Katholieke Universiteit Leuven

Departement Elektrotechniek

ESAT-SISTA/TR 1993-78I

Matrix Descriptor Formulation of a Large Class of Nonlinear Physical Dynamic Systems

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December 23, 1993

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Abstract

This paper is focussed on the determination of a descriptor formulation of a class of nonlinear physical dynamic systems. Based on bond graph methodology, general nonlinear physical systems are explored. It is not always useful to derive such a formulation; however, a closed form is obtainable for a certain class of them. It turns out that the result is a generalisation of the state space formulation of linear physical systems. This methodology is illustrated with an example.

1 Introduction

In most modern control design procedures the plant or system is described as a *linear state space form*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

where $\mathbf{x}$ is *the state vector*, $\mathbf{u}$ is *the input vector*, $\dot{\mathbf{x}}$ is *the state vector velocity*, $\mathbf{A}$ is *the state matrix* and $\mathbf{B}$ is *the input matrix*. In very unusual cases the real system is almost linear and a true linear state space form, Eqn. (1), can be found. If not, a linearisation of the system behaviour around a working point is considered for the system description.
More generally, the matrices $\mathbf{A}$ and $\mathbf{B}$ can be linear time dependent in order to include the changing system dynamics. But when using a state space description one assumes the system acts always *causal*, i.e. the propagation of the state vector $\mathbf{x}$ can be determined recursively as the input vector $\mathbf{u}$ is known. This is not always the case [5], [8]. Linear systems which do not obey Eqn. (1) are often called *degenerate*.
Moreover, general physical systems are nonlinear time varying, and most modeling procedures to obtain Eqn. (1) are not very well structured. The major contribution of this paper is to outline a structured derivation of a descriptor form for a large class of nonlinear systems. According to Tan [9] a *descriptor* or *singular system* can be described as follows

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2}$$

where $\mathbf{E}$ can be singular. Here the vector $\mathbf{x}$ is called *the descriptor vector*.
Therefore bond graphs, a structured modeling tool, are used to ensure that the derived dynamic model includes the necessary energetical system properties. The paper presented here uses the state equation generation procedure of the keynote paper [6], where the state equations are generated for linear and state dependent nonlinear multiport systems.
For most practical physical dynamic systems the field relations of the bond graph can be written down as matrix functions. For such systems, a closed matrix descriptor formulation is derived. It turns out that this equation involves a relation of the form

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\dot{\mathbf{u}} \tag{3}$$

where again $\mathbf{E}$ may be singular. This result deviates slightly from the definition of a descriptor system used in [9] (Eqn. (2)) or [5] and [10].

Mainly depending on the junction structure these dynamic equations can be written into an form without cross-terms. The proposed treatement also leads to an explicit determination of degeneracy (or singularity) of the observed physical dynamic system.
This paper is arranged as follows. In Section 2 nonlinear physical systems are described in a general bond graph representation, and a certain class of nonlinear physical systems is defined. In Section 3 the derivation of the nonlinear descriptor and output equations for this class is done with use of the bond graph representation. Depending on the linearity and time dependancy of certain parts of the bond graph of the physical system different subclasses are also treated. Finally, in Section 4 the results are illustrated with a two-dimensional mechanical system.

2 Nonlinear system description with bond graphs

While describing physical systems one can focus the attention to the two conservation principles, the *energy and power conservation law.* Bond graphs, shown in figure 1, visualize explicitely those two fundamental laws of nature. Here the physical system is divided into two parts. Every physical effect that obeys the power conservation law is put together in the *power conservation* part or field, and every physical effect that obeys the energy conservation law is put together in the *energy conservation* part or field.

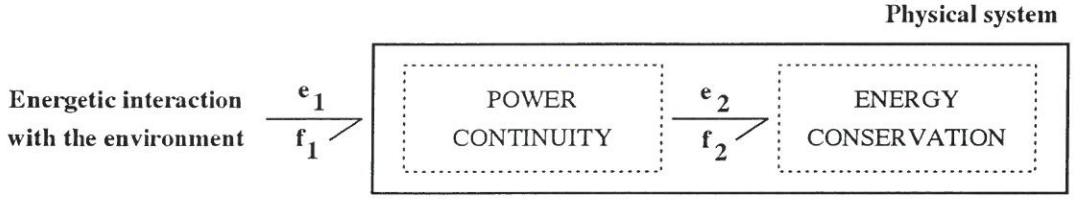

Figure 1: Schematic view of a physical system obeying the conservation laws. The power interaction in (index 2) and across the physical system border (index 1) is shown with a simple line or a harpoon. This harpoon is called *a power bond.* If a harpoon is used, as is done in the figure, then a reference direction of the energy flow is assumed.

Each power interaction involves two time dependent conjugate power variables (in general vectors), an effort $\mathbf{e}$ and a flow variable $\mathbf{f}$, since

$$P = \mathbf{e}^{\mathrm{T}}\mathbf{f}$$

For each power bond these two variables can be shown in the bond graph, the flow variable always stands on the side of the harpoon arrow. The ensemble of those varables describing the energetic and non energetic interaction between those fields and the environment are called the key vectors.

2.1 General definition of the nonlinear field relations and the key vectors

An exploded view of the physical system of figure 1 is shown in figure 2. Different bond graph elements are used in these fields. In general these elements can be characterized by implicit equations, as shown in the sequel.

1. The (energy) storage field is constructed of **C**- and **I**-elements. The difference between those two types lies in the fact that their constituent relation is different:

$$\begin{aligned} \text{C-element} &: \Phi_{\mathbf{C}}(\mathbf{e},\mathbf{q}) = 0 \\ \text{I-element} &: \Phi_{\mathbf{I}}(\mathbf{f},\mathbf{p}) = 0 \end{aligned}$$

   where $\mathbf{q}$ is the generalized displacement vector and $\mathbf{p}$ is the generalized momentum vector of the corresponding element. They can be calculated as the time integral of the vectors $\mathbf{f}$, resp. $\mathbf{e}$. In mechanics the stored energy in a **C**-element is called potential, while the stored energy in an **I**-element is called kinetic. In electrical circuits these energies are called electric, resp. magnetic.

2. The dissipation field is constructed of **R**- elements representing, according to the causality, a resistance or a conductance. The constituent relation of an **R**-element is

$$\Phi_{\mathbf{R}}(\mathbf{e},\mathbf{f}) = 0$$

3. The junction structure is constructed of 4 types power transducers, **0**, **1**, (M)**TF** ((modulated) transformer) and/or (M)**GY** ((modulated) gyrator) elements representing the different power conserving transformations.

4. The power sources are constructed of **Se**- (effort source) and **Sf**- (flow source) elements according to the causality involved. An effort source

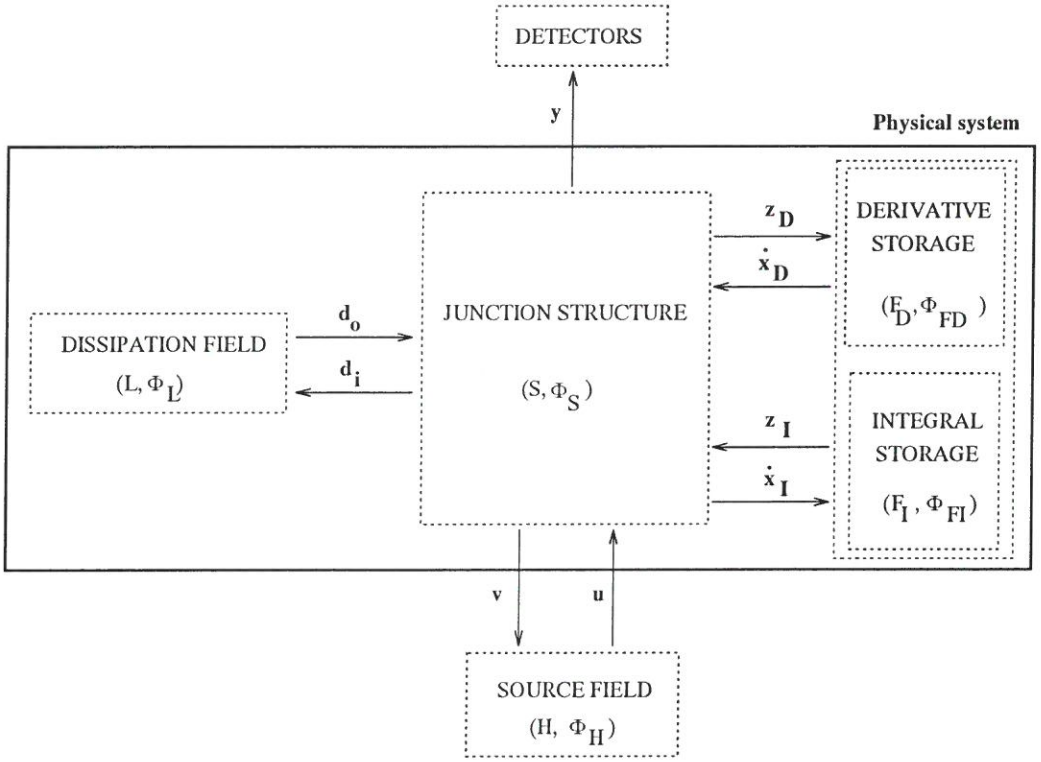

Figure 2: Visualisation of the key vectors of a physical system. The part of the system obeying the power conservation law is denoted as the JUNCTION STRUCTURE, the part of the system obeying the energy conservation law is partitioned in an independent part denoted as INTEGRAL STORAGE, and a dependent part denoted as DERIVATIVE STORAGE. The environment part involving energetic interaction is also partitioned in a part showing the free energy transformation to the thermal domain denoted as DISSIPATION, the DETECTORS representing powerless (or negligible) measures of physical variables, and the remaining part denoted as SOURCES. It is also possible to consider the dissipation field as a part of the system, as shown in the figure. Then the dissipation is a kind of *power hole* in the physical system.

determines the effort variable at the power port, and a flow source determines the flow variable at the power port. The constituent relation of a source-element is of course of the same form as a dissipation field,

$$\Phi_P(e, f) = 0$$

Because the detector elements involve no (or negligible) power interaction with the system, there are no bond graph elements representing their behaviour. The powerless interaction is visualized as a single signal flow vector $\mathbf{y}$.
The different key vectors involving power or powerless interaction with or within the physical system are ordered below:

1. The dissipation field input and output vector

$$\mathbf{d_i} \ , \ \mathbf{d_o}$$

2. The energy vector of the dynamic system is taken as descriptor vector $\mathbf{x}$. This vector can be partitioned as follows

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_I} \\ \mathbf{x_D} \end{bmatrix}$$

where $\mathbf{x_I}$ is the independent descriptor vector. This vector contains all I- and C-elements with integral causality. Its components are the generalized momenta and generalized displacements

$$\mathbf{x_I} = \begin{bmatrix} \vdots \\ p_{i_I} \\ \vdots \\ q_{j_I} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \int e_{i_I} dt \\ \vdots \\ \int f_{j_I} dt \\ \vdots \end{bmatrix} \begin{matrix} \\ \text{on I-element } i \text{ with integral causality} \\ \\ \text{on C-element } j \text{ with integral causality} \\ \\ \end{matrix}$$

and $\mathbf{x_D}$ the dependent descriptor vector. This vector contains all I- and C-elements with derivative causality. Its time derivative is defined as

$$\dot{\mathbf{x}}_\mathbf{D} = \begin{bmatrix} \vdots \\ e_{i_D} \\ \vdots \\ f_{j_D} \\ \vdots \end{bmatrix} \begin{matrix} \\ \text{on I-element } i \text{ with derivative causality} \\ \\ \text{on C-element } j \text{ with derivative causality} \\ \\ \end{matrix}$$

3. The coenergy vector $\mathbf{z}$. This vector can be partitioned as follows

$$\mathbf{z} = \begin{bmatrix} \mathbf{z_I} \\ \mathbf{z_D} \end{bmatrix}$$

where $\mathbf{z_I}$ is the independent coenergy vector. This vector contains all I- and C-elements with integral causality

$$\mathbf{z_I} = \begin{bmatrix} \vdots \\ f_{i_I} \\ \vdots \\ e_{j_I} \\ \vdots \end{bmatrix} \begin{array}{l} \\ \text{on I-element } i \text{ with integral causality} \\ \\ \text{on C-element } j \text{ with integral causality} \\ \\ \end{array}$$

and $\mathbf{z_D}$ the dependent coenergy vector. This vector contains all I- and C-elements with derivative causality

$$\mathbf{z_D} = \begin{bmatrix} \vdots \\ f_{i_D} \\ \vdots \\ e_{j_D} \\ \vdots \end{bmatrix} \begin{array}{l} \\ \text{on I-element } i \text{ with derivative causality} \\ \\ \text{on C-element } j \text{ with derivative causality} \\ \\ \end{array}$$

4. The source field input and output vector

$$\mathbf{v} \ , \ \mathbf{u}$$

5. The detector field input vector

$$\mathbf{y}$$

The power interaction with the different fields can be calculated with these vectors. For example, the instantaneous power interaction with the storage field with N independent and M dependent power ports is

$$P = \dot{\mathbf{x}}_\mathbf{I}^T \mathbf{z_I} + \dot{\mathbf{x}}_\mathbf{D}^T \mathbf{z_D} = \sum_{i=1}^{N} e_{iI} f_{iI} + \sum_{i=1}^{M} e_{iD} f_{iD}$$

2.2 General explicit field descriptions

As already shown, the general description of the fields can be done by different sets of implicit nonlinear equations. In most cases an explicit formulation of these field can be derived:

1. The dissipation field
$$\mathbf{d_o} = \Phi_{\mathbf{L}}(\mathbf{d_i}) \tag{4}$$

2. The source field
$$\mathbf{u} = \Phi_{\mathbf{H}}(\mathbf{v}) \tag{5}$$

3. The storage field
$$\begin{bmatrix} \mathbf{z_I} \\ \mathbf{x_D} \end{bmatrix} = \Phi_{\mathbf{F}}(\begin{bmatrix} \mathbf{x_I} \\ \mathbf{z_D} \end{bmatrix}) \tag{6}$$
If only one port storage elements exists, or if there are no mixed storage fields in the system involved, then the independent and dependent state vector $\mathbf{x_I}$ and $\mathbf{x_D}$ is not coupled, and Eqn. (6) can be written as
$$\begin{cases} \mathbf{z_I} = \Phi_{\mathbf{FI}}(\mathbf{x_I}) \\ \mathbf{x_D} = \Phi_{\mathbf{FD}}^{-1}(\mathbf{z_D}) \end{cases} \tag{7}$$

4. The junction structure field implicit equation is
$$\Phi_{\mathbf{S}}(\begin{bmatrix} \dot{\mathbf{x}}_\mathbf{I} \\ \mathbf{z_D} \\ \mathbf{d_i} \\ \mathbf{y} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathbf{z_I} \\ \dot{\mathbf{x}}_\mathbf{D} \\ \mathbf{d_o} \\ \mathbf{u} \end{bmatrix}) = \mathbf{0} \tag{8}$$
Where $\begin{bmatrix} \dot{\mathbf{x}}_\mathbf{I} \\ \mathbf{z_D} \\ \mathbf{d_i} \\ \mathbf{y} \\ \mathbf{v} \end{bmatrix}$ is the junction structure output vector, and $\begin{bmatrix} \mathbf{z_I} \\ \dot{\mathbf{x}}_\mathbf{D} \\ \mathbf{d_o} \\ \mathbf{u} \end{bmatrix}$ is the junction structure input vector. In case Eqn. (8) can be solved explicitely, the junction structure is called *solvable*, see [7]. Only solvable juction structures are considered here.

In a large class of physical systems these explicit nonlinear field functions can be written as matrix functions. In general these functions are not restricted to linear functions. Consequently

$$\mathbf{d_o} = \mathbf{L}\mathbf{d_i} \tag{9}$$

with $\mathbf{L}$ the dissipation field matrix;

$$\mathbf{u} = \mathbf{H}\mathbf{v} \tag{10}$$

with $\mathbf{H}$ the source field matrix;

$$\mathbf{z_I} = \mathbf{F_I}\mathbf{x_I} \tag{11}$$

with $\mathbf{F_I}$ the independent storage field matrix;

$$\mathbf{x_D} = \mathbf{F_D^{-1}}\mathbf{z_D} \tag{12}$$

with $\mathbf{F_D}$ the dependent storage field matrix. Finally

$$\begin{bmatrix} \dot{\mathbf{x}}_\mathbf{I} \\ \mathbf{z_D} \\ \mathbf{d_i} \\ \mathbf{v} \\ \mathbf{y} \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{z_I} \\ \dot{\mathbf{x}}_\mathbf{D} \\ \mathbf{d_o} \\ \mathbf{u} \end{bmatrix} \tag{13}$$

with $\mathbf{S}$ the junction structure field matrix. According to the dimension of the vectors involved, this matrix can be partitoned as follows

$$\mathbf{S} = \begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} & \mathbf{S_{13}} & \mathbf{S_{14}} \\ \mathbf{S_{21}} & \mathbf{S_{22}} & \mathbf{S_{23}} & \mathbf{S_{24}} \\ \mathbf{S_{31}} & \mathbf{S_{32}} & \mathbf{S_{33}} & \mathbf{S_{34}} \\ \mathbf{S_{41}} & \mathbf{S_{42}} & \mathbf{S_{43}} & \mathbf{S_{44}} \\ \mathbf{S_{51}} & \mathbf{S_{52}} & \mathbf{S_{53}} & \mathbf{S_{54}} \end{bmatrix} \tag{14}$$

Some of those submatrices obey certain properties or are related to each other. First the diagonal submatrices are all square. After doing the causal analysis it turns out that the derivative storage field is only a function of the independent state and input vector, thus

$$\begin{cases} \mathbf{S_{22}} = 0 \\ \mathbf{S_{23}} = 0 \end{cases} \tag{15}$$

It is always possible to direct all the power arrows towards or outwards the junction structure. Then it follows form the power conserving property of the junction structure, [6], that the matrix $\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$ is skew symmetric. Using this property and Eqn. (15), Eqn. (13) can be rewritten as

$$\begin{bmatrix} \dot{x}_I \\ z_D \\ d_i \\ v \\ y \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ -S_{12}^T & 0 & 0 & S_{24} \\ -S_{13}^T & 0 & S_{33} & S_{34} \\ -S_{14}^T & -S_{24}^T & -S_{34}^T & S_{44} \\ S_{51} & S_{52} & S_{53} & S_{54} \end{bmatrix} \cdot \begin{bmatrix} z_I \\ \dot{x}_D \\ d_o \\ u \end{bmatrix} \tag{16}$$

with the diagonal submatrices S_{11}, S_{33} and S_{44} skew symmetric.

3 Derivation of the nonlinear matrix descriptor and output equation

In the sequel the class of nonlinear systems for which the field relations can be written in matrix form are considered. No constraint is made on the time dependency or linearity of the matrices involved. For this class a matrix descriptor and matrix output equation form is deduced. Therafter several subclasses are explored.

3.1 Descriptor equation formulation

In order to derive the matrix descriptor equation the vector $\dot{x}_D$ and d_o has to be rewritten as a function of z_I and u. Putting Eqn. (16) (b) into Eqn. (12) yields after differentiation

$$\begin{aligned} \dot{x}_D &= -\left\{\dot{F}_D^{-1} S_{12}^T + F_D^{-1} \dot{S}_{12}^T\right\} z_I - F_D^{-1} S_{12}^T \dot{z}_I \\ &\quad + \left\{\dot{F}_D^{-1} S_{24} + F_D^{-1} \dot{S}_{24}\right\} u + F_D^{-1} S_{24} \dot{u} \end{aligned} \tag{17}$$

And putting Eqn. (16) (c) in Eqn. (9) yields

$$d_o = (I - L S_{33})^{-1} L \left[-S_{13}^T z_I + S_{34} u\right] \tag{18}$$

Of course, this equation only holds for an invertible $(\mathbf{I} - \mathbf{LS_{33}})$. For non-dissipative systems this is always true, since $\mathbf{L} = \mathbf{0}$. Finally the matrix descriptor formulation can be obtained by putting Eqns. (17), (18) and (11) into Eqn. (16) (a)

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x}_\mathbf{I} + \mathbf{B}\mathbf{u} + \mathbf{G}\dot{\mathbf{u}} \tag{19}$$

with

$$\mathbf{E} = \mathbf{I} + \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}\mathbf{F_I} \tag{20}$$

$$\begin{aligned}\mathbf{A} = {} & \left(\mathbf{S_{11}} - \mathbf{S_{12}}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{12}^T} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_{12}^T\right]\right. \\ & \left. -\mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \\ & -\mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}\dot{\mathbf{F}}_\mathbf{I}\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{B} = {} & \mathbf{S_{14}} + \mathbf{S_{12}}\left(\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{24}} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_{24}\right) \\ & +\mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}\end{aligned} \tag{22}$$

$$\mathbf{G} = \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{24}} \tag{23}$$

Basically $\mathbf{E}$ can become singular for certain values of $\mathbf{S_{12}}$, $\mathbf{F_D^{-1}}$ and $\mathbf{F_I}$. It is easy to see that $\mathbf{F_D^{-1}}$ plays a crucial role in the time behaviour of the descriptor vector. If no derivative causality appears in the storage field, then $\mathbf{E} \equiv \mathbf{I}$ and $\mathbf{G} \equiv \mathbf{0}$. As a consequence Eqn. (19) is in fact a state space description and the descriptor vector equals the state vector. Contrary, if a derivative causality appears in the storage field a state space description is only possible iff $\mathbf{E}$ is not singular, or

$$\det(\mathbf{I} + \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}) \neq 0 \tag{24}$$

However, the state vector is not more equal to the independent energetic physical states. An artificial state vector $\tilde{\mathbf{x}}$ can be defined as

$$\tilde{\mathbf{x}} = \mathbf{x}_\mathbf{I} - \mathbf{G}\mathbf{u}$$

Bringing this vector into Eqn. (19) yields the matrix state equation

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \tag{25}$$

with

$$\tilde{\mathbf{A}} = \mathbf{E}^{-1}\mathbf{A} \tag{26}$$

$$\tilde{\mathbf{B}} = \mathbf{E}^{-1}\left(\mathbf{B} + \mathbf{A}\mathbf{E}^{-1}\mathbf{G} + \dot{\mathbf{E}}\mathbf{E}^{-1} - \dot{\mathbf{G}}\right) \tag{27}$$

Notice that the nature of a system obeying state equations is totally different to those obeying descriptor equations [10] and thus influences further control schemes [9].
Notice also that the matrix descriptor form Eqn. (19) differs form its standard form defined in [5] and [9]. However when defining the block matrix

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{G} \end{bmatrix} \tag{28}$$

and and the artificial input vector

$$\tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} \tag{29}$$

Eqn. (19) can be rewritten in standard form

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x}_\mathbf{I} + \tilde{\mathbf{B}}\tilde{\mathbf{u}} \tag{30}$$

Consequently the set of independent energetical states can always serve as a descriptor vector. But this implies constraints on the artificial input vector $\tilde{\mathbf{u}}$, since its components cannot vary independently.

3.2 Output equation formulation

If $\mathbf{E}$ is not singular then the standard output equations can be derived. By putting Eqns. (17), (18) and (11) into Eqn. (16) (e), wherafter $\dot{\mathbf{x}}_\mathbf{I}$ is substituted by Eqn. (19), one obtains

$$\mathbf{y} = \mathbf{C}\mathbf{x}_\mathbf{I} + \mathbf{D}\mathbf{u} + \mathbf{Q}\dot{\mathbf{u}} \tag{31}$$

with

$$\mathbf{C} = \mathbf{C}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{A} \tag{32}$$

$$\mathbf{D} = \mathbf{D}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{B} \tag{33}$$

$$\mathbf{Q} = \mathbf{Q}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{G} \tag{34}$$

in which $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ are defined by Eqns. (20), (21), (22) and (23), and

$$\begin{aligned} \mathbf{C}^* = & \Big(\mathbf{S}_{51} - \mathbf{S}_{52}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathrm{T} + \mathbf{F}_\mathbf{D}^{-1}\dot{\mathbf{S}}_{12}^\mathrm{T}\right] \\ & -\mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^\mathrm{T}\Big)\mathbf{F}_\mathbf{I} \\ & -\mathbf{S}_{52}\mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathrm{T}\dot{\mathbf{F}}_\mathbf{I} \end{aligned} \tag{35}$$

$$\mathbf{P} = -\mathbf{S}_{52}\mathbf{F}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}}\mathbf{F}_{\mathrm{I}} \tag{36}$$

$$\mathbf{D}^{*} = \mathbf{S}_{54} + \mathbf{S}_{52}\left(\dot{\mathbf{F}}_{\mathrm{D}}^{-1}\mathbf{S}_{24} + \mathbf{F}_{\mathrm{D}}^{-1}\dot{\mathbf{S}}_{24}\right) + \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{37}$$

$$\mathbf{Q}^{*} = \mathbf{S}_{52}\mathbf{F}_{\mathrm{D}}^{-1}\mathbf{S}_{24} \tag{38}$$

Again, Eqn. (31) holds for an invertible $(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})$ only.

3.3 Some important remarks

In the matrices **A**, **B**, **C** and **D** time derivatives of some submatrices of the matrix **S** and $\mathbf{F}_{\mathrm{D}}^{-1}$ appear. In certain cases, like in two or three dimensional mechanics major parts of the the matrix **S** are coordinate transformation matrices. Then the submatrices of **S** can be written as functions of generalized coordinates, which can be seen as the time integral of some descriptor variable. For such systems Eqns. (19) and (31) contain cross-terms because products of descriptor variables with themselves and input variables appear. It is also possible that supplementary algebraic constraints arise in the calculation of some time dependent components of the matrices involved, mainly the junction structure matrix **S**. These constraints also stemm from the bond graph.
At last, it can be seen that the descriptor and output equation is independent of the source field constituent relation and thus is formally not a system property, as stated earlier.

3.4 Subclasses of nonlinear systems with field matrices

According to the behaviour of the different matrices involved in the descriptor and output equation formulation, different reduced descriptions can be found. Of course a combination of the results shown below is possible.

1. **There is no causal connection between the dependent storage field and the detectors.**
   Then in Eqn. (16) the submatrix $\mathbf{S}_{52} = \mathbf{0}$ and the output equation (31) becomes much simpler

   $$\mathbf{y} = \mathbf{C}\mathbf{x}_{\mathrm{I}} + \mathbf{D}\mathbf{u} \tag{39}$$

   with

   $$\mathbf{C} = \left(\mathbf{S}_{51} - \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^{\mathrm{T}}\right)\mathbf{F}_{\mathrm{I}} \tag{40}$$

   $$\mathbf{D} = \mathbf{S}_{54} + \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{41}$$

If supplementary there is no causal connection between the dissipation field and the detectors, then $\mathbf{S_{53}} = \mathbf{0}$ and the output matrices of Eqn. (39) become very simple

$$\mathbf{C} = \mathbf{S_{51}F_I} \tag{42}$$
$$\mathbf{D} = \mathbf{S_{54}} \tag{43}$$

2. **The storage field matrices are time invariant.**
If both the matrices $\mathbf{F_D^{-1}}$ and $\mathbf{F_I}$ are constant, then the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$, and the output equation matrices $\mathbf{C^*}$ and $\mathbf{D^*}$, Eqns. (21), (22), (35) and (37) simplify into

$$\mathbf{A} = \left(\mathbf{S_{11}} - \mathbf{S_{12}F_D^{-1}\dot{S}_{12}^T} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \tag{44}$$
$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{12}F_D^{-1}\dot{S}_{24}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \tag{45}$$
$$\mathbf{C^*} = \left(\mathbf{S_{51}} - \mathbf{S_{52}F_D^{-1}\dot{S}_{12}^T} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \tag{46}$$
$$\mathbf{D^*} = \mathbf{S_{54}} + \mathbf{S_{52}F_D^{-1}\dot{S}_{24}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \tag{47}$$

3. **The junction structure field matrix is time invariant.**
This is the case when no MTF of MGY elements appear in the junction structure. Then also the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$, and the output equation matrices $\mathbf{C^*}$ and $\mathbf{D^*}$, Eqns. (21), (22), (35) and (37) simplify into

$$\mathbf{A} = \left(\mathbf{S_{11}} - \mathbf{S_{12}\dot{F}_D^{-1}S_{12}^T} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} - \mathbf{S_{12}F_D^{-1}S_{12}^T\dot{F}_I} \tag{48}$$
$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{12}\dot{F}_D^{-1}S_{24}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \tag{49}$$
$$\mathbf{C^*} = \left(\mathbf{S_{51}} - \mathbf{S_{52}\dot{F}_D^{-1}S_{12}^T} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} - \mathbf{S_{52}F_D^{-1}S_{12}^T\dot{F}_I} \tag{50}$$
$$\mathbf{D^*} = \mathbf{S_{54}} + \mathbf{S_{52}\dot{F}_D^{-1}S_{24}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \tag{51}$$

4. **Only the dissipation and source field matrices are eventually not time invariant.**
Again the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$ and the output equation matrices $\mathbf{C^*}$ and $\mathbf{D^*}$, Eqns. (21), (22), (35) and (37) simplify into

$$\mathbf{A} = \left\{\mathbf{S_{11}} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right\}\mathbf{F_I} \tag{52}$$

$$\mathbf{B} = \mathbf{S}_{14} + \mathbf{S}_{13}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{53}$$

$$\mathbf{C}^* = \left\{\mathbf{S}_{51} - \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^{\mathrm{T}}\right\}\mathbf{F}_{\mathrm{I}} \tag{54}$$

$$\mathbf{D}^* = \mathbf{S}_{54} + \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{55}$$

Remark that formally this result is also valid for complete linear systems. However, here $\mathbf{L}$ can be time varying or state dependent.

5. **There is no causal connection between the sources and the dependent storage field.**
Then in Eqn. (16) the submatrix $\mathbf{S}_{24} = \mathbf{0}$, and as result the influence of the time derivative of the input vector $\mathbf{u}$ disappears

$$\mathbf{E}\dot{\mathbf{x}}_{\mathrm{I}} = \mathbf{A}\mathbf{x}_{\mathrm{I}} + \mathbf{B}\mathbf{u} \tag{56}$$

with

$$\mathbf{E} = \mathbf{I} + \mathbf{S}_{12}\mathbf{F}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}}\mathbf{F}_{\mathrm{I}} \tag{57}$$

$$\begin{aligned}\mathbf{A} = {} & \left(\mathbf{S}_{11} - \mathbf{S}_{12}\left[\dot{\mathbf{F}}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}} + \mathbf{F}_{\mathrm{D}}^{-1}\dot{\mathbf{S}}_{12}^{\mathrm{T}}\right]\right. \\ & \left. -\mathbf{S}_{13}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^{\mathrm{T}}\right)\mathbf{F}_{\mathrm{I}} \\ & -\mathbf{S}_{12}\mathbf{F}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}}\dot{\mathbf{F}}_{\mathrm{I}}\end{aligned} \tag{58}$$

$$\mathbf{B} = \mathbf{S}_{14} + \mathbf{S}_{13}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{59}$$

$$\tag{60}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x}_{\mathrm{I}} + \mathbf{D}\mathbf{u} \tag{61}$$

with

$$\mathbf{C} = \mathbf{C}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{A} \tag{62}$$

$$\mathbf{D} = \mathbf{D}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{B} \tag{63}$$

in which $\mathbf{E}$ is defined in Eqn. (57) and

$$\begin{aligned}\mathbf{C}^* = {} & \left(\mathbf{S}_{51} - \mathbf{S}_{52}\left[\dot{\mathbf{F}}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}} + \mathbf{F}_{\mathrm{D}}^{-1}\dot{\mathbf{S}}_{12}^{\mathrm{T}}\right]\right. \\ & \left. -\mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^{\mathrm{T}}\right)\mathbf{F}_{\mathrm{I}} \\ & -\mathbf{S}_{52}\mathbf{F}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}}\dot{\mathbf{F}}_{\mathrm{I}}\end{aligned} \tag{64}$$

$$\mathbf{P} = -\mathbf{S}_{52}\mathbf{F}_{\mathrm{D}}^{-1}\mathbf{S}_{12}^{\mathrm{T}}\mathbf{F}_{\mathrm{I}} \tag{65}$$

$$\mathbf{D}^* = \mathbf{S}_{54} + \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{66}$$

The sources of such a system can be called *natural* because they don't force a derivative causality to a part of the storage field.

6. **There is no dependent storage field in the physical system.** This is the case when the bond graph can be made fully causal. As already shown in the previous section it turns out that the descriptor and output equations, Eqns. (19) and (31), changes fundamentally; they reduce into a state space description

$$\dot{\mathbf{x}}_\mathrm{I} = \mathbf{A}\mathbf{x}_\mathrm{I} + \mathbf{B}\mathbf{u} \tag{67}$$

with

$$\mathbf{A} = \left\{\mathbf{S}_{11} - \mathbf{S}_{13}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^\mathrm{T}\right\}\mathbf{F}_\mathrm{I} \tag{68}$$

$$\mathbf{B} = \left\{\mathbf{S}_{14} + \mathbf{S}_{13}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34}\right\} \tag{69}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x}_\mathrm{I} + \mathbf{D}\mathbf{u} \tag{70}$$

with

$$\mathbf{C} = \left\{\mathbf{S}_{51} - \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{13}^\mathrm{T}\right\}\mathbf{F}_\mathrm{I} \tag{71}$$

$$\mathbf{D} = \mathbf{S}_{54} + \mathbf{S}_{53}(\mathbf{I} - \mathbf{L}\mathbf{S}_{33})^{-1}\mathbf{L}\mathbf{S}_{34} \tag{72}$$

Only in this case the time derivative of the physical energetic states is a function of itself and the power input variable.
The same formulation is also obtained in [6], where the linearity of the physical system was demanded.

4 An example

A two-dimensional mechanical example is taken as example.

4.1 Physical description

The physical model of a ball and beam is shown in figure 3. The ball can freely roll on the beam, and the beam angle can be set by a stepper motor. The inertial frame of reference is denoted as XYZ, the frame of reference attached to the ball with its origin in the center of mass of the ball is denoted as xyz.

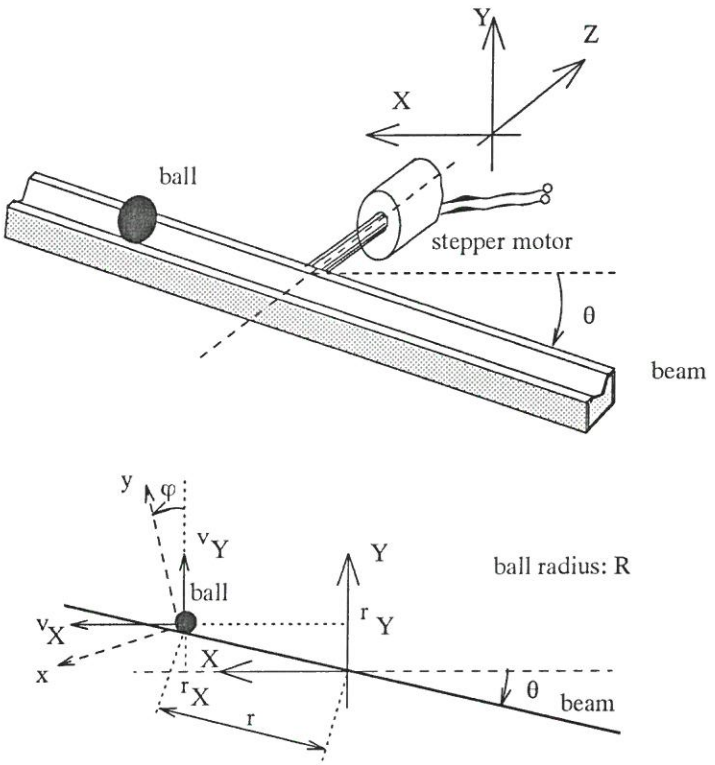

Figure 3: (u) mechanical model of the ball and beam, (l) key geometric quantities of the mechanical model. The position of the ball can be determined by the polar coordinates of the center of gravity of the ball, r and θ, and by its angular position, φ. The rectangular components of the velocity of the center of gravity of the ball in the inertial frame are v_X and v_Y.

In Appendix A the bond graph derivation is done in much detail. In order to formulate the descriptor and output equations, first the bond graph has to be made causal, wherafter the field matrices and descriptor equation are calculated. This will be done in the next subsections.

4.2 Field matrices of the ball

In figure 4 the causal bond graph of the mechanical system is partitioned as was done in figure 2. The causal analysis can be done structured [4], even automated [2]. At the same time all the power arrows are directed outward the junction structure. Therefore only the flow source S_f assignment statement has been multiplied with -1. A causal analysis of the proposed bond graph shows that it cannot be made full causal. Two 1-port storage elements are in derivative causality, so a dependent storage field arises. Also the junction structure consists of time varying elements (**MTF**), and thus **S** is not a constant matrix. From this causal bond graph the components of the key vectors can be found,

$$\begin{cases} \mathbf{d_o} = \begin{bmatrix} e_6 \end{bmatrix} \\ \mathbf{d_i} = \begin{bmatrix} f_6 \end{bmatrix} = \begin{bmatrix} \dot{r} \end{bmatrix} \\ \mathbf{x} = \begin{bmatrix} \mathbf{x_I} \\ \mathbf{x_D} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \\ \mathbf{z} = \begin{bmatrix} \mathbf{z_I} \\ \mathbf{z_D} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} v_Y \\ v_X \\ \dot{\varphi} \end{bmatrix} \\ \mathbf{u} = \begin{bmatrix} e_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} m_b.g \\ \dot{\theta} \end{bmatrix} \\ \mathbf{v} = \begin{bmatrix} f_4 \\ e_5 \end{bmatrix} \\ \mathbf{y} = \begin{bmatrix} f_8 \end{bmatrix} = \begin{bmatrix} \dot{r} \end{bmatrix} \end{cases} \tag{73}$$

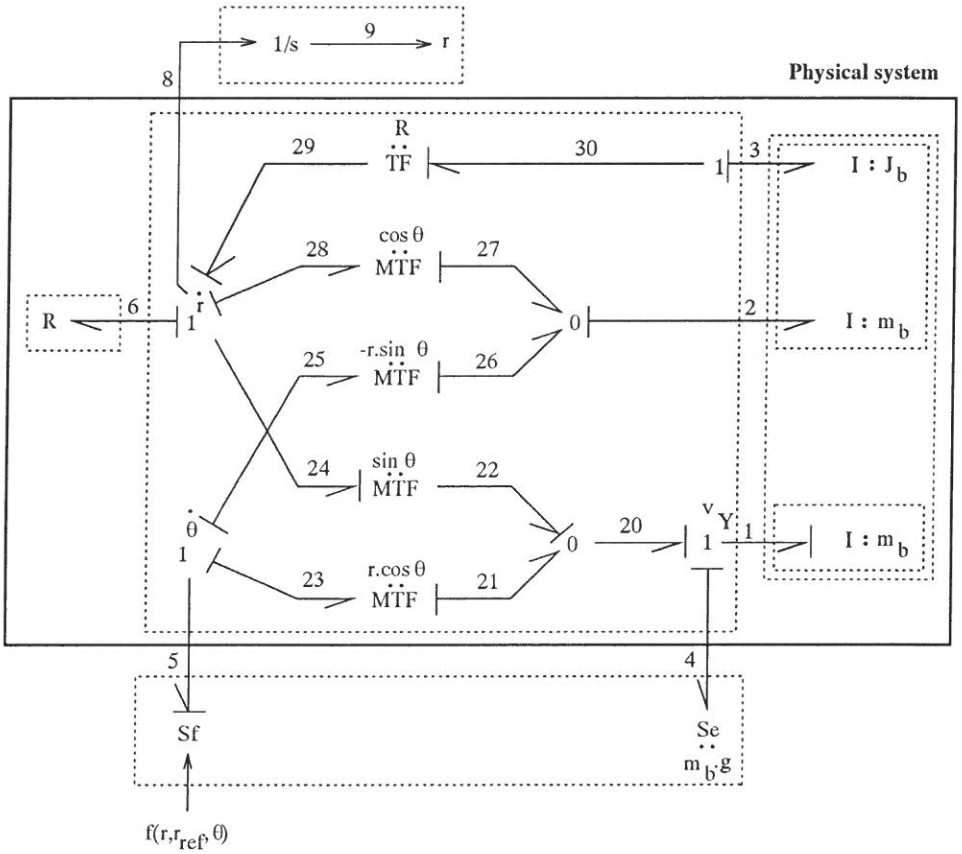

Figure 4: The causal bond graph of the ball. The different bond graph elements are put in their respective field. The external bonds, i.e. bonds that stab through the junction structure field, are numbered form 1 to 6, the internal bonds are numbered form 20 to 30. The causal stroke, or *effort arrow*, denotes the direction of the effort variable at each power port. Simply spoken the causal assignment determines the way of calculating the effort variables and thus the dynamic behaviour of the physical system. The major objective of a causal analysis is thus to calculate as many as possible storage elements in integral form, which provide numerically the most stable calculation algorithms. Clearly only the lower storage element is in integral causality. Therefore, if the physical system would be linear, then the energetic order of the system is 1.

Also the junction structure matrix S can be calculated, see Appendix B for a detailed calculation:

$$\mathbf{S} = \left[\begin{array}{c|ccc|cc} 0 & -\frac{1}{\tan\theta} & -\frac{1}{R.\sin\theta} & -\frac{1}{\sin\theta} & -1 & 0 \\ \hline \frac{1}{\tan\theta} & 0 & 0 & 0 & 0 & -\frac{r}{\sin\theta} \\ \frac{1}{R.\sin\theta} & 0 & 0 & 0 & 0 & -\frac{r}{R.\tan\theta} \\ \hline \frac{1}{\sin\theta} & 0 & 0 & 0 & 0 & -\frac{r}{\tan\theta} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{r}{\sin\theta} & \frac{r}{R.\tan\theta} & \frac{r}{\tan\theta} & 0 & 0 \\ \hline \frac{1}{\sin\theta} & 0 & 0 & 0 & 0 & -\frac{r}{\tan\theta} \end{array}\right] \tag{74}$$

It can be verified that this matrix indeed obeys the properties as stated in Eqn. (16). Also the storage field matrices can be disposed

$$\mathbf{F_I} = \left[\; m_b^{-1} \;\right] \tag{75}$$

$$\mathbf{F_D^{-1}} = \left[\begin{array}{cc} m_b & 0 \\ 0 & J_b \end{array}\right] \tag{76}$$

For the dissipation field it is hard to find an exact matrix equation description. From Eqn. (89) it is possible to derive the following trivial dissipation field input-output matrix field equation

$$e_6 = \underbrace{\left[\; \frac{c.m_b.g}{R.f_6} \;\right]}_{\mathbf{L}} . f_6 \tag{77}$$

However, the corresponding dissipation field matrix **L** can only be used for $f_6 \neq 0$. A way to solve this matrix calculation problem is to define the dissipation field matrix with an extra switch function

$$\begin{array}{lll} \text{if } f_6 \neq 0 & \text{then} & \mathbf{L} = \left[\; \frac{c.m_b.g}{R.f_6} \;\right] \\ & \text{else} & \mathbf{L} = 0 \end{array} \tag{78}$$

or to leave the exact dissipation beahviour and define some linear or cubic matrix field relation with a zero output when $f_6 = 0$. For example

$$e_6 = \underbrace{\left[\; a.f_6^2 \;\right]}_{\mathbf{L}} . f_6 \tag{79}$$

with a some variable.

4.3 Structured derivation of the descriptor and output equations

Descriptor equation formulation. From the previous matrices it follows that the storage field matrices are constant and thus the descriptor formulation Eqn. (19) with matrices Eqns. (44), (45), (23) and (20) can be taken

$$\mathbf{E}\begin{bmatrix} \dot{p}_1 \end{bmatrix} = \mathbf{A}\begin{bmatrix} p_1 \end{bmatrix} + \mathbf{B}\begin{bmatrix} m_b.g \\ \dot{\theta} \end{bmatrix} + \mathbf{G}\begin{bmatrix} 0 \\ \ddot{\theta} \end{bmatrix} \tag{80}$$

Since also $\mathbf{S_{33}} = \mathbf{0}$ the descriptor matrices are

$$\begin{aligned} \mathbf{E} &= \left(\mathbf{I} + \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}\mathbf{F_I}\right) && (81)\\ &= \frac{1 + \dfrac{J_b}{m_b.R^2}}{\sin^2\theta} \\ \mathbf{A} &= \left(\mathbf{S_{11}} - \mathbf{S_{12}}\mathbf{F_D^{-1}}\dot{\mathbf{S}}_{12}^{\mathbf{T}} - \mathbf{S_{13}}\mathbf{L}\mathbf{S_{13}^T}\right)\mathbf{F_I} && (82)\\ &= \begin{bmatrix} \dfrac{\dot{\theta}.\cos\theta.\left(m_b.R^2 + J_b\right) - L.R^2.\sin\theta}{m_b.R^2.\sin^3\theta} \end{bmatrix} \\ \mathbf{B} &= \mathbf{S_{14}} + \mathbf{S_{12}}\mathbf{F_D^{-1}}\dot{\mathbf{S}}_{24} + \mathbf{S_{13}}\mathbf{L}\mathbf{S_{34}} && (83)\\ &= \begin{bmatrix} -1 & \dfrac{\sin 2\theta.\left(\dot{r}.\left(m_b.R^2 + J_b\right) + L.r.R^2\right) - 2.\dot{\theta}.r.\left(J_b + m_b.R^2.\cos^2\theta\right)}{2.R^2.\sin^3\theta} \end{bmatrix} \\ \mathbf{G} &= \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{24}} && (84)\\ &= \begin{bmatrix} 0 & \dfrac{r.\cos\theta.\left(m_b.R^2 + J_b\right)}{R^2.\sin^2\theta} \end{bmatrix} \end{aligned}$$

Clearly $\mathbf{E}$ depends on the ball inertia properties and the beam angle position. It can be verified that $\mathbf{E} \neq \mathbf{0}$ and thus a state space form Eqn. (25), can be calculated by means of Eqn. (26) and (27). However the state vector has no physical interpretation. This derivation is left for the reader.
It can be seen that the descriptor equation (80) contains cross-terms. This is due to the appearance in $\mathbf{A}$ and $\mathbf{B}$ of the dissipation field matrix (in fact a scalar) L (Eqn. (77) or (79), with use of Eqn. (90)), the input vector component $\dot{\theta}$ and the output vector component $\dot{r}$. The necessity of this last one in the calculation of the descriptor equation can be explained as an algebraic constraint.

Output equation formulation. Since $\mathbf{E} \neq \mathbf{0}$ the output equations can

be derived. It follows that the junction structure submatrices $\mathbf{S_{52}} = \mathbf{0}$ and $\mathbf{S_{53}} = \mathbf{0}$, and hence the output equation formulation Eqn. (39) with matrices Eqn. (42) and (43) are considered, or

$$\begin{bmatrix} \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{1}{m_b . \sin\theta} \end{bmatrix} \cdot \begin{bmatrix} p_1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{r}{\tan\theta} \end{bmatrix} \cdot \begin{bmatrix} m_b . g \\ \dot{\theta} \end{bmatrix} \qquad (85)$$

5 Conclusions

In this paper nonlinear systems are modeled and analyzed with bond graphs. It has been shown that a large class of nonlinear systems can be described with matrix field functions. The main results of this paper are closed formulae for the descriptor and output equations for this class of nonlinear systems. The method also provides a necessary condition for degeneracy of the descriptor equation. From these results different subclasses of nonlinear systems are treated, and an example of such a subclass is given.
It should be stressed that the descriptor variables all have a physical meaning since they are in fact the generalized momenta and displacements of the independent storage field. As a consequence this approach can be seen as a generalisation of the classical derivation of the state space formulation of linear systems [6] for which the junction structure matrix $\mathbf{S}$ is constant or only dependent on the state vector $\mathbf{x_I}$.
A second contribution of this paper is the conclusion that a descriptor form defined by [5] or [9] can be found. However, the derived equations in this paper differ in two ways:

1. When using the independent energetical states as descriptor vector, the input vector is constrained and can be ordered such that a block input matrix can be found.

2. It should be clear that a descriptor formulation without cross-terms is not always guaranteed.

These differences depend not on the proposed method, but on the physical system involved; especially on the existence of a dependent storage field, **MTF**- and **MGY**- transducer moduli in the junction structure matrix and the dissipation field characteristics.
The main advantage of this method is that the whole effort to derive the descriptor and output equations of physical dynamic systems:

- the modeling phase,
- the computational causality assignment,
- the determination of the descriptor variables and key vectors,
- the determination of the descriptor form,
- the transformation into a state space form and the necessary condition to do so,

can be done structured when bond graphs are used. It is clear that the calculation by hand of the descriptor and output equations is perfectly possible, although the major benifit lies in programming the derived equations in a symbolic calculation package like *Maple* or *Mathematica.*
This all can be advantageous in modern control applications for which a state space description (linear - not linear, constant - time varying) is needed. As a consequence it is for example possible to derive exactly the differences between the linear and the proposed exact nonlinear state space description. This can be beneficial in robust control schemes.

A Bond graph generation of the ball

The word bond graph of the ball and beam is shown in figure 5.
The stepper motor has a dynamic range with exceeds far the range of interest. Thus, in order to model the open loop system, it is sufficient to consider only the ball dynamics.
In the holonomic case a procedure based on a geometric approach is used [4][1]:

1. **Determination of the significant displacement vectors**
   - the vector of the kinematic displacements or *generalized coordinates*, $\mathbf{q_k}$, can be chosen in order to implement easily the physical effects in the bond graph

$$\mathbf{q_k} = \begin{bmatrix} \varphi \\ r \\ \theta \end{bmatrix} \tag{86}$$

[1] Another method based on the multiport methodology can be found in [1].

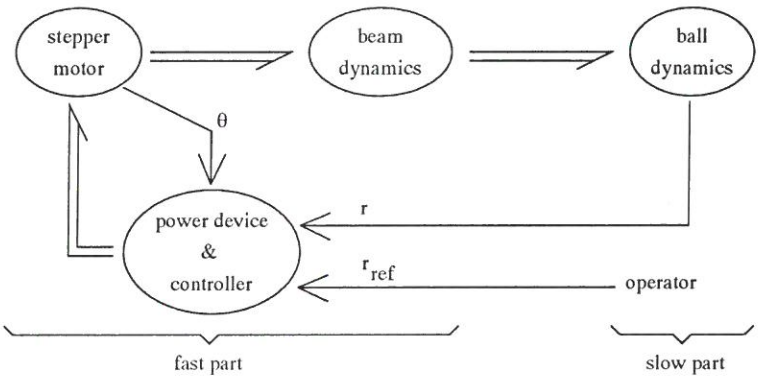

Figure 5: the ball and beam in a multibond graph notation. The fast part of the system is dimensioned such that the beam immediately follows the desired angular position θ. The controlled and reference position of the ball, r and r_{ref}, are the two other signals entering the controller.

- the vector of the I-field displacements can be written as a function of the kinematic displacments, $\mathbf{q_I} = \mathbf{\Phi_{Ik}(q_k)}$. The time derivative of this vector $\dot{\mathbf{q}}_\mathbf{I}$ must represent properly the kinetic co-energy of the system, $\mathbf{v_I}$. From figure 3 (d) it can be shown that

$$\dot{\mathbf{q}}_\mathbf{I} = \mathbf{v_I} = \begin{bmatrix} \dot{\varphi} \\ v_X \\ v_Y \end{bmatrix}$$

  Thus

$$\mathbf{q_I} = \mathbf{\Phi_{Ik}(q_k)} = \begin{bmatrix} \varphi \\ r_X \\ r_Y \end{bmatrix} = \begin{bmatrix} \varphi \\ r.\cos\theta \\ r.\sin\theta \end{bmatrix} \tag{87}$$

- the vector of the C-field displacements, $\mathbf{q_C} = \mathbf{\Phi_{Ck}(q_k)}$, must properly define the potential energy of the system. Because, beside the gravity, there is no potential energy field available in the system, this vector does not have to be considered.

2. **Calculation of the velocity transformations.**
   This can be done by differentiating the I-field displacement vector, Eqn. (87)

$$\dot{\mathbf{q}}_\mathbf{I} = \left[\frac{\partial \mathbf{\Phi_{Ik}(q_k)}}{\partial \mathbf{q_k}}\right] . \dot{\mathbf{q}}_\mathbf{k} = \mathbf{T_{Ik}(q_k)} . \dot{\mathbf{q}}_\mathbf{k}$$

where $\mathbf{T_{Ik}(q_k)}$ is a velocity transformation matrix. The i-th component of this vector can be calculated as

$$\dot{q}_{Ii} = \sum_{j=1}^{N} \left[\frac{\partial \Phi_{\mathbf{Ik}i}(\mathbf{q_k})}{\partial q_{kj}} \right] . \dot{q}_{kj}$$

Straightforward calculation of this leads to

$$\mathbf{T_{Ik}(q_k)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -r.\sin\theta \\ 0 & \sin\theta & r.\cos\theta \end{bmatrix} \tag{88}$$

3. **Establishing the junction structure.**
The general junction structure can be derived by direct observation of the velocity transformation matrix, Eqn. (88). This is shown in figure 6. Because this matrix relates velocities, transformers can be used for this dependency. Transformers with ratio 0 are for sake of brevity not shown.

   From the power coserving property it follows that the velocity transformations also contain the force transformations [3].

4. **Append the remaining elements to the junction structure, and simplify it.**
In figure 7 the following physical effecs are attached to the junction structure.

   (a) storage effects. Here only the inertia effects of the ball are considered. They are the mass m_b of the ball, and the moment of inertia about its center of mass J_b.

   (b) dissipation effects. The contact between the ball and beam involves a small rolling friction force

   $$F_r = \frac{c}{R} . G \tag{89}$$

   with c the coefficient of rolling friction of the ball on the beam, R the ball diameter, and $G = m_b g$ the ball gravity force. Notice that this rolling friction force is only valid for a roll on a horizontal plane. However, the error made for an inclined plane is minor.

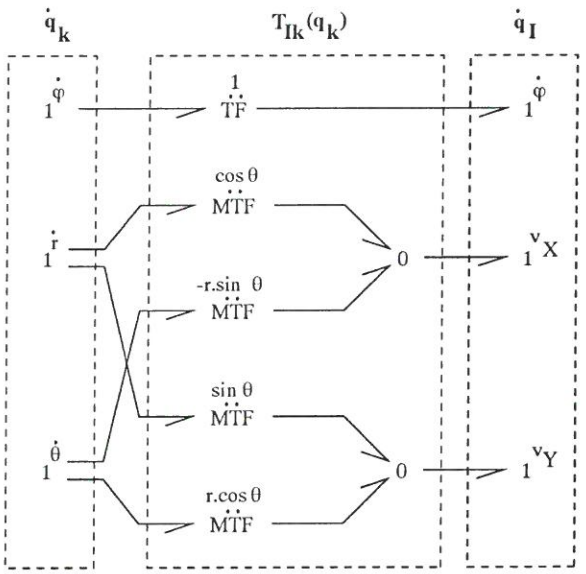

Figure 6: Scalar bond graph notation of the junction structure of the ball. The two vectors $\mathbf{q_I}$ and $\mathbf{q_k}$ are shown as an array of 1-junctions. The first one is placed right, the last one is placed left. The MTF and TF bond graph elements in between those two arrays represents the non zero elements of the velocity transformation matrix $\mathbf{T_{Ik}(q_k)}$. If the matrix element is a constant a TF bond graph element is used; if not, a MTF element is used. In this bond graph it is assumed that the energy flow direction (harpoon arrows) is also the direction of the flow variable.

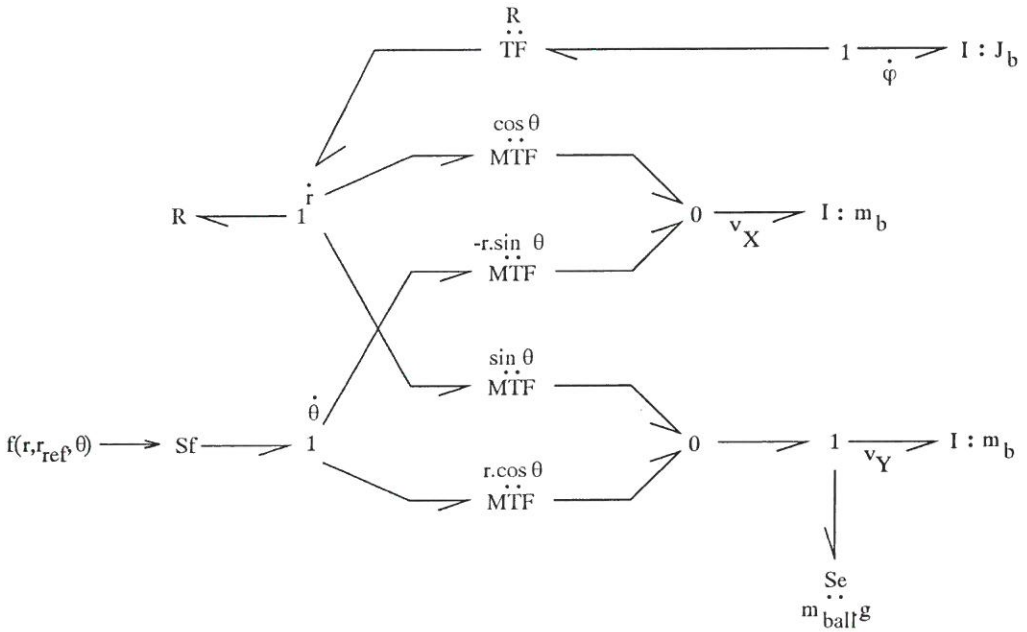

Figure 7: The bond graph of the ball. All the unnecessary bond graph elements are discarted. Remark that a sign constraint obeys the existence of upper right 1-junction.

(c) sources. The external gravity force $m_b.g$ (watch the sign convention of the inertial frame for the right harpoon direction of this power port) can be included as an effort source S_e, and the power input form the fast part can be seen as a general modulated source S.

(d) supplementary geometric constraints not yet included in the model. Assume that the ball rolls without slipping on the beam. This implies the following velocity constraint

$$\dot{r} = R.\dot{\varphi}$$

This can be included as a transformer with constant ratio R. Another important velocity constraint, namely the assumption that there always will be contact between the ball and beam, is already included in the vector of I-field displacements, Eqn. (87).

B Derivation of the junction structure output equations

The junction structure matrix Eqn. (16) can be written down by visual inspection of the scalar causal junction structure shown in figure 4. Therefore the junction structure output variables are written in terms of the junction structure input variables:

$$\begin{aligned}
f_6 &= f_{31} = f_8 = f_{24} = \frac{1}{\sin\theta}.f_{22} \\
&= \frac{1}{\sin\theta}.(f_{20} - f_{21}) \\
&= \frac{1}{\sin\theta}.f_1 - \frac{r}{\tan\theta}.f_5 \qquad (90) \\
f_3 &= f_{30} \\
&= \frac{1}{R}.f_{24} \\
&= \frac{1}{R.\sin\theta}.f_1 - \frac{r}{R.\tan\theta}.f_5 \qquad (91) \\
f_2 &= f_{26} + f_{27} \\
&= -r.\sin\theta.f_5 + \cos\theta.f_{24} \\
&= \frac{1}{\tan\theta}.f_1 - \frac{r}{\sin\theta}.f_5 \qquad (92)
\end{aligned}$$

$$
\begin{aligned}
e_1 &= -e_4 + e_{20} = -e_4 + \frac{1}{\sin\theta}.(-e_6 - e_{28} + e_{29}) \\
&= -e_4 + \frac{1}{\sin\theta}.(-e_6 - \cos\theta.e_{27} + \frac{1}{R}.e_{30}) \\
&= -\frac{1}{\tan\theta}.e_2 - \frac{1}{R.\sin\theta}.e_3 - \frac{1}{\sin\theta}.e_6 - e_4 \qquad (93) \\
e_5 &= -e_{25} - e_{23} = r.\sin\theta.e_{26} - r.\cos\theta.e_{22} \\
&= r.\sin\theta.e_2 - \frac{r}{\tan\theta}.(-e_6 - e_{28} + e_{29}) \\
&= \frac{r}{\sin\theta}.e_2 + \frac{r}{R.\tan\theta}.e_3 + \frac{r}{\tan\theta}.e_6 \qquad (94)
\end{aligned}
$$

and $f_4 = f_1$.

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