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Computational Complexity of Minimum Rank Problems involving LMIs*

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In recent years it has been shown how low order controller design can be formulated as minimum rank problems involving linear matrix inequalities (LMIs) ([6]):

$$\min_x \text{ Rank } F(x) \tag{1}$$

$$\text{subject to } \begin{cases} F(x) \geq 0 \\ C(x) \geq 0, \end{cases}$$

where $F(x)$ and $C(x)$ are symmetric matrix functions:

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \ldots + x_n F_n,$$

$$C(x) = C_0 + x_1 C_1 + x_2 C_2 + \ldots + x_n C_n,$$

with $F_i = F_i^t \in \mathbb{R}^{n_f \times n_f}$ and $C_i = C_i^t \in \mathbb{R}^{n_c \times n_c}$.

In [6], it is shown how output feedback stabilization, low order H_∞ design and low order μ-design with constant scalings can be formulated in this framework. However, several other problems can also be formulated in this way (See e.g. [1]). To solve these problems heuristic algorithms have been developed based on concepts for convex optimization ([3],[4],[5]).

In this paper we will study the computational complexity of the general minimum rank problem as formulated in (1). To apply computational complexity theory, the minimum rank problem is formulated as a *rank decision problem* (The answer is "yes" or "no".). We show that this decision problem is NP-hard, i.e. at least as hard as the NP-complete problems.

To prove the NP-hardness the following procedure is used. First it is shown that *zero-one integer programming* can be cast as a minimum rank

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problem. Since zero-one integer programming is NP-complete, the rank decision problem must be NP-hard.

The zero-one integer programming problem can be formulated as:

> Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, is there an n-vector x with all entries belonging to $\{0, 1\}$ that satisfies $Ax \leq b$?

It is known that this problem is NP-complete ([7]). This problem can be linked to a decision problem involving the rank of $F(x)$ (In this text it is called the rank decision problem.):

> Given $k \in \mathbb{N}$, and linear matrix inequalities $F(x) \geq 0$ and $C(x) \geq 0$ does there exist an $x \in \mathbb{R}^n$ such that the rank of $F(x) \leq k$, and which satisfies the constraints $F(x) \geq 0$ and $C(x) \geq 0$?

The following theorem shows how the zero-one integer programming problem can be cast as a decision problem on the rank of $F(x)$:

Theorem 1 *Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, define*

$$F(x) = \begin{pmatrix} diag(x) & 0 \\ 0 & I_n - diag(x) \end{pmatrix}$$

and

$$C(x) = diag(b) - \sum_{i=1}^{n} diag(A_{.i} x_i),$$

where $A_{.i}$ is the ith column of A.
Then there exists an x such that rank $F(x) \leq n$, $F(x) \geq 0$ and $C(x) \geq 0$ if and only if x is an n-vector x with entries belonging to $\{0, 1\}$, satisfying $Ax \leq b$.

Now it is also easy to prove that the rank decision problem is NP-hard.

Theorem 2 *Checking if there exists an x such that rank $F(x) \leq k$, $F(x) \geq 0$ and $C(x) \geq 0$, is NP-hard.*

The minimum rank problem is at least as difficult to solve as the rank decision problem, since the solution of the rank decision problem immediately follows from the solution of the minimum rank problem.

We also refer to [2], where a similar procedure is used to prove the NP-hardness of the μ calculation.

In fact it can be shown that the rank decision problem is NP-complete. The only thing left to show is that it belongs to the class of NP problems. If an x is given, we can check in polynomial time if the x satisfies the convex constraints $F(x) \geq 0$ and $C(x) \geq 0$. This is a convex feasibility problem that can be solved in polynomial time. Checking if rank $F(x) \leq k$, can also be done in polynomial time.

As the minimum rank problems are NP-hard, we do not look for algorithms that are efficient in all cases. In most practical cases the heuristic

algorithms give a reduced rank solution $F(x)$. However, it is not guaranteed that the minimal rank is reached. Nevertheless, although we have proved that the general minimum rank problem is NP-hard, we have not proved that each of the particular problems we consider, are NP-hard. We hope to include more details on the latter subject in the final version of the article.

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