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Descriptor Equations and Classification of Nonlinear Physical Dynamic Systems

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Abstract

This paper focusses on the determination of a descriptor formulation of nonlinear physical dynamic systems. Based on bond graph methodology it is shown that a closed descriptor formulation is obtainable if the field relations can be written down as explicit matrix functions. It turns out that the result is a generalisation of the state space formulation of linear physical systems. According to the behaviour of the matrices involved, different reduced descriptions can be found.

The benefit of the proposed method is threefold: first, the derivation of the governing equations can be automated; second, the method gives a better understanding of the dynamic behaviour of nonlinear systems; and at last, the derived equations can be used in robust control schemes, where structured upper bounds on the state matrices have to be found. This methodology is illustrated with an example.

1 Introduction

In most modern control design procedures the plant or system is described as a *linear time invariant state space form*

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}.\mathbf{x}(\mathbf{t}) + \mathbf{B}.\mathbf{u}(\mathbf{t}) \tag{1}$$

where $\mathbf{x}(\mathbf{t})$ is *the state vector*, $\mathbf{u}(\mathbf{t})$ is *the input vector*, $\dot{\mathbf{x}}(\mathbf{t})$ is *the state vector velocity*, $\mathbf{A}$ is *the system matrix* and $\mathbf{B}$ is *the input matrix*. However, exept in unusual cases, the real system is non-linear time varying and most often, a linearisation of the system behaviour around a working point is considered to arrive at Eqn. (1). Remark that when using such a state space description one assumes the system acts always *causal*, i.e. the propagation of the state vector $\mathbf{x}(\mathbf{t})$ can be determined recursively as the input vector $\mathbf{u}(\mathbf{t})$ is known. This is not always the case [3], [8]. Systems which do not obey Eqn. (1) are often called *non-causal* or *degenerate*. In literature (e.g. [9], [3] and [10]) such linear (or linearized) systems are described with the following equation

$$\mathbf{E}(\mathbf{t}).\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{t}).\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{t}).\mathbf{u}(\mathbf{t}) \tag{2}$$

where $\mathbf{E}(\mathbf{t})$ can be singular (Therefore such systems are also called *singular*). The vector $\mathbf{x}(\mathbf{t})$ is called *the descriptor vector*.

The major contribution of this paper is to outline a structured derivation of a similar descriptor form for a large class of nonlinear systems. Because also controller design is of interest, it is advantageuous to maintain as much as possible the same representation as for linear systems, Eqn. (2). It turns out that this equation involve a relation of the form

$$\mathbf{E}(\mathbf{x},\mathbf{u},\mathbf{t}).\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{x},\mathbf{u},\mathbf{t}).\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{x},\mathbf{u},\mathbf{t}).\mathbf{u}(\mathbf{t}) + \mathbf{G}(\mathbf{x},\mathbf{u},\mathbf{t}).\dot{\mathbf{u}}(\mathbf{t}) \tag{3}$$

The explicit knowledge of these matrices could be beneficial in robust controller schemes. This in fact is the major motivation of our research.

Therefore we will use bond graphs, a structured modeling tool, which ensures that the derived dynamic model includes the necessary energetical system properties. The paper presented here uses the state equation generation procedure of the keynote paper [5], where the state equations are generated for linear and state dependent nonlinear multiport systems.

For most practical physical dynamic systems the field relations of the bond graph can be written down explicitely as matrix functions. For such systems, a closed matrix descriptor form Eqn. (3) is derived. Mainly depending on the junction structure these dynamic equations can be written into an form without cross-terms (i.e. with $\mathbf{E(t)}$, $\mathbf{A(x,t)}$, $\mathbf{B(u,t)}$ and $\mathbf{G(u,t)}$) The proposed treatment also leads to an explicit determination of degeneracy (or singularity) of the observed physical dynamic system.

This paper is arranged as follows. In Section 2 nonlinear physical systems are described in a general bond graph representation, and a certain class of nonlinear physical systems is defined. In Section 3 the derivation of the nonlinear matrix descriptor and output equation for this class is done with use of the bond graph representation. Depending on the linearity and time dependancy of certain parts of the bond graph of the physical system different subclasses are treated in Section 4. Finally, in Section 5 the results are illustrated with a two-dimensional mechanical system.

2 Nonlinear system description with bond graphs

Bond graphs can be seen as a graphical tool that visualises the energetic construction of a physical system [6]. While describing a physical system one can focus attention to the *energy*

conservation and power continuity property of subparts of the system. Bond graphs, shown in figure 1, visualize explicitely those two properties. Here the physical system is divided into two parts. Every physical effect that obeys the power continuity is put together in the *power continuity* part or field, and every physical effect that obeys the energy conservation is put together in the *energy conservation* part or field. Each power interaction P involves two time dependent conjugate power variables (or vectors), an effort $\mathbf{e}$ and a flow variable $\mathbf{f}$, since

$$P = \mathbf{e}^{\mathbf{T}}\mathbf{f}$$

For each power bond these two variables can be shown in the bond graph, the flow variable always stands on the side of the harpoon arrow.

An enlarged view of a physical system is shown in figure 2. This partitioning of a bond graph can be done after converting the bond graph into a standard and augmented form [2]. The standard form of a bond graph classifies each power bond as *external* or *internal*, the augmenting of a bond graph involves a computational causality assignment procedure. This conversion can be automated [1], and results in the determination of the *key vectors* and the *field assignment statements*. Based on these vectors, Rosenberg [5] has outlined a method for deriving the state and output equations of linear and state dependent nonlinear systems.

2.1 The key vectors

The key vectors are the output variables of the different fields and involve power and powerless interaction with or within the physical system. For convenience they are ordered below:

1. *The dissipation field input and output vector:*

$$\mathbf{d_i} \ , \ \mathbf{d_o}$$

2. The energy vector of the dynamic system is taken as *descriptor vector* $\mathbf{x}$. This vector can be partitioned as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_I} \\ \mathbf{x_D} \end{bmatrix}$$

where $\mathbf{x_I}$ is the independent descriptor vector. Its components are the generalized momenta on I-elements with integral causality, and generalized displacements on C-elements with integral causality:

$$\mathbf{x_I} = \begin{bmatrix} \vdots \\ p_{i_I} \\ \vdots \\ q_{j_I} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \int e_{i_I} dt \\ \vdots \\ \int f_{j_I} dt \\ \vdots \end{bmatrix} \begin{matrix} \\ \text{on I-element } i \text{ with integral causality} \\ \\ \text{on C-element } j \text{ with integral causality} \\ \end{matrix}$$

The time derivative of this vector is the integral storage field input vector.

$\mathbf{x_D}$ is the dependent descriptor vector. The components of the time derivative of this vector are the derivative storage field output components:

$$\dot{\mathbf{x}}_\mathbf{D} = \begin{bmatrix} \vdots \\ e_{i_D} \\ \vdots \\ f_{j_D} \\ \vdots \end{bmatrix} \begin{matrix} \\ \text{on I-element } i \text{ with derivative causality} \\ \\ \text{on C-element } j \text{ with derivative causality} \\ \end{matrix}$$

3. *The co-energy vector* $\mathbf{z}$. This vector can be partitioned as follows:

$$\mathbf{z} = \begin{bmatrix} \mathbf{z_I} \\ \mathbf{z_D} \end{bmatrix}$$

where $\mathbf{z_I}$ is the independent coenergy vector. Its components are the integral storage field output components:

$$\mathbf{z_I} = \begin{bmatrix} \vdots \\ f_{i_I} \\ \vdots \\ e_{j_I} \\ \vdots \end{bmatrix} \begin{matrix} \\ \text{on I-element } i \text{ with integral causality} \\ \\ \\ \text{on C-element } j \text{ with integral causality} \\ \\ \end{matrix}$$

and $\mathbf{z_D}$ the dependent coenergy vector. Its components are the derivative storage field input components:

$$\mathbf{z_D} = \begin{bmatrix} \vdots \\ f_{i_D} \\ \vdots \\ e_{j_D} \\ \vdots \end{bmatrix} \begin{matrix} \\ \text{on I-element } i \text{ with derivative causality} \\ \\ \\ \text{on C-element } j \text{ with derivative causality} \\ \\ \end{matrix}$$

4. *The source field input and output vector:*

$$\mathbf{v} \ , \ \mathbf{u}$$

5. *The detector field input vector:*

$$\mathbf{y}$$

6. *The junction structure input and output vector:*

$$\mathbf{j_i} = \begin{bmatrix} \mathbf{z_I} \\ \dot{\mathbf{x}}_\mathbf{D} \\ \mathbf{d_o} \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{j_o} = \begin{bmatrix} \dot{\mathbf{x}}_\mathbf{I} \\ \mathbf{z_D} \\ \mathbf{d_i} \\ \mathbf{v} \\ \mathbf{y} \end{bmatrix}$$

2.2 The field assignment statements

Different bond graph elements are used in the fields shown in figure 2. The implicit constituent relation of such a field can be characterized by assembling the underlying implicit element relations. For most nonlinear fields an explicit matrix function can be derived. In general these functions are not restricted to linear functions, as shown in the sequel.

1. *The (energy) storage field* is constructed of **C**- and **I**-elements. Their constituent relation is different:

$$\begin{aligned} \mathbf{C}\text{-element} &: \mathbf{\Phi_C}(\mathbf{e}, \mathbf{q}) = 0 \\ \mathbf{I}\text{-element} &: \mathbf{\Phi_I}(\mathbf{f}, \mathbf{p}) = 0 \\ \text{mixed } \mathbf{IC}\text{-element} &: \mathbf{\Phi_{IC}}(\mathbf{f_1}, \mathbf{p_1}, \mathbf{e_2}, \mathbf{q_2}) = 0 \end{aligned}$$

where $\mathbf{q}$ is the generalized displacement vector and $\mathbf{p}$ is the generalized momentum vector of the corresponding element. They can be calculated as the time integral of the vectors $\mathbf{f}$, resp. $\mathbf{e}$.

Provided explicit relations of the storage elements exist, we may assemble them into

the following explicit field relation:

$$\begin{bmatrix} \mathbf{z_I} \\ \mathbf{x_D} \end{bmatrix} = \mathbf{\Phi_F}\left(\begin{bmatrix} \mathbf{x_I} \\ \mathbf{z_D} \end{bmatrix} \right) \tag{4}$$

If only one port storage elements exists, or if there are no mixed causality storage fields in the system involved, then the independent and dependent state vector $\mathbf{x_I}$ and $\mathbf{x_D}$ is not coupled, and Eqn. (4) can be written as:

$$\begin{cases} \mathbf{z_I} = \mathbf{\Phi_{FI}}(\mathbf{x_I}) \\ \mathbf{x_D} = \mathbf{\Phi_{FD}^{-1}}(\mathbf{z_D}) \end{cases} \tag{5}$$

In a large class of physical systems these explicit nonlinear field functions can be written as matrix functions:

$$\begin{cases} \mathbf{z_I} = \mathbf{F_I x_I} \\ \mathbf{x_D} = \mathbf{F_D^{-1} z_D} \end{cases} \tag{6}$$

with $\mathbf{F_I}$ the independent storage field matrix and $\mathbf{F_D}$ the dependent storage field matrix. Notice that these matrices could depend on $\mathbf{x_I}$ and $\mathbf{z_D}$, but for reason of simplicity this dependancy is not shown explicitely. This will also be done with the subsequent matrices.

2. *The dissipation field* is constructed of **R**- elements representing, according to the causality, a resistance or a conductance. The constituent relation of an **R**-element is

$$\mathbf{\Phi_R}(\mathbf{e}, \mathbf{f}) = 0$$

Provided explicit relations of the dissipative elements exist, we may assemble them into the following explicit field relation:

$$\mathbf{d_o} = \mathbf{\Phi_L}(\mathbf{d_i}) \tag{7}$$

In a large class of physical systems this explicit nonlinear field function can be written as a matrix function:

$$\mathbf{d_o} = \mathbf{L}\mathbf{d_i} \tag{8}$$

with $\mathbf{L}$ the dissipation field matrix.

3. *The junction structure* is constructed of 4 types power transducers, **0**, **1**, **(M)TF** ((modulated) transformer) and/or **(M)GY** ((modulated) gyrator) elements representing the different power conserving transformations. The junction structure field implicit equation is:

$$\mathbf{\Phi_S}(\mathbf{j_i}, \mathbf{j_o}) = \mathbf{0} \tag{9}$$

In case Eqn. (9) can be solved explicitely, the junction structure is called *solvable* [7]. Only solvable juction structures are considered here. In a large class of physical systems this explicit nonlinear field function can be written as a matrix function:

$$\mathbf{j_o} = \mathbf{S}\mathbf{j_i} \tag{10}$$

with $\mathbf{S}$ the junction structure field matrix. According to the dimension of the vectors $\mathbf{j_i}$ and $\mathbf{j_o}$, this matrix can be partitoned as follows:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} & \mathbf{S_{13}} & \mathbf{S_{14}} \\ \mathbf{S_{21}} & \mathbf{S_{22}} & \mathbf{S_{23}} & \mathbf{S_{24}} \\ \mathbf{S_{31}} & \mathbf{S_{32}} & \mathbf{S_{33}} & \mathbf{S_{34}} \\ \mathbf{S_{41}} & \mathbf{S_{42}} & \mathbf{S_{43}} & \mathbf{S_{44}} \\ \mathbf{S_{51}} & \mathbf{S_{52}} & \mathbf{S_{53}} & \mathbf{S_{54}} \end{bmatrix} \tag{11}$$

Some of those submatrices obey certain properties or are related to each other. First the diagonal submatrices are all square. After doing the causal analysis it turns out

that the derivative storage field is only a function of the independent state and input vector, thus:

$$\begin{cases} \mathbf{S_{22}} = 0 \\ \mathbf{S_{23}} = 0 \end{cases} \tag{12}$$

It is always possible to direct all the power arrows towards or outwards the junction structure. Then it follows form the power conserving property of the junction structure, [5], that the matrix $\begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} & \mathbf{S_{13}} & \mathbf{S_{14}} \\ \mathbf{S_{21}} & \mathbf{S_{22}} & \mathbf{S_{23}} & \mathbf{S_{24}} \\ \mathbf{S_{31}} & \mathbf{S_{32}} & \mathbf{S_{33}} & \mathbf{S_{34}} \\ \mathbf{S_{41}} & \mathbf{S_{42}} & \mathbf{S_{43}} & \mathbf{S_{44}} \end{bmatrix}$ is skew symmetric. Using this property and Eqn. (12), Eqn. (10) can be rewritten as:

$$\dot{\mathbf{j}}_o = \begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} & \mathbf{S_{13}} & \mathbf{S_{14}} \\ -\mathbf{S_{12}^T} & \mathbf{0} & \mathbf{0} & \mathbf{S_{24}} \\ -\mathbf{S_{13}^T} & \mathbf{0} & \mathbf{S_{33}} & \mathbf{S_{34}} \\ -\mathbf{S_{14}^T} & -\mathbf{S_{24}^T} & -\mathbf{S_{34}^T} & \mathbf{S_{44}} \\ \mathbf{S_{51}} & \mathbf{S_{52}} & \mathbf{S_{53}} & \mathbf{S_{54}} \end{bmatrix} \cdot \dot{\mathbf{j}}_i \tag{13}$$

with the diagonal submatrices $\mathbf{S_{11}}$, $\mathbf{S_{33}}$ and $\mathbf{S_{44}}$ skew symmetric.

4. *The power sources* are constructed of **Se**- (effort source) and **Sf**- (flow source) elements according to the causality involved. An effort source determines the effort variable at the power port, and a flow source determines the flow variable at the power port. The constituent relation of a source-element is of course of the same form as a dissipation field:

$$\mathbf{\Phi_H}(\mathbf{e}, \mathbf{f}) = 0$$

Provided explicit relations of the source elements exist, we may assemble them into the following explicit field relation:

$$\mathbf{u} = \mathbf{\Phi_H}(\mathbf{v}) \tag{14}$$

In a large class of physical systems this explicit nonlinear field function can be written as a matrix function:

$$\mathbf{u} = \mathbf{H}\mathbf{v} \tag{15}$$

with $\mathbf{H}$ the source field matrix.

5. Because the detector elements involve no (or negligible) power interaction with the system, there are no bond graph elements representing their behaviour.

3 Derivation of the nonlinear matrix descriptor and output equation

In the sequel the class of nonlinear systems for which the field relations can be written in explicit matrix form are considered. No constraint is made on the time dependency or linearity of the matrices involved.

3.1 Descriptor equation formulation

In order to derive the matrix descriptor equation the vector $\dot{\mathbf{x}}_\mathbf{D}$ and $\mathbf{d_o}$ has to be rewritten as a function of $\mathbf{z_I}$ and $\mathbf{u}$. Putting Eqn. (13) (b) into Eqn. (6) (b) yields after differentiation:

$$\begin{aligned} \dot{\mathbf{x}}_\mathbf{D} &= -\left\{\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathbf{T} + \mathbf{F}_\mathbf{D}^{-1}\dot{\mathbf{S}}_{12}^\mathbf{T}\right\}\mathbf{z_I} - \mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathbf{T}\dot{\mathbf{z}}_\mathbf{I} \\ &\quad + \left\{\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S}_{24} + \mathbf{F}_\mathbf{D}^{-1}\dot{\mathbf{S}}_{24}\right\}\mathbf{u} + \mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{24}\dot{\mathbf{u}} \end{aligned} \tag{16}$$

Remark that $\dot{\mathbf{F}}_{\mathbf{D}}^{-1}$ denotes the time derivative of the inertia matrix $\mathbf{F}_{\mathbf{D}}^{-1}$. Putting Eqn. (13) (c) in Eqn. (8) yields:

$$\mathbf{d_o} = (\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\,\mathbf{L}\left[-\mathbf{S}_{\mathbf{13}}^{\mathbf{T}}\mathbf{z_I} + \mathbf{S_{34}}\mathbf{u}\right] \tag{17}$$

Of course, this equation only holds for an invertible $(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})$. For non-dissipative systems this is always true, since $\mathbf{L} = \mathbf{0}$. Finally the matrix descriptor formulation can be obtained by putting Eqns. (16), (17) and (6) into Eqn. (13) (a):

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x_I} + \mathbf{B}\mathbf{u} + \mathbf{G}\dot{\mathbf{u}} \tag{18}$$

with

$$\mathbf{E} = \mathbf{I} + \mathbf{S_{12}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S}_{\mathbf{12}}^{\mathbf{T}}\mathbf{F_I} \tag{19}$$

$$\begin{aligned}\mathbf{A} = {} & \Big(\mathbf{S_{11}} - \mathbf{S_{12}}\left[\dot{\mathbf{F}}_{\mathbf{D}}^{-1}\mathbf{S}_{\mathbf{12}}^{\mathbf{T}} + \mathbf{F}_{\mathbf{D}}^{-1}\dot{\mathbf{S}}_{\mathbf{12}}^{\mathbf{T}}\right] \\ & -\mathbf{S_{13}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S}_{\mathbf{13}}^{\mathbf{T}}\Big)\,\mathbf{F_I} \\ & -\mathbf{S_{12}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S}_{\mathbf{12}}^{\mathbf{T}}\dot{\mathbf{F}}_\mathbf{I}\end{aligned} \tag{20}$$

$$\begin{aligned}\mathbf{B} = {} & \mathbf{S_{14}} + \mathbf{S_{12}}\left(\dot{\mathbf{F}}_{\mathbf{D}}^{-1}\mathbf{S_{24}} + \mathbf{F}_{\mathbf{D}}^{-1}\dot{\mathbf{S}}_{\mathbf{24}}\right) \\ & +\mathbf{S_{13}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S_{34}}\end{aligned} \tag{21}$$

$$\mathbf{G} = \mathbf{S_{12}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S_{24}} \tag{22}$$

Basically $\mathbf{E}$ can become singular for certain values of $\mathbf{S_{12}}$, $\mathbf{F}_{\mathbf{D}}^{-1}$ and $\mathbf{F_I}$.

It is easy to see that $\mathbf{F}_{\mathbf{D}}^{-1}$ plays a crucial role in the time behaviour of the descriptor vector. If no derivative causality appears in the storage field, then $\mathbf{E} \equiv \mathbf{I}$ and $\mathbf{G} \equiv \mathbf{0}$. As a consequence Eqn. (18) is in fact a state space description and the descriptor vector equals the state vector. This also means that the definition of causality given in the Introductory Section is in accordance with the causal analysis in bond graphs. Contrary, if a derivative causality

appears in the storage field a state space description is only possible iff $\mathbf{E}$ is not singular, or

$$\det(\mathbf{I} + \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}) \neq 0 \tag{23}$$

However, the state vector is not more equal to the independent energetic physical states. An artificial state vector $\tilde{\mathbf{x}}$ can be defined as:

$$\tilde{\mathbf{x}} = \mathbf{x_I} - \mathbf{Gu}$$

Bringing this vector into Eqn. (18) yields the matrix state equation:

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \tag{24}$$

with

$$\tilde{\mathbf{A}} = \mathbf{E^{-1}A} \tag{25}$$

$$\tilde{\mathbf{B}} = \mathbf{E^{-1}}\left(\mathbf{B} + \mathbf{AE^{-1}G} + \dot{\mathbf{E}}\mathbf{E^{-1}} - \dot{\mathbf{G}}\right) \tag{26}$$

Notice that the nature of a system obeying state equations is totally different to those obeying descriptor equations [10] and thus influences further control schemes [9].

Notice also that the matrix descriptor form Eqn. (18) differs form its standard form defined in [3] and [9]. However when defining the block matrix

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{G} \end{bmatrix} \tag{27}$$

and the artificial input vector

$$\tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} \tag{28}$$

Eqn. (18) can be rewritten in *standard form:*

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x}_\mathbf{I} + \tilde{\mathbf{B}}\tilde{\mathbf{u}} \tag{29}$$

Consequently the set of independent energetical states can always serve as a descriptor vector. But this implies constraints on the artificial input vector $\tilde{\mathbf{u}}$, since its components cannot vary independently.

Similary, if one defines the block matrices

$$\tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{E} & -\mathbf{G} \end{bmatrix} \tag{30}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} \tag{31}$$

and the artificial descriptor vector

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x_I} \\ \mathbf{u} \end{bmatrix} \tag{32}$$

Eqn. (18) can be rewritten in *autonomuous standard form:*

$$\tilde{\mathbf{E}}\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} \tag{33}$$

Therefore the artificial descriptor vector is constructed of the independent state and input vector.

3.2 Output equation formulation

If $\mathbf{E}$ is not singular then the *standard output equation* can be derived. By putting Eqns. (16), (17) and (6) into Eqn. (13) (e), wherafter $\dot{\mathbf{x}}_\mathbf{I}$ is substituted by Eqn. (18), one obtains

$$\mathbf{y} = \mathbf{C}\mathbf{x_I} + \mathbf{D}\mathbf{u} + \mathbf{Q}\dot{\mathbf{u}} \tag{34}$$

with

$$\mathbf{C} = \mathbf{C}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{A} \tag{35}$$

$$\mathbf{D} = \mathbf{D}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{B} \tag{36}$$

$$\mathbf{Q} = \mathbf{Q}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{G} \tag{37}$$

in which $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ are defined by Eqns. (19), (20), (21) and (22), and

$$\begin{aligned}\mathbf{C}^* &= \left(\mathbf{S_{51}} - \mathbf{S_{52}}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathbf{T} + \mathbf{F}_\mathbf{D}^{-1}\dot{\mathbf{S}}_{12}^\mathbf{T}\right]\right. \\ &\quad \left. -\mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS}_{13}^\mathbf{T}\right)\mathbf{F_I} \\ &\quad -\mathbf{S_{52}}\mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathbf{T}\dot{\mathbf{F}}_\mathbf{I} \end{aligned} \tag{38}$$

$$\mathbf{P} = -\mathbf{S_{52}}\mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{12}^\mathbf{T}\mathbf{F_I} \tag{39}$$

$$\begin{aligned}\mathbf{D}^* &= \mathbf{S_{54}} + \mathbf{S_{52}}\left(\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{24}} + \mathbf{F}_\mathbf{D}^{-1}\dot{\mathbf{S}}_{24}\right) \\ &\quad +\mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \end{aligned} \tag{40}$$

$$\mathbf{Q}^* = \mathbf{S_{52}}\mathbf{F}_\mathbf{D}^{-1}\mathbf{S_{24}} \tag{41}$$

Again, Eqn. (34) holds for an invertible $(\mathbf{I} - \mathbf{LS_{33}})$ only.

3.3 Some other important remarks

In the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ time derivatives of some submatrices of the matrix $\mathbf{S}$ and $\mathbf{F}_\mathbf{D}^{-1}$ appear. In certain cases, like in two or three dimensional mechanics major parts of the matrix $\mathbf{S}$ are coordinate transformation matrices. Then the submatrices of $\mathbf{S}$ can be written as functions of generalized coordinates, which can be seen as the time integral of some descriptor variable. For such systems Eqns. (18) and (34) contain cross-terms because products of descriptor variables with themselves and input variables appear.

It is also possible that supplementary algebraic constraints arise in the calculation of some time dependent components of the matrices involved, mainly the junction structure matrix $\mathbf{S}$. These constraints also stem from the bond graph.

4 Subclasses of nonlinear systems with field matrices

According to the behaviour of the different matrices involved in the descriptor and output equation formulation, different reduced descriptions can be found. Of course a combination of the results shown below is possible.

1. **There is no causal connection between the dependent storage field and the detectors.**

   Then in Eqn. (13) the submatrix $\mathbf{S_{52}} = \mathbf{0}$ and the output equation (34) becomes much simpler

$$\mathbf{y} = \mathbf{C}\mathbf{x_I} + \mathbf{D}\mathbf{u} \tag{42}$$

   with

$$\mathbf{C} = \left(\mathbf{S_{51}} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S}_{\mathbf{13}}^{\mathbf{T}}\right)\mathbf{F_I} \tag{43}$$

$$\mathbf{D} = \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S_{34}} \tag{44}$$

   If in addition there is no causal connection between the dissipation field and the detectors, then $\mathbf{S_{53}} = \mathbf{0}$ and the output matrices of Eqn. (42) become very simple

$$\mathbf{C} = \mathbf{S_{51}}\mathbf{F_I} \tag{45}$$

$$\mathbf{D} = \mathbf{S_{54}} \tag{46}$$

2. **The storage field matrices are time invariant.**

   If both the matrices $\mathbf{F_D^{-1}}$ and $\mathbf{F_I}$ are constant, then the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$, and the output equation matrices $\mathbf{C}^*$ and $\mathbf{D}^*$, Eqns. (20), (21), (38) and (40) simplify into

$$\mathbf{A} = \left(\mathbf{S_{11}} - \mathbf{S_{12}}\mathbf{F_D^{-1}}\dot{\mathbf{S}}_{\mathbf{12}}^{\mathbf{T}} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S}_{\mathbf{13}}^{\mathbf{T}}\right)\mathbf{F_I} \tag{47}$$

$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{12}F_D^{-1}\dot{S}_{24}} + \mathbf{S_{13}(I - LS_{33})^{-1}LS_{34}} \tag{48}$$

$$\mathbf{C^*} = \left(\mathbf{S_{51}} - \mathbf{S_{52}F_D^{-1}\dot{S}_{12}^T} - \mathbf{S_{53}(I - LS_{33})^{-1}LS_{13}^T}\right)\mathbf{F_I} \tag{49}$$

$$\mathbf{D^*} = \mathbf{S_{54}} + \mathbf{S_{52}F_D^{-1}\dot{S}_{24}} + \mathbf{S_{53}(I - LS_{33})^{-1}LS_{34}} \tag{50}$$

3. **The junction structure field matrix is time invariant.**

This is the case when no **MTF** of **MGY** elements appear in the junction structure. Then also the descriptor equation matrices **A** and **B**, and the output equation matrices $\mathbf{C^*}$ and $\mathbf{D^*}$, Eqns. (20), (21), (38) and (40) simplify into

$$\mathbf{A} = \left(\mathbf{S_{11}} - \mathbf{S_{12}\dot{F}_D^{-1}S_{12}^T} - \mathbf{S_{13}(I - LS_{33})^{-1}LS_{13}^T}\right)\mathbf{F_I} - \mathbf{S_{12}F_D^{-1}S_{12}^T\dot{F}_I} \tag{51}$$

$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{12}\dot{F}_D^{-1}S_{24}} + \mathbf{S_{13}(I - LS_{33})^{-1}LS_{34}} \tag{52}$$

$$\mathbf{C^*} = \left(\mathbf{S_{51}} - \mathbf{S_{52}\dot{F}_D^{-1}S_{12}^T} - \mathbf{S_{53}(I - LS_{33})^{-1}LS_{13}^T}\right)\mathbf{F_I} - \mathbf{S_{52}F_D^{-1}S_{12}^T\dot{F}_I} \tag{53}$$

$$\mathbf{D^*} = \mathbf{S_{54}} + \mathbf{S_{52}\dot{F}_D^{-1}S_{24}} + \mathbf{S_{53}(I - LS_{33})^{-1}LS_{34}} \tag{54}$$

4. **The storage and junction structure field matrices are time invariant.**

Again the descriptor equation matrices **A** and **B** and the output equation matrices $\mathbf{C^*}$ and $\mathbf{D^*}$, Eqns. (20), (21), (38) and (40) simplify into

$$\mathbf{A} = \left\{\mathbf{S_{11}} - \mathbf{S_{13}(I - LS_{33})^{-1}LS_{13}^T}\right\}\mathbf{F_I} \tag{55}$$

$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{13}(I - LS_{33})^{-1}LS_{34}} \tag{56}$$

$$\mathbf{C^*} = \left\{\mathbf{S_{51}} - \mathbf{S_{53}(I - LS_{33})^{-1}LS_{13}^T}\right\}\mathbf{F_I} \tag{57}$$

$$\mathbf{D^*} = \mathbf{S_{54}} + \mathbf{S_{53}(I - LS_{33})^{-1}LS_{34}} \tag{58}$$

Remark that formally this result is also valid for linear systems. However, here $\mathbf{L}$ can be time varying or state dependent.

5. **There is no causal connection between the sources and the dependent storage field.**

Then in Eqn. (13) the submatrix $\mathbf{S_{24}} = \mathbf{0}$, and as result the influence of the time derivative of the input vector $\mathbf{u}$ disappears

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x}_\mathbf{I} + \mathbf{B}\mathbf{u} \tag{59}$$

with

$$\mathbf{E} = \mathbf{I} + \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}\mathbf{F_I} \tag{60}$$

$$\begin{aligned}\mathbf{A} = {} & \left(\mathbf{S_{11}} - \mathbf{S_{12}}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{12}^T} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_{12}^\mathbf{T}\right]\right. \\ & \left. -\mathbf{S_{13}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S_{13}^T}\right)\mathbf{F_I} \\ & -\mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{12}^T}\dot{\mathbf{F}}_\mathbf{I}\end{aligned} \tag{61}$$

$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{L}\mathbf{S_{33}})^{-1}\mathbf{L}\mathbf{S_{34}} \tag{62}$$

(63)

and

$$\mathbf{y} = \mathbf{C}\mathbf{x}_\mathbf{I} + \mathbf{D}\mathbf{u} \tag{64}$$

with

$$\mathbf{C} = \mathbf{C}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{A} \tag{65}$$

$$\mathbf{D} = \mathbf{D}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{B} \tag{66}$$

in which $\mathbf{E}$ is defined in Eqn. (60) and

$$\begin{aligned}\mathbf{C}^* &= \left(\mathbf{S_{51}} - \mathbf{S_{52}}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S}_{\mathbf{12}}^\mathbf{T} + \mathbf{F}_\mathbf{D}^{-1}\dot{\mathbf{S}}_{\mathbf{12}}^\mathbf{T}\right]\right. \\ &\quad \left. -\mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS}_{\mathbf{13}}^\mathbf{T}\right)\mathbf{F_I} \\ &\quad -\mathbf{S_{52}}\mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{\mathbf{12}}^\mathbf{T}\dot{\mathbf{F}}_\mathbf{I}\end{aligned} \tag{67}$$

$$\mathbf{P} = -\mathbf{S_{52}}\mathbf{F}_\mathbf{D}^{-1}\mathbf{S}_{\mathbf{12}}^\mathbf{T}\mathbf{F_I} \tag{68}$$

$$\mathbf{D}^* = \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \tag{69}$$

The sources of such a system can be called *natural* because they don't force a derivative causality to a part of the storage field.

6. **There is no dependent storage field in the physical system.**

This is the case when the bond graph can be made fully causal. As already shown in the previous section it turns out that the descriptor and output equations, Eqns. (18) and (34), changes fundamentally; they reduce into a state space description

$$\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x}_\mathbf{I} + \mathbf{B}\mathbf{u} \tag{70}$$

with

$$\mathbf{A} = \left\{\mathbf{S_{11}} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS}_{\mathbf{13}}^\mathbf{T}\right\}\mathbf{F_I} \tag{71}$$

$$\mathbf{B} = \left\{\mathbf{S_{14}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}\right\} \tag{72}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x}_\mathbf{I} + \mathbf{D}\mathbf{u} \tag{73}$$

with

$$\mathbf{C} = \left\{\mathbf{S_{51}} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS}_{\mathbf{13}}^\mathbf{T}\right\}\mathbf{F_I} \tag{74}$$

$$\mathbf{D} = \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}} \tag{75}$$

Only in this case the time derivative of the physical energetic states is a function of itself and the power input variable.

The same formulation is also obtained in [5], where the linearity of the physical system was demanded.

5 An example

A two-dimensional mechanical example, a ball and beam, is taken as example. The physical model of the system is shown in figure 3. The ball can freely roll on the beam, and the beam angle can be set by a stepper motor. The inertial frame of reference is denoted as XYZ, the frame of reference attached to the ball with its origin in the center of mass of the ball is denoted as xyz.

In [4] the bond graph derivation and the causal analysis is done in much detail. The key vectors, field matrices and the descriptor and output equation are calculated in the next subsections.

5.1 Key vectors and field matrices of the ball

In figure 4 the causal bond graph of the mechanical system is partitioned as was done in figure 2. The causal analysis can be done structured [2], even automated [1]. A causal analysis of the proposed bond graph shows that it cannot be made full causal. Two 1-port storage elements are in derivative causality, so a dependent storage field arises. Also the junction structure consists of time varying elements (**MTF**), and thus **S** is not a constant

matrix. From this causal bond graph the components of the key vectors can be found:

$$\begin{cases} \mathbf{d_o} = \begin{bmatrix} e_6 \end{bmatrix} \\ \mathbf{d_i} = \begin{bmatrix} f_6 \end{bmatrix} = \begin{bmatrix} \dot{r} \end{bmatrix} \\ \mathbf{x} = \begin{bmatrix} \mathbf{x_I} \\ \mathbf{x_D} \end{bmatrix} = \left[\begin{array}{c} p_1 \\ \hline p_2 \\ p_3 \end{array}\right] \\ \mathbf{z} = \begin{bmatrix} \mathbf{z_I} \\ \mathbf{z_D} \end{bmatrix} = \left[\begin{array}{c} f_1 \\ \hline f_2 \\ f_3 \end{array}\right] = \left[\begin{array}{c} v_Y \\ \hline v_X \\ \dot{\varphi} \end{array}\right] \\ \mathbf{u} = \begin{bmatrix} e_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} m_b.g \\ \dot{\theta} \end{bmatrix} \\ \mathbf{v} = \begin{bmatrix} f_4 \\ e_5 \end{bmatrix} \\ \mathbf{y} = \begin{bmatrix} f_8 \end{bmatrix} = \begin{bmatrix} \dot{r} \end{bmatrix} \end{cases} \tag{76}$$

Also the junction structure matrix $\mathbf{S}$ can be calculated [4]:

$$\mathbf{S} = \left[\begin{array}{c|cc|c|cc} 0 & -\frac{1}{\tan\theta} & -\frac{1}{R.\sin\theta} & -\frac{1}{\sin\theta} & -1 & 0 \\ \hline \frac{1}{\tan\theta} & 0 & 0 & 0 & 0 & -\frac{r}{\sin\theta} \\ \frac{1}{R.\sin\theta} & 0 & 0 & 0 & 0 & -\frac{r}{R.\tan\theta} \\ \hline \frac{1}{\sin\theta} & 0 & 0 & 0 & 0 & -\frac{r}{\tan\theta} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{r}{\sin\theta} & \frac{r}{R.\tan\theta} & \frac{r}{\tan\theta} & 0 & 0 \\ \hline \frac{1}{\sin\theta} & 0 & 0 & 0 & 0 & -\frac{r}{\tan\theta} \end{array}\right] \tag{77}$$

It can be verified that this matrix indeed obeys the properties as stated in Eqn. (13). Also the storage field matrices can be disposed:

$$\mathbf{F_I} = \left[\begin{array}{c} m_b^{-1} \end{array}\right] \tag{78}$$

$$\mathbf{F_D^{-1}} = \left[\begin{array}{cc} m_b & 0 \\ 0 & J_b \end{array}\right] \tag{79}$$

The dissipation field includes only a small rolling friction. It is possible to derive the following trivial dissipation field input-output matrix equation

$$e_6 = \underbrace{\left[\begin{array}{c} \frac{c.m_b.g}{R.f_6} \end{array}\right]}_{\mathbf{L}} . f_6 \tag{80}$$

However, the corresponding dissipation field matrix $\mathbf{L}$ can only be used for $f_6 \neq 0$. A way to solve this matrix calculation problem is to define the dissipation field matrix with an extra

switch function

$$\begin{aligned} \text{if } f_6 \neq 0 \quad \text{then} \quad & \mathbf{L} = \left[\frac{c.m_b.g}{R.f_6} \right] \\ \text{else} \quad & \mathbf{L} = 0 \end{aligned} \tag{81}$$

or to leave the exact dissipation behaviour and define some linear or cubic matrix field relation with a zero output when $f_6 = 0$. For example

$$e_6 = \underbrace{\left[a.f_6^2 \right]}_{\mathbf{L}} . f_6 \tag{82}$$

with a some variable.

5.2 Structured derivation of the descriptor and output equations

Descriptor equation formulation.

From the previous matrices it follows that the storage field matrices are constant and thus the descriptor formulation Eqn. (18) with matrices Eqns. (47), (48), (22) and (19) can be taken

$$\mathbf{E} \begin{bmatrix} \dot{p}_1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} p_1 \end{bmatrix} + \mathbf{B} \begin{bmatrix} m_b.g \\ \dot{\theta} \end{bmatrix} + \mathbf{G} \begin{bmatrix} 0 \\ \ddot{\theta} \end{bmatrix} \tag{83}$$

Since also $\mathbf{S_{33}} = \mathbf{0}$ the descriptor matrices are

$$\begin{aligned} \mathbf{E} &= \left(\mathbf{I} + \mathbf{S_{12}} \mathbf{F_D^{-1}} \mathbf{S_{12}^T} \mathbf{F_I} \right) \\ &= \frac{1 + \dfrac{J_b}{m_b.R^2}}{\sin^2 \theta} \end{aligned} \tag{84}$$

$$\begin{aligned} \mathbf{A} &= \left(\mathbf{S_{11}} - \mathbf{S_{12}} \mathbf{F_D^{-1}} \dot{\mathbf{S}}_{\mathbf{12}}^{\mathbf{T}} - \mathbf{S_{13}} \mathbf{L} \mathbf{S_{13}^T} \right) \mathbf{F_I} \\ &= \left[\frac{\dot{\theta}. \cos\theta. \left(m_b.R^2 + J_b \right) - L.R^2. \sin\theta}{m_b.R^2. \sin^3 \theta} \right] \end{aligned} \tag{85}$$

$$\begin{aligned} \mathbf{B} &= \mathbf{S_{14}} + \mathbf{S_{12}} \mathbf{F_D^{-1}} \dot{\mathbf{S}}_{\mathbf{24}} + \mathbf{S_{13}} \mathbf{L} \mathbf{S_{34}} \\ &= \left[-1 \quad \frac{\sin 2\theta. \left(\dot{r}. \left(m_b.R^2 + J_b \right) + L.r.R^2 \right) - 2.\dot{\theta}.r. \left(J_b + m_b.R^2. \cos^2 \theta \right)}{2.R^2. \sin^3 \theta} \right] \end{aligned} \tag{86}$$

$$\begin{aligned} \mathbf{G} &= \mathbf{S_{12}}\mathbf{F_D^{-1}}\mathbf{S_{24}} \\ &= \begin{bmatrix} 0 & \dfrac{r.\cos\theta.\left(m_b.R^2+J_b\right)}{R^2.\sin^2\theta} \end{bmatrix} \end{aligned} \tag{87}$$

Clearly $\mathbf{E}$ depends on the ball inertia properties and the beam angle position. It can be verified that $\det(\mathbf{E}) \neq \mathbf{0}$ and thus a state space form Eqn. (24), can be calculated by means of Eqn. (25) and (26). However the state vector has no physical interpretation. This derivation is left for the reader.

It can be seen that the descriptor equation (83) contains cross-terms. This is due to the appearance in $\mathbf{A}$ and $\mathbf{B}$ of the dissipation field matrix (in fact a scalar) L (Eqn. (80) or (82), with use of Eqn. (77) (g)), the input vector component $\dot{\theta}$ and the output vector component $\dot{r}$. The necessity of this last one in the calculation of the descriptor equation can be explained as an algebraic constraint.

Output equation formulation.

Since $\det(\mathbf{E}) \neq \mathbf{0}$ the output equations can be derived. It follows that the junction structure submatrices $\mathbf{S_{52}} = \mathbf{0}$ and $\mathbf{S_{53}} = \mathbf{0}$, and hence the output equation formulation Eqn. (42) with matrices Eqn. (45) and (46) are considered, or

$$\begin{bmatrix} \dot{r} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{m_b.\sin\theta} \end{bmatrix} \cdot \begin{bmatrix} p_1 \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{r}{\tan\theta} \end{bmatrix} \cdot \begin{bmatrix} m_b.g \\ \dot{\theta} \end{bmatrix} \tag{88}$$

6 Conclusions

In this paper nonlinear systems are modeled and analyzed with bond graphs. It has been shown that a large class of nonlinear systems can be described with matrix field functions. The main results of this paper are closed formulae for the descriptor and output equations for

this class of nonlinear systems. The method also provides a necessary condition for degeneracy of the descriptor equation. From these results different subclasses of nonlinear systems are treated, and an example of such a subclass is given.

It should be stressed that the descriptor variables all have a physical meaning since they are in fact the generalized momenta and displacements of the independent storage field. As a consequence this approach can be seen as a generalisation of the classical derivation of the state space formulation of linear systems [5] for which the junction structure matrix $\mathbf{S}$ is constant or only dependent on the state vector $\mathbf{x_I}$.

A second contribution of this paper is the conclusion that a descriptor form defined by [3] or [9] can be found. However, the derived equations in this paper differ in two ways:

1. When using the independent energetical states as descriptor vector, the input vector is constrained.

2. It should be clear that a descriptor formulation without cross-terms is not always guaranteed.

These differences depend not on the proposed method, but on the physical system involved; especially on the existence of a dependent storage field, **MTF**- and **MGY**- transducer moduli in the junction structure matrix and the dissipation field characteristics.

The main advantage of this method is that the whole effort to derive the descriptor and output equations of physical dynamic systems:

- the modeling phase,

- the computational causality assignment,

- the determination of the descriptor variables and key vectors,

- the determination of the descriptor form,
- the transformation into a state space form and the necessary condition to do so,

can be done structured when bond graphs are used. It is clear that the calculation by hand of the descriptor and output equations is perfectly possible, although the major benifit lies in programming the derived equations in a symbolic calculation package like *Maple* or *Mathematica.*

This all can be advantageous in modern control applications for which a state space description (linear - not linear, constant - time varying) is needed. As a consequence it is for example possible to derive exactly the differences between the linear and the proposed exact nonlinear state space description. This structured knowledge can be beneficial in robust control schemes, and is subject of further research.

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List of Figures

1 Schematic view of a physical system. The power interaction in (index 2) and across the physical system border (index 1) is shown with a simple line or a harpoon. This harpoon is called *a power bond.* If a harpoon is used, as is done in the figure, then a reference direction of the energy flow is assumed. 29

2 Visualisation of the key vectors of a physical system. The part of the system obeying the power continuity is denoted as the JUNCTION STRUCTURE, the part of the system obeying the energy conservation is partitioned in an independent part denoted as INTEGRAL STORAGE, and a dependent part denoted as DERIVATIVE STORAGE. The environment part involving energetic interaction is also partitioned in a part showing the free energy transformation to the thermal domain denoted as DISSIPATION, the DETECTORS representing powerless (or negligible) measures of physical variables, and the remaining part denoted as SOURCES. 30

3 (u) mechanical model of the ball and beam, (l) key geometric quantities of the mechanical model. The position of the ball can be determined by the polar coordinates of the center of gravity of the ball, r and θ, and by its angular position φ. The rectangular components of the velocity of the center of gravity of the ball in the inertial frame are v_X and v_Y. 31

4 The causal bond graph of the ball. The different bond graph elements are put in their respective field. The external bonds, i.e. bonds that stab through the junction structure field, are numbered form 1 to 6, the internal bonds are numbered form 20 to 30. The causal stroke, or *effort arrow*, denotes the direction of the effort variable at each power port. Simply spoken the causal assignment determines the way of calculating the effort variables. The major objective of a causal analysis is to calculate as many as possible storage elements in integral form, which provide numerically the most stable calculation algorithms. Clearly only the lower storage element is in integral causality. Therefore, if the physical system would be linear, then the energetic order of the system is 1. 32

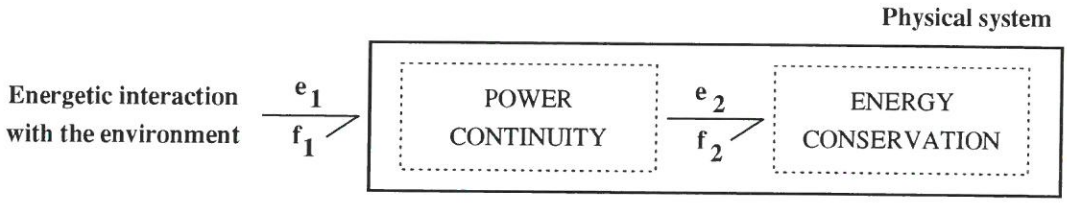


Figure 1: Schematic view of a physical system. The power interaction in (index 2) and across the physical system border (index 1) is shown with a simple line or a harpoon. This harpoon is called *a power bond.* If a harpoon is used, as is done in the figure, then a reference direction of the energy flow is assumed.

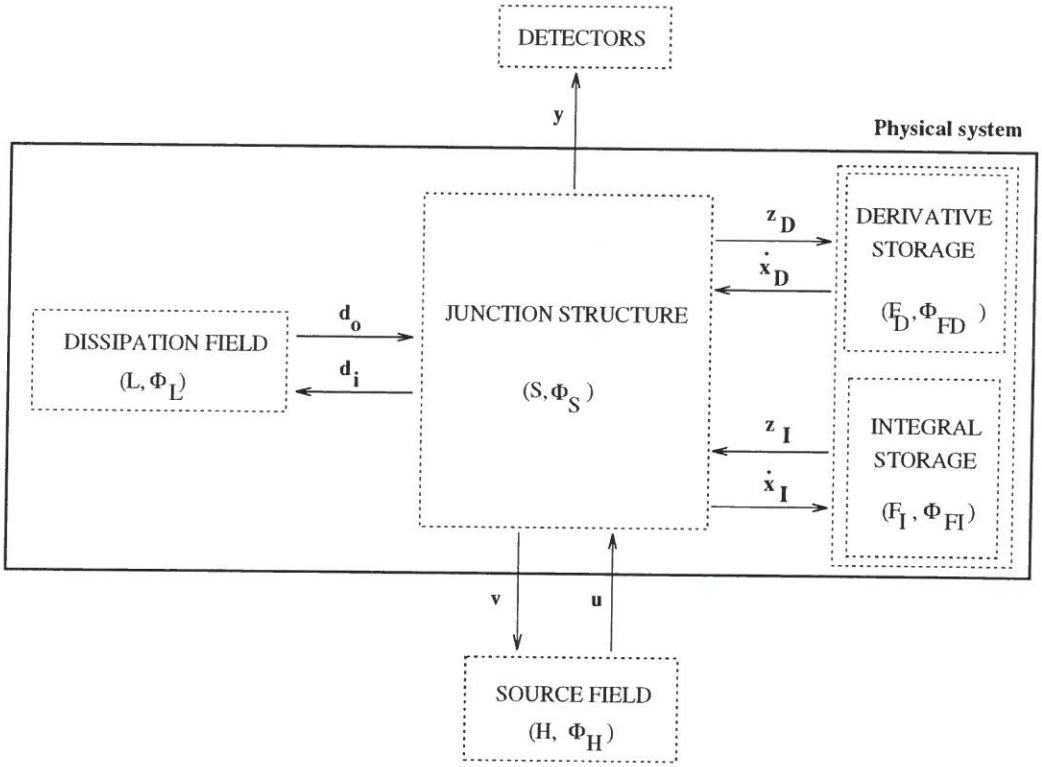


Figure 2: Visualisation of the key vectors of a physical system. The part of the system obeying the power continuity is denoted as the JUNCTION STRUCTURE, the part of the system obeying the energy conservation is partitioned in an independent part denoted as INTEGRAL STORAGE, and a dependent part denoted as DERIVATIVE STORAGE. The environment part involving energetic interaction is also partitioned in a part showing the free energy transformation to the thermal domain denoted as DISSIPATION, the DETECTORS representing powerless (or negligible) measures of physical variables, and the remaining part denoted as SOURCES.

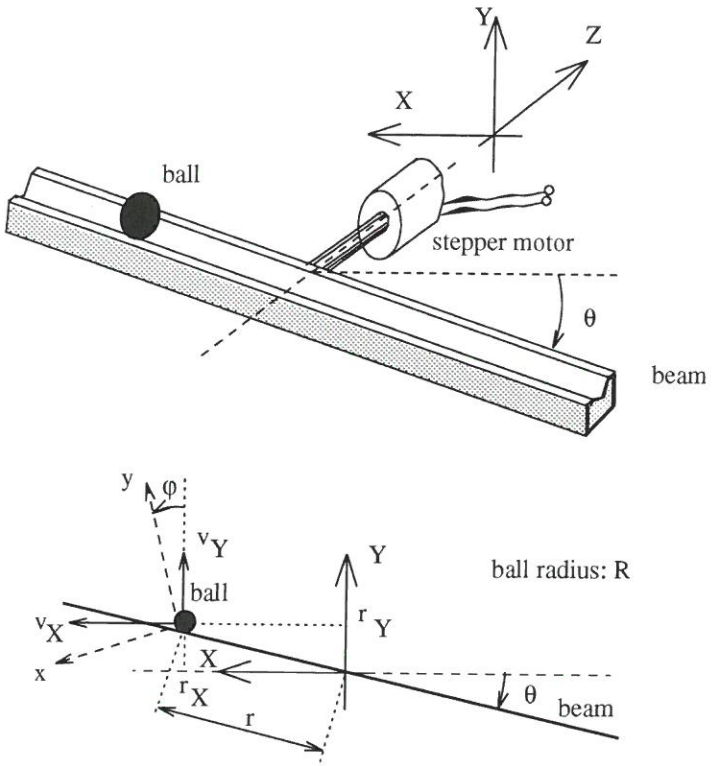


Figure 3: (u) mechanical model of the ball and beam, (l) key geometric quantities of the mechanical model. The position of the ball can be determined by the polar coordinates of the center of gravity of the ball, r and θ, and by its angular position φ. The rectangular components of the velocity of the center of gravity of the ball in the inertial frame are v_X and v_Y.

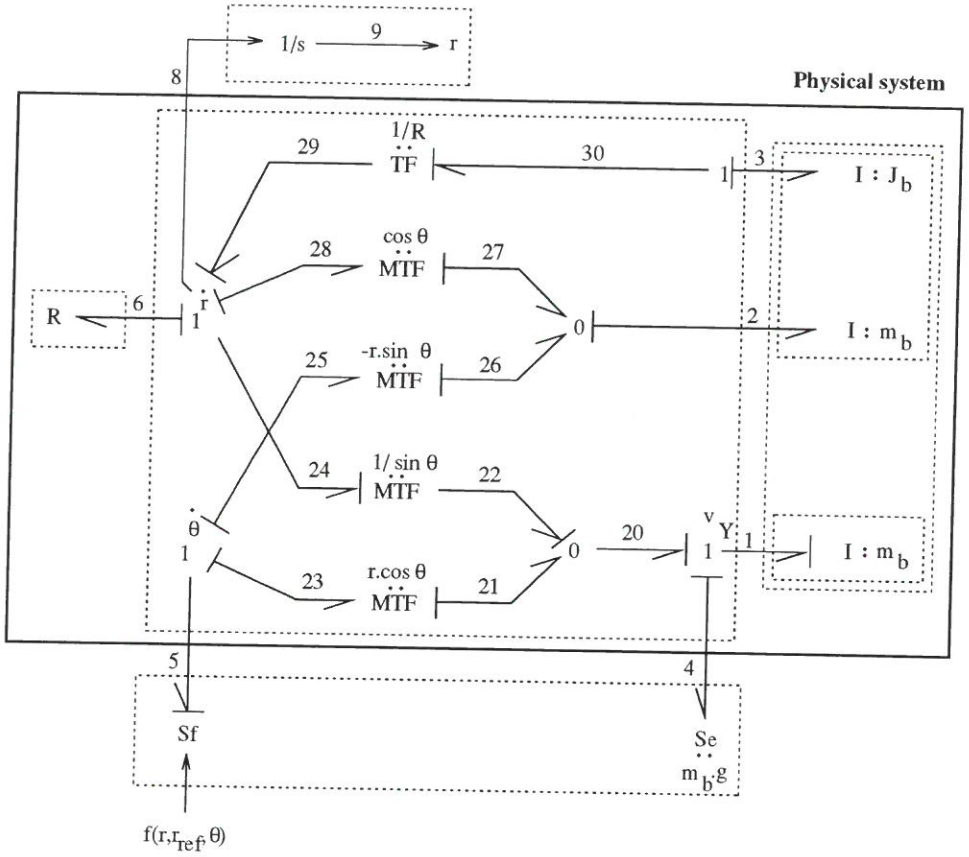


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