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The opposite of analytic centering for solving minimum rank problems in control and identification *

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Abstract

Recently it has been shown that certain control problems can be formulated as minimum rank problems with linear matrix inequalities. Also in identification there are problems that can be formulated in a similar way. In this paper a heuristic algorithm for solving these problems is presented. It is illustrated with some numerical examples.

keywords: minimum rank problems, low order controller design, linear matrix inequalities

1 Introduction

The last few years it has been shown that a lot of problems in control can be formulated as convex optimization problems, which have the advantage that there exists only one minimum. Fast algorithms exist for solving them ([2],[6]). However, most output feedback problems cannot be formulated as convex optimization problems unless an approximation of the controller is used ([1]). Recently, however, El Ghaoui and Gahinet have shown how some of these problems can be reformulated as minimum rank problems with linear matrix inequalities ([4]). An additional advantage is that the order of the controller needs not be specified a priori.
In this paper we present a heuristic algorithm to solve these problems. It is based on concepts of convex optimization with positive definite linear matrix inequalities. However, the updating direction is the opposite of the one used for convex optimization. In fact, looking for the minimum rank can be stated as searching for the maximum over a convex set.
Not only problems in control can be formulated in this way. A well-known problem in system identification is the Frisch scheme ([5]). There the rank has to be minimized by subtracting a diagonal matrix with positive elements from a given matrix. These problems can easily be solved using the proposed algorithm. An example is shown.
An outline of the algorithm is presented in section 2. An example of the Frisch scheme is given in section 3. In section 4, an example is shown for the design of an H_∞ controller. The conclusions are summarized in section 5.

2 The algorithm

The problem can be formulated as follows:

$$\min_{x \in \mathbb{R}^n} \text{Rank } F(x) \quad \text{subject to } F(x) \geq 0, C(x) > 0 \quad (1)$$

where

$$\begin{aligned} F(x) &= F_0 + x_1 F_1 + x_2 F_2 + \ldots + x_n F_n \\ C(x) &= C_0 + x_1 C_1 + x_2 C_2 + \ldots + x_n C_n \end{aligned}$$

with $F_i = F_i^t$ and $C_i = C_i^t$ for $i = 1 \ldots n$.
Of course, the rank of a matrix is a discontinuous function of its elements.
Consider first the problem when there are no constraints $C(x) > 0$. As the heuristic method to solve the minimum rank is mainly an inversion of an analytic center method for convex optimization, we'll first explain the main concepts of the latter. In [2] Boyd and El Ghaoui consider convex optimization under LMI constraints. $F(x)$ has to be in the cone of positive definite matrices. Therefore, the following barrier function is introduced: $-\log\det F(x)$. The minimum of this function is called the analytic center of the set of inequalities represented by $F(x) \geq 0$. It is also shown how the Newton direction can be easily computed by solving a least squares problem.
This barrier function has some nice properties for convex optimization ([6]). However, it can also be used to find the minimum rank. Assume that $F(x) \in \mathbb{R}^{p \times p}$ has $p-1$ eigenvalues 1 and one small eigenvalue $\epsilon > 0$. Then $-\log\det F(x) = -\log\epsilon$. If there are $p-2$ eigenvalues equal to 1 and 2 small eigenvalues ϵ_1 and ϵ_2 then $-\log\det F(x) = -\log\epsilon_1 - \log\epsilon_2$, which is larger. Thus the higher the value of the barrier function, the lower the rank of $F(x)$. The intuitive idea is then to solve minimum rank problems by finding the maximum of the barrier function. As the optimization is done over the set of positive (semi)definite matrices, the optimal solutions will be at the boundaries.
The outline of the algorithm is very basically as follows: The algorithm starts at the analytic center of the constraints, which can be computed as explained in [2]. The analytic center is then slightly perturbed. To find the updating direction, the Newton method is used as explained in [2]. However, instead of going in the direction of the analytic center, the opposite direction is taken. In this way we try to trace back the path that comes from the highest point on the boundary. Using an appropriate stepsize, it can be guaranteed that the update is again feasible.
The algorithm is heuristic and the optimum is not always found. However, in many cases good results are obtained.

3 The Frisch scheme

The Frisch scheme can be formulated as follows ([3],[5]):

> Given a real positive definite symmetric $n \times n$ matrix A, find D nonnegative definite diagonal such that rank$(A - D)$ is minimized with $A - D \geq 0$.

The interpretation is that A is the covariance matrix of some data, while D is the noise covariance matrix, which is assumed to be diagonal (uncorrelated noise among channels), with different noise variance for each channel. We want to minimize the rank to find as many linear relations as possible between the measured channels.
This problem is fully understood when the minimum achievable rank is $n-1$. Necessary and sufficient conditions are described in [5]. However, the problem remains unsolved when the minimum rank is lower than $n-1$. No algorithm is available yet to solve these problems. It

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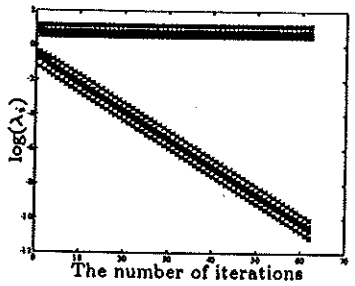


Figure 1: The eigenvalues values of $A - D$ versus the number of iterations for an $A \in \mathbb{R}^{8\times 8}$. The x-axis is the number of iterations of the algorithm outlined in section 2. The y-axis is the logarithm of the eigenvalues values. 5 eigenvalues tend to zero. Thus the resulting rank of $A - D$ is 3.

is obvious that the Frisch scheme can be formulated as a problem of the form (1).
We will illustrate this with an example. To check if the algorithm finds the correct solution we have constructed a full rank positive definite matrix A by adding positive elements to the diagonal of a positive semidefinite matrix of rank 3. A was an 8×8 matrix. The minimal rank of $A - D$ is thus 3.
In figure 1, the eigenvalues of $A - D$ are plotted as a function of the number of iterations of the algorithm described in section 2. 5 eigenvalues tend to 0. This implies that the resulting rank of $A - D$ is 3. The matrix D found by the algorithm is exactly the one added to the rank deficient matrix.

4 Low order H_∞ design

In [4] El Ghaoui and Gahinet show how the H_∞ control problem can be stated as a minimum rank problem:

> Consider a proper plant $P(s)$ of order n and with a minimal realization:
>
> $$P = \left(\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ \hdashline C_2 & D_{21} & 0 \end{array}\right)$$
>
> Under the assumptions of [4], a k^{th} order γ-suboptimal controller exists if and only if there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ such that
>
> $$AR + RA^t + RC_1^tC_1R + \gamma^{-2}B_1B_1^t - B_2B_2^t < 0$$
>
> $$A^tS + SA + SB_1B_1^tS + \gamma^{-2}C_1^tC_1 - C_2^tC_2 < 0$$
>
> $$\begin{pmatrix} R & \gamma^{-1}I \\ \gamma^{-1}I & S \end{pmatrix} \geq 0$$
>
> $$\text{Rank} \begin{pmatrix} R & \gamma^{-1}I \\ \gamma^{-1}I & S \end{pmatrix} \leq n + k$$

The quadratic matrix inequalities can easily be transformed in linear ones (e.g. [2]). It is clear that the order of the controller is not specified in advance. The order of the controller can differ with different values of γ. We will illustrate this with an example. A random stable 5^{th} order plant was constructed that satisfies the assumptions. The low order H_∞ controller was computed for several values of γ. In figure 2 the order of the controller is shown as a function of γ. From standard H_∞ techniques, we know that the lowest possible γ is 1.3573. It can be seen that the order of the controller increases as γ approaches the optimal (lowest) γ. When γ is large enough a static output feedback is sufficient to obtain the desired γ.

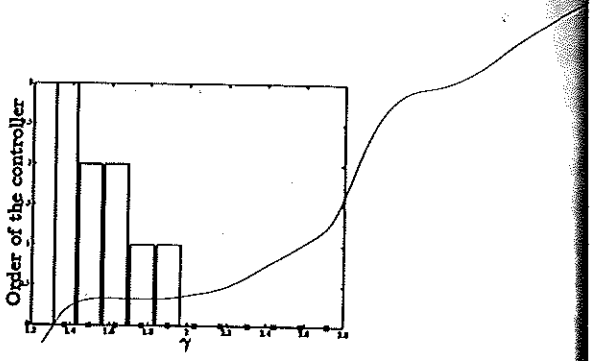


Figure 2: The order of the controller for different values of γ. The stars indicate the values of γ for which a low order H_∞ suboptimal controller is computed. A 0^{th} order controller is a static output feedback. Apparently, the higher the value of γ, the lower the necessary order of the controller is.

5 Conclusions

In this paper we have presented an algorithm for solving minimum rank problems subject to linear matrix inequalities. The algorithm is heuristic and based on some concepts of convex optimization. We don't claim that the algorithm will always find the optimum. However, in many cases it does. In any case, a reduced order solution is always found. Still, a lot of research has to be done. There are a lot of parameters that can be tuned and their influence is not known yet. Two examples were given, that show possible applications of the algorithm.

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