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ON A CLASS OF FRACTAL MATRICES. III. LIMIT STRUCTURES AND HIERARCHICAL ITERATED FUNCTION SYSTEMS

ANDRÉ M. BARBÉ*
Katholieke Universiteit Leuven, Department of Electrical Engineering, Kard. Mercierlaan 94, B-3001 Leuven, Belgium

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Fractrices (fractal matrices, excess-matrices) were defined in an earlier paper, Part I, as a special class of integer matrices with multifractal features. In contrast with Part II where a quantitative analysis of these objects was performed from a finitist constructivist point of view, the present Part III discusses the corresponding genuine limit fractal version. First this is done by a standard limit analysis which eventually leads to a (nonstandard) hyperreal-number view on this limit. Then a related hierarchical iterated function system is constructed whose attractor is the very same limit. This limit is either a hierarchical Zeno or Cantor set, or a space filling set. Its fractal dimension function is investigated.

1. Introduction

In Part I [Barbé, 1992] fractal matrices were introduced as objects obtained through the recursion

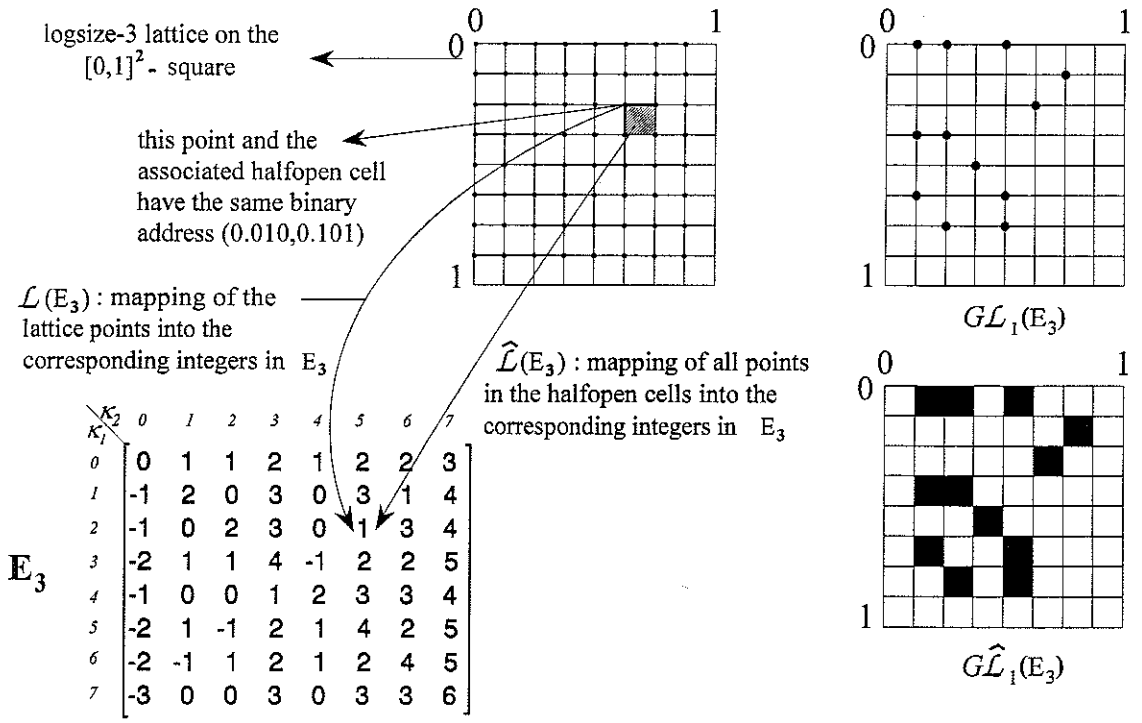
$$E_{m+1} = E_1 \oplus E_m\,, \quad m = 1, 2, 3, \ldots\,, \qquad (1)$$

where $\oplus$ is the bigsum operation[1] and E_1 is an integer-valued *generator matrix* of arbitrary dimension p and arbitrary size $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p)$, σ_i being the size in the ith dimension. Figure 1 shows E_3 for a two-dimensional example with size $\boldsymbol{\sigma} = (2, 2)$. E_m has size σ_i^m in direction i, m is the so-called *logsize* (logarithmic size) of E_m. E_m has elements $e_m(\boldsymbol{\kappa})$, where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_p)$ is a coordinate or address vector. Address counting starts at 0, and so $0 \leq \kappa_i < \sigma_i^m - 1$ for E_m. E itself refers to the matrix of unbounded size obtained when m keeps growing$^{(I)}$.[2]

The name "fractrix" was coined in Part II [Barbé, 1993] as a contraction between "fractal" and "matrix". As already mentioned in Part I, these objects form an extension of a class of similar objects (but restricted to two dimensions, while the matrix entries belong to a finite set) investigated earlier by Shallit & Stolfi [1989] and Salon [1987]. Von Haeseler *et al.* [1993] recently introduced related *substitution matrices* in the context of certain cellular automata, a field from which also emerged our occupation with fractrices, although from a totally different point of departure (Barbé [1989]).

Whereas Part I dealt with structural self-similar properties, Part II concentrated on fractal dimensions of these objects from a finitist constructivist point of view: the fractrices were considered as discrete objects that grew *outwardly* with increasing m. Part I already mentioned the so-called latticed

*Email: Andre.Barbe@esat.kuleuven.ac.be
[1]Recall: $A \oplus B$ is the matrix formed by replacing each element a in A by $a + B$, where $a + B$ is the matrix obtained by adding a to all elements of B.
[2]The superscripts (I) and (II) will be used as shorthand references to Part I and Part II respectively.



K_1 \ K_2	0	1	2	3	4	5	6	7
0	0	1	1	2	1	2	2	3
1	-1	2	0	3	0	3	1	4
2	-1	0	2	3	0	1	3	4
3	-2	1	1	4	-1	2	2	5
4	-1	0	0	1	2	3	3	4
5	-2	1	-1	2	1	4	2	5
6	-2	-1	1	2	1	2	4	5
7	-3	0	0	3	0	3	3	6

Fig. 1. E_3, $\mathcal{L}(E_3)$, $\hat{\mathcal{L}}(E_3)$, $G\mathcal{L}_1(E_3)$ and $G\hat{\mathcal{L}}_1(E_3)$ for a two-dimensional example with generator $E_1 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

versions $\mathcal{L}(E_m)$ of these matrices, geometrical objects obtained by squeezing a graphical representation of the E_m-matrices into the p-dimensional unit hypercube $[0, 1]^p$, thereby creating an *inwardly* growing fractrix for increasing m. We will give here strict definitions of $\mathcal{L}(E_m)$ and related graphical representations. This paper will deal with these objects when m becomes infinite. As might have become clear from Parts I and II, this is not a trivial matter.

Definitions 1.1. (Illustrated with a two-dimensional example in Fig. 1):

- The p-dimensional, $\boldsymbol{\sigma}$-adic logsize-m lattice L_m on $[0, 1]^p$ is the lattice with cell-size σ_i^{-m} in direction i, whose nodes have the σ_i-adic coordinate $0.c_{i1}c_{i2}__c_{im} \triangleq \sum_{j=1}^{m} c_{ij}\sigma_i^{-j}$ $(0 \le c_{ij} \le \sigma_i - 1)$ in direction i. The global address of an arbitrary lattice point is given by its p coordinates:

$$\begin{array}{lll} 1^{st} \text{ coordinate} & 0.c_{11}c_{12}__c_{1m}\,, & (0 \le c_{1j} < \sigma_1) \\ 2^{nd} \text{ coordinate} & 0.c_{21}c_{22}__c_{2m}\,, & (0 \le c_{2j} < \sigma_2) \\ \vdots & \vdots & \vdots \\ p^{th} \text{ coordinate} & 0.c_{p1}c_{p2}__c_{pm}\,. & (0 \le c_{pj} < \sigma_p) \end{array} \qquad (2)$$

  This is written in short as: $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m$ $(\mathbf{0} \le \mathbf{c}_j < \boldsymbol{\sigma})$, where $\boldsymbol{c}_j \triangleq$ columnvector $(c_{1j}, c_{2j}, \ldots, c_{pj})$($\boldsymbol{c}_j$ is the jth *conjunction vector* of the $\boldsymbol{\sigma}$-base representation of the point's address$^{(I)}$).
- Each logsize-m lattice point with address $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m$ also carries an *elementary cell*: the set of points with representation $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m***\ldots$, where $***\ldots$ denotes an arbitrary infinite sequence of conjunctions. So, Eq. (2) represents as well the address of this cell.
- $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m$ is the *m-resolution representation* of any point with real-number representation $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m***\ldots$
- $\mathcal{L}(E_m)$ is the mapping of all logsize-m lattice points into $\mathbb{Z}$ (the set of integers), according to

$$\boxed{\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m \to \sum_{j=1}^{m} e_1(\boldsymbol{c}_j)} \qquad (3)$$

  where $e_1(\boldsymbol{c}_j)$ is the element in the generator E_1 whose coordinates are given by the components of the conjunction vector $\boldsymbol{c}_j$ (cf. Part I, Property 2.3: fundamental decomposition property). The sum in Eq. (3) is called the *weight of the address* at its left, and will be denoted by $w(\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2__\boldsymbol{c}_m)$.

- $\hat{\mathcal{L}}(E_m)$ is the mapping of all points in $[0, 1]^p$ into $\mathbb{Z}$, such that each point in an elementary logsize-m cell with address $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2 __ \boldsymbol{c}_m$ maps into the weight of this address. Or: it maps each point into the weight of its m-resolution representation.
- $\mathcal{L}_k(E_m)$ and $\hat{\mathcal{L}}_k(E_m)$ are the subsets of $\mathcal{L}(E_m)$ and of $\hat{\mathcal{L}}(E_m)$ that map into k.
- $G\mathcal{L}_k(E_m)$ and $G\hat{\mathcal{L}}_k(E_m)$ are the domains of $\mathcal{L}_k(E_m)$ and $\hat{\mathcal{L}}_k(E_m)$ respectively, and also refer to the graphical representation thereof. That is, $G\mathcal{L}_k(E_m)$ refers to the picture that highlights the L_m-lattice points in $[0, 1]^p$ whose address-weight equals k, while $G\hat{\mathcal{L}}_k(E_m)$ highlights all cells with address-weight k (see Fig. 1).

Now, what happens in the limit when m grows without bounds (see Fig. 2)? It just seems to be a natural extension of Eq. (3) that we should be interested in the mapping of all points from $[0, 1]^p$ into $\mathbb{Z}$, through their σ-adic representation $\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{c}_3\ldots$, as given by

$$\boxed{\mathbf{0}.\boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{c}_3\ldots \rightarrow e_1(\boldsymbol{c}_1)+e_1(\boldsymbol{c}_2)+e_1(\boldsymbol{c}_3)+\cdots \triangleq \lim_{n\rightarrow\infty}\sum_{k=1}^{n} e_1(c_k)} \quad (4)$$

Hereby it is clear that the presupposed range $\mathbb{Z}$ of this mapping should be augmented with $+\infty$, $-\infty$ in case the right-hand side of Eq. (4) grows without bounds, and with $\star$ to indicate all other cases of nonconvergence in the classical sense. Whatever the case may be, we will always speak of "convergence" with respect to an element of $\mathbb{Z}_\star \triangleq \mathbb{Z}\cup\{+\infty, -\infty, \star\}$.

However, as will become clear in Sec. 2, this limit mapping is not a proper one. It will turn out to be multivalued. Moreover, when we define $\mathcal{F}_k$ as the set of all points in $[0, 1]^p$ that have a σ-adic representation that maps into k according to Eq. (4), i.e., for all $k \in \mathbb{Z}\cup\{+\infty, -\infty, \star\}$:

$$\boxed{\mathcal{F}_k \triangleq \{0.\boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{c}_3\ldots | e_1(\boldsymbol{c}_1)+e_1(\boldsymbol{c}_2)+e_1(\boldsymbol{c}_3)+\cdots = k\}}, \quad (5)$$

then $\mathcal{F}_k$ is only a subset of the *complete set* $\mathcal{G}_k$ of all points that map in the limit into k. As a matter of fact, $\mathcal{G}_k$ will emerge as the closure of $\mathcal{F}_k$, while it can quite naturally be described as the so-called standard part of σ-adic hyperrational numbers with global weight k.

In Sec. 3, we will discuss the nature of the $\mathcal{F}_k$ and $\mathcal{G}_k$ sets by considering the Hutchinson operators

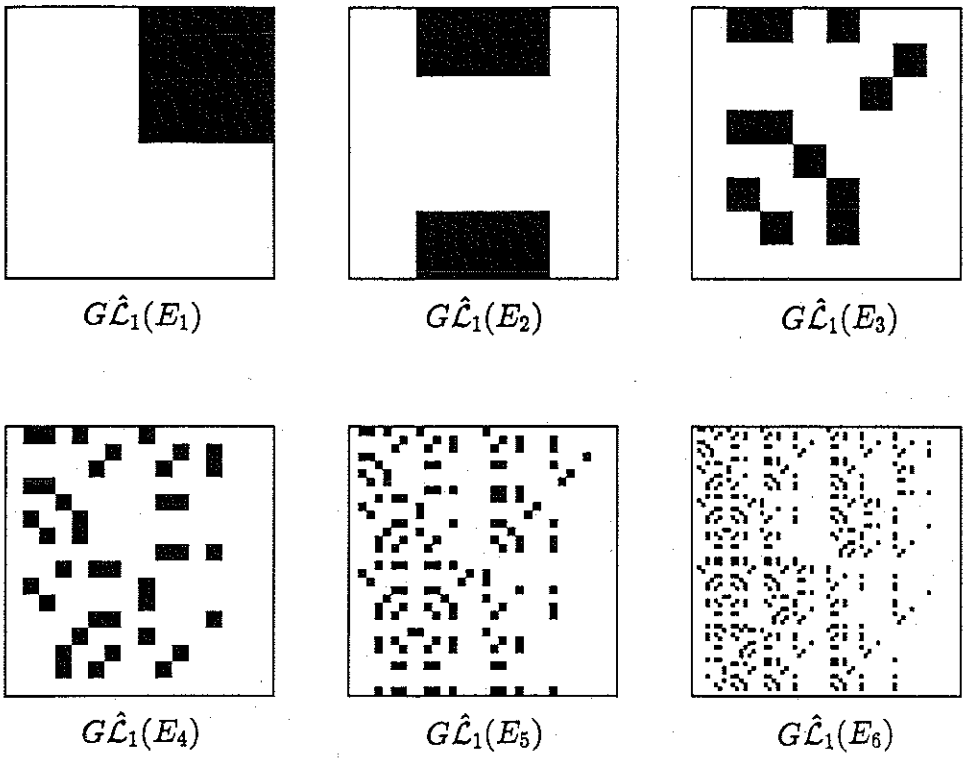


Fig. 2. The evolution of $G\hat{\mathcal{L}}_1(E_m)$ for $m = 1, \ldots, 6$, for the generator $E_1 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. What is the limit-set when $m \rightarrow \infty$?

for which these sets are invariant sets. The corresponding hierarchical iterated function systems (HIFS) will be presented, and it will be shown that they have the associated $\mathcal{G}_k$-set as attractor. A surprising situation occurs when the generator is of *mixed* type, i.e. contains integers of both signs. In this case the attractor of HIFS completely blackens the whole $[0, 1]^p$-space. In the *nonmixed* case, where the elements in the generator are nonnegative or nonpositive, the $\mathcal{F}_k$ and $\mathcal{G}_k$ form the so-called hierarchical Zeno and Cantor sets. Also, the expanding HIFS for the construction of the k-indicator matrix of E (i.e. the matrix indicating which elements in E equal $k^{(II)}$) will be introduced. The behavior of the HIFS will also be studied under limited resolution: the resulting images are compositions of certain $G\hat{\mathcal{L}}_k(E_r)$, where r is related to the resolution. In all this, a fundamental distinction between mixed and nonmixed cases will become apparent.

In Sec. 4, we will reconsider the fractal dimension of the $\mathcal{F}_k$ and $\mathcal{G}_k$-sets in the context of the findings in Sec. 3.

Recall that for the discussion in Parts I and II, we always assumed that the first entry $e_1(\mathbf{0})$ in the generator matrix E_1 was 0 (without loss of generality). In what follows, we will only require that an *arbitrary* element in the generator be 0. That at least one element in the generator should be 0 is necessary in order to find points in $[0, 1]^p$ that map in a convergent way into finite weights under the rule of Eq. (4).

2. About $\lim_{m \to \infty} G\mathcal{L}_k(E_m)$: How Many Numbers Does A Real Number Hide?

In this section, we will restrict ourselves to the one-dimensional case without loss of generality. The higher-dimensional cases are dealt with by an almost trivial componentwise extension, or by reinterpreting the p-dimensional $\boldsymbol{\sigma}$-adic addresses as 1-dimensional σ_π-adic addresses as discussed in Appendix A.

Notations 2.1.

- $0.c_1c_2c_3\ldots$ will refer to a general σ-adic representation of a real number (σ fixed but nonspecified). A specific value for σ will be indicated by using a specific superscript above the "σ-adic" point. For example,

$$0^3.1012\ (\text{triadic}), \quad 0^2.c_1c_2c_3\ldots(\text{binary}).$$

- Indices, weights, elements of matrices, regular fractions will always be written in the decimal system.
- "..." at the end of a symbol string indicates an unlimited continuation of the string in either an unspecified way, or in a way that continues the structure of the preceding string in a "natural" fashion. This will generally be clear from the context.
- $a^l(bcd)^m$ represents the string consisting of l a's followed by m repetitions of the substring bcd.
- a string ending on $\overline{bcd}$ means that bcd is repeated endlessly.
- $\tilde{c}_l = c_1c_2 __ c_l$ indicates a connected string of l digits.
- $w(c_1c_2 __ c_l) \triangleq \sum_{i=1}^{l} e_1(c_i)$ is the *weight* of the string $c_1c_2 __ c_l$: it is also the value of the element with address $c_1c_2 __ c_l$ in E_l. It will occur that we refer to the empty string s_ϕ. Consistency then requires that $w(s_\phi) = 0$.
- $\mathbb{N}$ indicates the set of natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$.

2.1. *Ambiguities in σ-adic representations of real numbers*

We will for some time consider the example of the simple one-dimensional generator $E_1 = [0\ 1]$ (so $\sigma = 2$, and we should consider binary representations using the digits 0 and 1 for which the given generator implies that $e_1(0) = 0$ and $e_1(1) = 1$). To what $\mathcal{F}_k$ does the real (rational) number $\frac{1}{2}$ belong? According to Eq. (5), we find that $\frac{1}{2} = 0^2.1000\ldots$ belongs to $\mathcal{F}_1$. But, $\frac{1}{2}$ also has an alternative representation: $\frac{1}{2} = 0^2.01111\ldots$, and this will make it belong to $\mathcal{F}_\infty$.

In the general case of a p-dimensional generator of size $\boldsymbol{\sigma}$, any point $\boldsymbol{a} = (a_1, a_2, \ldots, a_p) \in [0, 1]^p$, for which at least one of the components a_j has a σ_j-adic representation ending on an unlimited string of 0's, will have multiple representations. Indeed, both σ_j-adic representations $0.c_1c_2 __ c_m\overline{0}$ and $0.c_1c_2 __ (c_m - 1)\overline{(\sigma_j - 1)}$ designate the same real number. In fact, we find the points with multiple representations at the boundaries of the cells of all L_m-lattices. The multiplicity equals 2^q, where q is the number of components of $\boldsymbol{a}$ that have a twofold representation.

The ambiguous σ-adic representation of certain points in the $[0, 1]^p$-continuum destroys the unique correspondence "point $\leftrightarrow$ address $\leftrightarrow$ weight" that

exists in $\mathcal{L}(E_m)$ for finite m. This phenomenon will be called the **collapse into ambiguity**. Either we try to restore the unique correspondence, e.g. by banning certain representations (e.g. those ending on $\bar{1}$ in the example),[3] or at least we try to explain this phenomenon. We will start by presenting this peculiar fact in its full extent.

Let us first recall why $0^2.1000\ldots$ and $0^2.0111\ldots$ are two different representations of the unique real number $\frac{1}{2}$. The reason for this is that any σ-adic representation $0.c_1c_2c_3\ldots$ is meant to represent the real number associated with the limit

$$\lim_{m\to\infty} \sum_{i=1}^{m} \frac{c_i}{\sigma^i}, \tag{6}$$

which is actually the limit of the associated sequence of σ-rational numbers[4] $\langle 0.c_1,\ 0.c_1c_2,\ 0.c_1c_2c_3,\ldots\rangle = \langle \sum_{i=1}^{m} \frac{c_i}{\sigma^i}\rangle_{m\in\mathbb{N}_0}$. The "..."-symbol appearing in the σ-adic real number representation as well as in the associated sequence, denotes that the string (sequence) is to be continued unlimitedly [i.e. the length of the string and the number of terms in the sequence become *potentially infinite* (a kind of variable that grows without bounds)]. We adhere the view that the limit itself is never reached by increasing m in Eq. (6) as long as $m \in \mathbb{N}$, although one gets ever closer to it.[5] The limit itself is reached only when m becomes *actually infinite*, and this requires a kind of mental jump, because there is no gradual transition from the potential to the actual infinity (see e.g. Dauben [1990], Rucker [1982] for comments on the meaning, history and controversies concerning these concepts. See also Hazewinkel [1988], under the entries "abstraction of actual infinity," "abstraction of potential realizability," "infinity").

[3]Forbidding certain representations was already suggested by Cantor when he felt the need for it (see Dauben [1990], p. 56, reporting this). See also Schroeder [1992], rejecting triadic representations ending on $1\bar{0}$ or on $1\bar{2}$ to avoid ambiguity in deciding whether a point $\in [0, 1]$ belongs to the triadic Cantor set C by looking at its triadic representation.

[4]σ-rationals are numbers that can be written as a regular fraction whose denominator is a natural power of σ.

[5]We fall back on the ageold Zeno's paradox (5th century B.C.). In its dichotomy version, it states: in order to go from 0 to 1, one has to get first at $\frac{1}{2}$, then at $\frac{3}{4}$, $\frac{7}{8}$, etc ... (at any stage the remaining interval is halved). The sequence $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$,... never ends and so 1 will never be reached (see, for example W. Salmon [1970]).

How does the limit of $\mathcal{L}(E_m)$ looks like? Let's go back to the origin of the mapping expressed in Eq. (4). Figure 3 shows the points in $\mathcal{L}(E_m)$ that form the sequence associated with $0^2.1000\ldots$ (green dot sequence) and with $0^2.0111\ldots$ (red dot sequence):

$$\begin{aligned} &\langle s_1(i)\rangle_{i\in\mathbb{N}_0}:\ 0^2.1,\ 0^2.10,\ 0^2.100,\ 0^2.1000,\ldots \quad (a)\\ &\langle s_2(i)\rangle_{i\in\mathbb{N}_0}:\ 0^2.0,\ 0^2.01,\ 0^2.011,\ 0^2.0111,\ldots \quad (b)\end{aligned} \tag{7}$$

Both sequences have limit $\frac{1}{2}$. The associated weight sequences ws are

$$\begin{aligned} &\langle ws_1(i)\rangle_{i\in\mathbb{N}_0}:\ 1,\ 1,\ 1,\ 1,\ldots \quad (a)\\ &\langle ws_2(i)\rangle_{i\in\mathbb{N}_0}:\ 0,\ 1,\ 2,\ 3,\ldots \quad (b)\end{aligned} \tag{8}$$

with respective limits 1 and ∞. So, it seems natural to say that in the limit the real point $\frac{1}{2}$ belongs to $\mathcal{F}_1$ [through 7(a), 8(a)] as well as to $\mathcal{F}_\infty$ [through 7(b), 8(b)].

Viewed from the same perspective, however, we can find infinitely many other meaningful sequences of points over the $G\mathcal{L}(E_m)$ in the interval [0, 1] that converge to $\frac{1}{2}$ and that have associated converging weight sequences. The blue and yellow dot sequences in Fig. 3 provide an example. They respectively correspond to the following sequences ($*$ is the don't care symbol):

$$\begin{aligned} &\langle s_3(i)\rangle_{i\in\mathbb{N}_0}: *,\ 0^2.11,\ 0^2.101,\ 0^2.1001,\ 0^2.10001,\ldots,\\ &\langle s_4(i)\rangle_{i\in\mathbb{N}_0}: *,\quad *,\quad 0^2.111,\ 0^2.1011,\ 0^2.10011,\ldots,\end{aligned} \tag{9}$$

with associated weight sequences

$$\begin{aligned} &\langle ws_3(i)\rangle_{i\in\mathbb{N}_0}:\ *,\ 2,\ 2,\ 2,\ 2,\ldots,\\ &\langle ws_4(i)\rangle_{i\in\mathbb{N}_0}:\ *,\ *,\ 3,\ 3,\ 3,\ldots.\end{aligned}$$

So, $\langle s_3\rangle$ and $\langle s_4\rangle$ also converge towards the point $\frac{1}{2}$, but with weights 2 and 3. It can be said that *in the limit*, $\frac{1}{2}$ maps into 2 and 3, although $\frac{1}{2} \notin \mathcal{F}_2, \mathcal{F}_3$ [as $0^2.1000$ and $0^2.0111\ldots$ are the only 2-adic representations of $\frac{1}{2}$ that can be used in the $\mathcal{F}_k$-definition of Eq. (5)].

We can still go one step further and consider the sequence $\langle s_5(i)\rangle_{i\in\mathbb{N}_0}$ defined by

$$s_5(i) = 0^2.10^{i-1}\tilde{c}^{(i)}_{m_i}, \tag{10}$$

whereby we recall that $\tilde{c}^{(i)}_{m_i}$ is a string of digits of length m_i, and so $s_5(i)$ denotes a point of the L_{m_i+i}-lattice (the upper index (i) in $\tilde{c}^{(i)}_{m_i}$ denotes an ith

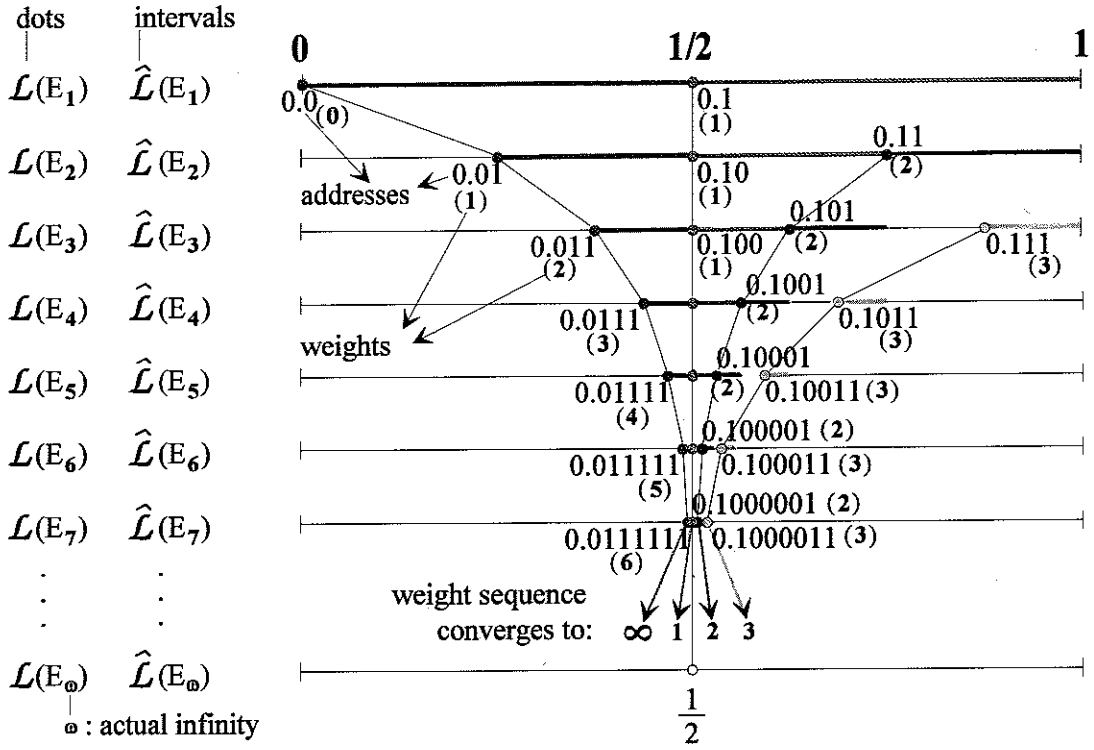


Fig. 3. Different sequences over $\mathcal{L}(E_m)$ that converge to $\frac{1}{2}$ with different weights (generator [0 1]).

item). The corresponding weight sequence is given by

$$ws_5(i) = w(s_5(i)) = 1 + w(\tilde{c}_{m_i}^{(i)}). \qquad (11)$$

$\langle s_5(i)\rangle$ certainly converges to $\frac{1}{2}$, and its associated weight sequence converges to a value in $\mathrm{N}_0 \cup \{+\infty\}$ dependent on the specific $\tilde{c}_{m_i}^{(i)}$. Actually, one can select the $\tilde{c}_{m_i}^{(i)}$ in such a way that $\langle w(\tilde{c}_{m_i}^{(i)})\rangle$ eventually converges to *any* value in $\mathrm{N}_0 \cup \{+\infty\}$. For example, let the $\tilde{c}_{m_i}^{(i)}$ be any sequence with k 1's among any number of 0's, then $\langle ws(i)\rangle$ converges to $1+k$. We can conclude that in the limit, $\frac{1}{2}$ maps into any k, $k \in \mathrm{N}_0 \cup \{+\infty\}$. So, the original twofold ambiguity of $\frac{1}{2}$ has been extended to an *unlimited ambiguity*. Banning certain σ-adic representations has lost every sense in the present context. Observe that when $\tilde{c}_{m_i}^{(i)}$ is taken to be the empty string in Eqs. (10)–(11), we obtain the sequences $\langle s_1\rangle$ and $\langle ws_1\rangle$ associated with the standard 2-adic representation $\frac{1}{2} = 0^2.100\ldots$.

We are now prepared for a more general treatment of the concepts that were introduced in the example above.

Notations 2.2. $\mathcal{Z}_\sigma$, $\mathcal{Z}_\sigma^0$, $\mathcal{Z}_\sigma^+$, $\mathcal{Z}_\sigma^-$.

$\mathcal{Z}_\sigma \triangleq \{0, 1, \ldots, \sigma-1\}$, the set of σ-adic digits, $\mathcal{Z}_\sigma^0$, $\mathcal{Z}_\sigma^+$, $\mathcal{Z}_\sigma^-$ are the subsets of $\mathcal{Z}_\sigma$ containing those digits that, according to a given generator, have zero, positive or negative weights respectively.

It will be useful to consider *three main categories of generator matrices*:

(a) *Single-zero positive* (SZP)-type generators: generators with *one* zero-element and all other elements positive.
(b) *Multiple-zero positive* (MPZ)-type generators: generators with nonnegative elements, more than one of which is 0.
(c) *Mixed* (M)-type generators: generators with elements of both signs and at least one element is 0.

The properties of mixedness and positiveness of the generators are inherited by the set of weights considered in E_m, $\mathcal{L}(E_m)$, $\mathcal{F}_k$, $\mathcal{G}_k$. We need not pay special attention to the *negative* counterparts of (a) and (b) (the SZN- and MZN-type generators), as these are convertible to the SZP and MZP cases through a change of sign in the generator elements. The cases (a) and (b) together will be called *nonnegative* (NN)-type generators.

Recall also from Parts I and II the concept of the *range* $R(E_m)$ as being the set of all integer values that appear in E_m (for a given but further unspecified generator). Recall that $R(E_k) \subset R(E_l)$ when $k < l$, and that $R(E) \triangleq \lim_{m\to\infty} R(E_m)$. When the generator is *reduced* (i.e. when the g.c.d. of its elements is 1, an assumption which is held throughout without loss of generality) it holds that $R(E) = \mathbb{Z}$ for mixed generators and $R(E) = \mathbb{N}\backslash T$ for nonnegative generators, where T is a finite subset of $\mathbb{N}_0$. It follows that any σ-adic representation as appearing in Eqs. (4) and (5) can only take a value in $R(E)_\star \triangleq R(E) \cup \{+\infty\}$ for the NN case and in $R(E)_\star \triangleq \mathbb{Z} \cup \{+\infty, -\infty, \star\}$ in the M case.

Definition 2.1. A point $a \in [0, 1]$ ($[0, 1]^p$ for the p-dimensional case) maps in the limit into the value k ($k \in R(E)_\star$), whenever there is a sequence of points in $[0, 1]^{(p)}$ that converges to point a and for which the corresponding weight sequence converges to k.

Remark. The weight sequence is an integer sequence, and convergence of an integer sequence $\langle w(i)\rangle_{i\in\mathbb{N}_0}$ to $k \in \mathbb{Z}$ means that there is an $M \in \mathbb{N}_0$ such that $w(i) = k$ for all $i > M$.

The next two properties, however trivial they may seem now, are important because they show that most points in $[0, 1]$ map in the limit into an unlimited number of values.

Property 2.1. *Let* $a = 0.a_1a_2a_3\ldots \in \mathcal{F}_k$, $k \in R(E)_\star$ *and consider the sequence* $\langle s(i)\rangle_{i\in\mathbb{N}_0}$ *defined by*

$$s(i) = 0.a_1a_2 __ a_i\tilde{c}^{(i)}_{m_i}, \quad (m_i \in \mathbb{N}), \tag{12}$$

where $\langle\tilde{c}^{(i)}_{m_i}\rangle_{i\in\mathbb{N}_0}$ *is a sequence such that* $\langle w(\tilde{c}^{(i)}_{m_i})\rangle_{i\in\mathbb{N}_0}$ *converges to* $l \in R(E)_\star$ *(it is always possible to find such a sequence). Then point* a *maps in the limit also in* $(k +_\star l) \in R(E)_\star$, *whereby*

$$\begin{aligned} k+_\star l &= k+l \in R(E) \text{ when } k, l \in R(E) && \text{(a)}\\ &= +\infty \text{ when } k \in R(E)\cup\{+\infty\}, l = +\infty \\ &\quad \text{or when } k = +\infty, l \in R(E)\cup\{+\infty\} && \text{(b)}\\ &= -\infty \text{ when } k \in R(E)\cup\{-\infty\}, l = -\infty \\ &\quad \text{or when } k = -\infty, l \in R(E)\cup\{-\infty\} && \text{(c)}\\ &= \text{to be determined for the specific} \\ &\quad \text{situation in all other cases} && \text{(d)} \end{aligned}$$

(for the nonmixed cases, only (a), (b) *or* (c) *apply).*

Notations 2.3. When $a = 0.a_1a_2a_3,\ldots$, the truncation $0.a_1a_2 __ a_i$ will further be denoted by $0.a\rfloor_i$ and a term like the one in Eq. (12) will be written as $s(i) = 0.a\rfloor_i\tilde{c}^{(i)}_{m_i}$.

Property 2.2. *For the M-case, every point* $a = 0.a_1a_2a_3\ldots \in [0, 1]^{(p)}$ *maps in the limit into all* $k \in \mathbb{Z}\cup\{+\infty, -\infty, \star\}$.

Proof. Consider the sequence $\langle s(i)\rangle_{i\in\mathbb{Z}_0}$, where $s(i) = 0.a\rfloor_i\tilde{c}^{(i)}_{m_i}$, $m_i \in \mathbb{N}$ and such that $ws(i) = w(s(i)) = w(0.a\rfloor_i) + w(\tilde{c}^{(i)}_{m_i}) = k$, for any fixed $k \in \mathbb{Z}\cup\{+\infty, -\infty, \star\}$. This is always possible, no matter how $\langle w(0.a\rfloor_i)\rangle_{i\in\mathbb{N}_0}$ evolves. ■

Example 2.1. Consider the generator $[0\ 1\ -1]$ and the point a with triadic representation $a = 0^3\cdot 10121\overline{212}$ (which is the unique triadic representation of the rational number $\frac{29}{72}$). Construct the sequence $\langle s(i) = 0.a\rfloor_i\rangle$. Then $\langle s(i)\rangle$ converges to a and the associated weight sequence is $\langle ws(i)\rangle = \langle w(0.a\rfloor_i)\rangle = \langle 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots\rangle$. As this weight sequence does not converge in the classical sense and it neither grows without bounds, we conclude that $a \in \mathcal{F}_\star$. However, when we take arbitrary but fixed $k, l \in \mathbb{N}$ and define

$$s(i) = 0.a\rfloor_i 2^{k+2}1^{l+1} \quad \text{for } i = 2n,\ n \in \mathbb{N}_0,$$

$$s(i) = 0.a\rfloor_i 2^{k+2}1^{l} \quad \text{for } i = 2n+1,\ n \in \mathbb{N}_0,$$

then $\langle s(i)\rangle$ also converges to a, but the associated weight sequence is $\langle 1, 1, 2, 1, 2, 1, 2, 1, 2, \ldots\rangle + \langle l-k-2, l-k-1, l-k-1, l-k-2, l-k-1, \ldots\rangle = \langle l-k-1, l-k, l-k, l-k, \ldots\rangle$ and converges to $(l-k)$. So in the limit, l and k being arbitrary, point $a = \frac{29}{72}$ maps into any value $z = (l-k) \in \mathbb{Z}$. If we take $s(i) = 0.a\rfloor_i 1^i$, then $\langle s(i)\rangle$ converges to a, but its associated weight sequence converges to $+\infty$. Take $s(i) = 0.a\rfloor_i 2^i$ and its weight sequence converges to $-\infty$. So a maps in the limit into all $k \in \mathbb{Z}\cup\{-\infty, \infty, \star\}$.

Now take the point with triadic representation $b = 0\cdot 21^32^21^42^31^5\ldots$ which corresponds definitely to a "true" real (i.e. nonrational) point. Its straightforward weight sequence under the given generator is

$$\langle -1, 0, 1, 2, 1, 0, 1, 2, 3, 4, 3, 2, 1, 2, 3, \\ 4, 5, 6, 5, 4, 3, 2, 3, 4, 5, 6, 7, 8, 7, \ldots\rangle$$

and converges to $+\infty$ in the classical sense, so $b \in \mathcal{F}_\infty$. Define the sequence $\langle s(i)\rangle$ with

$$s(i) = 0.b\rfloor_i(b\rfloor_i)_{1\leftrightarrow 2}2^k1^l,$$

where $(b\rfloor_i)_{1\leftrightarrow 2}$ is obtained from the $b_1 b_2 __ b_i$ string by replacing every 1 in it by 2, and vice versa. Then $\langle s(i)\rangle$ converges to b and the associated weight sequence converges to $(l-k)$ as $w(0.b\rfloor_i(b\rfloor_i)_{1\leftrightarrow 2}) = 0$ for the given generator. It should be clear that actually an unlimited number of different sequences can be constructed that converge to any given point in [0, 1], with an associated weight sequence that may converge to any $k \in \mathbb{Z} \cup \{+\infty, -\infty, \star\}$. So we conclude that for this example, the set $\mathcal{G}_k$ of all points that map in the limit into a given k is just [0, 1].

2.2. *Properties of $\mathcal{F}_k$ and $\mathcal{G}_k$*

Definition 2.2. For any $k \in R(E)_\star$, $\mathcal{G}_k$ is the set of all points $a \in [0,1]^{(p)}$ that map in the limit into k. Or, to put it more formally,

$$\boxed{\begin{aligned}\mathcal{G}_k \triangleq \Big\{ & c | c = 0.c_1c_2c_3\ldots \in [0,1] \text{ and such that there is a sequence} \\ & \langle s(i)\rangle_{i\in\mathbb{N}_0} = \langle 0.c\rfloor_i \tilde{q}^{(i)}_{m_i}\rangle_{i\in\mathbb{N}_0} \text{ whose corresponding weight-sequence} \\ & \left\langle \sum_{k=1}^{i} e_1(c_k) + w(\tilde{q}^{(i)}_{m_i}) \right\rangle_{i\in\mathbb{N}_0} \text{converges to } k \Big\}. \end{aligned}} \tag{13}$$

We put this in contrast with the reformulated definition for $\mathcal{F}_k$ [cf. Eq. (5)]

$$\text{for } k \in R(E)_\star: \quad \boxed{\begin{aligned}\mathcal{F}_k \triangleq \Big\{ & c | c = 0 \cdot c_1c_2c_3\ldots \in [0,1] \text{ such that} \\ & \langle w(0.c\rfloor_i)\rangle = \left\langle \sum_{k=1}^{i} e_1(c_k) \right\rangle_{i\in\mathbb{N}_0} \text{converges to } k \Big\}. \end{aligned}} \tag{14}$$

In what follows, we will restrict our attention mainly to $\mathcal{F}_k$ and $\mathcal{G}_k$ for finite k, i.e. for $k \in R(E)$. The next three properties establish the relationship between the $\mathcal{F}_k$ and $\mathcal{G}_k$ sets.

Property 2.3.

(A) *In the NN-case:*

$$\textit{for } k \in R(E): \boxed{\mathcal{G}_k = \bigcup_{j\in R(E)\cap[k-R(E)]} \mathcal{F}_j}, \tag{15}$$

where $k - R(E) \triangleq \{k - s | s \in R(E)\}$.

(B) *In the M-case:*

$$\textit{for } k \in R(E) = \mathbb{Z}: \boxed{\mathcal{G}_k = [0,1]^{(p)}}, \tag{16}$$

Proof

A. (a) We first prove that $\mathcal{G}_k \subset \bigcup_{j\in R(E)\cap[k-R(E)]} \mathcal{F}_j$ by showing that any element of $\mathcal{G}_k$, $k \in R(E)$, belongs to some $\mathcal{F}_j$, $j \in R(E) \cap [k - R(E)]$. Let $c = 0.c_1c_2c_3\ldots \in \mathcal{G}_k$, then there exists a sequence $\langle 0.c\rfloor_i \tilde{q}^{(i)}_{m_i}\rangle$ with corresponding weight sequence $\langle w(0.c\rfloor_i q^{(i)}_{m_i})\rangle = \langle \sum_{k=1}^{i} e_1(c_k)\rangle + \langle w(\tilde{q}^{(i)}_{m_i})\rangle$ which converges to k. As k is finite and as all digits in c and in $\tilde{q}^{(i)}_{m_i}$ have nonnegative weights, the two sequences in the decomposition above must converge separately to some fixed j and l respectively, with

$$j + l = k. \tag{17}$$

That $\langle \sum_{k=1}^{i} e_1(c_k)\rangle$ is the standard weight sequence associated with $c = 0.c_1c_2c_3\ldots$ and it converges to j means that $c \in \mathcal{F}_j$. As j and l are weights of strings of digits, $j \in R(E)$ and $l \in R(E)$. But $l \in R(E)$ also implies that $j \overset{(17)}{=} (k - l) \in [k - R(E)]$. It follows that, globally, $j \in R(E) \cap [k - R(E)]$.

(b) We now prove that $\bigcup_{j\in R(E)\cap[k-R(E)]} \mathcal{F}_j \subset \mathcal{G}_k$ by showing that when $c = 0.c_1c_2c_3\ldots \in \mathcal{F}_j$, $j \in R(E)$, c also belongs to $\mathcal{G}_k$ for all $k \in j + R(E)$. Construct the sequence $\langle 0.c\rfloor_i \tilde{q}\rangle$, where $\tilde{q}$ is a constant string of digits. Then $\langle 0.c\rfloor_i \tilde{q}\rangle$ converges to c and its associated weight sequence converges to $j + w(\tilde{q})$. So $c \in \mathcal{G}_{j+w(\tilde{q})}$ by definition. As $\tilde{q}$ can be selected so that $w(\tilde{q})$ may take any value in $R(E)$, it follows that $c \in \mathcal{G}_k$ for all $k \in j + R(E)$.

B. This is a direct consequence of the definition of $\mathcal{G}_k$ and of Property 2.2. ■

Remark (For the NN case). When $R(E) = \mathbb{Z}^+$, then $R(E) \cap [k - R(E)]$ equals the integer interval $[0, k]$, and $\mathcal{G}_k = \bigcup_{0 \le j \le k} \mathcal{F}_j$. Otherwise $R(E) \cap [k - R(E)]$ is a proper subset of the integer interval $[0, k]$.

Property 2.4.

(A) *For SZP:* $\bigcup_{j \in R(E) \cap [k - R(E) \setminus \{0\}]} \mathcal{F}_j$ *is the set of accumulation points of* $\mathcal{F}_k$.

(B) *For MZP:* $\bigcup_{j \in R(E) \cap [k - R(E)]} \mathcal{F}_j$ *is the set of accumulation points of* $\mathcal{F}_k$.

(C) *For both SZP and MZP, the closure of* $\mathcal{F}_k$ *is given by*

$$\boxed{\overline{\mathcal{F}}_k = \bigcup_{j \in R(E) \cap [k - R(E)]} \mathcal{F}_j} \qquad (18)$$

and it follows that

$$\boxed{\overline{\mathcal{F}}_k = \mathcal{G}_k}. \qquad (19)$$

Proof. We use the property that a point x is an accumulation (limit) point of a set of points X iff there is a point $y \in X$, $y \neq x$ in any (arbitrary small) open neighborhood of x of the form $(x - \frac{1}{\sigma^M}, x + \frac{1}{\sigma^M})$, $M > M_0 \in \mathbb{N}$.

A. SZP case:

(a) Any $x \in \mathcal{F}_j$, $j \in R(E)$ is of the form

$$x = 0.x_1 x_2 x_3 __ x_m \nu_1 \nu_2 \nu_3 \ldots, \text{ with } \nu_i \in \mathcal{Z}_\sigma^0$$

and

$$\sum_{i=1}^{m} e_1(x_i) = j\,.$$

Consider an open neighborhood $B_{x,M}$ around x : $B_{x,M} \triangleq (x - \frac{1}{\sigma^M}, x + \frac{1}{\sigma^M})$, $M > m$. Then, it is always possible to find a point $x_{\alpha_{M,q}} = 0.x_1 __ x_m \nu_1 __ \nu_{M-m} \alpha_1 \alpha_2 \alpha_3 \ldots \neq x$ such that $x_{\alpha_{M,q}} \in B_{x,M}$ and such that $\sum e_1(\alpha_i) = q$ for any $q \in R(E) \setminus \{0\}$ ($q = 0$ would imply $x_{\alpha_{M,q}} = x$ in this SZP case, but not in the MZP case where the corresponding condition turns into $q \in R(E)$). So, any arbitrary small neighborhood $B_{x,M}$ contains a point $x_{\alpha_{M,q}} \in \mathcal{F}_{j+q}$ for any $q \in R(E) \setminus \{0\}$. This implies that $x \in \mathcal{F}_j$ is a limit point of all $\mathcal{F}_{j+q}$, $q \in R(E) \setminus \{0\}$, and, as a consequence, $\mathcal{F}_j$ itself is a set of limit points of all these $\mathcal{F}_{j+q}$. Equivalently, for a fixed $k \in R(E)$, $\mathcal{F}_k$ has as limit points all points in $\mathcal{F}_j$, for all $j = k - q$ with $q \in R(E) \setminus \{0\}$ and $j \in R(E)$. So $\bigcup_{j \in R(E) \cap [k - R(E) \setminus \{0\}]} \mathcal{F}_j$ is a set of limit points of $\mathcal{F}_k$.

(b) We still have to show that no point x outside $\bigcup_{j \in R(E) \cap [k - R(E) \setminus \{0\}]} \mathcal{F}_j$ can be a limit point of $\mathcal{F}_k$. Such a point x must necessarily belong to some $\mathcal{F}_l$, with $l \in R(E) \cup \{+\infty\}$ but $l \notin R(E) \cap [k - R(E)]$, and so $l \in R(E) \cup \{+\infty\} \setminus [k - R(E)]$. x being an accumulation point of $\mathcal{F}_k$ would require the existence of an $\alpha_1 \alpha_2 \alpha_3 \ldots$-string with weight $(k - l) \in R(E) \cup \{+\infty\}$ in the $x_{\alpha_{M,q}}$-representation above. This would imply $l \in [k - R(E)] \cup \{-\infty\}$, contradicting an earlier condition on l.

B. MZP case: same as for the SZP case, but taking into account the remark given under ***A.***(a) that the SZP-related condition $q \in R(E) \setminus \{0\}$ should be replaced by $q \in R(E)$.

C. Equation (18) follows from using parts ***A*** and ***B*** of this property in the fact that the closure of a set contains all its points and its limits points. Equation (19) follows from Eqs. (15) and (18). ■

Property 2.5. *For the M case, any* $x \in [0, 1]^{(p)}$ *is a limit point for any set* $\mathcal{F}_j (j \in \mathbb{Z} \cup \{+\infty, -\infty, \star\}$. *So* $\overline{\mathcal{F}}_j = \overline{\mathcal{F}}_\infty = \overline{\mathcal{F}}_{-\infty} = \overline{\mathcal{F}}_\star = [0, 1]^{(p)}$, *and it follows that* $\overline{\mathcal{F}}_k = \mathcal{G}_k$.

Proof. Let $x = 0.c_1 c_2 c_3 \ldots$ be any number $\in [0, 1]^{(p)}$, then it is always possible to find a number of the form $x = 0.c\rfloor_M \tilde{q} \in (x - \frac{1}{\sigma^M}, x + \frac{1}{\sigma^M})$ such that $w(0.c\rfloor_M \tilde{q}) \in R(E)_\star = \mathbb{Z} \cup \{+\infty, -\infty, \star\}$, and this holds no matter how large M is. ■

Property 2.6.

$$\boxed{\text{For } k \in R(E) : \lim_{m \to \infty} G\mathcal{L}_k(E_m) = \mathcal{G}_k} \qquad (20)$$

$$\boxed{\text{and} \quad \lim_{m \to \infty} G\hat{\mathcal{L}}_k(E_m) = \mathcal{G}_k} \qquad (21)$$

Proof

A. $\lim G\mathcal{L}_k(E_m)$

(a) By definition, $\lim_{m \to \infty} G\mathcal{L}_k(E_m)$ is the set of points $x \in [0, 1]^{(p)}$ such that any open neighborhood $B_{x,M} \triangleq (x - \frac{1}{\sigma^M}, x + \frac{1}{\sigma^M})$ contains a point $p_m \in G\mathcal{L}_k(E_m)$ for all m belonging to some *infinite* subset of $\mathbb{N}_0$.

(b) Any point $x \in 0.x_1x_2x_3\ldots \in \mathcal{G}_k$ is, by definition, associated with a sequence $\langle s_x(i)\rangle = \langle 0.x\rfloor_i \tilde{q}^{(i)}_{m_i}\rangle$ such that $w(0.x\rfloor_i \tilde{q}^{(i)}_{m_i}) = k$ for $i > M_0$. Observe that then $s_x(i) \in B_{x,i}$, and that $s_x(i) \in G\mathcal{L}_k(E_{m_i+i})$ for $i > M_0$. It follows that any neighborhood $B_{x,M}$ contains a point $s_x(i) \in G\mathcal{L}_k(E_{m_i+i})$ for all $i > \max(M_0, M)$, and so $x \in \lim_{m\to\infty} G\mathcal{L}_k(E_m)$ [according to (a)].

(c) We still have to show that any $x = 0.x_1x_2x_3\ldots \in \lim_{m\to\infty} G\mathcal{L}_k(E_m)$ also belongs to $\mathcal{G}_k$. Indeed, every neighborhood $B_{x,M}$ of x contains a point $s_x(M)$ belonging to some $G\mathcal{L}_k(E_m)$ (with $m > M$). This point can always be written as $s_x(M) = 0.x\rfloor_M \tilde{q}^{(M)}_{m_M}$, while $w(0.x\rfloor_M \tilde{q}^{(M)}_{m_M}) = k$. The sequence $\langle s_x(M)\rangle_{M\in\mathbb{N}_0}$ then converges to x and has an associated weight sequence that converges to k, implying that $x \in \mathcal{G}_k$.

B. $\lim G\hat{\mathcal{L}}_k(E_m)$

The proof proceeds along the same line as under ***A***, where $G\mathcal{L}_k(E_m)$ should be replaced by $G\hat{\mathcal{L}}_k(E_m)$ while use can be made of the fact that $G\mathcal{L}_k(E_m) \subset G\hat{\mathcal{L}}_k(E_m)$. Only part (c) needs some extra attention. Let $x = 0.x_1x_2x_3\ldots \in \lim_{m\to\infty} G\hat{\mathcal{L}}_k(E_m)$, then any $B_{x,M}$-neighborhood contains a point $s^\star_x(M)$ that belongs to $G\hat{\mathcal{L}}_k(E_m)$ for some $m > M$. This point can always be expressed as $s^\star_x(M) = 0.x\rfloor_M \tilde{q}^{(M)}_{m_M} {*}{*}{*}\ldots$, where "$***\ldots$" denotes a string of arbitrary digits and $w(0.x\rfloor_M \tilde{q}^{(M)}_{m_M}) = k$. Now $\langle s^\star_x(M)\rangle_{M\in\mathbb{N}_0}$ converges to x together with $\langle s_x(M)\rangle = \langle 0.x\rfloor_M \tilde{q}^{(M)}_{m_M}\rangle$ which has a weight-sequence converging to k associated with it. So $x \in \mathcal{G}_k$. ■

2.3. *How many numbers? What numbers?*

In this section, we return to the somewhat enigmatic question from the heading of Sec. 2: "How many numbers does a real number hide?" The meaning of this question, and its answer, become readily clear when we move into the realm of the *hyperreal* (HR) or *nonstandard* numbers. In this number system which is an extension of the real number system, a hyperreal number is just a sequence $\langle a_n\rangle_{n\in\mathbb{N}_0}$ of real numbers a_n, and it can be treated like an ordinary real number under arithmetic operations, in function definitions and in relational operators. Actually, the relationship between a hyperreal number and a sequence is a bit more involved, and is dealt with in more detail in Appendix B. As the hyperreal formalism will not be used in the remainder of the text, we will only present here a very intuitive explanation of how the hyperreal vocabulary fits rather naturally some of the concepts encountered in the previous sections. It is based on that very idea that sequences can be treated as numbers in the HR-system.

Our question can then be interpreted as: "how many sequences can be associated with a real number?" The answer should be clear from our previous observations: there are infinitely many different sequences that converge to a given real number. In the HR-context, these sequences are hyperreal numbers that are *infinitely close* to that real number. But, under the real number resolution (which is even an unpracticable limit resolution), they cannot be distinguished from the real number itself.

We have met earlier the situation where different sequences converge to a given real number r, while their associated weight sequences converge to different integer-values k. In the HR-language this translates as: different hyperreal numbers that are infinitely close to r and that map into different integer-values k. Under real number resolution, all these hyperreal numbers that are infinitely close to r are just indistinguishably hidden in that single real point r, but the different k-values into which they map remain distinguishable. This explains the collapse into ambiguity, a phenomenon that does not occur in the hyperreal domain. Appendix B explores this view on a more rigorous basis, thereby also reformulating the $\mathcal{F}_k$ and $\mathcal{G}_k$ definitions.

3. Hutchinson Operators and Hierarchical IFS for $\mathcal{F}_k$, $\mathcal{G}_k$

3.1. *The Hutchinson operator*

Again, for the sake of simplicity and without loss of conceptual generality, we will only deal with the one-dimensional case ($p = 1$). The generalization to higher dimensions is almost trivial [everything should be extended componentwise or should be reinterpreted in σ_π-adic address space (see Appendix A)].

A generator

$$E_1 = [e(0)e(1)e(2)\cdots e(\sigma-1)] \quad (\text{size } \sigma,\ e(i) \in \mathbb{Z})$$

produces specific point sets $\mathcal{F}_k$, $\mathcal{G}_k$ over the interval [0, 1], as defined in Sec. 2.2. Now, divide [0, 1] into

σ equally-sized subintervals, numbered from 0 to $\sigma - 1$, where

$$a\text{th interval } (a \in \mathcal{Z}_\sigma) \triangleq \left[\frac{a}{\sigma}, \frac{a+1}{\sigma}\right].$$

For any $a \in \mathcal{Z}_\sigma$, define the shift-operations symbolized by $0.a\mathcal{F}_k$, $0.a\mathcal{G}_k$ as

$$\begin{aligned} 0.a\mathcal{F}_k &\triangleq \{0.ac_1c_2c_3\ldots | 0.c_1c_2c_3\ldots \in \mathcal{F}_k\}, \\ 0.a\mathcal{G}_k &\triangleq \{0.ac_1c_2c_3\ldots | 0.c_1c_2c_3\ldots \in \mathcal{G}_k\}. \end{aligned} \tag{22}$$

Let $\mathcal{Q}_k$ stand for either $\mathcal{F}_k$, $\mathcal{G}_k$ (but fixed in the context). When considering $\mathcal{Q}_k$ as a geometrical point set on its support [0, 1], $0.a\mathcal{Q}_k$ makes a copy of this point set while reducing it by a factor σ, and places this reduced version in the ath interval. This establishes the following equivalence:

$$0.a\mathcal{Q}_k \leftrightarrow \frac{a}{\sigma} + \frac{1}{\sigma}\mathcal{Q}_k, \tag{23}$$

where the obvious symbolism is used. In this interpretation, $0.a\mathcal{Q}_k$ clearly manifests itself as a *contractive mapping*. It is clear from the definitions that

$$0.a\mathcal{Q}_k \subseteq \mathcal{Q}_{k+e(a)}. \tag{24}$$

By induction from Eq. (22), expressions such as $0.a_l\ldots a_3a_2a_1\mathcal{Q}_k$ are easily interpreted, and it follows that

$$0.a_l\ldots a_3a_2a_1\mathcal{Q}_k \subseteq \mathcal{Q}_{k+e(a_1)+e(a_2)\ldots+e(a_l)}. \tag{25}$$

The following property is an immediate consequence of Eq. (24) reinterpreted as $0.a\mathcal{Q}_{k-e(a)} \subset \mathcal{Q}_k$ and of the fact that any point $c = 0.c_1c_2c_3\ldots \in \mathcal{Q}_k$ is such that $c_1 \in \mathcal{Z}_\sigma$ and $c \in 0.c_1\mathcal{Q}_{k-e(c_1)}$.

Property 3.1. *Given the generator* $E_1 = [e(0)e(1)e(2)\cdots e(\sigma-1)]$, *the* $\mathcal{Q}_k$ *satisfy the collage (union of contraction mappings):*

$$\begin{aligned} \forall k \in R(E) : \mathcal{Q}_k &= 0.00\mathcal{Q}_{k-e(0)} \cup 0.1\mathcal{Q}_{k-e(1)} \\ &\quad \cup 0.2\mathcal{Q}_{k-e(2)}\cdots \\ &\quad \cup 0.(\sigma-1)\mathcal{Q}_{k-e(\sigma-1)} \\ &\overset{(23)}{=} \bigcup_{a=0}^{\sigma-1}\left[\frac{a}{\sigma} + \frac{1}{\sigma}\mathcal{Q}_{k-e(a)}\right]. \end{aligned} \tag{26}$$

Remark. Recall that in the M case, $R(E) = \mathbb{Z}$ and Eq. (26) holds for all $k \in \mathbb{Z}$. In the NN case, it may happen that while $k \in R(E)$, $k - e(j) \notin R(E)$ for some j; then $\mathcal{Q}_{k-e(j)} = \phi$ and the corresponding term in Eq. (26) does not contribute. (As an example, consider the generator [0 2 5] for which $R(E) = \{0, 2, 4, 5, 6, 7, \ldots\} = \mathbb{Z}^+\backslash\{1, 3\}$. With $k = 6 \in R(E)$, $k - e(2) = 1 \notin R(E)$, and $\mathcal{Q}_6$ satisfies $\mathcal{Q}_6 = 0.0\mathcal{Q}_6 \cup 0.1\mathcal{Q}_4$). A necessary and sufficient condition for Eq. (26) to be valid for all $k \in \mathbb{Z}$ in the NN case is that all $\mathcal{Q}_j$, $j \notin R(E)$ are forced to be the empty set ϕ. The sufficiency follows from the fact that $k \notin R(E)$ implies that $k - e(j) \notin R(E)$, and so $\mathcal{Q}_k = \phi$ implies $\mathcal{Q}_{k-e(j)} = \phi$ and vice-versa in Eq. (26). The necessity follows from the observation that if we make all $\mathcal{Q}_k = \phi$ when $k \notin R(E)$, except for a single $k = k_0 \notin R(E)$, then Eq. (26) would lead for this particular k_0 to $\mathcal{Q}_{k_0} = 0.a\mathcal{Q}_{k_0}$, where a is such that $e(a) = 0$. But this equation has two solutions: $\mathcal{Q}_{k_0} = \phi$ and $\mathcal{Q}_{k_0} = \{0.aaa\ldots\}$. The last one is unacceptable because $k_0 \notin R(E)$. Retaining it would also introduce unacceptable solutions for other $\mathcal{Q}_k$.

Definition 3.1. The set of collage equations Eq. (26) extended over all $k \in \mathbb{Z}$ will be called the *extended set* of collage equations. An *extended solution set* $\{\mathcal{Q}_k = \mathcal{M}_k | k \in \mathbb{Z}\}$ is *valid* if and only if $\mathcal{M}_k = \phi$ for $k \notin R(E)$. It is sufficient to refer to the *non empty-enforced* subset $\{\mathcal{Q}_k = \mathcal{M}_k | k \in R(E)\}$.

Let the collage of Eq. (26) operate on arbitrary sets $B_{k-e(i)} \subseteq [0, 1]$, then the result is a new set $A_k \subseteq [0, 1]$:

$$\begin{aligned} A_k &= 0.0B_{k-e(0)} \cup 0.1B_{k-e(1)}\cdots \\ &\quad \cup 0.(\sigma-1)B_{k-e(\sigma-1)}. \end{aligned} \tag{27}$$

When applied simultaneously to all B_j, $j \in \mathbb{Z}$, a set of images $\{A_j, j \in \mathbb{Z}\}$ emerges. The whole operation can symbolically be written using an infinite matrix. This is illustrated for the example associated with the generator $E_1 = [0\ 2\ 1\ -2]$ (implying the use of the quaternary number representation). Here, $A_k = 0.0B_k \cup 0.1B_{k-2} \cup 0.2B_{k-1} \cup 0.3B_{k+2}$ and the complete operation is symbolized by the following matrix product, featuring an infinite band Toeplitz matrix

$$
\begin{array}{l}\text{column:}\quad \cdots\ -4\ \ -3\ \ -2\ \ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ \cdots \\
\begin{bmatrix} \vdots \\ A_{-3} \\ A_{-2} \\ A_{-1} \\ A_0 \\ A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} & & & & & & & & & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \phi & 0.1 & 0.2 & 0.0 & \phi & 0.3 & \phi & \phi & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \phi & 0.1 & 0.2 & 0.0 & \phi & 0.3 & \phi & \phi & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \phi & 0.1 & 0.2 & 0.0 & \phi & 0.3 & \phi & \phi & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \phi & 0.1 & 0.2 & 0.0 & \phi & 0.3 & \phi & \phi & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \phi & 0.1 & 0.2 & 0.0 & \phi & 0.3 & \phi & \phi & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \phi & 0.1 & 0.2 & 0.0 & \phi & 0.3 & \phi & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ B_{-3} \\ B_{-2} \\ B_{-1} \\ B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix} . \quad (28)
\end{array}
$$

The matrix product yields terms like $0.aB_j$ which are to be interpreted as above [Eq. (23)], while addition (+) is replaced by set union (∪). The empty elements ϕ do not contribute as $\phi B_j \triangleq \phi$.

This double-infinite band matrix W is a generalized form of the matrix Hutchinson operator as introduced in Peitgen [1992]. In general, element $W_{ij}(i, j \in \mathbb{Z})$ depends on the specific size and on the elements of the generator, and satisfies

$$W_{ij} = 0.a \text{ iff } i = j - e(a) \text{ for } a \in \mathcal{Z}_\sigma$$
$$= \phi \text{ else}. \quad (29)$$

It is clear that $\mathcal{F}_k$, $\mathcal{G}_k$ are all empty sets for $k < 0$ in the SZP and MZP situation. In these cases, the matrix Hutchinson operator W can be reduced to a single-infinite band matrix.

All solutions to the collage equation (26) are invariant sets of the Hutchinson operator of Eq. (28). This means that when $\{\mathcal{M}_k | k \in R(E)\}$ is a solution of Eq. (26), then $B_k = \mathcal{M}_k$ implies that $A_k = \mathcal{M}_k$ for all $k \in R(E)$. Because $\{\mathcal{F}_k\}$, and $\{\mathcal{G}_k\}$ satisfy Eq. (26), it follows that they are *invariant sets* of the Hutchinson operator. Are there other invariant sets? Yes, there are two trivial ones:

(a) $B_k = \phi$ for all k, implies that all $A_k = \phi$, so $\{B_k = \phi | k \in R(E)\}$ is an invariant valid solution: it is a trivial, uninteresting one.
(b) $\{B_k = [0, 1]^{(p)} | k \in R(E)\}$ is also a trivial valid invariant solution for the M-case ($R(E) = \mathbb{Z}$). We already know from Eq. (16) that this solution equals $\{\mathcal{G}_k\}$ for this case.

However, as will be shown in the next property, there are still other invariant sets.

Property 3.2. *Let $\{\mathcal{Q}_k = \mathcal{M}_k | k \in R(E)\}$ with $\mathcal{M}_0 \neq \phi$, be a particular solution set of Eq. (26), then:*

(a) *In the NN case: $\{\mathcal{Q}_k = \cup_{j \in S_k} \mathcal{M}_j\}$ for any $S_k \subseteq R(E) \cap [k - R(E)]$ is also a solution. In particular, $\{\mathcal{Q}_k = \cup_{j \in R(E) \cap [k - R(E)]} \mathcal{F}_j\} \overset{(19)}{=} \{\mathcal{G}_k\}$ is a solution.*
(b) *In the M case: $\{\mathcal{Q}_k = \cup_{j \in S} \mathcal{M}_j\}$ for any $S \subseteq \mathbb{Z}$ is also a solution. In particular $\{\mathcal{Q}_k = \cup_{j \in \mathbb{Z}} \mathcal{F}_j\}$ is a solution.*

Proof. This is a consequence of two facts:

1. Let $\{\mathcal{Q}_k = \mathcal{M}_k | k \in R(E)\} \cup \{\mathcal{Q}_k = \mathcal{M}_k = \phi | k \notin R(E)\}$ be a particular extended solution. As Eqs. (26) are identical in form for all k and so are k-shift invariant, we expect the solution to be k-shift invariant as well, i.e. $\{\mathcal{Q}_k = \mathcal{M}_{k-l} | k \in \mathbb{Z}\}$ for any given $l \in \mathbb{Z}$ would also be a solution. This is true for the M case with reduced generators, as then $R(E) = \mathbb{Z}$. However, in the NN case the empty-enforced subsets are not necessarily invariant over arbitrary shifts. We show that they are if and only if $l \in R(E)$. Hereto, observe that an l-shift $\{\mathcal{M}_{k-l}\}$ of the $\{\mathcal{M}_k\}$ set is a valid solution for $\{\mathcal{Q}_k\}$ if and only if

$$\mathcal{Q}_k = \mathcal{M}_{k-l} \text{ for } k \in R(E)$$

and

$$\mathcal{Q}_k = \mathcal{M}_{k-l} = \phi \text{ for } k \notin R(E).$$

In order for the second condition to hold, it is necessary that $\mathcal{M}_{k-l}$ belongs to the empty-enforced sets, requiring that $k - l \notin R(E)$ or $k \notin l + R(E)$, for all $k \notin R(E)$. This is certainly

satisfied when $l \in R(E)$, as then $l + R(E) \subset R(E)$ and so k cannot belong to $l + R(E)$. It is also necessary that $l \in R(E)$. Indeed, suppose that $l \notin R(E)$, then $k - l \in R(E)$ for at least one $k \notin R(E)$, namely for $k = l$, and this would imply that $\mathcal{Q}_l = \mathcal{M}_0 \neq \phi$. But from the premises we know that $\mathcal{M}_0 = \phi$, and so there is a contradiction. So we conclude that if $\{\mathcal{Q}_k = \mathcal{M}_k | k \in R(E)\}$ is a solution, then $\{\mathcal{Q}_k = \mathcal{M}_{k-l} | k \in R(E)\}$ is also a solution if and only if $l \in R(E)$.

2. When $\{\mathcal{Q}_k = \mathcal{M}_{k,1} | k \in R(E)\}$ and $\{\mathcal{Q}_k = \mathcal{M}_{k,2} | k \in R(E)\}$ are valid solutions of Eq. (26), then so is their union $\{\mathcal{Q}_k = \mathcal{M}_{k,1} \cup \mathcal{M}_{k,2} | k \in R(E)\}$. This can be applied to any number of particular solutions from the set $\{\{\mathcal{Q}_k = \mathcal{M}_{k-l} | k \in R(E)\}, l \in R(E)\}$, what produces the solutions $\{\mathcal{Q}_k = \bigcup_{j \in S'_k} \mathcal{M}_j\}$ where $S'_k \subseteq [k - R(E)]$. But, as $\mathcal{M}_j = \phi$ for $j \notin R(E)$, the j-range in this expression can be restricted to the non empty-enforced subsets $\mathcal{M}_j$, so that it transforms into $\{\mathcal{Q}_k = \bigcup_{j \in S_k} \mathcal{M}_j\}$, $S_k \subseteq R(E) \cap [k - R(E)]$. (For the M case, where $R(E) = \mathbb{Z}$, this transforms into $S_k \subseteq \mathbb{Z}$).

The special solutions under (a) and (b) follow from the particular $\{\mathcal{Q}_k = \mathcal{F}_k\}$ solution set. ■

Remark. When $\{\mathcal{Q}_k = \cup_j \mathcal{M}_j\}$ is a combination of particular solutions $\mathcal{M}_j$, each subset $\mathcal{M}_j$ is auto-invariant under the kth Hutchinson operator: i.e. there is no transition between points in different $\mathcal{M}_j$ when the operator is applied recursively.

3.2. *The HIFS and its attractor*

Associated with the Hutchinson operator is a geometrically inspired dynamical hierarchical iterated function system (HIFS) defined over the line segment that corresponds with the real interval [0, 1], as introduced in Peitgen, [1992] as an extension of Barnsley's IFS (Barnsley, [1989]). (See Appendix A for the p-dimensional equivalent).

This HIFS is the time-dynamical version of Eq. (26) (we consider the extended version):

$$\boxed{\forall k \in \mathbb{Z} : \hat{\mathcal{Q}}_k(m) = \bigcup_{a=0}^{\sigma-1} 0.a\hat{\mathcal{Q}}_{k-e(a)}(m-1)\,.} \quad (30)$$

A general graphical representation is given in Fig. 4 for a single level k. The complete HIFS consists of an unlimited replication of this scheme in both upper and lower directions for the M-type case,

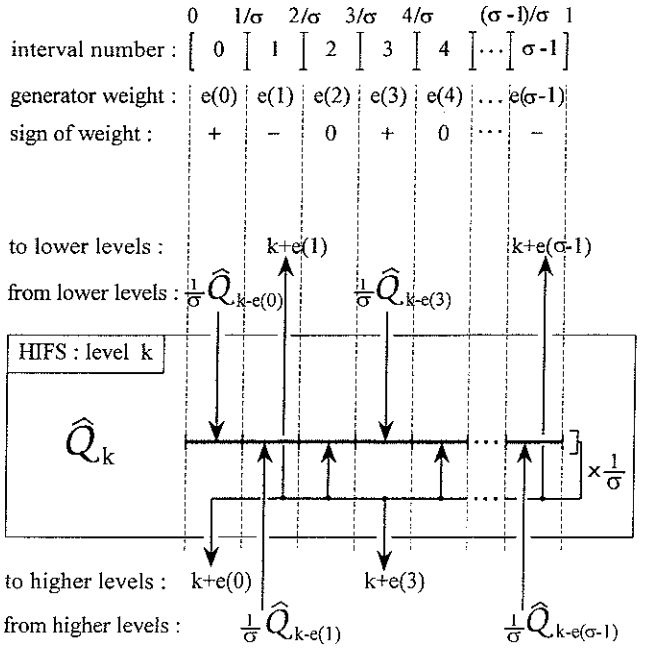


Fig. 4. Graphical representation of the level-k section of the HIFS for an arbitrary generator with positive, negative and zero weights. Zero weights give rise to feedback loops on the same level, positive weights lead to feedforward connections (connections from lower to higher k-levels) and negative weights provide feedbacks from higher to lower k-levels.

in the lower direction only for the SZP and MZP-type cases (starting at $k = 0$).

The HIFS of Eq. (30) starts with a set of initial conditions (images) $\{\hat{\mathcal{Q}}_k(0)\}$. We will now show that $\mathcal{G}_k$ is the attractor for $\hat{\mathcal{Q}}_k(m)$ in the HIFS.

Lemma 3.1.

(A) *When* $t \in [k - R(E)\backslash\{0\}]$, *then* $[t - R(E)] \subset [k - R(E)]$.

(B) *When* $t \in R(E) \cap [k - R(E)\backslash\{0\}]$, *then* $\mathcal{G}_t \subset \mathcal{G}_k$.

Proof

(A) $t \in [k - R(E)\backslash\{0\}]$ implies $t = k - j$, with $j \in R(E)\backslash\{0\}$. As a consequence, $[t - R(E)] = [k - j - R(E)] = [k - [j + R(E)]] \subset [k - R(E)]$, whereby the inclusion follows from the fact that $[j + R(E)] \subset R(E)$ under the condition $j \in R(E)\backslash\{0\}$.

(B) From (A) it follows that when $t \in R(E) \cap [k - R(E)\backslash\{0\}]$, then $[t - R(E)] \subset [k - R(E)]$ and so $R(E) \cap [t - R(E)] \subset R(E) \cap [k - R(E)]$. Use this in Eq. (15) to obtain

$$\mathcal{G}_t \overset{(15)}{=} \bigcup_{j \in R(E) \cap [t - R(E)]} \mathcal{F}_j \subset \bigcup_{j \in R(E) \cap [k - R(E)]} \mathcal{F}_j \overset{(15)}{=} \mathcal{G}_k\,.$$

■

Remark. When $t \notin R(E) \cap [k - R(E)\backslash\{0\}]$ (but $t \in R(E) \cap [0,\ k-1]$), then it does not necessarily follow that $\mathcal{G}_t \subset \mathcal{G}_k$. As an example, consider the generator matrix [0 2 5] for which $R(E) = \mathbb{Z}^+\backslash\{1, 3\}$. Then $\mathcal{G}_6 = \bigcup_{j\in\{0,2,4,6\}} \mathcal{F}_j \not\subset \mathcal{G}_7 = \bigcup_{j\in\{0,2,5,7\}} \mathcal{F}_j$, as $6 \notin \{0, 2, 5, 7\}$. Also $\mathcal{G}_4 = \bigcup_{j\in\{0,2,4\}} \mathcal{F}_j \not\subset \mathcal{G}_7$. But $\mathcal{G}_5 = \bigcup_{j\in\{0,5\}} \mathcal{F}_j \subset \mathcal{G}_7$ and $\mathcal{G}_2 = \bigcup_{j\in\{0,2\}} \mathcal{F}_j \subset \mathcal{G}_7$. As before, much of the technicalities of the proofs in the coming properties are due to the fact that in general we have to consider *nondense* generators (i.e. generators whose elements do not occupy a whole integer interval, so that $R(E) \neq \mathbb{Z}^+$).

Property 3.3. Convergence Theorem.
$\lim_{m\to\infty} \hat{\mathcal{Q}}_k(m)$ *for the HIFS given by Eq. (30) equals*

(A) $\mathcal{G}_k$ *in the SZP and MZP cases iff* $\hat{\mathcal{Q}}_0(0) \neq \phi$ *and* $\hat{\mathcal{Q}}_l(0) = \phi$ *for* $l \notin R(E)$.
(B) $\mathcal{G}_k = [0, 1]^{(p)}$ *in the M case when* $\hat{\mathcal{Q}}_l(0) \neq \phi$ *for at least one* l.

Proof

A. For the SZP and MZP cases:
Using Eq. (30) backwards, we obtain $\forall k \in \mathbb{Z}$:

$$\hat{\mathcal{Q}}_k(m) = \bigcup_{j\in R(E_m)} \{0.(a_1a_2 __ a_m)_j \hat{\mathcal{Q}}_{k-j}(0) \diamond |a_1a_2 __ a_m \in (\mathcal{Z}_\sigma)^m\},^{6} \quad (31)$$

where $0.(a_1a_2 __ a_m)_j$ indicates that $\sum_{i=1}^m e(a_i) = j$, so that necessarily $j \in R(E_m)$.

First observe that $k \notin R(E)$ implies that $\hat{\mathcal{Q}}_k(m) = \phi$ for all m. Indeed, $k \notin R(E)$ also means that $k - j \notin R(E)$ when $j \in R(E_m)$, and so all $\hat{\mathcal{Q}}_{k-j}(0)$ in Eq. (31) are empty under the assumptions given under A.

Now, select a specific $j = l$ set in Eq. (31) for which $\hat{\mathcal{Q}}_{k-l}(0) \neq \phi$, i.e.,

$$\mathcal{K}_l(m) = \{0.(a_1 __ a_m)_l \hat{\mathcal{Q}}_{k-l}(0)\}. \quad (32)$$

We show that $\lim_{m\to\infty} \mathcal{K}_l(m) = \mathcal{G}_l$, in two steps:

(a) *Any point of* $\mathcal{G}_l$ *is a limit point of* $\mathcal{K}_l(m)$.
Let $x = 0.x_1x_2x_3\ldots \in \mathcal{G}_l$, then according to Eq. (15), $x \in \mathcal{F}_j$ for some $j \in R(E) \cap [l - R(E)]$, and so x can be written as $x = 0.(x_1 __ x_N)_j x_{N+1}x_{N+2}\ldots$ where $w(x_{N+1}x_{N+2}\ldots) = 0$ for some $N \in \mathbb{N}_0$. Now construct the following subset of the open neighborhood $B_{x,M} = (x - \frac{1}{\sigma^M},\ x + \frac{1}{\sigma^M})$:

$$\mathcal{S}_{M,T,z} = \{0.(x_1 __ x_M __ x_N)_j(\eta_1 __ \eta_T)_{l-j} \diamond (\tilde{\nu}_z)_0 \hat{\mathcal{Q}}_{k-l}(0)\} \quad \text{when } M \leq N,$$

$$\mathcal{S}_{M,T,z} = \{0.(x_1 __ x_N __ x_M)_j(\eta_1 __ \eta_T)_{l-j} \diamond (\tilde{\nu}_z)_0 \hat{\mathcal{Q}}_{k-l}(0)\} \quad \text{when } M \geq N,$$

where $(\eta_1 __ \eta_T)_{l-j}$ is a string of digits $\eta_i \in \mathcal{Z}_\sigma$ of minimal length such that $w(\eta_1 __ \eta_T) = l-j$. (so T depends generally on $l - j$). Observe that it is always possible to find such a finite $(\eta_1 __ \eta_T)_{l-j}$ string because $(l - j) \in R(E)$ as a concequence of the fact that $l \in R(E)$ and $j \in R(E) \cap [l - R(E)]$ (otherwise $j \notin l - R(E)$). $(\tilde{\nu}_z)_0$ is a string of z digits with zero weight. It follows from the $\mathcal{S}_{M,T,z}$-defining equation above that $\mathcal{S}_{M,T,z} \subset \{0.(a_1 __ a_m)_l \hat{\mathcal{Q}}_{k-l}(0) | m = \max(M, N) + T + z\}$ and from Eq. (32) that $\mathcal{S}_{M,T,z} \subset \mathcal{K}_l(\max(M, N) + T + z)$, and this for all $z \in \mathbb{N}_0$. Together with $\mathcal{S}_{M,T,z} \subset B_{x,M}$ (by construction), this gives $B_{x,M} \cap \mathcal{K}_l(m) \neq \phi$ for all $m \geq \max(M, N) + T$, whatever the value of $M \in \mathbb{N}_0$. Hence x is a limit point of $\mathcal{K}_l(m)$ when $m \to \infty$.

(b) *No point outside* $\mathcal{G}_l$ *can be a limit point of* $\mathcal{K}_l(m)$.
Let $x = 0.x_1x_2x_3\ldots \notin \mathcal{G}_l$, then as a consequence of Eq. (15), $x \in \mathcal{F}_j$ for $j \in R(E) \cup \{+\infty\}$ but $j \notin R(E) \cap [l - R(E)]$ (or combined: $j \in R(E) \cup \{+\infty\}\backslash[l - R(E)]$). Then there is a finite N such that x can be written as $x = 0.(x_1 __ x_N)_j x_{N+1}x_{N+2}\ldots$ where $x_i \in \mathcal{Z}_\sigma^0$ for $i > N$. Let $m > N$, then $B_{x,M} = \{0.(x_1 __ x_M)_j\eta_1\eta_2\ldots | \eta_i \in \mathcal{Z}_\sigma\}$, cannot contain a point from $\mathcal{K}_l(m)$, $m > M$ because this would require that $w(0.(x_1 __ x_m)_j\eta_1 __ \eta_m) = l$, or $w(\eta_1 __ \eta_m) = l - j \in R(E_m)$. But this would imply that $j \in [l - R(E_m)] \subset [l - R(E)]$ is in conflict with the starting condition $j \in R(E) \cup \{+\infty\}\backslash[l - R(E)]$. So $B_{x,M} \cap \mathcal{K}_l(m) = \phi$ for all $m > M$ and for all M, and $x \in \mathcal{G}_l$ cannot be a limit point of $\mathcal{K}_l(m)$.

Both (a) and (b) together imply that $\mathcal{G}_l$ is the limit of the $\mathcal{K}_l(m)$-set for $m \to \infty$. Eq. (31) can now be written as the following union of *nonempty* $\mathcal{K}_j(m)$-sets:

$$\hat{\mathcal{Q}}_k(m) = \bigcup_{j\in\mathcal{V}_{m,k}} \mathcal{K}_j(m),$$

[6] The $\diamond$-sign indicates a continuation (concatenation) of the current expression. It should be discarded in its interpretation.

where $\mathcal{V}_{m,k} \triangleq R(E_m) \cap [k - R(E)] \cap \{n : \hat{\mathcal{Q}}_{k-n}(0) \neq \phi\}$. The restriction $[k - R(E)]$ leaves out of consideration all $\hat{\mathcal{Q}}_{k-j}(0)$ for which $k - j \notin R(E)$, as these sets are already enforced to be empty under the conditions of the theorem. So

$$\lim_{m\to\infty} \hat{\mathcal{Q}}_k(m) = \bigcup_{j\in\mathcal{V}_k} \mathcal{G}_j\,, \tag{33}$$

with $\mathcal{V}_k = R(E) \cap [k - R(E)] \cap \{n : \hat{\mathcal{Q}}_{k-n}(0) \neq \phi\}$.

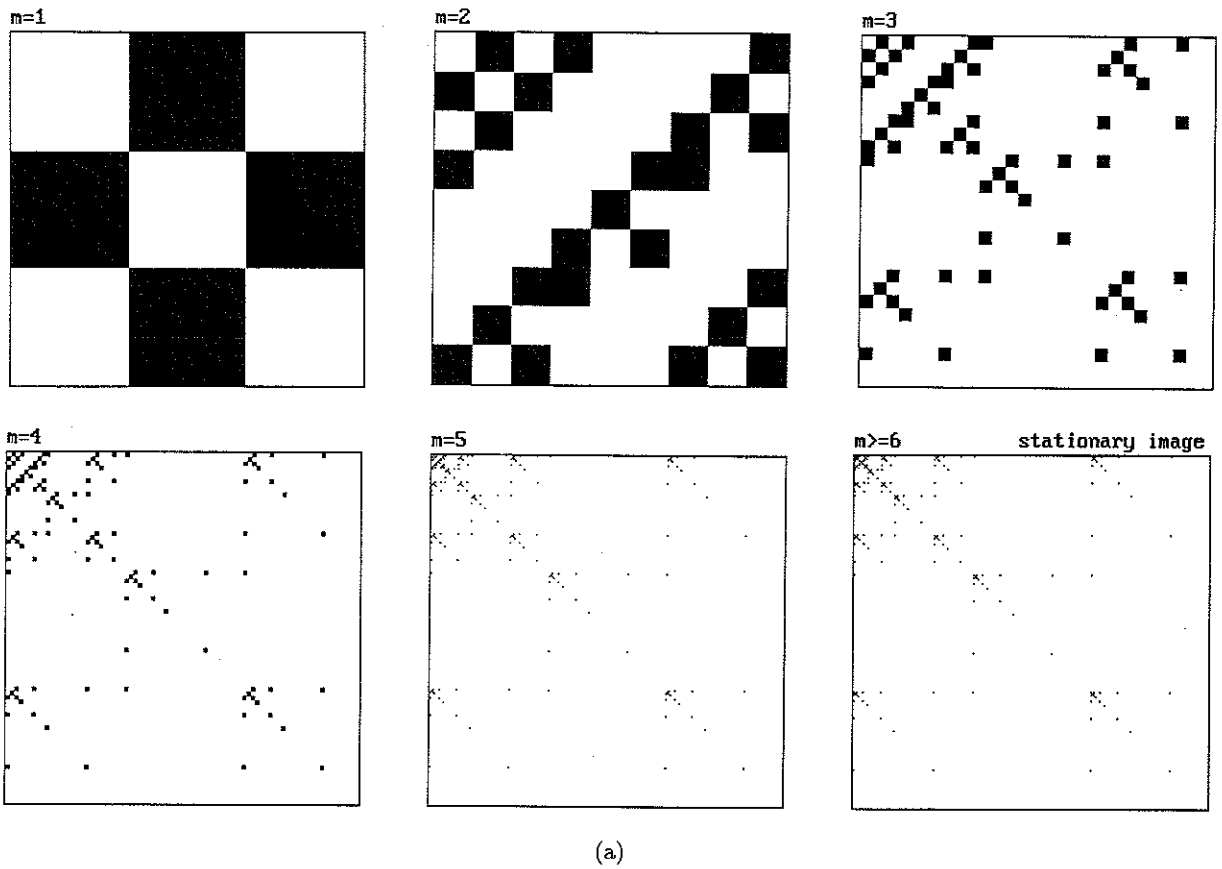
Observe that the necessary and sufficient condition for $\bigcup_{j\in\mathcal{V}_k} \mathcal{G}_j = \mathcal{G}_k$ is that $\hat{\mathcal{Q}}_{k-k}(0) = \hat{\mathcal{Q}}_0(0) \neq \phi$. Indeed, it then follows that $\bigcup_{j\in\mathcal{V}_k} \mathcal{G}_j = \mathcal{G}_k \bigcup_{j\in\mathcal{V}_k\setminus\{0\}} \mathcal{G}_j = \mathcal{G}_k$ as a consequence of Lemma 3.1.B. Otherwise if $\hat{\mathcal{Q}}_0(0) = \phi$, then $k \notin \mathcal{V}_k$ implying that $\bigcup_{j\in\mathcal{V}_k} \mathcal{G}_j \subset \mathcal{G}_k$.

B. For the M case

The proof runs along the same lines as under ***A***, but with $\mathcal{G}_l = [0, 1]$ for all l, so that $\lim_{m\to\infty} \mathcal{K}_l(m) = [0, 1]$ for all $l \in \mathbb{Z}$. Point (b) need no longer be considered. Equation (33) becomes $\lim_{m\to\infty} \hat{\mathcal{Q}}_k(m) = \bigcup_{j\in\mathcal{V}_k}[0, 1] = [0, 1](= \mathcal{G}_k)$ iff there is at least one $\hat{\mathcal{Q}}_l(0) \neq \phi$. ■

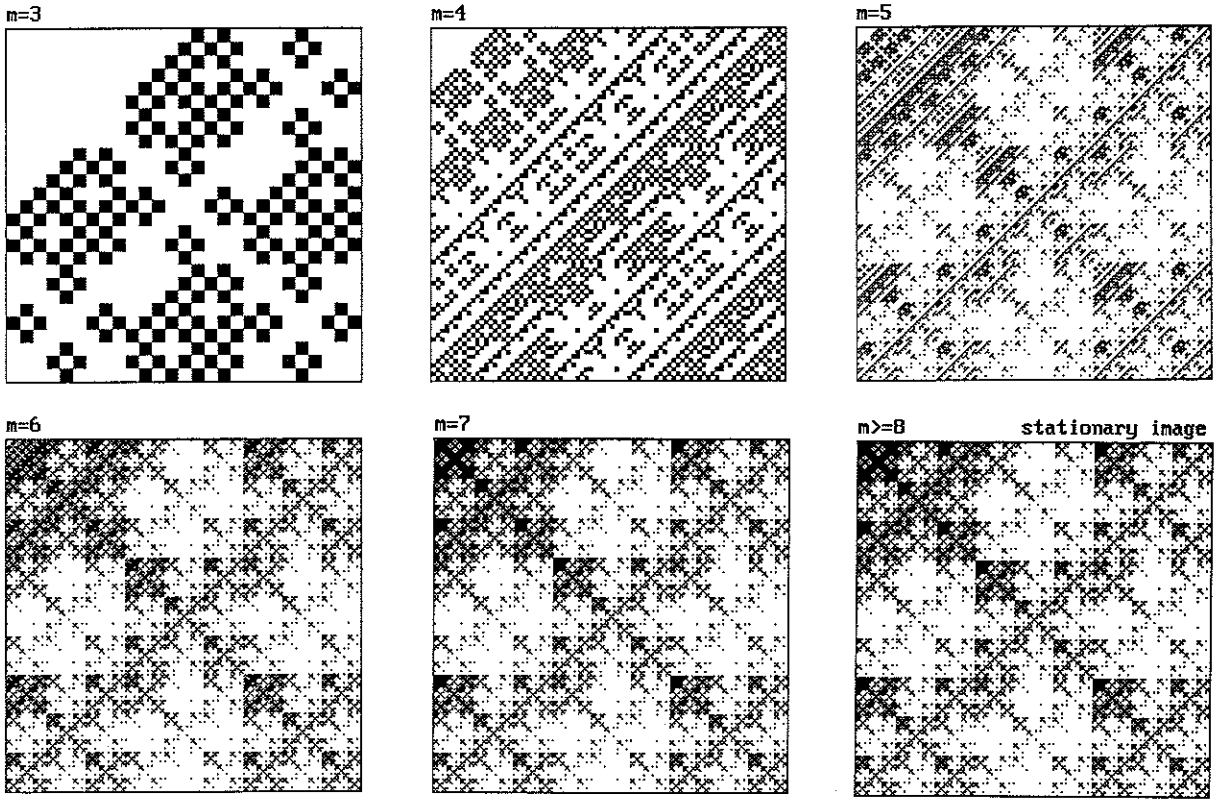
Remark. The HIFS [Eq. (30)] being a contractive operation, we could have applied the contraction mapping theorem (Peitgen [1992] referring to Hutchinson [1981], Barnsley [1989]). It states that $\hat{\mathcal{Q}}_k(j)$ converges in Hausdorff distance to a unique compact invariant attractor set, and this for all (compact) initial conditions. As only $[0, 1]^{(p)}$ itself for the M case, and $\mathcal{G}_k$ for the SZP and MZP cases are compact invariants for the HIFS, these are indeed the respective attractor sets.

Figures 5, 6 and 7 show the evolution of $\hat{\mathcal{Q}}_k(m)$ for different generator types, different k and different initial conditions, all under the restricted resolution of our display. How this limited resolution

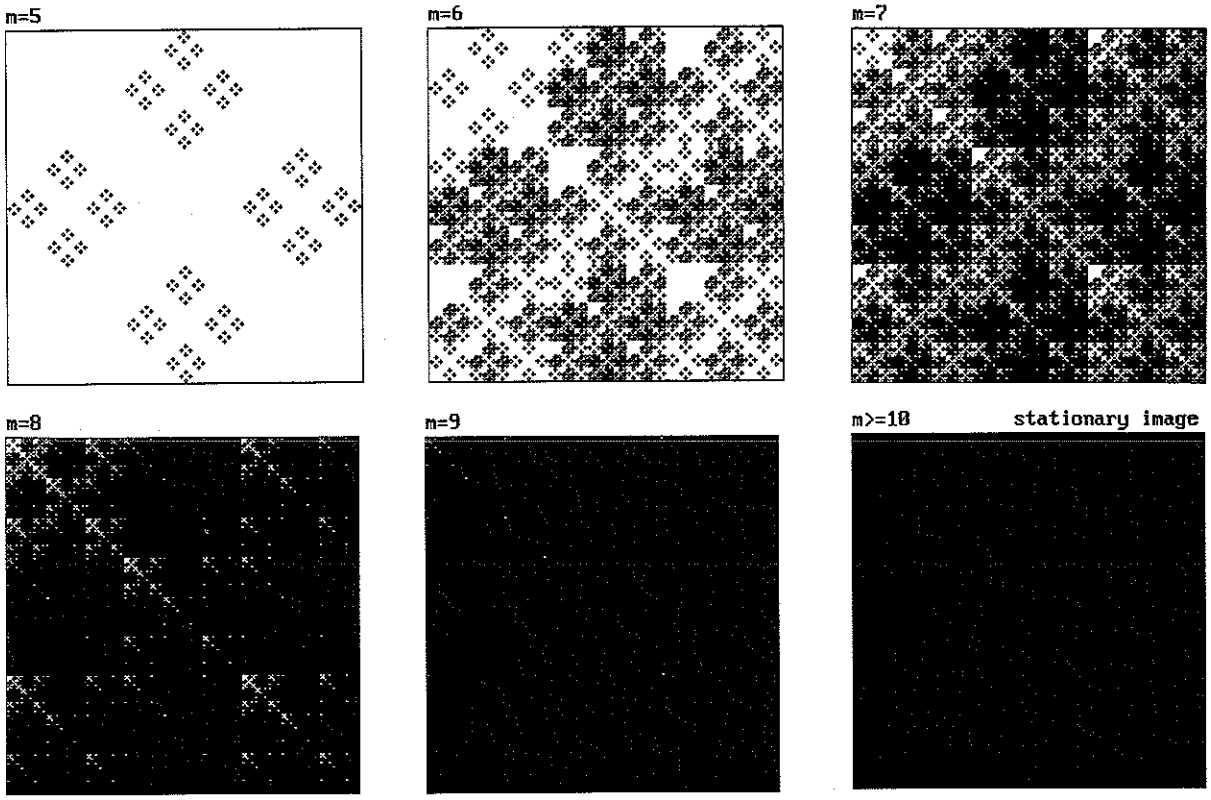


(a)

Fig. 5. Illustration of the evolution of the fractrix HIFS image $\hat{\mathcal{Q}}_k(m)$ for different k under limited resolution: the logsize-5 resolution display $\mathcal{S}_5\hat{\mathcal{Q}}_k(m)$ for the SZP generator $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (243 × 243 pixels over $[0, 1]^2$). The initial conditions are $\hat{\mathcal{Q}}_0(0) = [0, 1]^2$ and $\hat{\mathcal{Q}}_j(0) = \phi$ for $j \neq 0$. (a) $k = 2$, for m ranging from 1 to 6 ($m \geq 6$: stationary image, $m = 0$: empty image) (b) $k = 5$, for m ranging from 3 to 8 ($m \geq 8$: stationary image, $m \leq 2$: empty image) (c) $k = 10$, for m ranging from 5 to 10 ($m \geq 10$: stationary image, $m \leq 4$: empty image).



(b)



(c)

Fig. 5. (*Continued*)

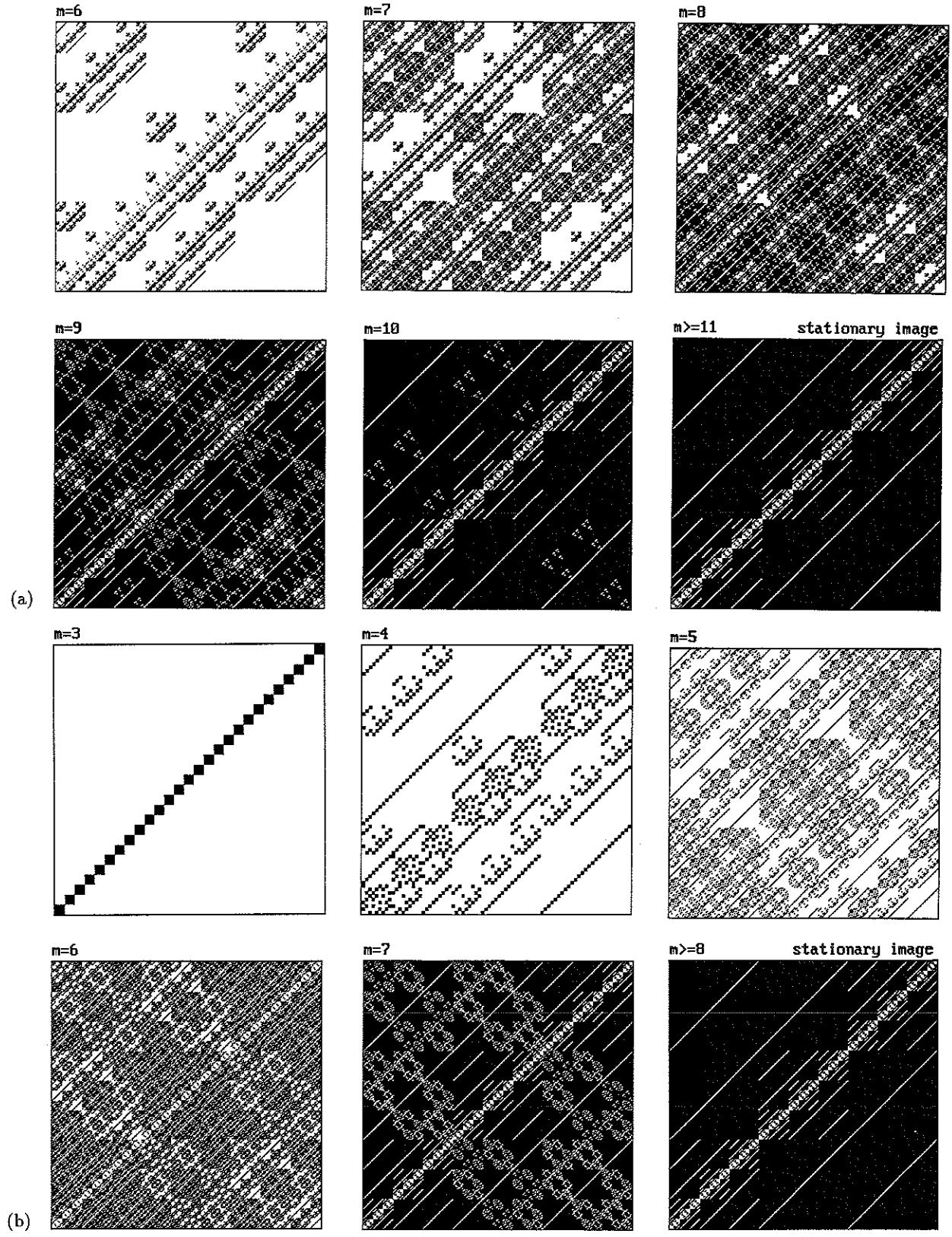
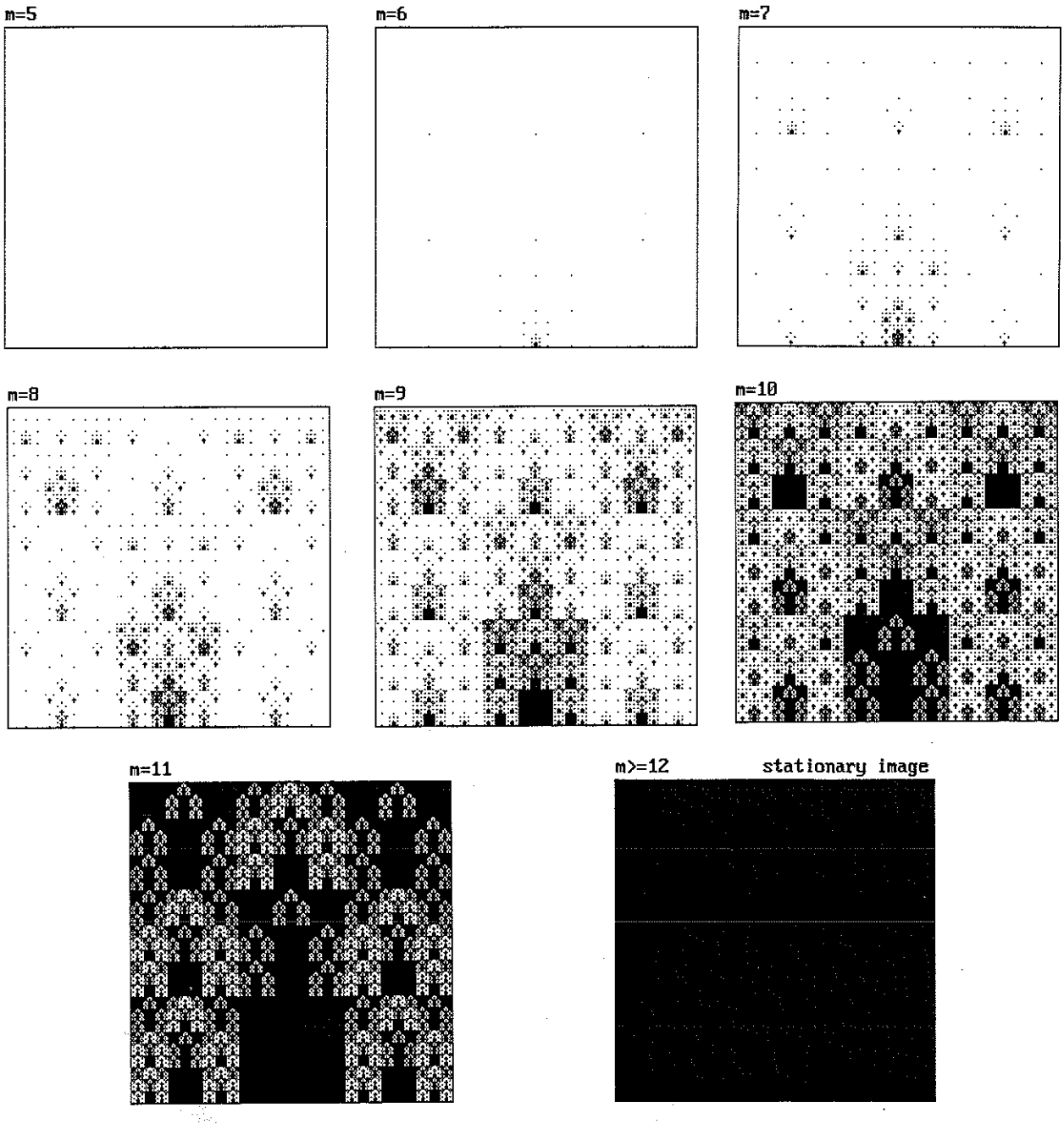


Fig. 6. The stationary image of a fractrix HIFS for a given level k is independent of the initial condition (as long as $\hat{\mathcal{Q}}_0(0) \neq \phi$): logsize-5 resolution display $\mathcal{S}_5\hat{\mathcal{Q}}_{21}(m)$ for the MZP generator $\begin{bmatrix} 2 & 0 & 5 \\ 0 & 5 & 2 \\ 5 & 2 & 0 \end{bmatrix}$ (a) Initial conditions: $\hat{\mathcal{Q}}_0(0) = [0, 1]^2$ and $\hat{\mathcal{Q}}_j(0) = \phi$ for $j \neq 0$ (empty image for $m < 6$, stationary image for $m \geq 11$). (b) Initial conditions: $\hat{\mathcal{Q}}_j(0) = [0, 1]^2$ for $j \in \mathcal{J} = \{0, 6, 7\}$ and $\hat{\mathcal{Q}}_j(0) = 0$ for $j \notin \mathcal{J}$ (empty image for $m < 3$, stationary image for $m \geq 8$).

affects the evolution and the outlook of the pictures will be investigated in depth in Sec. 3.5. Compare also with the corresponding colorful E_5 versions in Fig. 8. We will now first concentrate on the geometrical structure of the $\mathcal{F}_k$ and $\mathcal{G}_k$-sets.

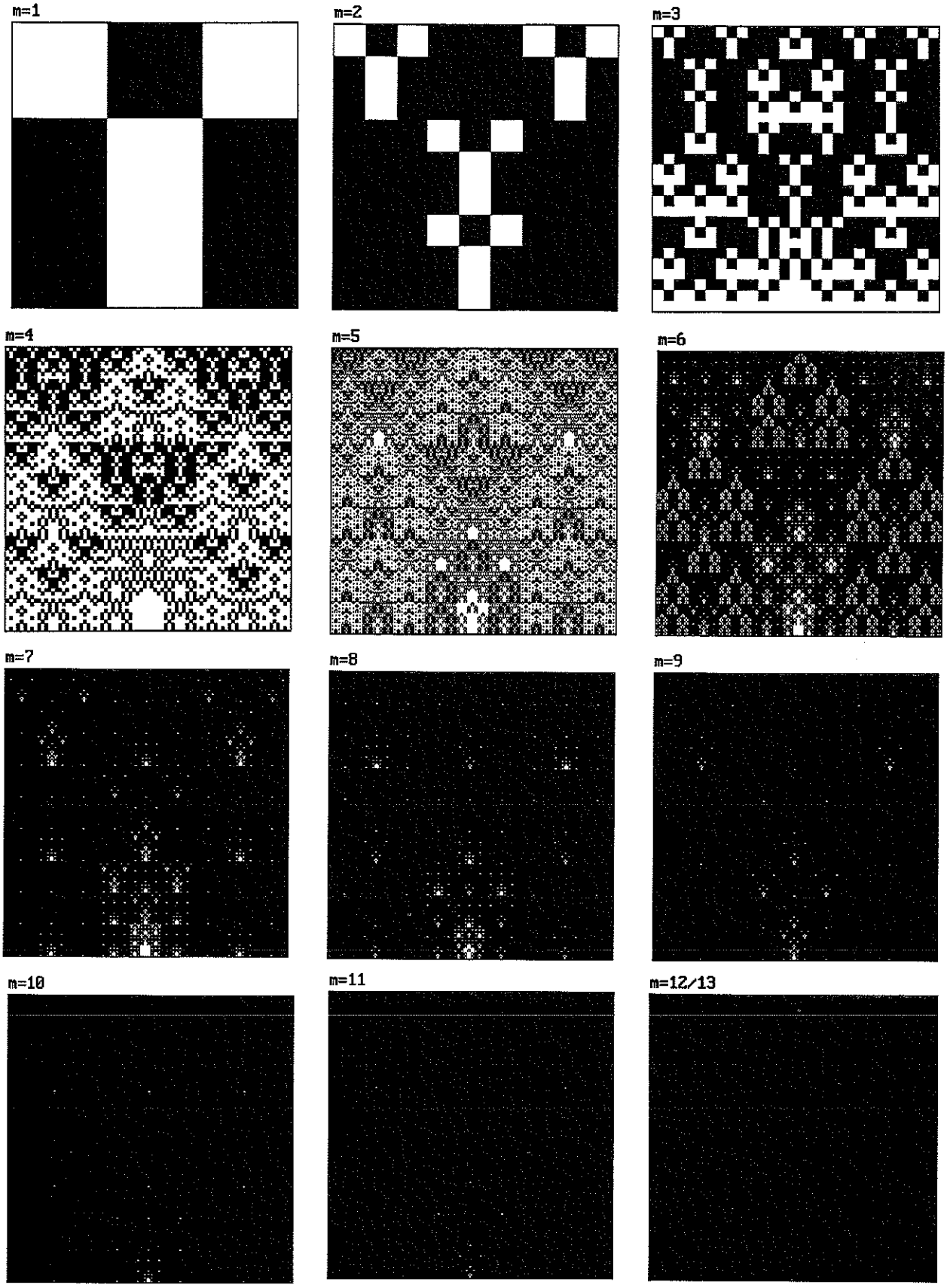
3.3. $\mathcal{F}_k$ and $\mathcal{G}_k$ as hierarchical Zeno and Cantor sets.

In its finite, *outwardly* growing version, a fractrix E becomes usually more interesting when one moves



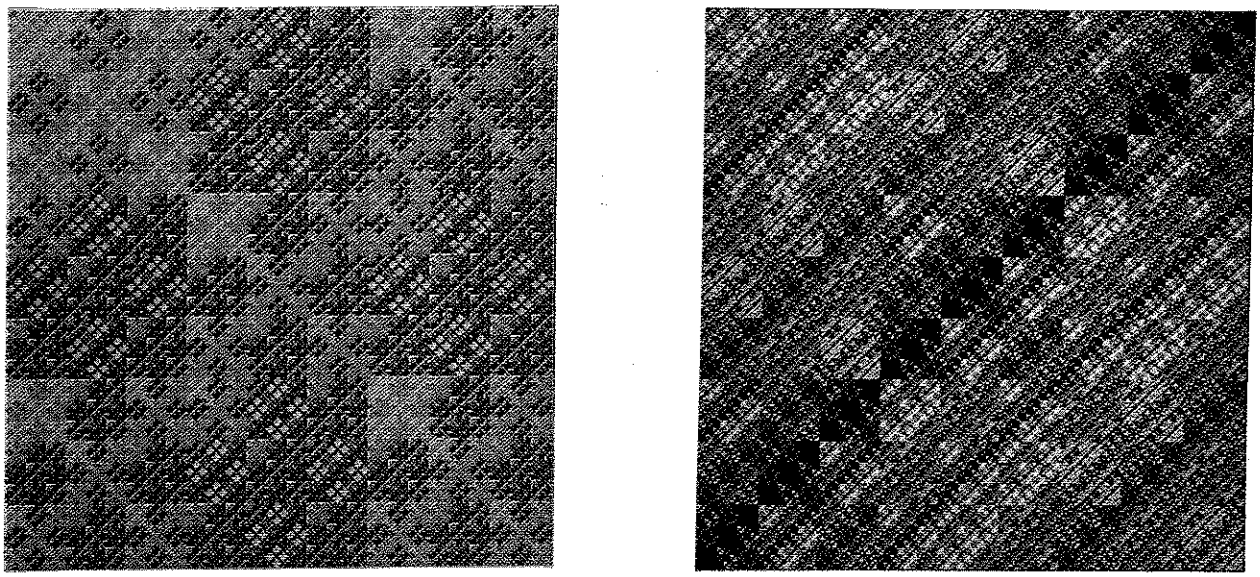
(a)

Fig. 7. Illustration of the evolution of the fractrix HIFS towards a saturated display for mixed-type generators. The logsize-5 resolution display $\mathcal{S}_5\hat{\mathcal{Q}}_k(m)$ for the generator $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 2 & -1 \end{bmatrix}$. The initial condition is $\hat{\mathcal{Q}}_j(0) = [0,\,1]^2$ for $j \in \mathcal{J} = \{-2, 0, 1\}$ and $\hat{\mathcal{Q}}_j(0) = \phi$ for $j \notin \mathcal{J}$. (a) $k = 10$ (empty image for $m < 5$, saturated for $m \geq 12$). (b) $k = -1$ (empty image for $m = 0$, saturated for $m \geq 14$).

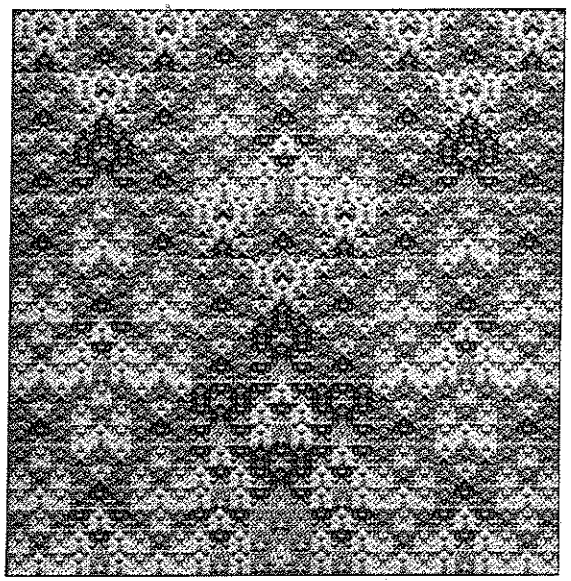


(b)

Fig. 7. (*Continued*)



(a) (b)



(c)

Fig. 8. Colorful E_5 versions based on the generators of Figs. 5, 6 and 7 (display resolution: 243×243 pixels). The generators are (a) $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 & 5 \\ 0 & 5 & 2 \\ 5 & 2 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 2 & -1 \end{bmatrix}$. The pixels in (b) that map into a value >16 are all black (by lack of sufficient colors). All pictures in Figs. 5, 6 and 7 for $m \geq 5$ are unions of several subsets of different colors in the respective colorful images above (see Sec. 3.5).

from the category SZP over MZP towards the M category. The reason for this is the fact that small elements are abundantly distributed over all parts of E in the M case, while this is less so in the MZP case and almost absent in the SZP case. However, the limit version of the *inwardly* growing counterpart is totally noninteresting for the M case as it completely blackens the whole $[0, 1]^{(p)}$-space. An

interesting structure only remains for the SZP and MZP cases. This structure will now be studied a bit closer.

A. Structure of $\mathcal{F}_k$ and $\mathcal{G}_k$ in the SZP case

Consider the example $E_1 = [2 \quad 0 \quad 1]$, implying *triadic* representations. Then $\mathcal{F}_0 = \{0.111\ldots\} = \{\frac{1}{2}\}$. According to Eq. (26), with $\mathcal{Q}_k = \mathcal{F}_k$, we find

$$\begin{aligned}\mathcal{F}_1 &= 0.0\mathcal{F}_{-1} \cup 0.1\mathcal{F}_1 \cup 0.2\mathcal{F}_0 \\ &= 0.1\mathcal{F}_1 \cup 0.2\mathcal{F}_0, \quad \text{as } \mathcal{F}_{-1} = \phi\,, \end{aligned} \tag{34}$$

So $0.2\mathcal{F}_0 = \{0.2\bar{1}\} \subseteq \mathcal{F}_1$, and by the recursion term $0.1\mathcal{F}_1$ in Eq. (34), it follows that $0.12\bar{1}, 0.112\bar{1}, \ldots \in \mathcal{F}_1$. In general, we have

$$\begin{aligned}\mathcal{F}_1 &= \{0.1^m 2\bar{1} | m \in \mathbb{N}\} = \left\{\frac{5}{6}, \frac{11}{18}, \frac{29}{54}, \frac{83}{162}\cdots\right\} \\ &= \left\{\frac{3^k+2}{2.3^k}\middle| k \geq 1\right\}. \end{aligned} \tag{35}$$

We will call such a set *Zeno-set*, in reference to Zeno's paradox where a set of similar nature appears: $\mathcal{Z} = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}\ldots\}$. This is the set of points in $[\frac{1}{2}, 1]$ that is visited when moving from $\frac{1}{2}$ to 1 by taking steps that halve the remaining distance to 1 (recall footnote 5). $\mathcal{F}_1$ is the set of points when going from $\frac{5}{6}$ to $\frac{1}{2}$, by moving over $\frac{2}{3}$ of the remaining interval at each step. Observe that neither $\mathcal{Z}$ nor $\mathcal{F}_1$ contains their limit points ($1, \frac{1}{2}$ respectively). But $\mathcal{F}_0 = \{\frac{1}{2}\}$ is the set of limit points of $\mathcal{F}_1$. According to Property 2.4, $\mathcal{G}_1 = \overline{\mathcal{F}}_1 = \mathcal{F}_0 \cup \mathcal{F}_1$ forms all points that map in the limit into 1.

Continuing, we find

$$\mathcal{F}_2 = 0.0\mathcal{F}_0 \cup 0.1\mathcal{F}_2 \cup 0.2\mathcal{F}_1\,. \tag{36}$$

From Eqs. (23) and (35), we obtain

$$\begin{aligned}0.2\mathcal{F}_1 &= \frac{2}{3} + \frac{1}{3}\left\{\frac{3^k+2}{2.3^k}\middle| k \geq 1\right\} \\ &= \left\{\frac{5.3^k+2}{2.3^{k+1}}\middle| k \geq 1\right\}, \\ 0.0\mathcal{F}_0 &= \frac{1}{3}\left\{\frac{1}{2}\right\} = \left\{\frac{1}{6}\right\}. \end{aligned}$$

So

$$\begin{aligned}\mathcal{F}_2' \triangleq 0.0\mathcal{F}_0 \cup 0.2\mathcal{F}_1 &= \left\{\frac{1}{6}\right\} \\ &\cup \left\{\frac{5.3^k+2}{2.3^{k+1}}\middle| k \geq 1\right\} \subseteq \mathcal{F}_2\,. \end{aligned} \tag{37}$$

From Eq. (36), we infer that $0.1\mathcal{F}_2' \subseteq \mathcal{F}_2$, implying that also $0.1\mathcal{F}_2' \cup \mathcal{F}_2' \subseteq \mathcal{F}_2$, and by induction we find that $\bigcup_{m\in\mathbb{N}} 0.1^m\mathcal{F}_2' \subseteq \mathcal{F}_2$.

Actually,

$$\mathcal{F}_2 = \bigcup_{m\in\mathbb{N}} 0.1^m\mathcal{F}_2'\,, \tag{38}$$

as this expression makes Eq. (36) an identity:

$$\begin{aligned}\mathcal{F}_2 &\overset{(38)}{=} \bigcup_{m\in\mathbb{N}} 0.1^m\mathcal{F}_2' \equiv \bigcup_{m\in\mathbb{N}_0} 0.1^m\mathcal{F}_2' \cup \mathcal{F}_2' \\ &\equiv \frac{1}{3}\bigcup_{m\in\mathbb{N}} 0.1^m\mathcal{F}_2' \cup \mathcal{F}_2' \\ &\overset{(37,38)}{\equiv} 0.1\mathcal{F}_2 \cup 0.0\mathcal{F}_0 \cup 0.2\mathcal{F}_1\,. \end{aligned}$$

From Eqs. (37) and (38), we obtain the explicit solution

$$\begin{aligned}\mathcal{F}_2 &= \left\{\frac{3^k-2}{2.3^k}\middle| k \geq 1\right\} \\ &\diamond \bigcup_{m\in\mathbb{N}}\left[\frac{3^{m-1}}{2.3^m} + \frac{1}{3^m}\left\{\frac{5.3^k+2}{2.3^{k+1}}\middle| k \geq 1\right\}\right], \end{aligned}$$

whereby $a + bS$, with S an arbitrary set of numbers, denotes the set obtained by multiplying each element of S by b and adding a. The set of limit points of $\mathcal{F}_2$ is obtained by letting $k \to \infty$, and gives

$$\left\{\frac{1}{2}\right\} \cup \left\{\frac{3^{m+1}+2}{2.3^{m+1}}\middle| m \geq 0\right\} = \left\{\frac{1}{2}, \frac{5}{6}, \frac{11}{18}, \frac{29}{54}, \cdots\right\}.$$

Observe that this equals $\mathcal{F}_0 \cup \mathcal{F}_1$ in conformity with Property 2.4, showing also that $\mathcal{G}_2 = \overline{\mathcal{F}}_2 = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$ is the set of all points that map in the limit into 2.

All further $\mathcal{F}_k$ can now be obtained in a similar way from its two predecessor sets $\mathcal{F}_{k-1}$ and $\mathcal{F}_{k-2}$, by using the second form of Property 3.1 for this example:

$$\mathcal{F}_k = \frac{1}{3}\mathcal{F}_{k-2} \cup \left(\frac{1}{3} + \frac{1}{3}\mathcal{F}_k\right) \cup \left(\frac{2}{3} + \frac{1}{3}\mathcal{F}_{k-1}\right).$$

This operation can be interpreted in an alternative way, as shown in Fig. 9. First, put a $\frac{1}{3}$-size copy of $\mathcal{F}_{k-2}$ in the interval $L_0 = [0, \frac{1}{3}]$ and a similar copy of $\mathcal{F}_{k-1}$ in $R_0 = [\frac{2}{3}, 1]$. Then make a $\frac{1}{3}$-size copy of these two intervals and put these copies in the intervals $L_1 = \frac{1}{3} + \frac{1}{3}[0, \frac{1}{3}]$ and $R_1 = \frac{1}{3} + \frac{1}{3}[\frac{2}{3}, 1]$. Then make a $\frac{1}{3}$-size copy of L_1 and R_1 and put these

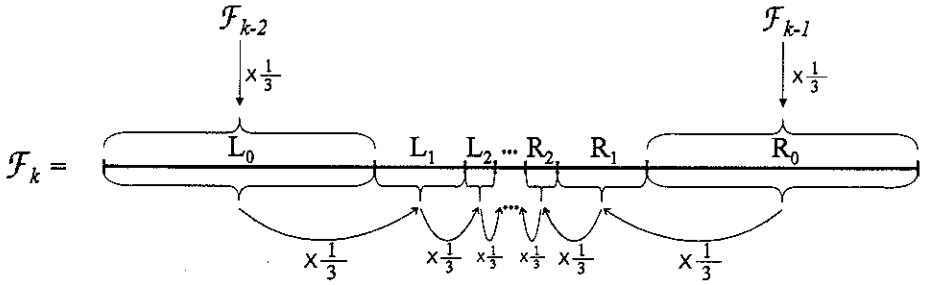


Fig. 9. $\mathcal{F}_k$ as a Zeno-nesting of Zeno-sets for $k \geq 1$ (generator [2 0 1]), to start with $\mathcal{F}_k = \phi$ for $k < 0$ and $\mathcal{F}_0 = \frac{1}{2}$. $\mathcal{F}_k$ does not contain its limit points. But $\mathcal{G}_k$, which is obtained by a similar scheme where $\mathcal{F}_{k-e(j)}$ is replaced by $\mathcal{G}_{k-e(j)}$ and by letting the Zenonian squeezing jump into the actual infinity, does!

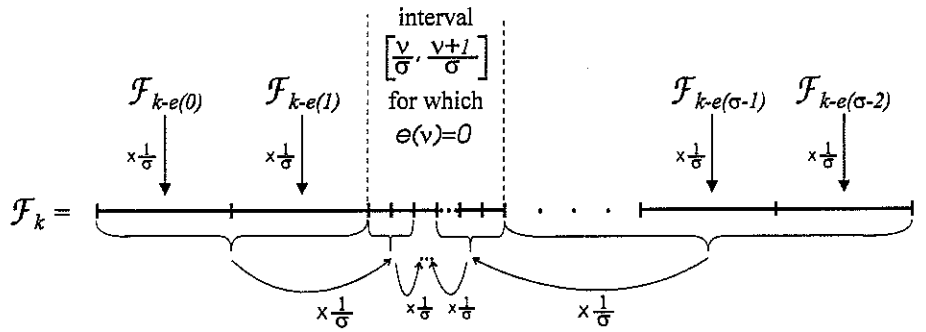


Fig. 10. Zeno-nesting for an arbitrary SZP-generator of size σ.

in $L_2 = \frac{1}{3} + \frac{1}{3}L_1$, $R_2 = \frac{1}{3} + \frac{1}{3}R_1$ etc. This recurrent copying corresponds to the feedback term $\frac{1}{3} + \frac{1}{3}\mathcal{F}_k$ in the last equation. The ultimate global result is $\mathcal{F}_k$ (under our assumption that such constructions do not reach their limit points under a potentially infinite continuation).

The similarity with the Zeno set is clear. As $\mathcal{F}_1$ was shown to be Zeno-like, $\mathcal{F}_2$ is a Zeno-nesting of Zeno-sets, $\mathcal{F}_3$ a Zeno-nesting of a Zeno-nesting of Zeno-sets, and for $\mathcal{F}_k$ we will find a $(k-1)$-fold Zeno-nesting. Sets like these will be called *hierarchical Zeno-sets*. All $\mathcal{F}_k$, $k \in \mathbb{N}_0$ are *infinite denumerable* sets. $\mathcal{G}_k$ is obtained by including the limit points of this *Zenonian squeezing*,[7] i.e. by adding $\bigcup_{j \in R(E) \cap [k-R(E) \setminus \{0\}]} \mathcal{F}_j$ to $\mathcal{F}_k$. It follows that when the $\mathcal{F}_{k-e(j)}$ are replaced by $\mathcal{G}_{k-e(j)}$ in the squeezing scheme, and we allow the scheme also to jump into the actual infinity (i.e. to include its limit points), it produces $\mathcal{G}_k$.

What about $\mathcal{F}_\infty$? Clearly $\mathcal{F}_\infty = [0, 1] \setminus \bigcup_{k \in \mathbb{N}} \mathcal{F}_k$, and it is a *nondenumerable* set whose closure is [0, 1].

[7] "Zenonian squeezing": a suggestive term found in Rucker [1989].

The scheme of Fig. 9 can be easily generalized for arbitrary SZP generators. This is shown in Fig. 10. The starting set $\mathcal{F}_0$ is given by the singleton $\{\frac{\nu}{\sigma-1} = 0^\sigma.\nu\nu\nu\ldots\}$, where ν is the only digit whose weight $e(\nu) = 0$.

B. Structure of $\mathcal{F}_k$ and $\mathcal{G}_k$ in the MZP case.

$\mathcal{F}_0$ is the set of points that have a σ-adic representation consisting only of digits with zero weight. Formally

$$\mathcal{F}_0 = \{0.\nu_1\nu_2\nu_3\ldots | \nu_i \in \mathcal{Z}_\sigma^0\}.$$

This set is a generalization of the triadic Cantor set $\mathcal{C}$, which is the $\mathcal{F}_0$ related to the generator [0 1 0]. It can be constructed as the limit of the geometrical recursion which deletes the intervals associated with positive weights in [0, 1] (instead of the middle-third, see Fig. 11(a). These $\mathcal{F}_0$-sets are *nondenumerable*, in contrast with the SZP case, and they are also *perfect*: they completely consist of their limit points. Now consider $\mathcal{F}_k$ for $k \in R(E)$: it consists of those points with a σ-adic representation that only contains a finite number of digits with nonzero weight. The sum total of these

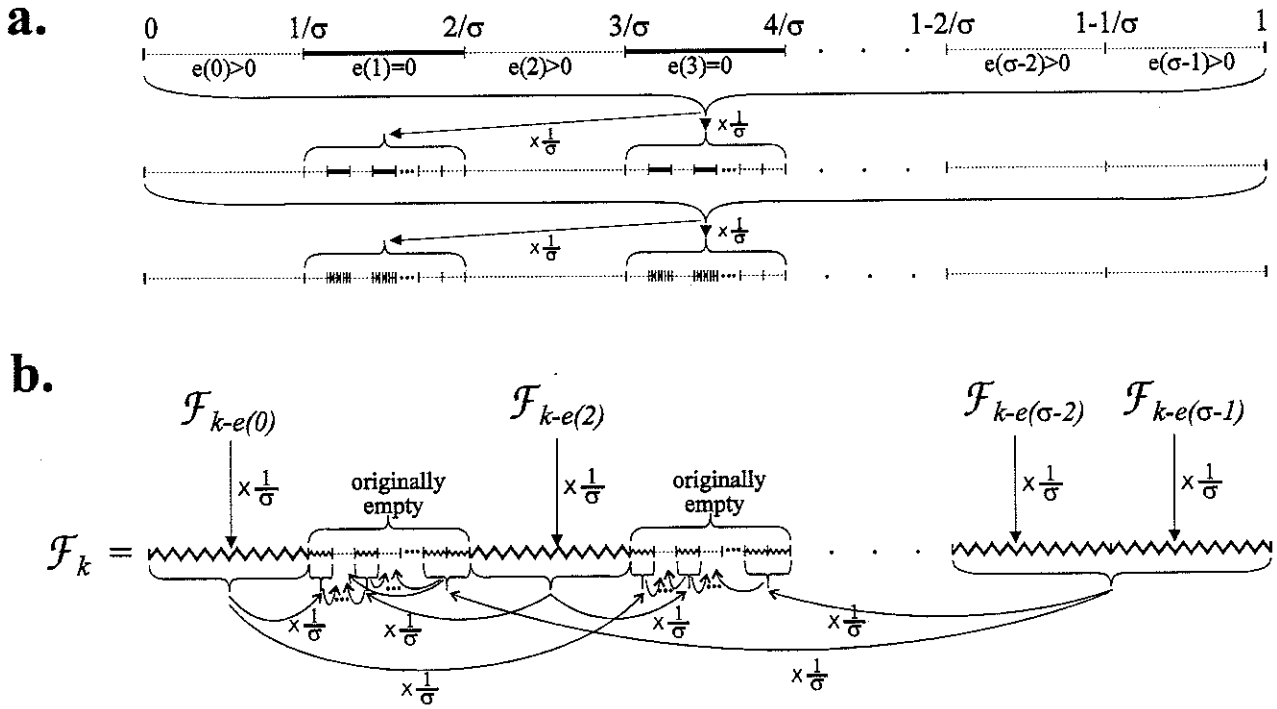


Fig. 11. (a) Construction of the $\mathcal{F}_0$ Cantor set: $\mathcal{F}_0$ is the limit of the procedure shown. (b) Construction of $\mathcal{F}_k$ through Zenonian squeezing of Cantor sets: start with the hierarchical Cantor sets $\mathcal{F}_{k-e(j)}, e(j) \neq 0$, and with empty subintervals ν_i for which $e(\nu_i) = 0$. Make $\frac{1}{\sigma}$ copies of what's already available in the remaining empty intervals. Do not include the limit points. $\mathcal{G}_k$ is obtained in a similar way, by replacing $\mathcal{F}_{k-e(j)}$ with $\mathcal{G}_{k-e(j)}$, and by including the limit.

nonzero weights must be k. Again, these $\mathcal{F}_k$ are also Cantor sets, but they are *nonperfect* as they do not contain all their limit points (Property 2.4.B). However, $\mathcal{G}_k = \bigcup_{j \in R(E) \cap [k - R(E)]} \mathcal{F}_j = \bar{\mathcal{F}}_k$ is perfect.

Again, as illustrated in Fig. 11(b), $\mathcal{F}_k$ can be obtained from the $\mathcal{F}_j$, $j < k$ by a Zenonian squeezing of these $\mathcal{F}_j$. This squeezing results from the self-referring part $\bigcup_{\nu_i \in \mathcal{Z}_\sigma^0} 0.\nu_i \mathcal{F}_k$ in the collage relationship

$$\mathcal{F}_k = \bigcup_{a_i \in \mathcal{Z}_\sigma^+} 0.a_i \mathcal{F}_{k-e(a_i)} \bigcup_{\nu_i \in \mathcal{Z}_\sigma^0} 0.\nu_i \mathcal{F}_k \,. \tag{39}$$

We give a formal proof, which is also valid for the SZP case.

Property 3.4. (*Zenonian squeezing property for the NN case*)

$$\text{Let } \mathcal{F}'_k = \bigcup_{a_i \in \mathcal{Z}_\sigma^+} 0.a_i \mathcal{F}_{k-e(a_i)} \,, \tag{40}$$

then $\mathcal{F}_k$ *is given by Zenonian squeezing*

$$\mathcal{F}_k = \mathcal{F}'_k \bigcup_{m \in \mathbb{N}_0, \nu_i \in \mathcal{Z}_\sigma^0} 0.\nu_1 __ \nu_m \mathcal{F}'_k \,. \tag{41}$$

Proof. Observe that Eq. (41) can be written as

$$\mathcal{F}_k = \mathcal{F}'_k \bigcup_{\nu_1 \in \mathcal{Z}_\sigma^0} 0.\nu_1 \mathcal{F}''_k \,, \tag{42}$$

where

$$\mathcal{F}''_k \triangleq \mathcal{F}'_k \bigcup_{(m-1) \in \mathbb{N}_0, \nu'_i \in \mathcal{Z}_\sigma^0} 0.\nu'_1 __ \nu'_{m-1} \mathcal{F}'_k \overset{(41)}{=} \mathcal{F}_k \,.$$

Using $\mathcal{F}''_k = \mathcal{F}_k$ and Eq. (40) in Eq. (42) produces the collage equation Eq. (39) as an identity. ■

So, $\mathcal{F}_k$ results from a hierarchical nesting of Cantor sets. Hence the name: *hierarchical Cantor sets*. $\mathcal{G}_k$ is again obtained as the limit of the Zenonian squeezing of a similar scheme as in Fig. 11(b), whereby the $\mathcal{F}_{k-e(a_i)}$ have to be replaced by the $\mathcal{G}_{k-e(a_i)}$.

3.4. *HIFS for expanding k-indicator matrices*

Let $I_k(m)$ denote the binary matrix that indicates where the value k appears in the p-dimensional matrix E_m: i.e. the logsize-m matrix where a k is

replaced by 1 and all other values by 0. This is called the *k-indicator matrix*. $GI_k(m)$ will denote the graphical representation, where each element in the matrix is symbolized by an elementary (p-dimensional) cell of fixed size, and where a cell is made black for a 1 and white for a 0 (or any other binary color scheme). Actually, $GI_k(m) = \sigma^m G\hat{\mathcal{L}}_k(E_m)$ if the cells in $GI_k(m)$ have unit size. Anyhow, $GI_k(m)$ and $G\hat{\mathcal{L}}_k(E_m)$ are isomorphic. The HIFS of Eq. (30) can now be reinterpreted as

$$\forall k \in \mathbb{Z} : I_k(m+1) = 0I_{k-e(0)}(m) \cup 1I_{k-e(1)}(m) \cdots \cup (\sigma-1)I_{k-e(\sigma-1)}(m)\,, \quad (43)$$

meaning that $I_k(m+1)$ is obtained by taking $I_{k-e(a)}(m)$ as the logsize-m subblock at address a in $I_k(m+1)$, and this for all $a \in \mathcal{Z}_\sigma$ (for the p-dimensional case, σ should actually be replaced by σ_π cf. Appendix A). The same relationship holds for the graphical representation

$$GI_k(m+1) = \bigcup_{a=0}^{\sigma-1} aGI_{k-e(a)}(m)\,.$$

The iterations should be started with $I_k(1)$, $k \in R(E)$, i.e. with the indicator matrices of the generator E_1. This is illustrated in Fig. 12 for an M-type one-dimensional example.

3.5. *Limited resolution views on the $\hat{\mathcal{Q}}_k(m)$ evolution*

Hitherto, we have been looking with real-number precision at all sets involved. However, in the practicable real world, this is a nonreachable limit resolution. Especially graphical representations will be restricted by much coarser finite resolutions. We will now explain how the low-resolution pictures of the evolution of $\hat{\mathcal{Q}}_k(m)$ as shown in the Figs. 5, 6 and 7 are obtained.

Suppose that we have a p-dimensional logsize-r display $\mathcal{S}_r$ over $[0, 1]^p$, i.e. a display whose pixels occupy the cells of the L_r-lattice over $[0, 1]^p$. Each pixel has a natural r-digit address $0.a_1a_2 __ a_r$ [in p-dimensional σ-adic or in one-dimensional σ_π-adic notation (see Appendix A for the equivalence)]. How will $\hat{\mathcal{Q}}_k(m)$ appear in such a display under the condition that a pixel will be highlighted (filled,

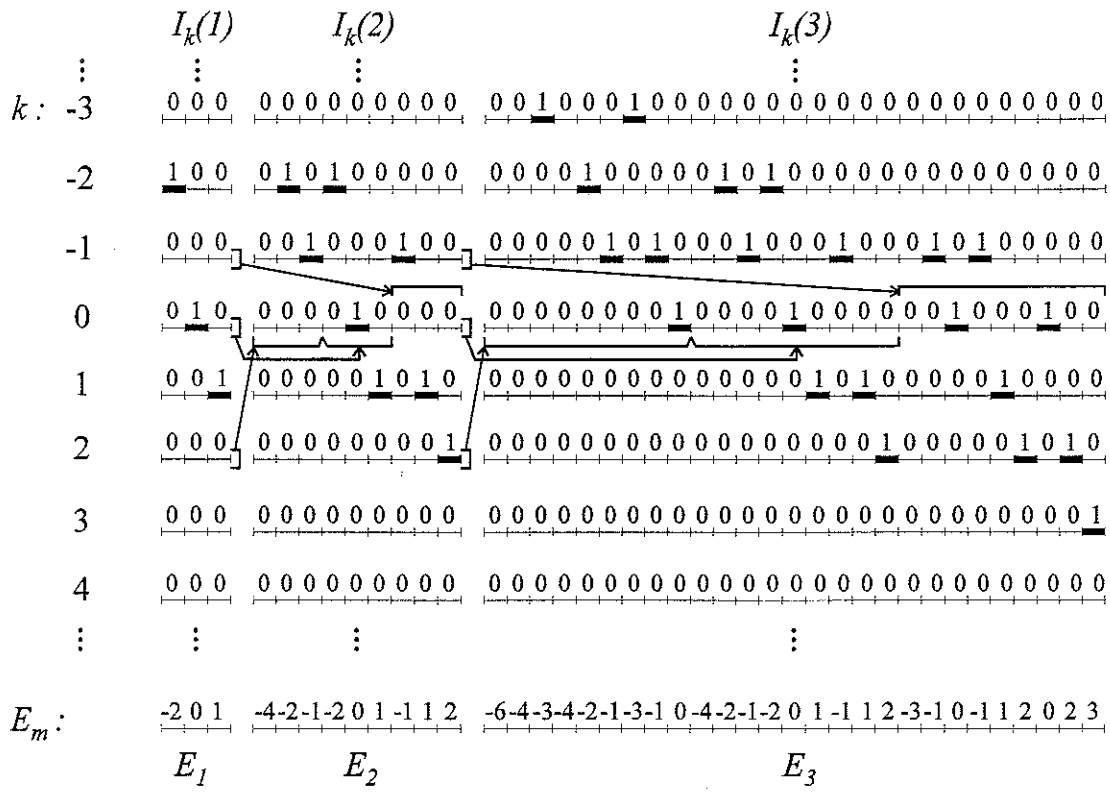


Fig. 12. The outwardly growing HIFS for obtaining $I_k(m)$ and $GI_k(m)$ illustrated for the generator $[-2\ 0\ 1] : I_k(m+1) = 0I_{k+2}(m) \cup 1I_k(m) \cup 2I_{k-1}(m)$. The copying-sequence is indicated with respect to level 0: it is the same (but vertically shifted) for all levels.

occupied, blackened) whenever it contains a point that belongs to $\hat{\mathcal{Q}}_k(m)$?

Property 3.5.

(A) *Let $\mathcal{J} \subset R(E)$ be the set of integers j for which the initial condition is $\hat{\mathcal{Q}}_j(0) = [0, 1]^{(p)}$, while $\hat{\mathcal{Q}}_i(0) = \phi$ for $i \notin \mathcal{J}$. Let $0 \in \mathcal{J}$. Then the r-resolution display of $\hat{\mathcal{Q}}_k(m)$, $k \in R(E)$, denoted by $\mathcal{S}_r\hat{\mathcal{Q}}_k(m)$ is obtained by highlighting the pixels with r-address $0.a_1a_2 __ a_r$ such that*

when $m \leq r$:

$$w(a_1a_2 __ a_m) = \sum_{i=1}^{m} e_1(a_i) = k - j \qquad (44)$$

for some $j \in \mathcal{J}$,

and,

when $m > r$:

$$w(a_1a_2 __ a_r) = \sum_{i=1}^{r} e_1(a_i) = k - j - s \qquad (45)$$

for some $j \in \mathcal{J}$ and some $s \in R(E_{m-r})$.

Or, equivalently: for $k \in R(E)$:

$$\mathcal{S}_r\hat{\mathcal{Q}}_k(m) = \bigcup_{j \in \mathcal{J}} G\hat{\mathcal{L}}_{k-j}(E_m) \quad \text{when } m \leq r\,, \qquad (46)$$

$$\mathcal{S}_r\hat{\mathcal{Q}}_k(m) = \bigcup_{\substack{j \in \mathcal{J} \\ s \in R(E_{m-r})}} G\hat{\mathcal{L}}_{k-j-s}(E_r) \quad \text{when } m > r\,. \qquad (47)$$

(B) *Moreover, $\mathcal{S}_r\hat{\mathcal{Q}}_k(m)$ reaches the stationary pattern*

$$\mathcal{S}_r\hat{\mathcal{Q}}_k = \bigcup_{u \in R(E_r) \cap [k - R(E)]} G\hat{\mathcal{L}}_u(E_r)\,, \qquad (48)$$

which is independent of m and $\mathcal{J}$. This pattern is obtained for

$$m \geq m_0(\mathcal{J}, r, k, E)\,, \qquad (49)$$

where $m_0(\mathcal{J}, r, k, E)$ is the minimal value of m for which

$$R(E_r) \cap [k - R(E)] \subseteq \{k - j - s | j \in \mathcal{J}, s \in R(E_{m-r})\} \qquad (50)$$

(stationarity condition).

The maximal value of m_0 occurs for $\mathcal{J} = \{0\}$, i.e.

$$m_0(\mathcal{J}, r, k, E) \leq m_0(\{0\}, r, k, E)\,. \qquad (51)$$

(C) *Let $e_{1\max}$ and $e_{1\min}$ be respectively the maximal and minimal values in the generator matrix E_1, then $m_0(\mathcal{J}, r, k, E)$ has the following lower bound:*

a. In the NN case:

$$m_0(\mathcal{J}, r, k, E) \geq r + \left\lceil \frac{k - j_{\max \leq k}}{e_{1\max}} \right\rceil, \qquad (52)$$

where $j_{\max \leq k} \triangleq \max\{j | j \in \mathcal{J}, j \leq k\}$.

It is sufficient that $R(E_{\min(r,m-r)})$ and *$\mathcal{J}$ are dense in order for this lower bound to be reached ($R(E_q)$ is dense when it equals the complete integer interval $[qe_{1\min}, qe_{1\max}]^{(I)}$, $\mathcal{J}$ is dense when it equals a complete integer interval).*

b. In the M case:

$$m_0(\mathcal{J}, r, k, E) \geq r + \max\left\{ \left\lceil \frac{k - j_{\max} + r|e_{1\min}|}{e_{1\max}} \right\rceil, \left\lceil \frac{re_{1\max} - k - |j_{\min}|}{|e_{1\min}|} \right\rceil \right\} \qquad (53)$$

and this lower bound is reached when $R(E_{\min(r,m-r)})$ and *$\mathcal{J}$ are dense ($j_{\max}$ and $j_{\min}$ are respectively the maximal and minimal value in $\mathcal{J}$).*

(D) *The stationary pattern of Eq. (48) fills the whole display:*

a. In the NN case: when $R(E_r) \cap [k - R(E)] = R(E_r)$, and hereto it is necessary that

$$k \geq re_{1\max} \qquad (54)$$

(this condition is also sufficient when $R(E_r)$ is dense).

b. In the M case: always, *and the general stationarity condition Eq. (50) reduces to*

$$R(E_r) \subseteq \{k - j - s | j \in \mathcal{J}, s \in R(E_{m-r})\}\,. \qquad (55)$$

Remark. All relationships except Eqs. (44) and (46) also hold for the condition $\hat{\mathcal{Q}}_j(0) \subset [0, 1]^{(p)}$ and $\hat{\mathcal{Q}}_j(0) \neq \phi$, instead of $\hat{\mathcal{Q}}_j(0) = [0, 1]^{(p)}$.

Proof

(A) Recall Eq. (31) which gives $\hat{\mathcal{Q}}_k(m)$ as a function of the initial conditions $\hat{\mathcal{Q}}_k(0)$. First, let $m \leq r$, then

$$\hat{\mathcal{Q}}_k(m) = \bigcup_{q \in R(E_m)} \{0.(a_1 a_2 __ a_m)_q \hat{\mathcal{Q}}_{k-q}(0) \diamond | a_1 __ a_m \in (\mathcal{Z}_\sigma)^m\},$$

meaning that a properly scaled copy of $\hat{\mathcal{Q}}_{k-q}(0)$ is placed in all those cells of the L_m-lattice whose address $\alpha_m = 0.a_1 a_2 __ a_m$ has weight $w(\alpha_m) = q$ [Recall that $(a_1 a_2 __ a_m)_q$ means that the $a_1 a_2 __ a_m$ string has weight $\sum_{i=1}^m \diamond e_1(a_i) = q$]. So, those L_m-cells with address α_m are highlighted for which $\hat{\mathcal{Q}}_{k-w(\alpha_m)}(0) = [0, 1]^{(p)}$: i.e. those for which $k - w(\alpha_m) \in \mathcal{J}$, or $w(\alpha_m) \in k - j$ for some $j \in \mathcal{J}$. In the r-resolution display $\mathcal{S}_r$, an L_m-cell ($m \leq r$) with specific address $0.a_1 a_2 __ a_m$ is covered by all pixels in $\mathcal{S}_r$ whose r-address has $0.a_1 a_2 __ a_m$ as most significant m-digit part. Therefore, all those pixels with addresses $0.a_1 a_2 __ a_m __ a_r$ for which $w(a_1 a_2 __ a_m) = k - j$ for some $j \in \mathcal{J}$ should be highlighted in displaying $\mathcal{S}_r \hat{\mathcal{Q}}_k(m)$. This proves Eq. (44).

When $m > r$, we have

$$\begin{aligned}\hat{\mathcal{Q}}_k(m) &= \bigcup_{q \in R(E_m)} \{0.(a_1 a_2 __ a_r a_{r+1} __ a_m)_q \\ &\quad \diamond \hat{\mathcal{Q}}_{k-q}(0) | a_1 __ a_m \in (\mathcal{Z}_\sigma)^m\} \\ &= \bigcup_{q \in R(E_m)} \{0.(a_1 a_2 __ a_r)_{w(\alpha_r)} \\ &\quad \diamond (a_{r+1} __ a_m)_{q - w(\alpha_r)} \\ &\quad \diamond \hat{\mathcal{Q}}_{k-q}(0) | a_1 __ a_m \in (\mathcal{Z}_\sigma)^m\}.\end{aligned}$$

Observe that $q - w(\alpha_r)$ is the weight of a string of $(m - r)$ digits, and so $q - w(\alpha_r) \in R(E_{m-r})$. A pixel with address $0.a_1 a_2 __ a_r$ will be highlighted when $\hat{\mathcal{Q}}_{k-q}(0)$ is not empty, i.e. when $k - q \in \mathcal{J}$. Combining the conditions $q - w(\alpha_r) \in R(E_{m-r})$ and $k - q \in \mathcal{J}$ produces $w(\alpha_r) = k - j - s$ for some $j \in \mathcal{J}$ and some $s \in R(E_{m-r})$. This proves Eq. (45).

The equivalence between Eqs. (44) and (46) follows because when $m \leq r$, $G\hat{\mathcal{L}}_k(E_m)$ highlights those pixels in S_r whose m-resolution address has weight k. Equations (45) and (47) are equivalent because $G\hat{\mathcal{L}}_k(E_r)$ highlights those pixels in S_r whose address has weight k.

(B) In Eq. (47), the indices $k - j - s$ that do not belong to $R(E_r)$ are related to terms that do not contribute as they are empty. So, we can write

$$\begin{aligned}\mathcal{S}_r \hat{\mathcal{Q}}_k(m) &\overset{(47)}{=} \bigcup_{u \in k - \mathcal{J} - R(E_{m-r})} G\hat{\mathcal{L}}_u(E_r) \\ &= \bigcup_{u \in R(E_r) \cap [k - \mathcal{J} - R(E_{m-r})]} G\hat{\mathcal{L}}_u(E_r),\end{aligned} \tag{56}$$

where

$$k - \mathcal{J} - R(E_{m-r}) \triangleq \{k - j - s | j \in \mathcal{J}, s \in R(E_{m-r})\}. \tag{57}$$

Consider $k - \mathcal{J} - R(E_{m-r})$. As $R(E_q) \subset R(E_t)$ when $q < t$, it follows that $k - \mathcal{J} - R(E_{m_1 - r}) \subset k - \mathcal{J} - R(E_{m_2 - r})$ when $m_1 < m_2$. So the set $k - \mathcal{J} - R(E_{m-r})$ grows without bounds when m increases, and as $R(E_r)$ is bounded, it must happen that from a certain value $m_0(\mathcal{J}, r, k, E)$ on, the additional elements that appear in $k - \mathcal{J} - R(E_{m-r})$ with growing m all fall outside $R(E_r)$. From that value m_0 on, the effective u-range $R(E_r) \cap [k - \mathcal{J} - R(E_{m-r})]$ in Eq. (56) no longer changes and $\mathcal{S}_r \hat{\mathcal{Q}}_k(m)$ becomes constant, independent of m. The u-range can then also be written as $R(E_r) \cap [k - \mathcal{J} - R(E)]$. But, as $\mathcal{J} \subset R(E)$, it follows that $[\mathcal{J} + R(E)] = R(E)$, and the u-range can be further simplified to $R(E_r) \cap [k - R(E)]$. This explains the Eqs. (48) and (49). m_0 itself is found as the lowest value of m for which the stationary index set $R(E_r) \cap [k - R(E)]$ becomes part of the general index set in Eq. (57). This produces the stationary condition of Eq. (50). Let $m_0(\{0\}, r, k, E)$ be this minimal value when $\mathcal{J} = \{0\}$, then $R(E_r) \cap [k \cap R(E)] \subseteq \{k - s | s \in R(E_{m_0 - r})\} \subseteq \{k - j - s | j \in \mathcal{J}, s \in R(E_{m_0 - r})\}$, showing that the stationarity condition for $\mathcal{J} \supset \{0\}$ is certainly reached for $m = m_0(\{0\}, r, k, E)$, although it *may* be reached for some $m < m_0(\{0\}, r, k, E)$. This explains Eq. (51).

(C) This follows from the fact that the stationarity condition Eq. (50) requires *at least* the following:

a. In the NN case:

$\min_{j\le k}\{k-j-s|j\in\mathcal{J},\, s\in R(E_{m-r})\} = k - j_{\max\le k} - (m-r)e_{1\max} \le \min R(E_r) = 0$. This condition is clearly sufficient when $R(E_r)$ *and* $R(E_{m-r})$ *and* $\mathcal{J}$ are dense. As for $R(E_r)$ and $R(E_{m-r})$, it is sufficient that the one with the smallest index is dense in order for both to be dense.

b. In the M case:

$$\min_{j,s}\{k-j-s|j\in\mathcal{J},\, s\in R(E_{m-r})\} = k - j_{\max} - (m-r)e_{1\max} \le r|e_{1\min}|$$

and

$$\max_{j,s}\{k-j-s|j\in\mathcal{J},\, s\in R(E_{m-r})\} = k + |j_{\min}| + (m-r)|e_{1\min}| \ge re_{1\max}.$$

Again, it is clear that this condition is sufficient when $R(E_{\min(r,m-r)})$ and $\mathcal{J}$ are dense.

(D) All pixels in $\mathcal{S}_r$ become highlighted in the stationary situation, when the stationary u-range in Eq. (48) coincides with $R(E_r)$, i.e. when $R(E_r)\cap[k-R(E)] = R(E_r)$. In the NN case, this requires at least that $k \ge re_{1\max}$ [Eq. (54)], as $\max R(E_r)\cap[k-R(E)] = k$ and $\max R(E_r) = re_{1\max}$. When $R(E_r)$ is dense, this is also sufficient as $R(E_r)\cap[k-R(E)] = [0,\, k]$ and $R(E_r) = [0,\, re_{1\max}]$. In the M case, the display saturation condition $R(E_r)\cap[k-R(E)] = R(E_r)$ is always satisfied in the stationary situation, as $R(E) = \mathbb{Z}$, and so $k - R(E) = \mathbb{Z}$. Hence the reduction of Eq. (50) into Eq. (55). ■

Example 3.1. These examples illustrate the essence of the previous property in a graphical way.

1. Consider the case where $R(E_1) = \{0,\, 2,\, 5\}$ and where we have a display of logsize $r = 5$ (this corresponds to the situation of Fig. 6). Observe that E_1 need not be specified: the results are valid for all generators with the same range, although the actual linear display size will depend on the generator dimension and size. Figure 13 displays $R(E_5)$ and $k - R(E)$ for $k = 21$. Those values that belong to both $R(E_5)$ and $k - R(E)$ are the ones that refer to the contributing terms in the stationary pattern $S_r\hat{\mathcal{Q}}_k(m)$ of Eq. (48). They are shown encircled. Also $k - R(E_q)$ is shown for $q = m-r = 1,\, 2,\, 3,\, 4,\, 5,\, 6$, whereby it is clear that $R(E_{q_1}) \subset R(E_{q_2})$ for $q_1 < q_2$. With $\mathcal{J} = \{0\}$, the stationarity condition Eq. (50) is reached when the encircled points of $R(E_5)\cap k - R(E)$ are covered by $k - R(E_q)$. From the figure it can be seen that this happens when $q \ge 6$, and so the stationary pattern is established from $m_0 = r + 6 = 11$ on (compare this to the lower bound of Eq. (52): $r + \lceil\frac{21}{5}\rceil = 10$). When $\mathcal{J} \supset \{0\}$, we have to consider all points $k - R(Eq) - j$, $j\in\mathcal{J}$. With $\mathcal{J} = \{0,\, 6,\, 7\}$, the encircled points are covered from $q = m - r = 3$ on. The stationary pattern is the same, but it is now reached from $m_0 = r + 3 = 8$ on (lower bound: $r + \lceil\frac{21-7}{5}\rceil = 8$). This corresponds to the observations in Fig. 6. It is easy to assess the stationarity conditions for other values of k by merely shifting the k-related dot patterns in the figure. So it becomes readily clear that all points of $R(E_5)$ get covered for $k \ge 27$, so that $\mathcal{S}_r\hat{\mathcal{Q}}_k(m)$ fills the whole display for $k \ge 27$ (for $k = 25$, the point 22 from $R(E_5)$ would not appear as index in Eq. (48); for $k = 26$, the index 25 would be missing). This graph also gives the decomposition of $\mathcal{S}_r\hat{\mathcal{Q}}_k(m)$ into l-indicator matrices over E_r for arbitrary $m > r$: it consists of those integers l corresponding to the vertical dotted lines on which a dot is found from the dot-set related to $k - E(R_{m-r})$. For example, when $\mathcal{J} = \{0\}$, $\mathcal{S}_5\hat{\mathcal{Q}}_{21}(9) = \cup_u G\hat{\mathcal{L}}_u(E_5)$ for $u\in\{4,\, 6,\, 7,\, 9,\, 10,\, 11,\, 12,\, 13,\, 14,\, 15,\, 16,\, 17,\, 19,\, 21\}$ [this is read from line $k - R(E_{m-r}) = k - R(E_{9-5}) = k - R(E_4)$, and the result corresponds to Eq. (47)].
2. Figure 14 shows a similar graph for the M case illustrated in Fig. 7(a) with $R(E_1) = \{-1,\, 0,\, 2\}$ and for $k = 10$, $r = 5$. As $R(E) = k - R(E) = \mathbb{Z}$, the stationarity condition is now given by Eq. (55). For $\mathcal{J} = \{-2,\, 0,\, 1\}$, $10 - R(E_q) - \mathcal{J}$ covers $R(E_5)$ from $q = 7$ on. With $q = m - 5$, this means that from $m_0 = 5 + 7 = 12$ on, the stationary image saturates the whole display. It is not difficult to see that for any k, stationarity occurs from $m_0 = 5 + \max(15 - k,\, \lceil\frac{k+11}{3}\rceil)$ on [this value is indeed greater than the lower bound from Eq. (53): $5 + \max(15 - k,\, \lceil\frac{k+5}{3}\rceil)$]. No matter what value k may have, the stationary image will always highlight the whole display. Figure 15 shows how $\mathcal{S}_5\hat{\mathcal{Q}}_{-1}(6)$ as appearing in Fig. 7(a) is the union of l-indicator sets of E_5 for $l\in\{-4,\, -3,\, -2,\, -1,\, 0,\, 1,\, 2\}$ (as given by Eq. (47): $m - r = 1$, and as can be derived from Fig. 14, for the case $k' = -1$).

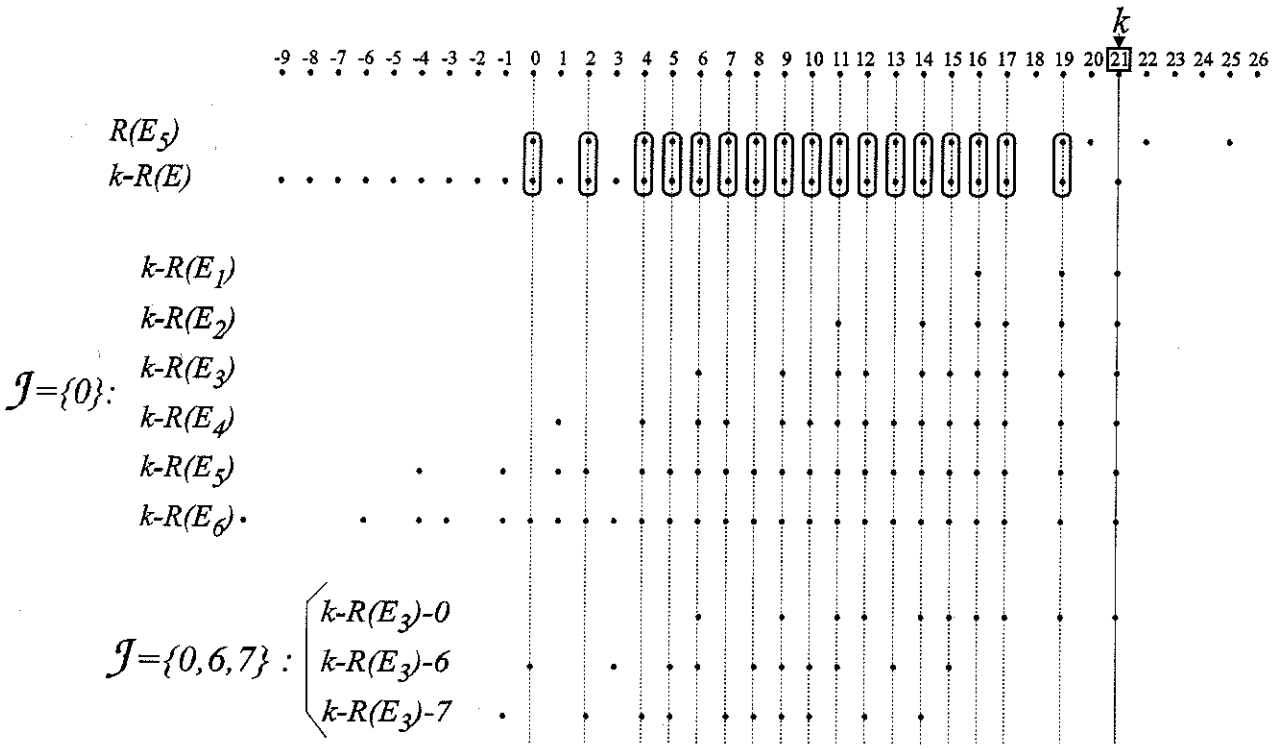


Fig. 13. Graphical interpretation of the stationarity condition Eq. (50), illustrated for the cases shown in Figs. 6(a) and 6(b): logsize-5 display, $R(E_1) = \{0, 2, 5\}$, $k = 21$. See Example 3.1.1.

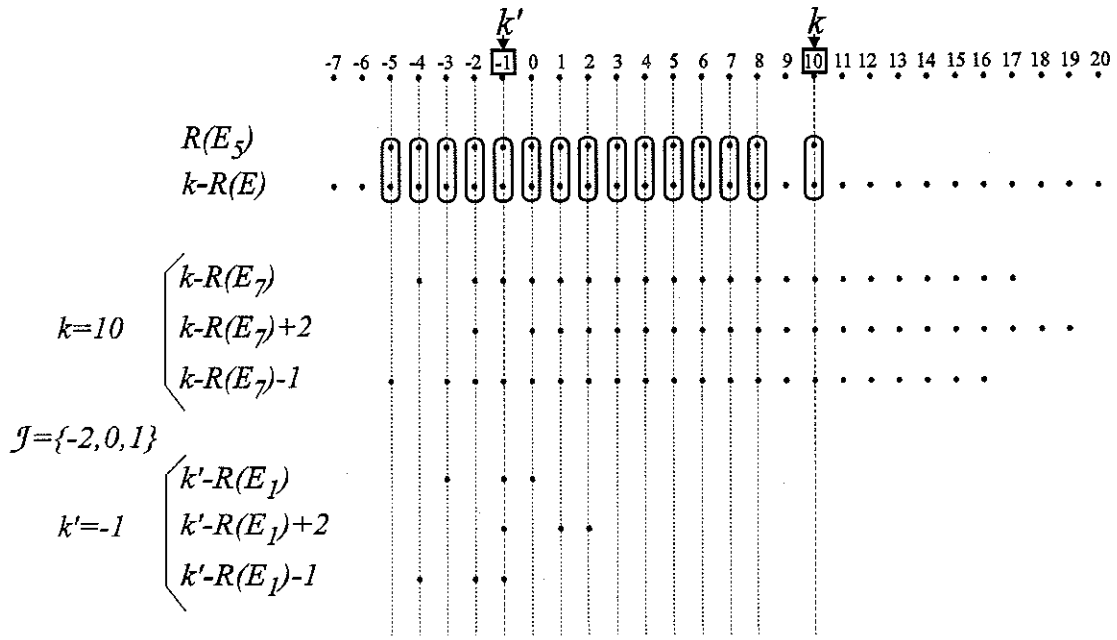
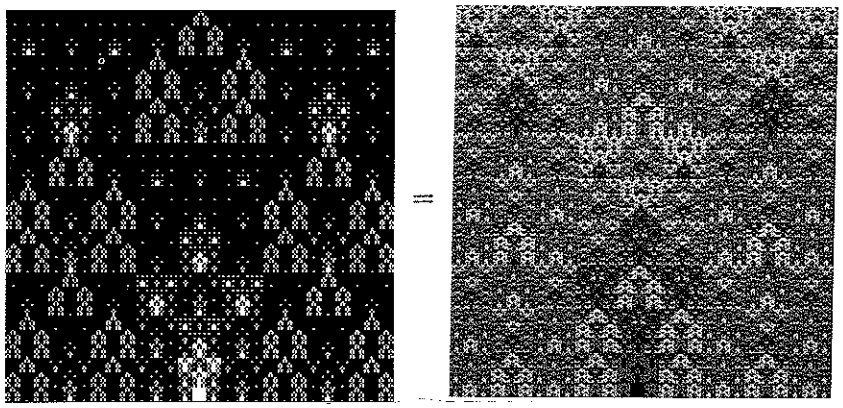
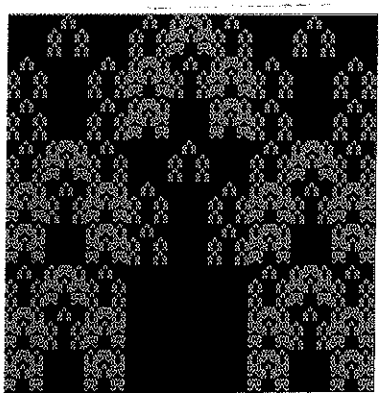


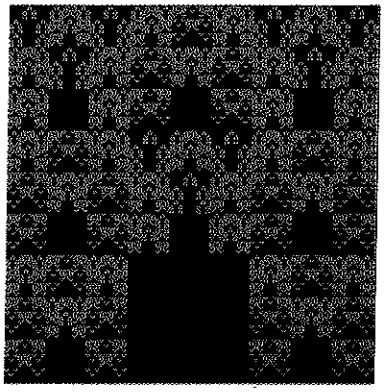
Fig. 14. Graphical determination of the stationary condition for the mixed-type case corresponding with Fig. 7(a): $R(E_1) = \{-1, 0, 2\}$, $r = 5$, $k = 10$, $\mathcal{J} = \{-2, 0, 1\}$. See Example 3.1.2.



$\mathcal{S}_5\hat{\mathcal{Q}}_{-1}(6)$ $\bigcup_{l\in[-4,2]} G\hat{\mathcal{L}}_l(E_5)$



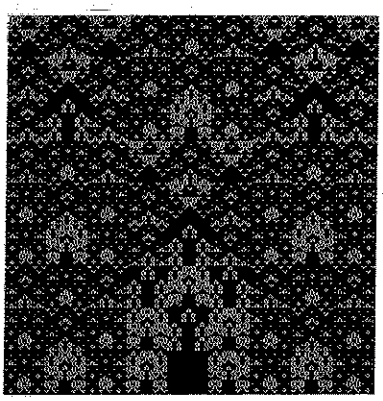
$G\hat{\mathcal{L}}_{-4}(E_5)$



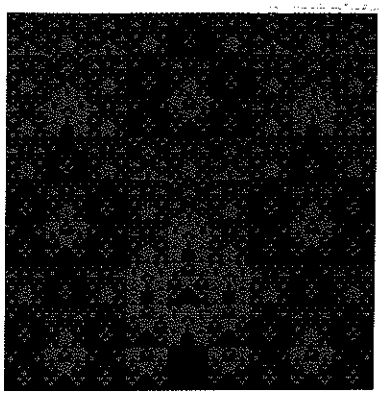
$G\hat{\mathcal{L}}_{-3}(E_5)$



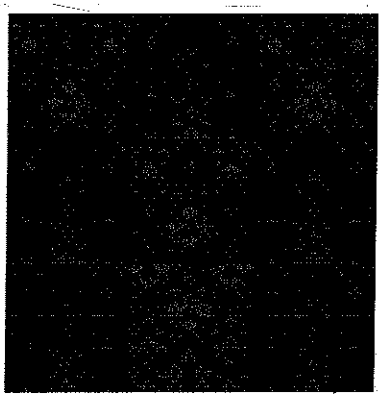
$G\hat{\mathcal{L}}_{-2}(E_5)$



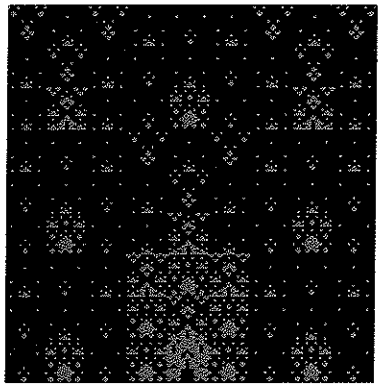
$G\hat{\mathcal{L}}_{-1}(E_5)$



$G\hat{\mathcal{L}}_0(E_5)$



$G\hat{\mathcal{L}}_1(E_5)$



$G\hat{\mathcal{L}}_2(E_5)$

Fig. 15. Decomposition of a logsize-r resolution view on $\hat{\mathcal{Q}}_k(m)$ into a set of l-indicator matrices over E_r: the black pixels in $\mathcal{S}_5\hat{\mathcal{Q}}_{-1}(6)$ correspond to the colored pixels in $\bigcup_{l\in[-4,2]} G\hat{\mathcal{L}}_l(E_5)$ [i.e. Eq. (46) with $\mathcal{J} = \{-2, 0, 1\}$], generator $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 2 & -1 \end{bmatrix}$. See Fig. 7(b), $m = 6$.

Remark. Once again, Property 3.5 uncovers a fundamental difference between the NN and M cases:

- In the NN case, it may happen that for k sufficiently large, an r-resolution display saturates [Eq. (54)] when the iteration has gone through a sufficient number of steps m_0 [Eq. (52), see for example Fig. 5(c)]. However, increasing the resolution r to a value $r > k/e_{1\max}$ [Eq. (54)] will reestablish the details on the display, and by a further increase of m (continuation of the iteration), the display will *never* become saturated again.
- In the M case, on the contrary, whatever the k-value considered and whatever the resolution r might be, the display will *always* saturate under continued iteration [Eq. (55), see Fig. 7].

4. Fractal dimension of $\mathcal{F}_k$ and $\mathcal{G}_k$

4.1. *m-resolution representation of $\mathcal{F}_k$ and $\mathcal{G}_k$*

Definition 4.1. The m-resolution graphical representation of $\mathcal{F}_k$ (or $\mathcal{G}_k$), denoted by $G\mathcal{F}_k(m)$ [or $G\mathcal{G}_k(m)$], highlights all L_m-cells that contain a point of $\mathcal{F}_k$ (or $\mathcal{G}_k$).

Property 4.1.

(A) *In the NN case, the m-resolution graphical representation of $\mathcal{F}_k$ and $\mathcal{G}_k$, $k \in R(E)$, is given by*

$$G\mathcal{F}_k(m) = \bigcup_{j\in R(E_m)\cap[k-R(E)]} G\hat{\mathcal{L}}_j(E_m)\,, \qquad (58)$$

$$G\mathcal{G}_k(m) = G\mathcal{F}_k(m)\,. \qquad (59)$$

(B) *In the MM case:*

$$G\mathcal{F}_k(m) = G\mathcal{G}_k(m) = [0,\,1]^{(p)}\,. \qquad (60)$$

Proof

A. NN case:

(a) Let $c = 0.c_1c_2c_3\ldots \in \mathcal{F}_k$. Then, by definition, $w(0.c_1c_2 __ c_m) \in R(E_m)$. But also: $w(0.c_1c_2 __ c_m) \in k - R(E)$, as $w(0.c_1 __ c_m) = j$ implies that $w(c_{m+1}c_{m+2}\ldots) = k - j \in R(E)$ and consequently $j \in k - R(E)$. So, $w(0.c_1c_2 __ c_m) \in R(E_m) \cap [k - R(E)]$. Conversely, any L_m-cell with an address $0.c_1c_2 __ c_m$ for which $w(0.c_1c_2 __ c_m) \in R(E_m)\cap[k-R(E)]$ contains at least one point $c = 0.c_1 __ c_mc_{m+1}\ldots$ that belongs to $\mathcal{F}_k$. The condition that $w(0.c_1c_2 __ c_m) \in k - R(E)$ is hereby necessary, because $w(c_{m+1}c_{m+2}\ldots) = k - w(0.c_1c_2 __ c_m) \in R(E)$. Hence, all points of $\mathcal{F}_k$ belong to $\bigcup_{j\in R(E_m)\cap[k-R(E)]} G\hat{\mathcal{L}}_j(E_m)$ and all black cells in $G\hat{\mathcal{L}}_j(E_m)$, $j \in R(E_m) \cap [k - R(E)]$ contain a point of $\mathcal{F}_k$. Hence Eq. (58) follows.

(b) Proof of Eq. (59). Equation (58) and Lemma 3.1.A together imply that for $t \in [k - R(E)\backslash\{0\}]$:

$$G\mathcal{F}_t(m) \subset G\mathcal{F}_k(m)\,. \qquad (61)$$

Ultimately, from $\mathcal{G}_k \overset{(15)}{=} \bigcup_{t\in R(E)\cap[k-R(E)]} \mathcal{F}_t$, we obtain

$$\begin{aligned} G\mathcal{G}_k(m) &= \bigcup_{t\in R(E)\cap[k-R(E)]} G\mathcal{F}_t(m) \\ &= G\mathcal{F}_k(m) \bigcup_{t\in R(E)\cap[k-R(E)\backslash\{0\}]} G\mathcal{F}_t(m) \\ &\overset{(61)}{=} G\mathcal{F}_k(m)\,. \end{aligned}$$

B. MM case:

When $c = 0.c_1c_2c_3\ldots \in \mathcal{F}_k$ or $\mathcal{G}_k$, $w(0.c_1 __ c_m)$ may take any value from $R(E_m)$, because in this case $R(E) = \mathbb{Z}$. Conversely, any L_m-cell contains at least one point that belongs to $\mathcal{F}_k$ or $\mathcal{G}_k$. So, the m-resolution representation of $\mathcal{F}_k$ and $\mathcal{G}_k$ is given by $\bigcup_{j\in R(E_m)} G\hat{\mathcal{L}}_j(E_m) = [0,\,1]^{(p)}$. ■

4.2. *The box counting fractal dimension function*

The m-resolution cells are the most natural covering boxes for $\mathcal{F}_k$ or $\mathcal{G}_k$. Recall from Part II the concept of *equivalent linear size* of the generator matrix: $\sigma_0 \triangleq (\sigma_1\sigma_2\ldots\sigma_p)^{1/p}$ (p being the "spatial dimension" of the matrix, σ_i being the size in the ith dimension). The equivalent linear size of an m-resolution cell is then σ_0^{-m} [or its logsize is $(-m)$], its volume is σ_0^{-mp}. Also recall that $v_m(k)$ denoted the number of elements in E_m that mapped into k. In the present context, $v_m(k)$ is also the number of "black" cells in $G\hat{\mathcal{L}}_k(E_m)$. Let $n_m(k)$ denote the number of logsize-m cells (boxes) needed to cover $\mathcal{F}_k$ (or $\mathcal{G}_k$). Then the box counting dimension $D(r;\,k)$ can be obtained from the evolution

of $n_m(k)$ when the equivalent linear size of the m-covering boxes goes to zero (or $m \to \infty$), as follows: for $r \in \mathbb{N}_0$, $k \in R(E)$:

$$\lim_{m\to\infty} \left\{ \left[\frac{\sigma_0^{-m}}{\sigma_0^{-(m+r)}} \right]^{D(r;k)} = \frac{n_{m+r}(k)}{n_m(k)} \right\},$$

or

$$D(r;\, k) \triangleq \frac{1}{r} \lim_{m\to\infty} \log_{\sigma_0} \frac{n_{m+r}(k)}{n_m(k)}. \qquad (62)$$

Property 4.2.

$$\boxed{D(r;\, k) = D(1;\, k) = \lim_{m\to\infty} \log_{\sigma_0} \frac{n_{m+1}(k)}{n_m(k)} \triangleq D_k}. \qquad (63)$$

Proof. Write $\frac{n_{m+r}(k)}{n_m(k)}$ in Eq. (62) as $\frac{n_{m+r}(k)}{n_{m+r-1}(k)} \cdot \frac{n_{m+r-1}(k)}{n_{m+r-2}(k)} \cdots \frac{n_{m+1}(k)}{n_m(k)}$ in order to obtain $D(r;\, k) = \frac{1}{r} \sum_{i=1}^{r} D(1;\, k) = D(1;\, k)$. ■

Remark. This property corresponds to Corollary 5.1 of Part II. It shows that the important quantity is actually $D(1;\, k)$ which we will rename D_k.

Property 4.3. *Let $v'_m(k) \triangleq v_m(k)/\sigma_0^{mp}$ be the relative number of k-values in E_m and let $R_k(E) \triangleq R(E) \cap [k - R(E)]$. Then the fractal dimension D_k is given by*

(a) for the NN case:

$$D_k = p - \lim_{m\to\infty} \left| \log_{\sigma_0} \frac{\sum_{j\in R_k(E)} v'_{m+1}(j)}{\sum_{j\in R_k(E)} v'_m(j)} \right|, \qquad (64)$$

(b) for the M case: $D_k = p$. (65)

Proof. From Property 4.1, it follows that the logsize-m cells that are needed to cover $\mathcal{F}_k$ are the highlighted ones in $\bigcup_{j\in R(E_m)\cap[k-R(E)]} G\hat{\mathcal{L}}_j(E_m)$ for the NNcase, and those in $\bigcup_{j\in R(E_m)} G\hat{\mathcal{L}}_j(E_m)$ for the M case. Because of the disjointed of $G\hat{\mathcal{L}}_j(E_m)$ for different j, the number of m-cells in these covering sets is $n_m(k) = \sum_{j\in R(E_m)\cap[k-R(E)]} v_m(j)$ for the NN case, and $n_m(k) = \sum_{j\in R(E_m)} v_m(j) = \sigma_0^{mp}$ for the M case. Use this in Eq. (63) to obtain

— for the NN case:

$$D_k = \lim_{m\to\infty} \log_{\sigma_0} \frac{\sum_{j\in R_k(E)} v_{m+1}(j)}{\sum_{j\in R_k(E)} v_m(j)}, \qquad (66)$$

where we use the fact that $\sum_{j\in R(E_m)\cap[k-R(E)]} v_m(j) = \sum_{j\in R(E)\cap[k-R(E)]} v_m(j)$ because $v_m(j) = 0$ when $j \notin R(E_m)$;

— for the M case:

$$D_k = \lim_{m\to\infty} \log_{\sigma_0} \frac{\sum_{j\in R(E_{m+1})} v_{m+1}(j)}{\sum_{j\in R(E_m)} v_m(j)}. \qquad (67)$$

Use the transformation $v'_m(k) = v_m(k)/\sigma_0^{mp}$ (implying that $\sum_{j\in R(E_m)} v'_m(k) = 1$, see Part II) in expressions (66) and (67) to obtain Eqs. (64) and (65). ■

Remarks. Equations (64) and (65) give here directly the limit values of the fractal dimension *functions* that were considered in Part II. Equation (64) corresponds to Eq. (55) in Part II (substitute $m-l-1$ there by m). Equation (65) corresponds to Eq. (57) in Part II (for $l \to \infty$). These limits are the classical fractal dimensions of the "genuine" fractals $\mathcal{F}_k$ and $\mathcal{G}_k$ over $[0, 1]^p$. As $\mathcal{G}_k$ completely fills the p-dimensional unit cube $[0, 1]^p$ in the M case, it is no surprise to find that this fractal dimension equals p. However, $\mathcal{F}_k(\mathcal{G}_k)$ is only a "proper" fractal for the NN cases, and have a classical fractal dimension strictly less than p. It will be shown, in the next property, that the fractal dimension of $\mathcal{F}_k(\mathcal{G}_k)$ is independent of k for a given generator. This shows once again how the fractal dimension functions introduced in Part II, which were related to the distribution of k-elements in the expanded fractal E-matrix, give a richer description of the peculiarities of the generated fractals. But it also stresses the difference between an outwardly grown fractrix and its inwardly grown equivalent, at least under a resolution which maximally discriminates between real numbers. Figure 16(b) shows $D_k(m)$, i.e. the nonlimit form of Eq. (65) in its dependence on m, for the MZP-fractrix Q that appeared in Fig. 12 of Part II [2×4 generator with elements (0, 0, 1, 2, 2, 2, 2, 2)]. Figure 16(a) shows this for an SZP-fractrix. Because of the nonuniformity of the $\mathcal{F}(\mathcal{G})$-fractals, the fractal dimension *function* $D_k(m)$ gives more

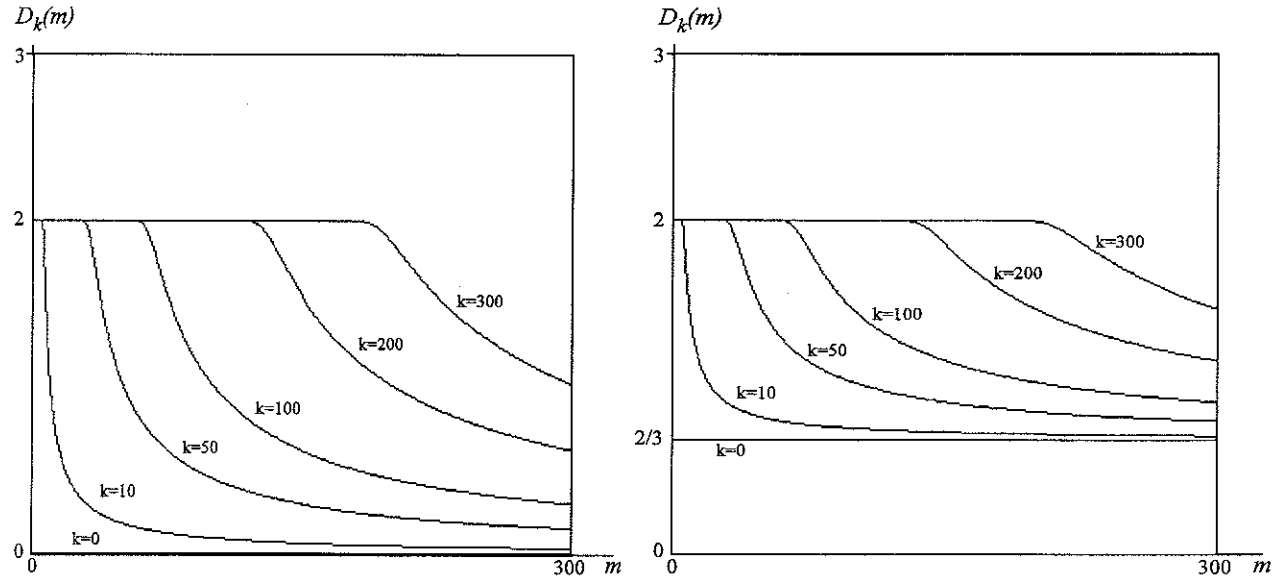


Fig. 16. The fractal dimension function $D_k(m)$ for (a) 2×4 generator with elements (0, 1, 1, 2, 2, 2, 2, 2), (b) 2×4 generator with elements (0, 0, 1, 2, 2, 2, 2, 2).

information than its limit value alone. $D_k(m)$ differentiates between the different $\mathcal{F}_k(\mathcal{G}_k)$, while D_k is independent of k. We now prove a few properties that can be observed from the examples in Fig. 16.

Property 4.4. *For the NN case, the fractal dimension function*

$$\boxed{D_k(m) \triangleq \log_{\sigma_0} \frac{n_{m+1}(k)}{n_m(k)} = \log_{\sigma_0} \frac{\sum_{j \in R_k(E)} v_{m+1}(j)}{\sum_{j \in R_k(E)} v_m(j)}}, \tag{68}$$

where $R_k(E) \triangleq R(E) \cap [k - R(E)]$,

satisfies

(a) for $k = 0$: $D_0(m) = \log_{\sigma_0} v_1(0)$ (69)

(b) for $k \in R(E) \backslash \{0\}$:

$$D_k(m) = p \text{ for } m \leq \left\lfloor \frac{k+1}{e_{1\max}} \right\rfloor - 1 \tag{70}$$

$$\text{and } D_k = \lim_{m \to \infty} D_k(m) = \log_{\sigma_0} v_1(0) \tag{71}$$

($e_{1\max} = \max R(E_1)$ = *maximal value in the generator matrix*).

Proof. Use $v_{m+1}(k) = \sum_{s \in R(E_1)} v_1(s) v_m(k - s)$ [see Part II, Eq. (14)], and the fact that $v_m(l) = 0$ for $l < 0$ to rewrite Eq. (68) as

$$D_k(m) = \log_{\sigma_0} \sum_{s \in R(E_1)} \left[v_1(s) \frac{\sum_{j \in [0, k-s] \cap R_k(E)} v_m(j)}{\sum_{j \in [0,k] \cap R_k(E)} v_m(j)} \right]. \tag{72}$$

(a) When $k = 0$, the only term in Eq. (72) that differs from 0 is the one for $s = 0$, yielding Eq. (69).

(b) Proof of Eq. (70): Observe that the global sum term in Eq. (72) can be rewritten as

$$\sum_{s \in R(E_1)} \left[v_1(s) \times \frac{\sum_{j \in [0, k-s] \cap R_k(E)} v_m(j)}{\sum_{j \in [0, k-s] \cap R_k(E)} v_m(j) + \sum_{j \in [k-s+1, k] \cap R_k(E)} v_m(j)} \right]. \tag{73}$$

The ratio of sums equals 1 for all s when the right sum in the denominator equals 0 for all s. This condition occurs when $v_m(j) = 0$ for all $j \geq k - e_{1\max} + 1$. As $v_m(j) = 0$ for $j > m e_{1\max}$, it suffices that $k - e_{1\max} + 1 > m e_{1\max}$ or $m \leq \lfloor \frac{k+1}{e_{1\max}} \rfloor - 1$. Then Eq. (73) equals $\sum_{s \in R(E_1)} v_1(s) = \sigma_0^p$ (= total number of elements in the generator), and $D_k(m) = \log_{\sigma_0} \sigma_0^p = p$. This proves Eq. (70).

Proof of Eq. (71):
We will show that the ratio of sums in Eq. (72) tends to 0 for $m \to \infty$ when $s \neq 0$. This follows from rewriting this ratio as

$$\left[1 + \frac{\sum_{j\in[k-s+1,k]\cap R_k(E)} v_m(j)}{\sum_{j\in[0,k-s]\cap R_k(E)} v_m(j)}\right]^{-1}, \qquad (74)$$

whereby we will prove that the order (in m) of the numerator in Eq. (74) strictly exceeds the order of the denominator for all nonzero $s \in R(E_1)$.

First observe that $v_m(k)$ is also the number of different m-resolution representations with weight k. Let $\{\alpha_i\}$ be the set of elements of the generator matrix that differ from 0, i.e. $\alpha_i \in R(E_1)\backslash\{0\}$, with $\alpha_1 < \alpha_2 < \alpha_3 \cdots$. Let δ_k be *a* decomposition of k over the α_i, i.e. $k = \#\alpha_1\cdot\alpha_1 + \#\alpha_2\cdot\alpha_2 \cdots + \#\alpha_l\cdot\alpha_l$, where $\#\alpha_i$ denotes the number of α_i in this particular decomposition. This corresponds to a particular m-resolution representation $0.c_1c_2c_3 __ c_m$, where a number $\#\alpha_i$ of the c-digits have weight α_i. Each term α_i in the decomposition can be obtained from $v_1(\alpha_i)$ different c-digits (those elementary addresses in E_1 that map into α_i). With $\mathcal{D}_k$ the set of all k-decompositions, it follows that

$$\begin{aligned} v_m(k) = \sum_{\delta_k\in\mathcal{D}_k} & [v_1(\alpha_1)]^{\#\alpha_1}\binom{m}{\alpha_1}\cdot[v_1(\alpha_2)]^{\#\alpha_2}\binom{m-\#\alpha_1}{\#\alpha_2}\cdots \\ & \cdot[v_1(\alpha_l)]^{\#\alpha_l}\binom{m-\#\alpha_1-\#\alpha_2\cdots-\#\alpha_{l-1}}{\#\alpha_l}\cdot[v_1(0)]^{m-\Sigma\#\alpha_i} \\ = [v_1(0)]^m \cdot \sum_{\delta_k\in\mathcal{D}_k} & \left[\frac{v_1(\alpha_1)}{v_1(0)}\right]^{\#\alpha_1}\cdots\left[\frac{v_1(\alpha_l)}{v_1(0)}\right]^{\#\alpha_l}\frac{m(m-1)\cdots(m-\sum\#\alpha_i+1)}{\#\alpha_1!\#\alpha_2!\cdots\#\alpha_l!} \end{aligned}$$

The order of $v_m(k)$ (in m) is determined by its maximal-order term. This is the term for which $\sum \#\alpha_i$ is maximal, so that $O[v_m(k)] = [v_1(0)]^m O[m^{\max_{\mathcal{D}_k}(\Sigma\#\alpha_i)}]$.

Let $v_m(q)$ be the highest order term in the denominator of the ratio in Eq. (74), then $v_m(q+\alpha_1)$ must belong to the numerator as $O[v_m(q+\alpha_1)] = mO[v_m(q)] > O[v_m(q)]$ (the equality follows from the fact that when q decomposes in $\sum \#\alpha_i\cdot\alpha_i$, whereby $\sum \#\alpha_i$ is maximal, then $q+\alpha_1$ decomposes as $\sum \#\alpha_i\cdot\alpha_i+\alpha_1$, whereby $\sum \#\alpha_i+1$ is maximal). So the numerator is certainly of higher order than the denominator, and this is what we wanted to prove and what ultimately leads to Eq. (71). ■

Remarks

- For the SZP case, $v_1(0) = 1$ and it follows that the limit fractal dimension $D_k = 0$. This is no surprise, as all $\mathcal{F}_k(\mathcal{G}_k)$ are countable Zeno-sets and fill the $[0, 1]^p$-cube rather sparsely.
- For the MZP case, the $\mathcal{F}_k(\mathcal{G}_k)$ are Cantor-type sets. The limit fractal dimension D_k depends completely on the number $v_1(0)$ of feedback loops at each level of the HIFS (= the number of 0's in the generator matrix).
- That D_k is independent of k should not be much of a surprise either, in view of the fact that $\mathcal{F}_k$ is a high order Zenonian squeezing of the original $\mathcal{F}_0$ which is a genuine Cantor set. Squeezing and adding squeezed copies of a fractal set does not change the fractal dimension. But it does influence the fractal dimension function.
- Equation (70) shows that for larger sizes of the covering boxes ("small" m, dependent on the chosen k), the fractal dimension function equals the topological dimension p. This is in agreement with our findings about the saturation of the display when looking at $\mathcal{G}_k$ with a resolution that is too low [cf. Eq. (54)].

5. Conclusion

We have studied the characteristics of the limit of inwardly grown fractrices, as is usual in the study of fractals. A dramatic collapse into ambiguity occurs when jumping into this limit form: points belonging to this limit map into an unlimited number of integer values. This stands in sharp contrast to the finitist constructivist point of view from Part II, where we always faced a discrete structure in

which each element was associated with a single integer. This discrete unique-value structure even prevails in all sets in the inwardly grown sequence that leads to the limit. A proper explanation of the ambiguity that emerges in the limit structure was given by placing this structure in a hyperreal context. Within the hyperreal realm, the ambiguity was absent: it arose by the restriction of the resolution to the real-number world. The reason behind this is the fact that each digit in the representation of a real number (used as an addressing system) is equally important in determining the associated weight, no matter what its position in the representation might be. This is completely opposed to what happens with the contribution of a digit to the real number itself: its contribution becomes less important the more it is positioned to the right in the symbol string that represents the number.

When restricting ourselves to a "real" view on the limit structure, we found two interesting sets that map into the integer $k \in R(E)$: $\mathcal{F}_k$ and $\mathcal{G}_k$. The $\mathcal{F}_k$ are open sets arising from a straightforward "infinitisation" of the mapping $0.c_1c_2 __ c_m \rightarrow \sum_{i=1}^{m} e_1(c_i)$ into $0.c_1c_2c_3 \ldots \rightarrow \sum_{i=1}^{\infty} e_1(c_i)$. $\mathcal{F}_k$ is actually a subset of the closed true limit set $\mathcal{G}_k$. Both $\{\mathcal{F}_k\}$ and $\{\mathcal{G}_k\}$ are invariant sets of a problem related Hutchinson operator. $\{\mathcal{G}_k\}$ is the unique attractor (limit) set of the corresponding hierarchical iterated function system. It was found that the structure of the sets $\mathcal{G}_k(\mathcal{F}_k)$ depends on the generator type. For mixed-type generators, which produce the most interesting patterns for the discrete version, $\mathcal{G}_k$ is the least interesting: it fills its whole support $[0, 1]^p$. When the generator is nonnegative and contains only one zero, the $\{\mathcal{F}_k\}$, $\{\mathcal{G}_k\}$ form a class of hierarchical Zeno-sets. When nonnegative with more zeroes, a class of more complex hierarchical Cantor sets appears.

We also discussed how the evolution of the HIFS images, which need an unlimited resolution for their description, appears under a finite resolution view: the resulting pictures can be decomposed into several k-indicator sets over a finite E_r [through their $G\hat{\mathcal{L}}_k(E_r)$-versions].

Expressions for the classical fractal dimension of $\mathcal{G}_k(\mathcal{F}_k)$ were derived. They depend on the generator dimension p and on its size, and on the distribution of the integers in the generator (in number, not in space). For the M-type generators, this fractal dimension equals the topological dimension p. For SZP-generators, the fractal dimension is 0, while it is between 0 and p for MZP-generators. Moreover, it is independent of k. Again, just as in Part II, a related fractal dimension function $D_k(m)$ has been defined. This has discriminating power for different k in the NN cases. This fractal dimension function for $\mathcal{G}_k$ complements the ones for the discrete fractal structures from Part II. The fractal dimension functions from Part II could even discriminate between different k-values for M-type fractrices, which is not the case for the one presented in this part. The overall conclusion is that the finitist constructions from Parts I and II are most basic.

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Appendix A
The One-Dimensional Equivalent of a p-Dimensional Fractrix

A.1. *Fractrices in 1-1 correspondence*

Consider a p-dimensional generator E_1 of size $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p)$. Let $\mathcal{Z}_{\sigma_i} = \{0, 1, 2, \ldots, \sigma_i - 1\}$ be the set of digits used in the σ_i-adic number representation. Then a p-tuple $\boldsymbol{c}_k = (c_{k1}, c_{k2}, \ldots, c_{kp})$ with $c_{ki} \in \mathcal{Z}_{\sigma_i}$ (this will be denoted by $\boldsymbol{c}_k \in \mathcal{Z}_{\boldsymbol{\sigma}}$) seems to be the "most natural" representation of the addresses of the elements in the generator matrix. This is illustrated in Fig. 17(a) for a two-dimensional generator. As there are $\sigma_\pi \triangleq \prod_{i=1}^{p} \sigma_i$ different elementary p-tuple addresses, each of these address vectors can be replaced in a 1-1 fashion by a σ_π-adic digit. So the σ-adic address $\mathbf{o}.\boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{c}_3 \ldots \boldsymbol{c}_M (\boldsymbol{c}_i \in \mathcal{Z}_{\boldsymbol{\sigma}})$ can be interpreted as a σ_π-adic

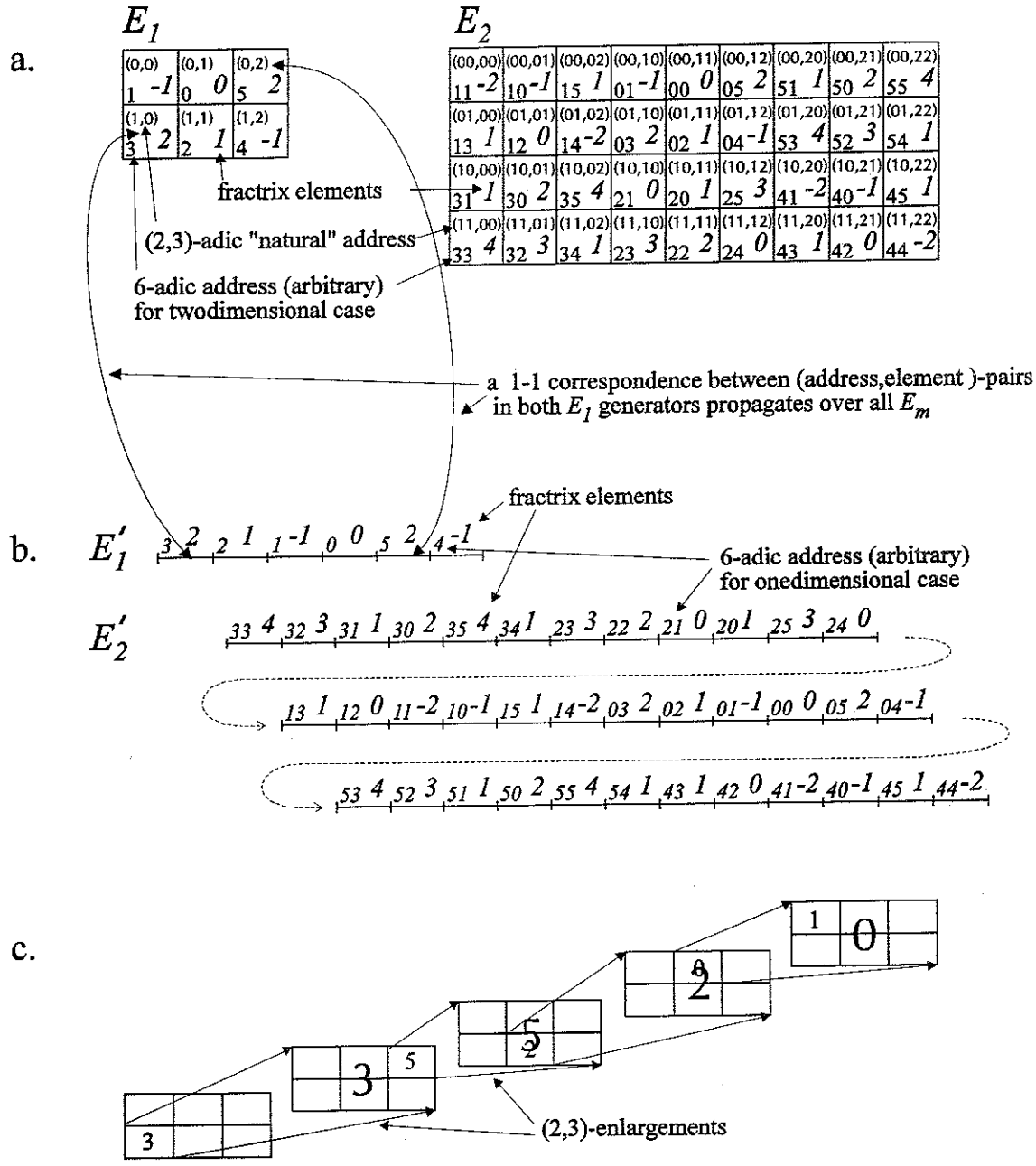


Fig. 17. (a) "natural" (2,3)-adic addresses and equivalent $2 \times 3 = 6$-adic addresses in a 2×3 generator matrix E_1 and in E_2. (b) A one-dimensional 2×3-adic counterpart of the matrices under (a). (c) Finding the cell with 6-adic address 0.35201 given the 6-adic addresses in the generator.

address $0.\beta_1\beta_2\beta_3 __ \beta_M (\beta_i \in \mathcal{Z}_{\sigma_\pi})$ through a 1-1 mapping from $\mathcal{Z}_{\boldsymbol{\sigma}}$ onto $\mathcal{Z}_{\sigma_\pi}$:

$$\mathcal{Z}_{\boldsymbol{\sigma}} \overset{1-1}{\leftrightarrow} \mathcal{Z}_{\sigma_\pi} : c_i \overset{1-1}{\leftrightarrow} \beta_i \quad \text{and} \quad e_1(c_i) = e_1(\beta_i)\,.$$

This also establishes a 1-1 correspondence between the p-dimensional generator E_1 and a one-dimensional generator E_1' with size σ_π, and also between the corresponding logsize-m matrices E_m and E_m', as shown in Figs. 17(a) and 17(b). Hereby, the element at $\boldsymbol{\sigma}$-adic address $\boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{c}_3 __ \boldsymbol{c}_M$ ($= \sigma_\pi$-adic address $\beta_1\beta_2 __ \beta_M$) in E_m corresponds to element at σ_π-adic address $\beta_1\beta_2 __ \beta_M$ in E_M'. Both have the value $\sum_{i=1}^{M} e_1(c_i) = \sum_{i=1}^{M} e_1(\beta_i)$.

As any 1-1 mapping between $\mathcal{Z}_{\boldsymbol{\sigma}}$ and $\mathcal{Z}_{\sigma_\pi}$ will do, there are actually $\sigma_\pi!$ possibilities for converting $\boldsymbol{\sigma}$-addresses into σ_π-addresses. Given a σ_π-address, it is straightforward to find the location of this element, by a stepwise nested subdivision of $[0, 1]^p$ in σ_π^m blocks and selection of the proper block according to the digits encountered [Fig. 17(c)]. Observe that digits have here a pure symbolical function: the p-vector symbols $\boldsymbol{c}_i$ are just replaced by simpler one character symbols. The nature of these symbols does not actually matter: just their positions within the addresses and within the generator are important.

We can also work in the opposite direction: a one-dimensional fractrix with generator size σ_π can be put into a 1-1 correspondence with any p-dimensional fractrix with generator size $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ for which $\sigma_\pi = \prod_{i=1}^{p} \sigma_i$. So, any factorization of σ_π leads to a higher dimensional equivalent of a one-dimensional fractrix. In this way, an equivalence class of fractrices with a 1-1 correspondence arises: it is the class of all fractrices obtained from generators that display the same set of integers, but in all possible different arrangements.

A.2. *The p-dimensional interpretation of the HIFS*

The reinterpretation of Eqs. (22)–(32) in the p-dimensional case is without problems under the following modifications:

- All addresses are to be interpreted as equivalent σ_π-adic addresses on the $[0, 1]^p$-cube.
- $0.a\mathcal{Q}_k$ means that a $\boldsymbol{\sigma}^{-1}$ copy of the p-dimensional $\mathcal{Q}_k$ is made (i.e. size reduction σ_i^{-1} in direction i), which is then placed in the block with σ_π-address a in $[0, 1]^p$ (this block has size $\boldsymbol{\sigma}^{-1}$).
- In Eqs. (26), (27), (28) and (30), σ should be replaced by σ_π.

Appendix B
Sequences and Hyperreal Numbers

The hyperreal (HR) number system extends the real number system and contains actual infinities and infinitesimals. It forms the basis of what is known as *nonstandard* analysis, the logical foundations of which were laid by A. Robinson [1966]. Hyperreal numbers can be introduced in a natural way by considering them as equivalence classes of sequences that map $\mathbb{N}_0$ into $\mathbb{R}$. The equivalence relation requires the concept of a so called *free ultrafilter* $\mathcal{U}$ over $\mathbb{N}_0$ (see, for example, Cutland [1988], Prestel [1990], Stroyan & Luxemburg [1976]). $\mathcal{U}$ is a set of subsets of $\mathbb{N}_0$ such that (a) $\phi \in \mathbb{N}_0$; (b) when $A,B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$; (c) when $A \subset B$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$; (d) for any $A \subset \mathbb{N}_0$, either $A \in \mathcal{U}$ or $\mathbb{N}_0 \backslash A \in \mathcal{U}$; (e) finite subsets of $\mathbb{N}_0$ do not belong to $\mathcal{U}$ (but, according to d., all their complements (cofinite sets) belong to $\mathcal{U}$). The ultrafilter concept is necessary in order for the HR-number system to form a totally ordered field under $(+, \cdot, >, =)$, hereby avoiding zero divisors and relating any two numbers by $=$ or $>$.

Two sequences $\langle a_n \rangle_{n\in\mathbb{N}_0}$ and $\langle b_n \rangle_{n\in\mathbb{N}_0}$ are considered equal when $\{n | a_n = b_n\} \in \mathcal{U}$. It follows that two sequences $\langle a_n \rangle_{n\in\mathbb{N}_0}$ and $\langle b_n \rangle_{n\in\mathbb{N}_0}$ for which there is a finite M such that $a_n = b_n$ for $n \geq M$ belong to the same equivalence class and are considered equal, no matter what the particular $\mathcal{U}$ might be.

So, any sequence $\langle s_n \rangle$, $s_n \in \mathbb{R}$ can be considered as a hyperreal number through its equivalence class membership (the subscript $n \in \mathbb{N}_0$ will henceforth be omitted). The set of all hyperreals will be denoted by ${}^*\mathbb{R}$. A standard real number $c \in \mathbb{R}$ is imbedded in ${}^*\mathbb{R}$ as the constant sequence $\langle c, c, c \ldots \rangle$ (actually, its corresponding equivalence class). We summarize a few elementary facts about HR-numbers. For a more extended treatment, we refer to the literature.

— Any arithmetic operations $(+, -, \cdot, /, \text{etc.})$ and functions that are defined on $\mathbb{R}$ are also defined on ${}^*\mathbb{R}$, by a componentwise application in the operand-sequences e.g. $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$, $\sqrt{|\langle a_n \rangle|} = \langle \sqrt{|a_n|} \rangle$.

— Order relationships ($=, <, \leq$) hold between two hyperreal numbers, whenever they hold componentwise for the related sequences over a subset of the ultrafilter $\mathcal{U}$. The $\mathcal{U}$-defining conditions imply that contradictory relationships over different subsets of $\mathcal{U}$ cannot occur, e.g. either $\{n|a_n = b_n\} \in \mathcal{U}$ or $\{n|a_n > b_n\} \in \mathcal{U}$, but not both. The choice determines the particular $\mathcal{U}$, or vice-versa. It follows that when $a_i > b_i$ for $i \geq M$, M finite, but $a_i = b_i$ for $i < M$, then $\langle a_i \rangle > \langle b_i \rangle$, as $\{i|i \geq M\} \in \mathcal{U}$. $^*\mathbb{R}$ is a completely ordered field (but non-Archimedian,[8] in contrast to $\mathbb{R}$).

— In the sense just defined, the HR-number $\varepsilon \triangleq \langle \frac{1}{n} \rangle$ is smaller than any positive real number r (as $\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \rangle < \langle r, r, r, \ldots \rangle$). Therefore, it is called an *infinitesimal* number. On the other hand, $\omega \triangleq \frac{1}{\varepsilon} = \langle n \rangle$ is larger than any natural number, therefore it is a positive *infinite* number. There are an unlimited number of ordered infinitesimals and infinities, for example,

$$\varepsilon_1 \triangleq \left\langle \frac{1}{n+1} \right\rangle < \varepsilon\,,$$

$$\varepsilon - \varepsilon_1 = \left\langle \frac{1}{n} \right\rangle - \left\langle \frac{1}{n+1} \right\rangle = \left\langle \frac{1}{n} - \frac{1}{n+1} \right\rangle$$

$$= \left\langle \frac{1}{n(n+1)} \right\rangle = \varepsilon_1 \varepsilon\,, \quad \text{and}$$

$$\varepsilon_1^2 < \varepsilon - \varepsilon_1 = \varepsilon_1 \varepsilon < \varepsilon^2\,,$$

where ε, ε_1, $\varepsilon - \varepsilon_1$, ε_1, ε, ε^2, ε_1^2 are all different infinitesimals. $\sqrt{\omega} \triangleq \langle \sqrt{n} \rangle < \omega$, $\sigma^\omega = \langle \sigma^n \rangle$ are infinite numbers, and so are the inverses of infinitesimals.

— Let $c = 0.c_1c_2c_3\ldots$ be the σ-adic representation of the real number c, then the HR-number $c_h \triangleq \langle 0.c\rfloor_n \rangle = \langle 0.c_1,\ 0.c_1c_2,\ 0.c_1c_2c_3, \ldots \rangle$ differs infinitesimally from c. Indeed: $c - c_h = \langle c, c, c, \ldots \rangle - \langle 0.c_1,\ 0.c_1c_2,\ 0.c_1c_2c_3, \ldots \rangle = \langle 0.0c_2c_3\ldots,\ 0.00c_3c_4\ldots,\ 0.000c_4c_5\ldots, \ldots \rangle < \langle \frac{1}{\sigma^n} \rangle$, and so $(c - c_h)$ is a positive infinitesimal number. When two hyperreal numbers a and b differ only infinitesimally (or: are *infinitely close* to each other), this will be indicated as $a \approx b$. $\approx$ is transitive.

[8] Archimedian order: let $a < b$, then there is an $n \in N$ such that $an > b$.

— All HR-numbers that are infinitely close to a real number c are said to form the *monad* of c, while c itself is called the *standard (real) part* of these numbers. Or: the standard part $st(x)$ of a hyperreal x is the real number that is infinitely close to x. The standard part of a hyperreal number is unique.

We are now prepared for a hyperreal vision on certain concepts from the previous sections. First notice that the binary representation $0.100\ldots$ of $\frac{1}{2}$, when seen as a hyperreal number through the associated sequence $\langle 0.1,\ 0.10,\ 0.100, \ldots \rangle$, equals the real number $\frac{1}{2}$, as $\langle 0.1,\ 0.10,\ 0.100, \ldots \rangle - \langle \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \ldots \rangle = \langle 0,\ 0,\ 0, \ldots \rangle = 0$. However, when the alternative representation $0.0111\ldots$ is seen as a HR-number through the sequence $\langle 0.0,\ 0.01,\ 0.011,\ 0.0111\ldots \rangle$, it differs infinitesimally from $\frac{1}{2}$, as $\langle \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}, \ldots \rangle - \langle 0.0,\ 0.01,\ 0.011, \ldots \rangle = \langle 0.1,\ 0.01,\ 0.001,\ 0.0001, \ldots \rangle$ is an infinitesimal. But $\frac{1}{2}$ is the standard part of $0.0111\ldots$ when it is seen as a hyperreal number. In order to distinguish between the real number c with σ-adic representation $0.c_1c_2c_3\ldots$ and the hyperreal number $\langle 0.c_1, 0.c_1c_2, 0.c_1c_2c_3, \ldots \rangle$, we will represent the latter by $\mathcal{H}0.c_1c_2c_3\ldots$. Actually, as all elements in this sequence are σ-rational numbers, $\mathcal{H}0.c_1c_2c_3\ldots$ is a *σ-adic hyperrational* number with standard part $c = 0.c_1c_2c_3\ldots$.

We will now extend the hyperreal σ-adic representation $\mathcal{H}0.c_1c_2c_3\ldots \triangleq \langle 0.c\rfloor_n \rangle$ to $\mathcal{H}0.c_1c_2c_3\ldots \tilde{q}_{m_\omega}^{(\omega)} \triangleq \langle 0.c\rfloor_n \tilde{q}_{m_n}^{(n)} \rangle$. Then observe the following:

— $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}_{m_\omega}^{(\omega)} - \mathcal{H}0.c_1c_2c_3\ldots = \langle 0.0^n \tilde{q}_{m_n}^{(n)} \rangle = \mathcal{H}0.000\ldots\tilde{q}_{m_\omega}^{(\omega)}$ = positive infinitesimal, or $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}_{m_\omega}^{(\omega)} \approx \mathcal{H}0.c_1c_2c_3\ldots$. As $\mathcal{H}0.c_1c_2c_3\ldots \approx 0.c_1c_2c_3\ldots$, it also follows from the transitivity of $\approx$ that $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}_{m_\omega}^{(\omega)} \approx 0.c_1c_2c_3\ldots$, and so $c = 0.c_1c_2c_3\ldots$ is the standard part of both $\mathcal{H}0.c_1c_2c_3\ldots$ and $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}_{m_\omega}^{(\omega)}$.

— The $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}_{m_\omega}^{(\omega)}$ representation is a true extension of the $\mathcal{H}0.c_1c_2c_3\ldots$ representation: the latter is obtained from the former by putting $\tilde{q}_{m_n}^{(n)} = s_\phi$ (i.e. the empty string) for all n.

— $0.c_1c_2c_3\ldots\tilde{q}_{m_\omega}^{(\omega)}$ can be considered as the "naturally extended" representation of the ωth term in the sequence $\langle 0.c\rfloor_n \tilde{q}_{m_n}^{(n)} \rangle$ when this sequence would be extended into the infinite hyperinteger index set.

In Sec. 2, we have been considering sequences $\langle s(n)\rangle = \langle 0.c\rfloor_n \tilde{q}^{(n)}_{m_n}\rangle$ with associated weight sequence $\langle ws(n)\rangle = w\langle 0.c\rfloor_n \tilde{q}^{(n)}_{m_n}\rangle \triangleq \langle w(0.c\rfloor_n \tilde{q}^{(n)}_{m_n})\rangle$. $\langle s(n)\rangle$ converges to $c = 0.c_1c_2c_3\ldots$, and when $\langle ws(n)\rangle$ converges simultaneously to a value $k \in \mathbb{Z}$, it was said that in the limit c maps into the value k. We are tempted to translate this in the HR context as: c is the standard part of the σ-adic hyperrational number $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}^{(\omega)}_{m_\omega}$ with weight k. However, in order for this to be true, it must be shown that this associated weight is k for any sequence that belongs to the equivalence class of $\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}^{(\omega)}_{m_\omega}$. We show that this is indeed the case. Let $\langle b_n\rangle$ belong to the same equivalence class as $\langle 0.c_n\tilde{q}^{(n)}_{m_n}\rangle$, then $I = \{n|b_n = 0.c\rfloor_n\tilde{q}^{(n)}_{m_n}\} \in \mathcal{U}$. It is clear that $\langle b_n\rangle$ has the same standard part c as $\langle 0.c\rfloor_n\tilde{q}^{(n)}_{m_n}\rangle$, because the subsequence $\langle b_n\rangle_{n\in I} = \langle 0.c\rfloor_n\tilde{q}^{(n)}_{m_n}\rangle_{n\in I}$ converges to c. But also the corresponding weight-sequence $\langle w(0.c\rfloor_n\tilde{q}^{(n)}_{m_n})\rangle_{n\in I}$ converges to k. As a matter of fact, this weight-subsequence equals k in the HR-sense, because $\{n|w(0.c\rfloor_n\tilde{q}^{(n)}_{m_n}) = k\}\cap I \in \mathcal{U}$ as the cofinite set $\{n|w(0.c\rfloor_n\tilde{q}^{(n)}_{m_n}) = k\} \in \mathcal{U}$. So, it is legal to speak about the weight of a σ-adic hyperrational number, because all members of a specific equivalence class of sequences have the same weight.

The following HR-language definition of $\mathcal{F}_k$ and $\mathcal{G}_k$, $k \in \mathbb{Z}$, should now be clear:

$$\mathcal{F}_k = \{st(\mathcal{H}0.c_1c_2c_3\ldots)|w(\mathcal{H}0.c_1c_2c_3\ldots) = k\},$$

$$\mathcal{G}_k = \{st(\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}^{(\omega)}_{m_\omega}) \times |w(\mathcal{H}0.c_1c_2c_3\ldots\tilde{q}^{(\omega)}_{m_\omega}) = k\},$$

and it follows that $\mathcal{F}_k \subset \mathcal{G}_k$. So, $\mathcal{G}_k$ is the set of real numbers which are standard parts of all σ-adic hyperrationals with weight k. $\mathcal{F}_k$ is similar, but is based on a more restricted set of σ-adic hyperrationals. The nice thing is that the HR view preserves the property that a given σ-adic hyperrational number has a unique weight. It is only because we restrict our perception to the real-number world that all σ-adic hyperrationals in the monad of a given real number c become indistinguishable from c, although their weights remain distinguishable. This explains the collapse into ambiguity: under real-number resolution, c maps into all weights of the σ-adic hyperrational numbers of its monad. The ultimate reason for this ambiguity is the fact that any sequence $\langle 0.c\rfloor_n\tilde{q}^{(n)}_{m_n}\rangle$ converges to $0.c_1c_2c_3\ldots$ no matter what $\langle\tilde{q}^{(n)}_{m_n}\rangle$ is, because the digits in $\tilde{q}^{(n)}_{m_n}$ contribute with decreasing importance in the associated sum $0.c\rfloor_n\tilde{q}^{(n)}_{m_n} = \sum_{j=1}^{n}\frac{c_j}{\sigma^j} + \frac{\tilde{q}^{(n)}_{m_n}}{\sigma^{n+m_n}}$ when n increases; while in the corresponding weight sequence, the way in which a given digit in $0.c\rfloor_n\tilde{q}^{(n)}_{m_n}$ contributes to the total weight $w(0.c\rfloor_n\tilde{q}_{m_n})$ is independent of the location of that digit in this string. And this is exactly why there is a need to distinguish between different $\langle 0.c\rfloor_n\tilde{q}^{(n)}_{m_n}\rangle$ sequences. In this sense, the hyperreal framework seems to be rather natural for studying the limiting process of fractrices. It would in principle be possible to reformulate all proofs concerning the limit forms of $G\mathcal{L}_k(E_m)$ and $\hat{\mathcal{Q}}_k(m)$ in a nonstandard (hyperreal) way.