is dependent on the 2 matrices, as can be seen in the figure.

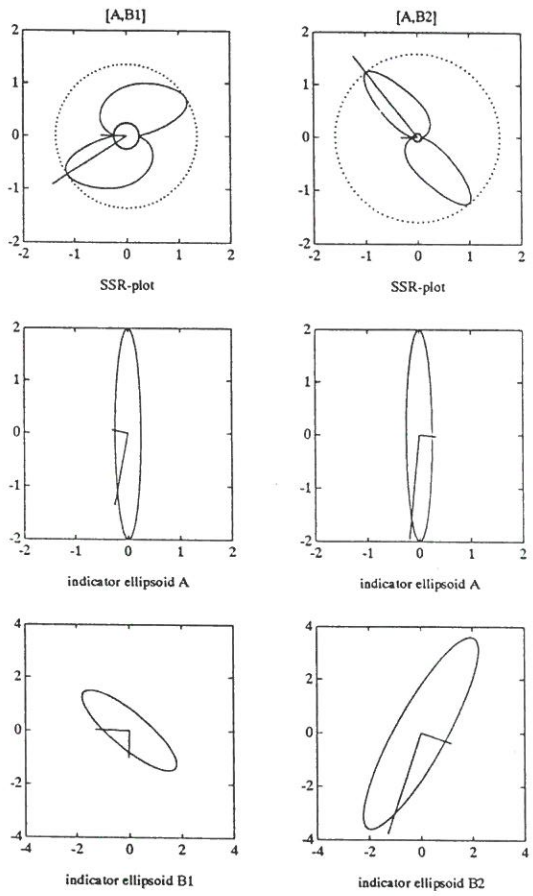

Figure 4: Left, the **SSR** and indicator ellipsoids of the matrix pair $[A,B_1]$ and right those of $[A,B_2]$. The circles have a radius corresponding to $\frac{\alpha_1}{\beta_1}$ and $\frac{\alpha_2}{\beta_2}$. Also shown are the column vectors x_1 and x_2 of X, which are the directions of maximal and minimal **SSR**. They are not orthogonal. Observe how the plot for the pair $[A,B_1]$ differs considerably from that of $[A,B_2]$. The lines on the plots of the indicator ellipsoids, are the directions of maximal and minimal **SSR** in the output space. In the output space these directions are orthogonal! However, they don't coincide with semi-axes of the indicator ellipsoids of each matrix separately.

3.4 Oriented energy of transfer matrices

Let us now consider complex transfer matrices, that are a function of frequency. The SVD of a transfer matrix is defined as:

$$y(s) = H(s)u(s) = U(s)S(s)V(s)^*u(s)$$

where $U(s), S(s)$ and $V(s)$ are such that at each s, $U(s)S(s)V(s)^*$ is the SVD of $H(s)$. The entries of $U(s)$ and $V(s)$ are in general not rational, even if $H(s)$ contains only rational entries.

Here we are only interested in the frequencies at the imaginary axis $s = j\omega$. If $H(s)$ has no poles on the imaginary axis, the singular values $\sigma_i(j\omega)$ are continuous, but in general non-differentiable functions of ω. However, the next theorem relates the singular values on the imaginary axis with real analytic functions:

Theorem 4 *There are real analytic functions $f_i : \mathsf{R} \to \mathsf{R}, i = 1, \ldots, m$, such that for all $\omega \in \mathsf{R}$,*

$$\{\sigma_1(H(j\omega)), \ldots, \sigma_m(H(j\omega))\} = \{|f_1(\omega)|, \ldots, |f_m(\omega)|\}$$

The functions $f(.)$ form an *unsigned singular value decomposition* (USVD) of the transfer matrix $H(s)$. Further details can be found in Boyd and Balakrishnan (1990) and De Moor and Boyd (1989).

It is also possible to show the oriented energy as a function of frequency. This is illustrated with an example.

Example:

Consider a 2 by 2 transfer function, which is for convenience, real on the imaginary axis:

$$H(j\omega) = \begin{pmatrix} 5 & \frac{8-\omega^2}{16.5-\omega^2} \\ \frac{19-\omega^2}{16.5-\omega^2} & \frac{8-\omega^2}{1.4-\omega^2} \end{pmatrix}$$

The oriented energy plot $u\|H(j\omega)u\|$ is shown in figure 5. u varies over all unit norm input vectors. In this case the vector represents the amplitudes and phases of sine waves with frequency ω at the inputs. Putting all these plots one behind the other we obtain the 3-dimensional plot shown in figure 6.

At $\omega = \sqrt{8} = 2.828$, H has a zero, which is obtained as a number $s \in \mathsf{C}$ where $H(s)$ drops rank. As can be seen on the corresponding plot, this does not imply that the output is zero. The output is zero only when the correct input direction is applied. In all other cases there is still an output signal. This directional aspect does not occur with SISO transfer functions which themselves become zero at a system zero.

3.5 Signal-to-signal ratio of transfer matrices

Let us now look at the signal-to-signal ratio of 2 transfer matrices as a function of frequency. Two different situations can be considered (see figure 7), either 2 systems with the same

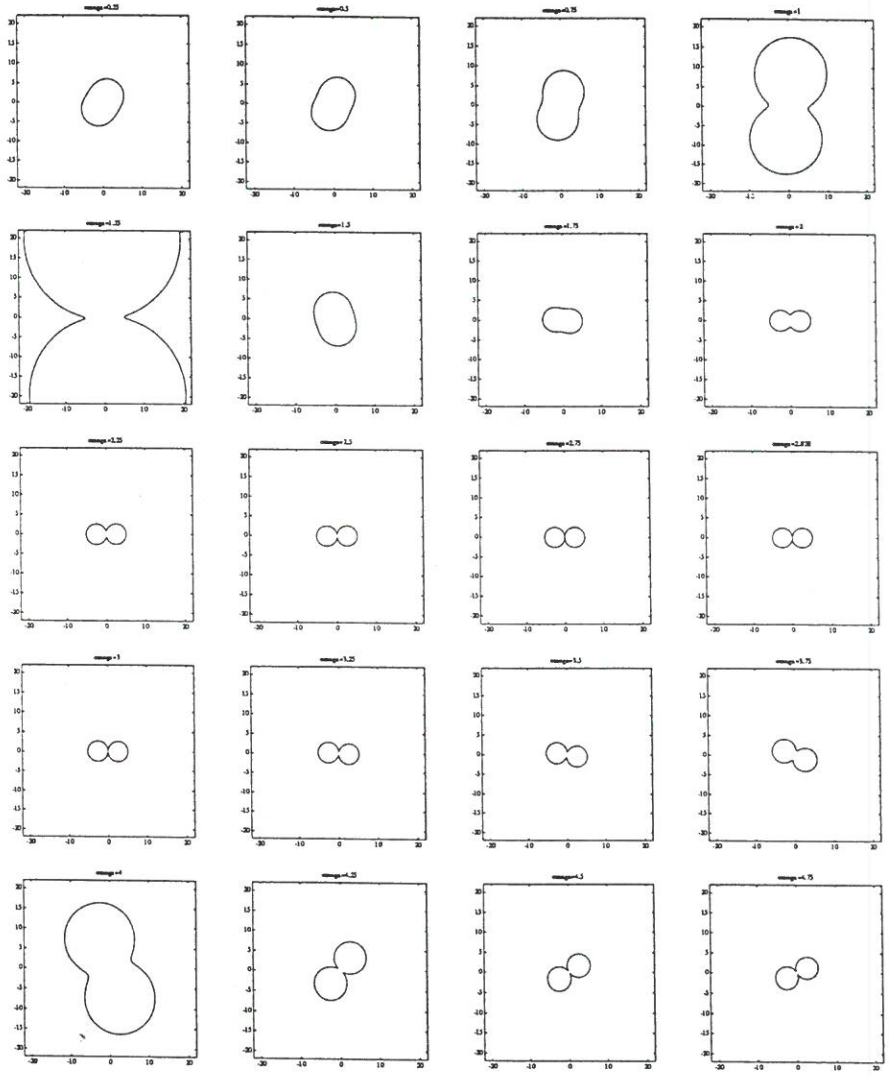

Figure 5: The oriented energy plots of $H(j\omega)$ for different values of ω. The direction and size of the largest amplification change drastically. The largest changes occur at the poles: $\omega = \sqrt{1.4}$ and $\omega = \sqrt{16.5}$. Observe also that even at a zero of the transfer matrix ($\omega = 2.828$) on the imaginary axis the output is non-zero unless the input has the correct orientation.

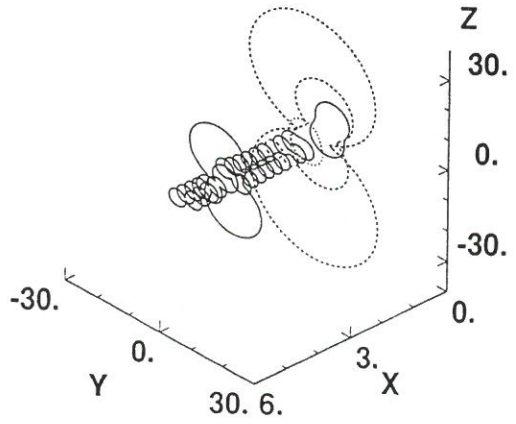

Figure 6: A 3D-plot of the oriented energy plot of a 2×2 transfer function. The X-axis is the frequency axis.

input and 2 outputs:

$$y_1 = H_1 u \qquad y_2 = H_2 u$$

or 2 inputs with the same output

$$y = H_1 u_1 + H_2 u_2$$

Consider first the case where the 2 transfer functions have the same input u but different outputs y_1 and y_2. For an interpretation assume that H_2 is a 'good' transfer matrix while H_1 is a disturbing one. Obviously, we would like the relative influence of u on y_1 to be small compared to its influence on y_2. In other words: At each frequency ω, we want the



Figure 7: Two interpretations of the oriented signal-to-signal ratio for matrices.

SSR $R_u[H_1(j\omega), H_2(j\omega)]$ as small as possible. Calculate the **QSVD** at each frequency:

$$y(s) = \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} H_1(s) \\ H_2(s) \end{pmatrix} u(s) = \begin{pmatrix} U_1(s)S_1(s)X^{-1}(s) \\ U_2(s)S_2(s)X^{-1}(s) \end{pmatrix} u(s)$$

The signal-to-signal ratio will be small if the Q-singular values are small for all frequencies. The **QSVD** reveals the size and the direction of the largest signal-to-signal ratio at each frequency. In control applications the output of the 'disturbing' transfer function H_1 has to be minimized relatively to the output of the 'good' transfer function H_2. The control engineer has to design a controller that minimizes the Q-singular values over all frequencies.

In the second situation there are different inputs u_1 and u_2 and one output y. In matrix notation this becomes:

$$y = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

For each frequency, define the **QSVD** of H_1 and H_2 as

$$H_1 = YS_1U^* = Y \begin{pmatrix} I & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^*$$

$$H_2 = YS_2V^* = Y \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} V^*$$

with an appropriate partitioning for all matrices. H_1 and H_2 have an equal number of rows. Their common factor is thus the left factor in the product. The right factors U and V are now orthogonal. Then we can write:

$$\begin{aligned} y &= y_1 + y_2 \\ &= H_1 u_1 + H_2 u_2 \\ &= YS_1U^*u_1 + YS_2V^*u_2 \end{aligned}$$

Partition Y, U and V according S_1 and S_2:

$$\begin{aligned} Y &= \begin{pmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{pmatrix} \\ U &= \begin{pmatrix} U_1 & U_2 & U_3 \end{pmatrix} \\ V &= \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} \end{aligned}$$

Now write the inputs u_1 and u_2 as

$$\begin{aligned} u_1 &= U_1p_1 + U_2p_2 + U_3p_3 \\ u_2 &= V_1q_1 + V_2q_2 + V_3q_3 \end{aligned}$$

for some vectors p_1, p_2, p_3, q_1, q_2 and q_3. Then

$$y = \begin{pmatrix} Y_1 & Y_2D_1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} 0 & Y_2D_2 & Y_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

or

$$\begin{aligned} y &= Y_1p_1 + Y_2(D_1p_2 + D_2q_2) + Y_3q_3 \\ &= \begin{pmatrix} Y_1 & Y_2 & Y_3 \end{pmatrix} \begin{pmatrix} p_1 \\ D_1p_2 + D_2q_2 \\ q_3 \end{pmatrix} \end{aligned}$$

From this we can conclude that there are directions in the output space which can only be influenced by the first input. In these directions the Q-singular values $\frac{\alpha_i}{\beta_i}$ are infinite. Other directions can only be controlled by the second input. In these directions the Q-singular values $\frac{\alpha_i}{\beta_i}$ are 0. Finally, there are directions that are controlled by both inputs.
The SSR is, in this case, a ratio of input signals. Suppose u_1 and u_2 are vectors of sinusoidal signals with the same frequency ω. Call u_{1_y} the input signal that produces an output signal in the direction y, when no input u_2 is present. Define u_{2_y} as the input signal u_2 that produces the same output y, when $u_1 = 0$. Then the SSR in the output direction y, is the ratio of the norms of the input signals u_{2_y} and u_{1_y}. A SSR value larger than 1 shows that the corresponding direction y is controlled relatively more by u_1 than by u_2. Or the same output signal y can be generated by a smaller signal u_1 than u_2.
For a control interpretation, suppose that u_2 is the 'good' input while u_1 is the disturbing one. If y has to be controlled mainly by u_1, a controller has to be designed that minimizes the Q-singular values of the pair $(H_1(j\omega), H_2(j\omega))$ at all frequencies.

Example:
As an example, take 2 transfer matrices in the first configuration (with the same input u):

$$H_1(s) = \begin{pmatrix} 5 & \frac{s^2+8}{s^2+16.5} \\ \frac{s^2+19}{s^2+16.5} & \frac{s^2+8}{s^2+1.4} \end{pmatrix}$$

$$H_2(s) = \begin{pmatrix} \frac{1}{s^2+2.5} & \frac{s^2+3}{s^2+5} \\ \frac{2}{s^2+13} & \frac{s^2+5}{s^2+10} \\ \frac{s^2+2}{s^2+7} & \frac{s^2+11}{s^2+19} \end{pmatrix}$$

The signal-to-signal ratio is plotted for different values of ω in figure 8. The maximal and minimal signal-to-signal ratio and direction vary quite drastically with changing ω. Notice that when H_2 becomes small, the signal-to-signal ratio becomes large (around $\omega = 1.25$). If H_1 has a zero on the imaginary axis, the SSR becomes zero in one direction.
The directions and sizes of the maximum and minimum SSR can be obtained using the QSVD. Recall once more that these directions are in general *not* orthogonal in the input space.

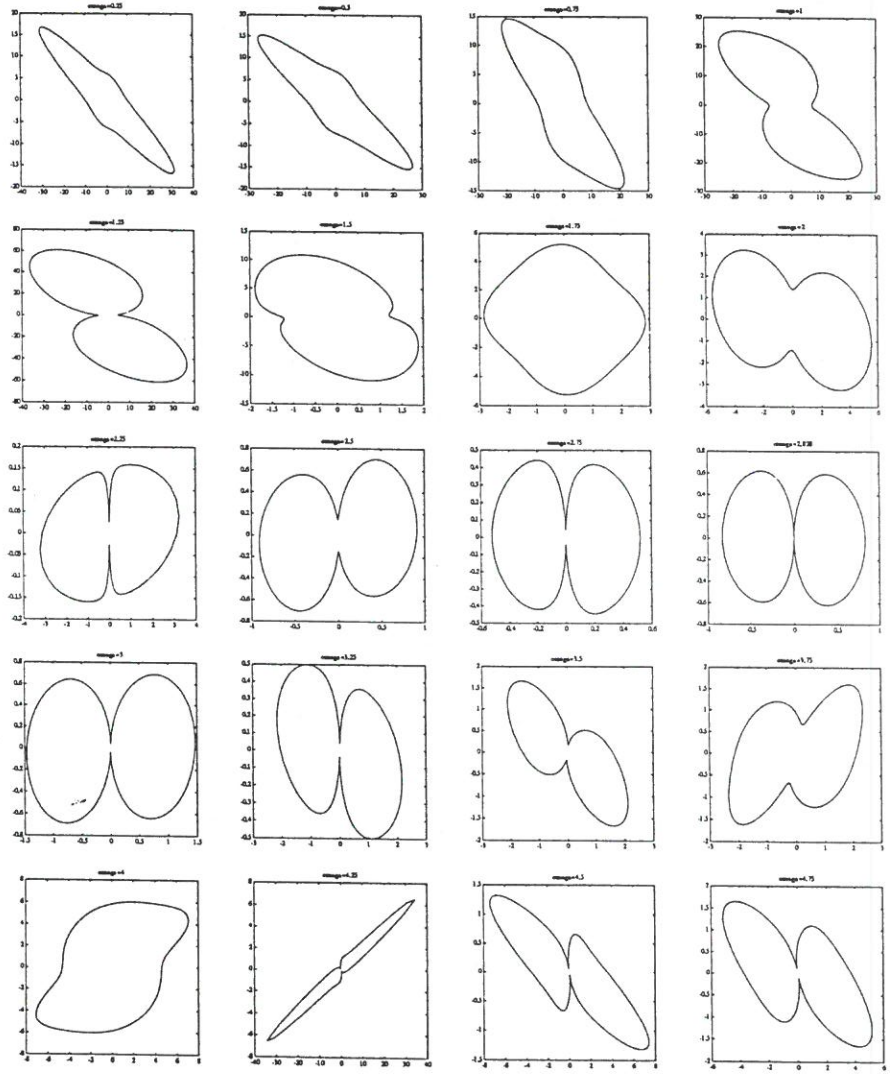

Figure 8: The SSR-plot of two transfer matrices with 2 inputs at different frequencies ω, for all unit norm input vectors. The direction and size of the SSR-plot can change drastically with the frequency. Note that the plots have different scales! Notice that the directions of maximal and minimal SSR are, in general, not orthogonal.

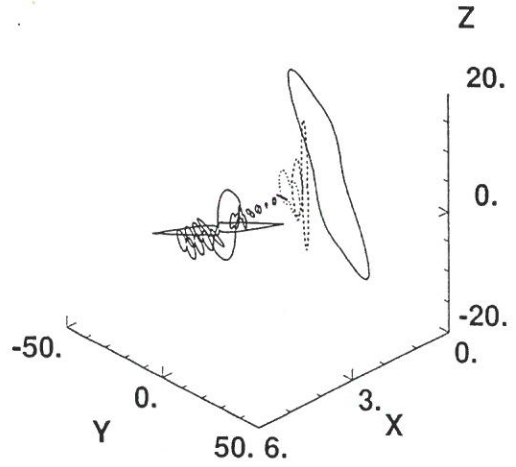

Figure 9: A 3D view of the signal-to-signal ratio of two transfer matrices. The X-axis is the frequency axis.

4 The H_{Q-2}-norm

In this section we introduce the H_{Q-2}-norm for two transfer matrices as a generalization of the classical H_2-norm. We show its connection with the H_2-norm and derive its calculation from the state space models of the transfer matrices. In this section we consider the first case with a common input vector for the 2 systems. However, everything can easily be extended to the dual case of 2 different inputs and a summation of the outputs.

4.1 Definition

The H_2-norm of a stable $l \times m$ transfer matrix $G(s) = C(sI - A)^{-1}B$ is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} trace\{G(j\omega)^* G(j\omega)\}d\omega$$

This can also be expressed in terms of its singular values:

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{i=1}^{\min(l,m)} \sigma_i(G(j\omega))^2 \, d\omega$$

The H_2-norm of a transfer matrix G can easily be calculated from its state space representation. If L_c denotes the controllability Grammian of G and L_o denotes its observability

Grammian, then they satisfy the following Lyapunov equations

$$AL_c + L_cA^t + BB^t = 0 \qquad A^tL_o + L_oA + C^tC = 0$$

and

$$\|G\|_2^2 = \text{trace}(CL_cC^t) = \text{trace}(B^tL_oB)$$

Similarly, we will define the H_{Q-2}-norm for a pair of matrices and give its calculation from a state space realization. Consider the setup with one set of inputs and two sets of outputs (situation 1) with state space model:

$$G(s) = \begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right)$$

where $G_1(s) = D_1 + C_1(sI - A)^{-1}B$ and $G_2(s) = D_2 + C_2(sI - A)^{-1}B$. If A is stable, we call the transfer matrix pair $(G_1(s), G_2(s))$ stable.

Definition 2 *The H_{Q-2}-norm of a stable transfer matrix pair $(G_1(s), G_2(s))$, with an equal number of inputs and $G_2(j\omega)$ of full column rank $\forall\omega \in \mathbb{R}$, is*

$$\left\|\frac{G_1}{G_2}\right\|_{Q-2}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} trace\left\{(G_2^*(j\omega)G_2(j\omega))^{-1}(G_1^*(j\omega)G_1(j\omega))\right\}d\omega$$

It is obvious that the H_{Q-2}-norm is only finite if $(G_2^*(\infty)G_2(\infty))^{-1}(G_1^*(\infty)G_1(\infty)) = 0$. As $G_2^*(\infty)G_2(\infty)$ has to be invertible, this implies $G_2(\infty)$ full column rank and $G_1(\infty) = 0$. The H_{Q-2}-norm can also be expressed in terms of the Q-singular values:

$$\begin{aligned}\left\|\frac{G_1}{G_2}\right\|_{Q-2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{i=1}^{r_a}\sigma_{qi}(G_1(j\omega), G_2(j\omega))^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{i=1}^{r_a}\left(\frac{\alpha_i(j\omega)}{\beta_i(j\omega)}\right)^2 d\omega\end{aligned}$$

where r_a is the generic rank of $G_1(j\omega)$. Observe that the full column rank assumption for G_2 implies that G_2 has no finite pure imaginary transmission zeros.

4.2 Relation with the H_2-norm.

Let us now investigate the relation between the H_2-norm and the H_{Q-2}-norm. Recall that the QSVD of a matrix pair (A, B) is related to the SVD of AB^{-1} if B is invertible:

$$AB^{-1} = US_aX^{-1}XS_b^{-1}V^* = U(S_aS_b^{-1})V^*$$

The same observation applies for transfer matrices with however one complication if G_2 is non-minimum phase. First, consider the special case where G_2 is stably invertible:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y_2 &= C_2x + D_2u \end{aligned}$$

so that

$$u = D_2^{\dagger}y_2 - D_2^{\dagger}C_2x$$

where $D_2^{\dagger}$ is the pseudo inverse of D_2. Substitution in the state space realization of G_1 yields (with $D_1 = 0$):

$$\begin{aligned} \dot{x} &= (A - BD_2^{\dagger}C_2)x + BD_2^{\dagger}y_2 \\ y_1 &= (C_1 - D_1D_2^{\dagger}C_2)x \end{aligned}$$

This is exactly the state space realization of $G_1G_2^{\dagger}$. Thus the H_{Q-2} norm of $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ can be computed from

$$\|G_1G_2^{\dagger}\|_2$$

More general, if G_2 contains zeros in the right half plane (non-minimum-phase zeros), it has no stable inverse. However, one can then compute a unique minimum phase spectral factor G_3 of $G_2^*G_2$, which has a stable inverse.

Lemma 2 *If $G_1(s)$ and $G_2(s)$ are stable, have an equal number of inputs and $G_2(s)$ is of full column rank on the $j\omega$-axis, then*

$$\left\| \frac{G_1}{G_2} \right\|_{Q-2}^2 = \|G_1G_3^{-1}\|_2^2$$

where G_3 is the spectral factor of $G_2^*G_2$:

$$G_2^*G_2 = G_3^*G_3$$

with G_3 as well as G_3^{-1} stable.

Suppose a stable state space realization of $G_1(s)$ and $G_2(s)$ is known:

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline C_1 & 0 \\ C_2 & D_2 \end{array}\right)$$

If $G_2(s)$ is of full column rank on the $j\omega$-axis and its state space realization $D_2 + C_2(sI - A)^{-1}B$ is minimal, then the H_{Q-2}-norm can easily be computed from the state space realization. First, the state space realization of G_3, the spectral factor of $G_2^*G_2$ has to be

obtained. Therefore, find the positive definite solution X of the following algebraic Riccati equation (Francis 1987):

$$X\underline{A} + \underline{A}^tX - X\underline{R}X + \underline{Q} = 0 \tag{1}$$

where

$$\begin{aligned} \underline{A} &= A - B(D_2^tD_2)^{-1}D_2^tC_2 \\ \underline{R} &= B(D_2^tD_2)^{-1}B^t \\ \underline{Q} &= C_2^tC_2 - C_2^tD_2(D_2^tD_2)^{-1}D_2^tC_2 \end{aligned}$$

The state space realization of $G_1G_3^{-1}$ is then

$$G_1G_3^{-1} = \left(\begin{array}{c|c} A - B(D_2^tD_2)^{-1}(D_2^tC_2 + B^tX) & B(D_2^tD_2)^{-\frac{1}{2}} \\ \hline C_1 & 0 \end{array}\right) \tag{2}$$

This function is stable. So, we can easily calculate the H_2-norm. This yields:

$$\left\| \frac{G_1}{G_2} \right\|_{Q-2}^2 = trace\left\{C_1QC_1^t\right\}$$

where Q is the solution of

$$A_zQ + QA_z^t + B(D_2^tD_2)^{-1}B^t = 0 \tag{3}$$

A_z is defined as

$$A_z = A - B(D_2^tD_2)^{-1}(D_2^tC_2 + B^tX)$$

If $D_2 + C_2(sI - A)^{-1}B$ is not a minimal state space realization of G_2, then first a minimal realization has to be obtained. (1) is still valid using for A, B, C_2 and D_2 a minimal realization of G_2. However, (2) and (3) cannot be used any more. For a state space realization of the spectral factor G_3, see e.g. Francis (1987).

Example:

Let G_1 and G_2 be 2×2 transfer functions with the following state space realization:

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right) = \left(\begin{array}{cccc|cc} -1.7248 & -0.5282 & 0.4624 & 1.4348 & -1.2735 & 175.6686 \\ 1.5608 & -3.0347 & 1.6567 & 0.1683 & -231.5284 & 122.1187 \\ 0.6891 & -0.6996 & -2.4053 & 0.6253 & -154.1046 & -31.8661 \\ -1.1154 & 0.5728 & 0.5063 & -0.1347 & -37.6361 & -29.0237 \\ \hline -166.7150 & 149.5111 & -76.6974 & 102.4094 & 0 & 0 \\ 161.0284 & 54.7933 & -84.4153 & -13.9756 & 0 & 0 \\ -1.2019 & -1.1340 & 0.1163 & -0.5071 & 0.4003 & 1.4079 \\ -0.2919 & -0.0209 & -0.6422 & -0.6246 & -0.6395 & 0.6615 \end{array}\right)$$

Applying the procedure described above, the H_{Q-2}-norm is 4.5521e+03.

5 The $L_{Q-\infty}$-norm and $H_{Q-\infty}$-norm

The L_∞-norm and H_∞-norm can also be generalized for the ratio of the outputs of 2 transfer functions. A level set algorithm to calculate the norm is also described.

5.1 Definition

Definition 3 *If $G_1(s)$ and $G_2(s)$ are two transfer matrices with an equal number of inputs, $G_1(s)$ has no imaginary poles and $G_2(s)$ is of full column rank on the $j\omega$-axis, then the $L_{Q-\infty}$-norm of the two transfer matrices is defined as*

$$\left\| \frac{G_1}{G_2} \right\|_{Q-\infty} = \sup_{\omega \in \mathbf{R}} \sigma_{q1}\left(G_1(j\omega), G_2(j\omega)\right)$$

Definition 4 *If $G_1(s)$ and $G_2(s)$ are two transfer matrices with an equal number of inputs, $G_1(s)$ is stable and $G_2(s)$ is stable and of full column rank on the right half plane, then the $H_{Q-\infty}$-norm of the transfer matrices is defined as*

$$\left\| \frac{G_1}{G_2} \right\|_{Q-\infty} = \sup_{\Re(s) \geq 0} \sigma_{q1}\left(G_1(s), G_2(s)\right)$$

When G_1 is stable and all the zeros of G_2 are in the open left half plane, then the $L_{Q-\infty}$ and $H_{Q-\infty}$-norm coincide.
The $H_{Q-\infty}$-norm is the largest ratio of the energies of the output signals, over all frequencies.
In this section we assume that G_2 is full rank on the $j\omega$-axis. Otherwise one of the Q-singular values would become $\frac{1}{0}$ or $\frac{0}{0}$. This would make the $H_{Q-\infty}$-norm ∞ or undefined. This property also corresponds to the condition that $G_2 \in \mathrm{RH}_\infty$. The functions have to be analytic in the right half plane.

Because G_2 is of full column rank, the Q-singular values will be equal to the singular values of $(G_1^t G_1 (G_2^t G_2)^{-1})^{\frac{1}{2}}$. Thus, all the properties on the singular values of a transfer matrix of Boyd and Balakrishnan (1990) are also valid for the Q-singular values, under the usual conditions. Some of these properties are (Boyd and Balakrishnan 1990, De Moor and Boyd 1989):

1. If $\sigma_{qi}\left(G_1(j\omega_0), G_2(j\omega_0)\right) > 0$ and has multiplicity one, then $\sigma_{qi}\left(G_1(j\omega), G_2(j\omega)\right) > 0$ is *real analytic* near ω_0.

2. There are real analytic functions f_i: $\mathbf{R} \to \mathbf{R}, i = 1, \ldots, m$, such that for all $\omega \in \mathbf{R}$,

   $$\{\sigma_{q1}\left(G_1(j\omega), G_2(j\omega)\right), \ldots, \sigma_{qm}\left(G_1(j\omega), G_2(j\omega)\right)\} = \{|f_1(\omega)|, \ldots, |f_m(\omega)|\}$$

   These functions f_i can be considered as *unordered, unsigned* 'Q-singular values'. We will illustrate this with an example.

Example:
The first plot in figure 10 shows the Q-singular values of a pair of transfer matrices. The functions $\sigma_{qi}(j\omega)$ are continuous but non-differentiable functions. The second plot shows continuous differentiable functions that can be interpreted as unordered, unsigned Q-singular values.

3. If $\sigma_{q1}\left(G_1(j\omega), G_2(j\omega)\right)$ has a local maximum at ω_M, then near ω_M, it has a continuous second derivative.

The H_∞-norm of a transfer function is smaller than γ, if there exists a positive definite solution to a certain Riccati equation. This is extended to the $H_{Q-\infty}$-norm.

Lemma 3 *Consider*

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline C_1 & 0 \\ C_2 & D_2 \end{array}\right)$$

If $G_2(s)$ is of full column rank and (A, C) is observable and (A, B) is controllable, where

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Then the following statements are equivalent:

(i)

$$\left\| \frac{G_1(s)}{G_2(s)} \right\|_{Q-\infty} < \gamma \tag{4}$$

(ii) $\exists X = X^t > 0$ and $\exists P = P^t > 0$ satisfying the Riccati equations

$$X\underline{A} + \underline{A}^t X - X\underline{R}X + \underline{Q} = 0 \tag{5}$$

$$PA_t + A_t^t P + \gamma^{-2} PB\underline{D}^{-1}B^t P + C_1^t C_1 = 0 \tag{6}$$

where

$$\begin{aligned} \underline{A} &= A - B\underline{D}^{-1}D_2^t C_2 \\ \underline{R} &= B\underline{D}^{-1}B^t \\ \underline{Q} &= C_2^t(I - D_2\underline{D}^{-1}D_2^t)C_2 \\ \underline{D} &= D_2^t D_2 \\ A_t &= A - B\underline{D}^{-1}(D_2^t C_2 + B^t X) \end{aligned}$$

such that $A_t + \gamma^{-2}B\underline{D}^{-1}B^t P$ is stable.

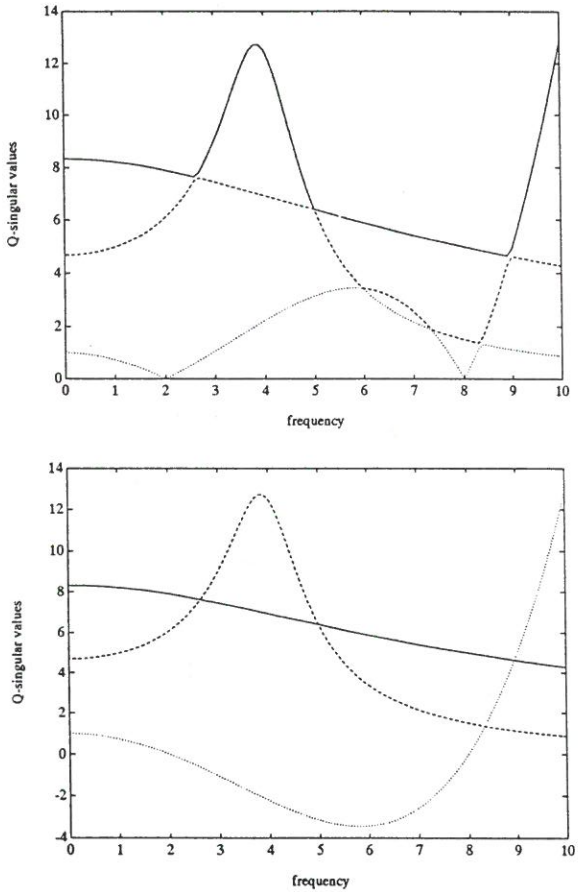

Figure 10: In the first plot the Q-singular values of a transfer matrix pair are plotted. The second plot are real analytic functions f_i, such that the set of absolute values at each frequency is equal to the Q-singular values. Observe that the functions f_i are allowed to become negative.

(iii) $\exists X = X^t > 0$ *and* $\exists S = S^t > 0$ *satisfying the Riccati equations*

$$X\underline{A} + \underline{A}^t X - X\underline{R}X + \underline{Q} = 0 \tag{7}$$

$$A_t S + S A_t^t + \gamma^{-2} S C_1^t C_1 S + B\underline{D}^{-1} B^t = 0 \tag{8}$$

such that $A_t + \gamma^{-2} S C_1^t C_1$ *is stable.*

Proof: A direct extension of the H_∞-case because we can find an invertible spectral factor G_3 of $G_2^* G_2$.

□

In discrete time similar definitions can be given. The $j\omega$-axis is replaced by the unit circle and the right half plane by the region outside the unit disk.

5.2 A level set algorithm for $L_{Q-\infty}$-norm

In Boyd and Balakrishnan (1990) a quadratically convergent algorithm for computing the L_∞-norm is described. This algorithm can be generalized to compute the $L_{Q-\infty}$-norm.

The main idea exploited in Boyd and Balakrishnan (1990) is the following:
Let σ be a singular value of the transfer matrix $G(j\omega)$ at a certain frequency ω. Then:

$$\begin{aligned} (D + C(j\omega I_n - A)^{-1}B)v &= u\sigma \\ (D^t + B^t(-j\omega I_n - A^t)^{-1}C^t)u &= v\sigma \end{aligned}$$

for two vectors u and v. The idea is now to determine the *level set* of σ, i.e. the set of values ω for which σ is a singular value. Further it is shown that the level set follows from the pure imaginary eigenvalues of a certain Hamiltonian matrix.

Here, we will follow a similar approach. Let (α, β) with $\alpha^2 + \beta^2 = 1$ be a quotient singular value pair of the pair of transfer matrices $G_1(j\omega)$ and $G_2(j\omega)$ at a certain frequency ω where

$$G(j\omega) = \begin{pmatrix} G_1(j\omega) \\ G_2(j\omega) \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (j\omega I_n - A)^{-1} B$$

Then, from Lemma 1 we find that there must exist 3 vectors u, v and x such that

$$\begin{aligned} (D_1 + C_1(j\omega I_n - A)^{-1}B)x &= u\alpha \\ (D_2 + C_2(j\omega I_n - A)^{-1}B)x &= v\beta \\ (D_1^t + B^t(-j\omega I_n - A^t)^{-1}C_1^t)u\beta &= (D_2^t + B^t(-j\omega I_n - A^t)^{-1}C_2^t)v\alpha \end{aligned}$$

with $\alpha^2 + \beta^2 = 1$. Now call

$$\begin{aligned} y &= (j\omega I_n - A)^{-1}Bx \\ p &= (-j\omega I_n - A^t)^{-1}C_1^t u \\ q &= (-j\omega I_n - A^t)^{-1}C_2^t v \end{aligned}$$

then

$$\begin{pmatrix} -\alpha I & 0 & D_1 \\ 0 & -\beta I & D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -C_2 \\ -B^t\beta & B^t\alpha & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ y \end{pmatrix}$$

$$\begin{pmatrix} -j\omega I_n - A^t & 0 & 0 \\ 0 & -j\omega I_n - A^t & 0 \\ 0 & 0 & -j\omega I_n + A \end{pmatrix}\begin{pmatrix} p \\ q \\ y \end{pmatrix} = \begin{pmatrix} C_1^t & 0 & 0 \\ 0 & C_2^t & 0 \\ 0 & 0 & -B \end{pmatrix}\begin{pmatrix} u \\ v \\ x \end{pmatrix}$$

We would now like to eliminate the vector $(u^t\ v^t\ x)^t$ which obviously requires the non-singularity of the matrix

$$\begin{pmatrix} -\alpha I & 0 & D_1 \\ 0 & -\beta I & D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}$$

It is straightforward to show that this matrix is nonsingular if (α, β) is not a quotient singular value pair of (D_1, D_2). This can be seen as follow. Let the **QSVD** of D_1, D_2 be given as

$$\begin{aligned} D_1 &= U_1 S_1 X^{-1} \\ D_2 &= U_2 S_2 X^{-1} \end{aligned}$$

where (α_i, β_i) are the quotient singular value pairs. Then

$$\begin{pmatrix} -\alpha I & 0 & D_1 \\ 0 & -\beta I & D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix} = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & X^{-t} \end{pmatrix}\begin{pmatrix} -\alpha I & 0 & S_1 \\ 0 & -\beta I & S_2 \\ S_1^t\beta & -S_2^t\alpha & 0 \end{pmatrix}\begin{pmatrix} U_1^t & 0 & 0 \\ 0 & U_2^t & 0 \\ 0 & 0 & X^{-1} \end{pmatrix}$$

While the left and right factors are clearly non-singular, the determinant of the middle factor will be a product of determinants of the form

$$det\begin{pmatrix} -\alpha & 0 & \alpha_i \\ 0 & -\beta & \beta_i \\ \alpha_i\beta & -\beta_i\alpha & 0 \end{pmatrix} = \alpha_i^2\beta^2 - \alpha^2\beta_i^2$$

Hence, the determinant is only zero when (α, β) coincides with a quotient singular value pair of D_1, D_2.

Elimination of the vector $(u^t\ v^t\ x^t)^t$ now results in the eigenvalue problem

$$\left[\begin{pmatrix} A^t & 0 & 0 \\ 0 & A^t & 0 \\ 0 & 0 & -A \end{pmatrix} + \begin{pmatrix} C_1^t & 0 & 0 \\ 0 & C_2^t & 0 \\ 0 & 0 & -B \end{pmatrix}\begin{pmatrix} -\alpha I & 0 & D_1 \\ 0 & -\beta I & D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -C_2 \\ -B^t\beta & B^t\alpha & 0 \end{pmatrix}\right]\begin{pmatrix} p \\ q \\ y \end{pmatrix}$$

$$= \begin{pmatrix} p \\ q \\ y \end{pmatrix}(-j\omega)$$

Define M_γ as

$$M_\gamma = \begin{pmatrix} A^t & 0 & 0 \\ 0 & A^t & 0 \\ 0 & 0 & -A \end{pmatrix} + \begin{pmatrix} C_1^t & 0 & 0 \\ 0 & C_2^t & 0 \\ 0 & 0 & -B \end{pmatrix}\begin{pmatrix} -\alpha I & 0 & D_1 \\ 0 & -\beta I & D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -C_2 \\ -B^t\beta & B^t\alpha & 0 \end{pmatrix} \tag{9}$$

The pure imaginary eigenvalues of M_γ are the frequencies at which there is a Q-singular value $\gamma = \frac{\alpha}{\beta}$. In other words: $-j\omega$ is an eigenvalue of

$$M_\gamma x = -j\omega x$$

if and only if for some i, $\sigma_{qi}(G_1(j\omega), G_2(j\omega)) = \gamma$.

A similar derivation applies in the case of discrete transfer matrices. Then a generalized eigenvalue problem has to be solved:

Define the matrices:

$$V_\gamma = \begin{pmatrix} I & 0 & 0 \\ 0 & A^t & 0 \\ 0 & 0 & A^t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ C_1^t & 0 & 0 \\ 0 & C_2^t & 0 \end{pmatrix}\begin{pmatrix} \alpha I & 0 & -D_1 \\ 0 & \beta I & -D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1}\begin{pmatrix} C_1 & 0 & 0 \\ C_2 & 0 & 0 \\ 0 & -B^t\beta & B^t\alpha \end{pmatrix}$$

$$W_\gamma = \begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha I & 0 & -D_1 \\ 0 & \beta I & -D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1}\begin{pmatrix} C_1 & 0 & 0 \\ C_2 & 0 & 0 \\ 0 & -B^t\beta & B^t\alpha \end{pmatrix}$$

Suppose $\gamma = \frac{\alpha}{\beta}$ is not a Q-singular value of the matrix pair (D_1, D_2), $\gamma \in \mathbf{R}$, and T is the sampling time. Then $e^{j\omega T}$ is a generalized eigenvalue of

$$V_\gamma x = e^{j\omega T} W_\gamma x$$

if and only if for some i, $\sigma_{qi}(G_1(e^{j\omega T}), G_2(e^{j\omega T})) = \gamma$.

As in Boyd and Balakrishnan (1990) the level set described above of can be used in a quadratically convergent algorithm for computing the $L_{Q-\infty}$-norm.

Example:
Take the same transfer matrices as in the example of section 4.2. We want to find the $H_{Q-\infty}$-norm with the quadratically convergent algorithm. The algorithm looks for the frequency intervals where the largest quotient value is larger than the level γ. At the midpoints of these intervals, the largest **QSV** is calculated. The maximum **QSV** at all the midpoints is taken as the new level γ for a new iteration. The solution was found in 3 iterations within a relative accuracy of $1e-4$. These are shown in figure 11. The first plot just shows the quotient singular values as function of the frequency. The next plots show the different iterations. The dotted line is level γ. The first level is at the $\sigma_{q1}(0)$. The plus-signs indicate the calculated largest **QSV** at the midpoint of an interval. The plus-sign of the maximum **QSV**s at all the midpoints is replaced by a star. The resulting $L_{Q-\infty}$-norm is 508.9. This occurs at $\omega = \omega_m = 136.9$ rad/sec. Using the **QSVD** the direction of the worst case input signal can be found:

$$x_1 = \begin{pmatrix} 0.0026 - 0.0015i \\ 0.0023 + 0.0005i \end{pmatrix}$$

From ω_m and x_1 we find that the worst case input signal is:

$$u(t) = \begin{pmatrix} 0.0030\sin(136.9t - 0.5143) \\ 0.0024\sin(136.9t + 0.2069) \end{pmatrix}$$

6 Conclusions

We have shown how the **QSVD** can be used to define norms on transfer matrices, relative to other transfer matrices. The H_2-norm was extended to the H_{Q-2}-norm, and the H_∞-norm to the $H_{Q-\infty}$-norm. Two different situations are possible:

1. The two systems have the same input but different outputs.
2. The two systems have different inputs and the global output is the sum of the two individual outputs.

The H_{Q-2}-norm is a measure for the energy in the first output signal relative to the energy in the second signal, when white noise is present at the input (situation 1). In that case the $H_{Q-\infty}$-norm is the worst case energy amplification relative to energy amplification of the second system.

For the second situation, the norms give the relative influence of the first input on the output compared to the influence of the second input.

The **QSVD** is only one member of a whole set of decompositions of two and more matrices (De Moor and Golub 1989). So, one might also define other norms for more than 2 transfer matrices. However, in that case the interpretation of the result gets complicated.

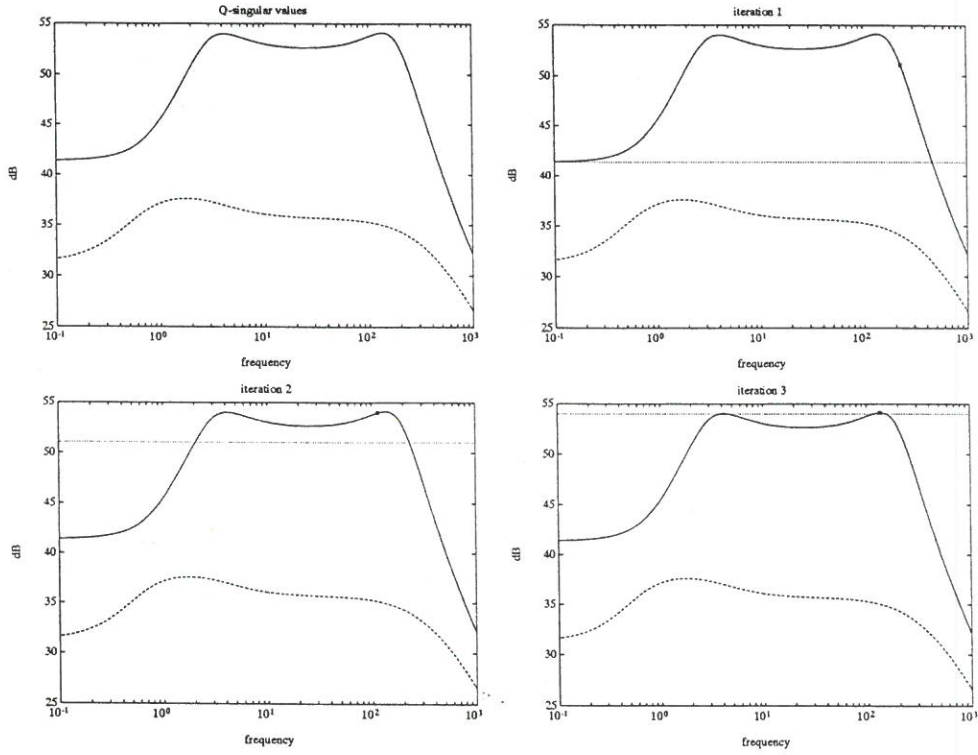

Figure 11: An illustration of the quadratically convergent algorithm for calculating the $H_{Q-\infty}$-norm. The first plot shows the Q-singular values. The next plots show the algorithm at work. The dotted line is the level γ. The plus-signs indicate the largest quotient singular values at the midpoints of the intervals. The plus-sign the maximum quotient singular value at all the midpoints is replaced by a star. Only at the third iteration, there are 2 intervals. The largest **QSV**s at the midpoints are indicated by the plus-signs. The maximum of the two is replaced by a star.

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