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A Nonlinear Control Scheme for Helix (Internal Report)

Yi Cheng Bart De Moor

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The proposed nonlinear control scheme is based on the state space exact linearization method. The controller consists mainly two parts: state space linearization and model following.

1 Helix Model

The Helix model is based on [1] and can be rewritten as following:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

where vector x is,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \alpha \\ \dot{\alpha} \\ \theta \\ \dot{\theta} \\ F_\alpha \\ F_\theta \end{pmatrix} \tag{3}$$

vector field $f(x)$ is,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \end{pmatrix} = \begin{pmatrix} x_2 \\ a_1 \cos(x_1) + a_2 \cos(x_1) \sin(x_1) x_4^2 + a_3 x_2 + a_4 x_5 \\ x_4 \\ a_5 x_4 / \cos^2(x_1) + a_6 x_6 / \cos(x_1) \\ a_7 x_5 \\ a_8 x_6 \end{pmatrix} \tag{4}$$

the input matrix $g(x)$ is

$$g = (g_1 \quad g_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & a_{10} \end{pmatrix} \tag{5}$$

where

$$\begin{aligned} a_1 &= -M_w g l_m / J_\alpha \\ a_2 &= -1 \\ a_3 &= -d_\alpha / J_\alpha \\ a_4 &= a_6 = l / J_\alpha \\ a_5 &= -d_\theta / J_\alpha \\ a_7 &= -1/\tau_{m\alpha} \\ a_8 &= -1/\tau_{m\theta} \\ a_9 &= k_\alpha / \tau_{m\alpha} \\ a_{10} &= k_\theta / \tau_{m\theta} \end{aligned} \tag{6}$$

and $h(x)$ is

$$h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \theta \end{pmatrix} \tag{7}$$

2 State space linearization

The theoretical background of state space linearization may be found in the literature[2]. A rather intuitive way is used here to derive the state space feedback controller for Helix. The method is to differentiate the output y w.r.t time until the the input u appears in the derivatives of y.
First, differentiating output y in (2) and using (1), we have

$$\dot{y} = \frac{\partial h(x)}{\partial x}\dot{x} = \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x) u \tag{8}$$

By defining operator(Lie derivative of λ w.r.t f)

$$L_f \lambda(x) = \sum_{i=1}^{n} \frac{\partial \lambda(x)}{\partial x_i} f_i(x) \tag{9}$$

both λ and f defined on R^n, Eq. (8) becomes

$$\dot{y} = \begin{pmatrix} L_f h_1(x) \\ L_f h_2(x) \end{pmatrix} + \begin{pmatrix} L_{g_1} h_1(x) & L_{g_2} h_1(x) \\ L_{g_1} h_2(x) & L_{g_2} h_2(x) \end{pmatrix} u \tag{10}$$

Using Eq. (4,5), we obtain

$$L_f h_1(x) = \sum_{i=1}^{6} \frac{\partial h_1(x)}{\partial x_i} f_i = x_2 \tag{11}$$

$$L_f h_2(x) = \sum_{i=1}^{6} \frac{\partial h_2(x)}{\partial x_i} f_i = x_4 \tag{12}$$

$$\begin{array}{l} L_{g_1} h_1(x) = \sum_{i=1}^{6} \frac{\partial h_1(x)}{\partial x_i} g_{1i} = 0 \\ L_{g_1} h_2(x) = \sum_{i=1}^{6} \frac{\partial h_2(x)}{\partial x_i} g_{1i} = 0 \\ L_{g_2} h_1(x) = \sum_{i=1}^{6} \frac{\partial h_1(x)}{\partial x_i} g_{2i} = 0 \\ L_{g_2} h_2(x) = \sum_{i=1}^{6} \frac{\partial h_2(x)}{\partial x_i} g_{2i} = 0 \end{array} \tag{13}$$

Note that $L_{g_i} h_j(x) = 0$, $1 \leq i, j \leq 2$, so

$$\dot{y} = \begin{pmatrix} L_f h_1(x) \\ L_f h_2(x) \end{pmatrix} \tag{14}$$

and we need to differentiate $\dot{y}$ further.

As above, differentiating Eq. 14 and using Eq. (1), we have

$$y^{(2)} = \frac{\partial L_f h(x)}{\partial x}[f(x) + g(x)u] = \begin{pmatrix} L_f^2 h_1(x) \\ L_f^2 h_2(x) \end{pmatrix} + \begin{pmatrix} L_{g_1} L_f h_1(x) & L_{g_2} L_f h_1(x) \\ L_{g_1} L_f h_2(x) & L_{g_2} L_f h_2(x) \end{pmatrix} u \tag{15}$$

With Eq. (4,5,11,12,13), following results can be obtained

$$L_f^2 h_1(x) = \sum_{i=1}^{6} \frac{\partial L_f h_1(x)}{\partial x_i} f_i = a_1 \cos(x_1) + a_2 \cos(x_1) sin(x_1) x_4^2 + a_3 x_2 + a_4 x_5 \tag{16}$$

$$L_f^2 h_2(x) = \sum_{i=1}^{6} \frac{\partial L_f h_2(x)}{\partial x_i} f_i = a_5 \frac{x_4}{\cos^2(x_1)} + a_6 \frac{x_6}{\cos(x_1)} \tag{17}$$

$$\begin{array}{l} L_{g_1} L_f h_1(x) = \sum_{i=1}^{6} \frac{\partial L_f h_1(x)}{\partial x_i} g_{1i} = 0 \\ L_{g_1} L_f h_2(x) = \sum_{i=1}^{6} \frac{\partial L_f h_2(x)}{\partial x_i} g_{1i} = 0 \\ L_{g_2} L_f h_1(x) = \sum_{i=1}^{6} \frac{\partial L_f h_1(x)}{\partial x_i} g_{2i} = 0 \\ L_{g_2} L_f h_2(x) = \sum_{i=1}^{6} \frac{\partial L_f h_2(x)}{\partial x_i} g_{2i} = 0 \end{array} \tag{18}$$

Note that $L_{g_i} L_f h_j(x) = 0$, $1 \leq i, j \leq 2$, so

$$y^{(2)} = \begin{pmatrix} L_f^2 h_1(x) \\ L_f^2 h_2(x) \end{pmatrix} \tag{19}$$

and we need again to differentiate $y^{(2)}$.

Follow the steps as above, it is easy to get following results for $y^{(3)}$

$$y^{(3)} = \begin{pmatrix} L_f^3 h_1(x) \\ L_f^3 h_2(x) \end{pmatrix} + \begin{pmatrix} L_{g_1} L_f^2 h_1(x) & L_{g_2} L_f^2 h_1(x) \\ L_{g_1} L_f^2 h_2(x) & L_{g_2} L_f^2 h_2(x) \end{pmatrix} u \tag{20}$$

where

$$L_f^3 h_1(x) = \sum_{i=1}^{6} \frac{\partial L_f^2 h_1(x)}{\partial x_i} f_i = [-a_1 \sin(x_1) + a_2 \cos(2x_1) x_4^2][x_2]$$

$$+[a_3][a_1 \cos(x_1) + a_2 \sin(x_1) \cos(x_1) x_4^2 + a_4 x_5 + a_3 x_2]$$

$$+[2a_2 \cos(x_1) sin(x_1) x_4][a_5 \frac{x_4}{\cos^2(x_1)} + a_6 \frac{x_6}{\cos(x_1)} + [a_4][a_7 x_5]$$

$$L_f^3 h_2(x) = \sum_{i=1}^{6} \frac{\partial L_f^2 h_2(x)}{\partial x_i} f_i = [a_5 \frac{x_4 \sin(x_1)}{\cos^3(x_1)} + a_6 \frac{x_6 \sin(x_1)}{\cos^2(x_1)}][x_2]$$

$$+[a_5 \frac{1}{\cos^2(x_1)}][a_5 \frac{x_4}{\cos^2(x_1)} + a_6 \frac{x_6}{\cos(x_1)}] + [a_6 \frac{1}{\cos(x_1)}][a_8 x_6]$$

$$L_{g_1} L_f^2 h_1(x) = \sum_{i=1}^{6} \frac{\partial L_f^2 h_1(x)}{\partial x_i} g_{1i} = a_4 a_9$$

$$L_{g_2} L_f^2 h_1(x) = \sum_{i=1}^{6} \frac{\partial L_f^2 h_1(x)}{\partial x_i} g_{2i} = 0$$

$$L_{g_1} L_f^2 h_2(x) = \sum_{i=1}^{6} \frac{\partial L_f^2 h_2(x)}{\partial x_i} g_{1i} = 0$$

$$L_{g_2} L_f^2 h_2(x) = \sum_{i=1}^{6} \frac{\partial L_f^2 h_2(x)}{\partial x_i} g_{2i} = [a_6 \frac{1}{\cos(x_1)}][a_{10}]$$

Finally, we have $L_{g_1} L_f^2 h_1(x) \neq 0$ and $L_{g_2} L_f^2 h_2(x) \neq 0$, and more than that, if defining

$$A = \begin{pmatrix} L_{g_1} L_f^2 h_1(x) & L_{g_2} L_f^2 h_1(x) \\ L_{g_1} L_f^2 h_2(x) & L_{g_2} L_f^2 h_2(x) \end{pmatrix} = \begin{pmatrix} a_4 a_9 & 0 \\ 0 & a_6 a_{10} \frac{1}{\cos(x_1)} \end{pmatrix}$$

then A is nonsingular.

The last step is to take the control law as following:

$$u = A^{-1} \begin{pmatrix} -L_f^3 h_1(x) + v_1 \\ -L_f^3 h_2(x) + v_2 \end{pmatrix} \tag{21}$$

and put it into Eq. (20), then

$$y^{(3)} = v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{22}$$

Now we have already linearized the system with new input v. To show that, take coordinates transformation:

$$z_{ij} = L_f^{j-1} h_i(x), 1 \leq j \leq 3, 1 \leq i \leq 2$$

then we have following linear system:

$$\begin{aligned} \dot{z}_{11} &= z_{12} \\ \dot{z}_{12} &= z_{13} \\ \dot{z}_{13} &= v_1 \\ \dot{z}_{21} &= z_{22} \\ \dot{z}_{22} &= z_{23} \\ \dot{z}_{23} &= v_2 \end{aligned} \tag{23}$$

and

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_{11} \\ z_{21} \end{pmatrix} \tag{24}$$

Remark 1: The linearization method used here is for local linearization. However, as we have seen, the conditions for linearization: $L_{g_i} L_f^j h_k(x) = 0$ for $1 \leq i, j, k \leq 2$ and A is nonsingular are satisfied almost everywhere for x on R^6 except $x_1 = n\pi + \pi/2$, $n = 1, 2, ...$ (where the system is not well-defined). So the linearization for Helix is global.

Remark 2: From Eq.(23,24), we can see that the system is not only linearized but also decoupled completely such that the control design for the linearized system may be down by designing a controller for each SISO subsystem.

3 Model Following Control Design

Let system-1(with input v_1 and output y_1) and system-2 (with input v_2 and y_2 follow two models as following respectively:

$$y_{m1}^{(3)} + a_{11} y_{m1}^{(2)} + a_{12} y_{m1}^{(1)} + a_{13} y_{m1} = b_1 u_{c1} \tag{25}$$

$$y_{m2}^{(3)} + a_{21} y_{m2}^{(2)} + a_{22} y_{m2}^{(1)} + a_{23} y_{m2} = b_2 u_{c2} \tag{26}$$

where u_{c1} and u_{c2} are reference inputs. Take the control laws:

$$v_1 = y_{m1}^{(3)} + a_{11}(y_{m1}^{(2)} - L_f^2 h_1(x)) + a_{12}(y_{m1}^{(1)} - L_f h_1(x)) + a_{13}(y_{m1} - y_1) \tag{27}$$

$$v_2 = y_{m2}^{(3)} + a_{21}(y_{m2}^{(2)} - L_f^2 h_2(x)) + a_{22}(y_{m2}^{(1)} - L_f h_2(x)) + a_{23}(y_{m2} - y_2) \tag{28}$$

Define $e_1 = y_1 - y_{m1}$ and $e_2 = y_2 - y_{m2}$, from Eq. (22,27,28), we have two error models:

$$e_1^{(3)} + a_{11} e_1^{(2)} + a_{12} e_1^{(1)} + a_{13} e_1 = 0$$

$$e_2^{(3)} + a_{21}e_2^{(2)} + a_{22}e_2^{(1)} + a_{23}e_1 = 0$$

These error models show that if the given reference models are asymptotically stable then the errors between the outputs of the models and the outputs of the controlled systems will asymptotically go to zero.

Combining Eq. (27,28) with Eq. (21), the total control law is:

$$u = A^{-1}\begin{pmatrix} y_{m1}^{(3)} - L_f^3 h_1(x) + a_{11}(y_{m1}^{(2)} - L_f^2 h_1(x)) + a_{12}(y_{m1}^{(1)} - L_f h_1(x)) + a_{13}(y_{m1} - y_1) \\ y_{m2}^{(3)} - L_f^3 h_2(x) + a_{21}(y_{m2}^{(2)} - L_f^2 h_2(x)) + a_{22}(y_{m2}^{(1)} - L_f h_2(x)) + a_{23}(y_{m2} - y_2) \end{pmatrix} \tag{29}$$

4 Simulation

Simulation for Helix and control system is done with MATRIXX. The parameters in Helix model are chosen as $J_\alpha = 0.5$, $l = 0.62$, $M_m g l_m = 3$, $d_\alpha = d_\theta = 0.1$, $\tau_{m\alpha} = \tau_{m\theta} = 0.5$ and $K_\alpha = K_\theta = 1$. The two models are chosen as

$$H_1(s) = H_2(s) = \frac{64}{(s+4)^3} \tag{30}$$

Fig. 1-4 show the simulation results. Helix outputs in Fig. 1 and Fig. 2 exactly follow the models without any transients because the initial states of the system are chosen such that $y_i^{(j)} = y_{mi}^{(j)} = 0$, $1 \le i \le 2, 0 \le j \le 2$.

References

[1] The helix project, the control group, April 3,1992.

[2] Alberto Isodori, *Nonlinear Control Systems An Introduction*, 1989.