94-17

Stability Analysis of continuous time power iteration for computing lower bound of μ^*

Yi Cheng Bart De Moor†
ESAT - Department of Electrical Engineering
Katholieke Universiteit Leuven
Kardinaal Mercierlaan 94 B-3001 Heverlee Belgium
email: cheng.yi@esat.kuleuven.ac.be, bart.demoor@esat.kuleuven.ac.be

Abstract

In this paper, stability of a continuous time power iteration algorithm for computing a lower pound of structured singular value or μ is analyzed, via the principle of stability in the first approximation. Typically, all stable equilibriums of the algorithm for two extreme cases of structured singular value: spectral radius and maximum eigenvalue, are completely characterized.

1 Introduction

For computing a lower bound of the structure singular value μ, a power iteration algorithm was developed for the purely complex case in [1]. However it suffers problems of limit cycling. A continuous time power iteration algorithm was proposed in [2], as a counterpart of the power iteration algorithm, and the results are promising, see [2] for detail. However, the stability property of the continuous time power iteration were not analyzed mathematically.

In this paper, stability of the continuous time power iteration algorithm is analyzed around equilibrium points, via the principle of stability in the first approximation. Some necessary conditions for a stable equilibrium point are obtained, typically, all stable equilibriums of the algorithm for two extreme cases of structured singular value: spectral radius and maximum eigenvalue, are completely characterized.

The balance of the paper is organized as follows. In section 2, the framework of the continuous time power iteration algorithm for the lower bound of μ is given. The main

*This text presents research results obtained within the framework of the Belgian programme on interuniversity attraction poles initiated by the Belgian State - Prime Minister's Office - Science Policy Programming (IUAP-17). The scientific responsibility is assumed by its authors.

†Bart De Moor is a research associate of the N.F.W.O. (Belgian National Fund for Scientific Research)

results on stability analysis of the algorithm for general μ cases are presented in section 3. These results are specialized to two extreme μ cases: spectral radius and maximum singular value in section 4. Finally, some conclusion remarks are made in section 5. Appendix A and B give the proves of some equalities used in previous sections.

2 Preliminary

Let a matrix $M \in \mathbb{C}^{n\times n}$ be given. For the definition of the structured singular value of matrix M, $\mu(M)$, there is a underling block structure $\mathbf{\Delta}$ defined as follows:

$$\mathbf{\Delta} := \{\text{diag}\,[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_{m_f}] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\}$$

where s and f are nonnegative integers, representing the number of the repeated scalar blocks and the number of the full blocks; $r_1, \cdots, r_s$ and $m_1, \cdots, m_f$ are the size of the corresponding blocks. Then the definition of $\mu(M)$[1] is

Definition 1 *For $M \in \mathbb{C}^{n\times n}$, $\mu(M)$ is defined*

$$\mu(M) := \frac{1}{\min_{\Delta\in\mathbf{\Delta}}\{\bar{\sigma}(\Delta) : \det(I + M\Delta) = 0\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I + M\Delta$ singular, then $\mu(M) = 0$

Let the block structure contains two blocks: a repeated scalar block ($s = 1$) and a full block ($f = 1$) with dimensions of $r_1 = r$ and $m_1 = m$ ($r + m = n$) respectively, then there exist nonzero vectors z_{10}, w_{10}, b_{10}, $a_{10} \in C^r$ and z_{20}, w_{20}, b_{20}, $a_{20} \in C^m$ with $a_{10}^* w_{10} \neq 0$ satisfying

$$\left.\begin{aligned} \beta a_0 = M b_0, \quad z_{10} = \frac{w_{10}^* a_{10}}{|w_{10}^* a_{10}|} w_{10}, \quad z_{20} = \frac{\|w_{20}\|}{\|a_{20}\|} a_{20} \\ \beta w_0 = M^* z_0, \quad b_{10} = \frac{a_{10}^* w_{10}}{|a_{10}^* w_{10}|} a_{10}, \quad b_{20} = \frac{\|a_{20}\|}{\|w_{20}\|} w_{20} \end{aligned}\right\} \tag{1}$$

where β is a lower bound of $\mu(M)$ and $a_0 = [a_{10}^t\ a_{20}^t]^t$, $b_0 = [b_{10}^t\ b_{20}^t]^t$, $w_0 = [w_{10}^t\ w_{20}^t]^t$ and $z_0 = [z_{10}^t\ z_{20}^t]^t$, see [1] for details.

Now by eliminating vector z_0 and b_0 and separating real and imaginary parts, we can rewrite (1) as[2]:

$$AD(x_0)x_0 = \beta x_0 \tag{2}$$

where vector $x_0 \in \mathbb{R}^{4n}$ and matrix $A \in \mathbb{R}^{4n \times 4n}$ are defined as

$$x_0 := \begin{bmatrix} a_{1r0} \\ a_{2r0} \\ w_{1r0} \\ w_{2r0} \\ a_{1i0} \\ a_{2i0} \\ w_{1i0} \\ w_{2i0} \end{bmatrix} \quad A := \begin{bmatrix} 0 & M_r & 0 & -M_i \\ M_r^t & 0 & M_i^t & 0 \\ 0 & M_i & 0 & M_r \\ -M_i^t & 0 & M_r^t & 0 \end{bmatrix} \tag{3}$$

with $a_{1r0} = \text{real}(a_{10})$, $a_{2r0} = \text{real}(a_{20})$, $w_{1r0} = \text{real}(w_{10})$, $w_{2r0} = \text{real}(w_{20})$, $a_{1i0} = \text{imag}(a_{10})$, $a_{2i0} = \text{imag}(a_{20})$, $w_{1i0} = \text{imag}(w_{10})$, $w_{2i0} = \text{imag}(w_{20})$, $M_r = \text{real}(M)$ and $M_i = \text{imag}(M)$, and the matrix function $D(x_0) \in \mathbb{R}^{4n \times 4n}$ is defined as:

$$D(x_0) := \begin{bmatrix} 0 & 0 & d_r^r(x_0)I_r & 0 \\ 0 & d_f(x_0)I_m & 0 & 0 \\ d_r^r(x_0)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & d_f^{-1}(x_0)I_m \\ 0 & 0 & d_r^i(x_0)I_r & 0 \\ 0 & 0 & 0 & 0 \\ -d_r^i(x_0)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -d_r^i(x_0)I_r & 0 \\ 0 & 0 & 0 & 0 \\ d_r^i(x_0)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_r^r(x_0)I_r & 0 \\ 0 & d_f(x_0)I_m & 0 & 0 \\ d_r^r(x_0)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & d_f^{-1}(x_0)I_m \end{bmatrix} \tag{4}$$

with $d_r^r(x_0) := \text{real}(d_r(x_0))$, $d_r^i(x_0) := \text{imag}(d_r(x_0))$, and

$$d_r(x_0) := \frac{w_{10}^* a_{10}}{|w_{10}^* a_{10}|} \quad d_f(x_0) := \frac{\|w_{20}\|}{\|a_{20}\|} \tag{5}$$

Note that matrices A and $D(x_0)$ are both symmetric and the matrix $D(x_0)$ here contains all structure information. This formulation can be easily generalized to the cases of $s > 1$ and $f > 1$.

For deriving the continuous time power iteration algorithm, we define the following state vectors a_{1r}, w_{1r}, a_{1i}, $w_{1i} \in \mathbb{R}^r$, and a_{2r}, w_{2r}, a_{2i}, $w_{2i} \in \mathbb{R}^m$, and $a_1 := a_{1r} + ja_{1i}$,

$a_2 := a_{2r} + ja_{2i}$, $w_1 := w_{1r} + jw_{1i}$ and $w_2 := w_{2r} + jw_{2i}$. Finally we define another state vector x:

$$x := \begin{bmatrix} a_{1r} \\ a_{2r} \\ w_{1r} \\ w_{2r} \\ a_{1i} \\ a_{2i} \\ w_{1i} \\ w_{2i} \end{bmatrix} \tag{6}$$

Note that the a-vectors and w-vectors can be expressed by vector x. For convenience, we shall frequently switch between vector x and a-vectors and w-vectors. All these state vectors are corresponding to those constant vectors (with subscript "0") used in (1)-(5). The continuous time power iteration algorithm in [2] is then presented in the following differential equation:

$$\dot{x} = (I - xx^t)AD(x)x \tag{7}$$

where matrix A and matrix function $D(\cdot)$ are the same defined in (3) and (4) . If we find an equilibrium point of (7), x_0, with $\|x_0\| = 1$, then (7) becomes (2), with $\beta = x_0^t AD(x_0)x_0$, and thus we find a lower bound of the structured singular value of matrix M.

One property of the differential equation of (7) is that vector x has a constant norm of 1 if the initial value of the norm is 1. We can see this by taking the derivative of $\|x\|^2$:

$$\frac{d\|x\|^2}{dt} = 2x^t\dot{x} = 2x^t(I - xx^t)AD(x)x = 2(1 - x^tx)x^tAD(x)$$

So we have

$$\left.\frac{d\|x\|^2}{dt}\right|_{\|x\|=1} = 0$$

Thus if we take the initial value $\|x(0)\| = 1$, $\|x\|$ is constant and equal to 1. Actually we can see this property by another way. Suppose vector $y(t)$ solves the following differential equation:

$$\frac{dy}{dt} = AD(y)y \tag{8}$$

then $x = y/\|y\|$ solves differential equation of (7), for

$$\frac{dx}{dt} = \frac{d}{dt}\frac{y}{\sqrt{y^ty}} = \frac{AD(y)y}{\|y\|} - \frac{y^tAD(y)y}{\|y\|^3} = (I - xx^t)AD(x)x$$

So the norm of x is constant and equal to 1. In all of the analysis, we assume $\|x\| = 1$. Further we also assume that $D(x)$ is well defined, by which we mean a_1, a_2, w_1, w_2 are nonzero vectors and $w_1^*a_1 \neq 0$.

3 Stability analysis of equilibrium points

This section will give the main results on stability analysis at the equilibrium points. The basic method used here is the so called *Principle of stability in the First Approximation*[4], which is summarized as the following. Consider a nonlinear system:

$$\frac{dx}{dt} = f(x) \tag{9}$$

Suppose $f(x_0) = 0$, that is x_0 is an equilibrium point of the system (9), and also suppose that $f(x)$ is continuously differentiable. Let F be the Jacobian matrix of $f(x)$ at $x = x_0$, that is:

$$F = \left[\frac{\partial f}{\partial x}\right]_{x=x_0}$$

Then

(i) if all the eigenvalues of F are in the left half complex plane, then $x = x_0$ is an asymptotically stable equilibrium point of (9). This condition is also a sufficient condition for $x = x_0$ being an exponentially stable equilibrium point of (9).

(ii) if one or more eigenvalues of F are in the right half plane, then $x = x_0$ is an unstable equilibrium point of (9).

Note that this principle does not completely cover the analysis of the local stability of the equilibrium $x = x_0$, because nothing can be inferred about the asymptotic properties of (9) when some eigenvalue of F has zero real part, the so called critical case. In such a case, the stability of the equilibrium is determined by the high order terms that are neglected in the linearization.

3.1 Jacobian matrix

For differential equation (7),

$$f(x) = (1 - xx^t)AD(x)x$$

According to the definition, the Jacobian matrix F of this $f(x)$ at $x = x_0$ is:

$$F = \left\{\frac{\partial f_i(x)}{\partial x_j}\right\}_{x=x_0}$$

where f_i and x_j are the ith element of vector $f(x)$ and j element of the vector x. Now we can calculate partial derivative $\frac{\partial f_i(x)}{\partial x_j}$ as follows.

$$\begin{aligned}\frac{\partial f_i(x)}{\partial x_j} &= e_i^t\left[\frac{\partial}{\partial x_j}(1 - xx^t)AD(x)x\right] \\ &= e_i^t\left[-x^tAD(x)xe_j - xx^tD(x)Ae_j + (1 - xx^t)AD(x)e_j + (1 - xx^t)A\frac{\partial D}{\partial x_j}x\right] \end{aligned} \tag{10}$$

where e_i and e_j are the ith and jth standard base vectors in $\mathbb{R}^{4n}$ respectively. The partial derivative term of $\frac{\partial D(x)}{\partial x_j}$ multiplied by x can be expressed as follows:

$$\frac{\partial D(x)}{\partial x_j} x = D(x)\Delta(x)D(x)e_j \tag{11}$$

where

$$\Delta(x) = \left(\frac{p_r x x^t p_r^t}{s_r(x)} - \frac{p_f x x^t p_f}{s_f(x)}\right) \tag{12}$$

with

$$p_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_f \end{bmatrix}$$

$$s_r(x) = |w_1^* a_1| \quad \text{and} \quad s_r(x) = \|w_2\|\|a_2\|$$

The prove of (11) is in Appendix A. Now (10) becomes:

$$\begin{aligned} \frac{\partial f_i(x)}{\partial x_j} &= e_i^t \Big[-x^t AD(x)xI - xx^t D(x)A + (1 - xx^t)AD(x) + \\ &\quad (1 - xx^t)AD(x)\Delta(x)D(x)\Big] e_j \end{aligned}$$

and the Jacobian matrix F then is:

$$\begin{aligned} F &= \Big[-x^t AD(x)xI - xx^t D(x)A + (1 - xx^t)AD(x) + \\ &\quad (1 - xx^t)AD(x)\Delta(x)D(x)\Big]_{x=x_0} \end{aligned} \tag{13}$$

3.2 The eigenvalue decomposition of Jacobian matrix

Define two matrices P_r and P_f as follows:

$$P_r = \begin{bmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_n \\ I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \end{bmatrix} \quad P_f = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & -I_n \end{bmatrix} \tag{14}$$

then it is proved in Appendix B that the following equation holds:

$$AD(x)P_rP_f = P_rP_fAD(x) \tag{15}$$

(15) shows that vector $P_rP_fx_0$ is also an eigenvector of matrix $AD(x_0)$ with eigenvalue β, for x_0 is an equilibrium point of (7), so it is an eigenvector of $AD(x_0)$ with eigenvalue β, and we have

$$AD(x_0)P_rP_fx_0 = P_rP_fAD(x_0)x_0 = \beta P_rP_fx_0$$

and it is easy to check that x_0 and $P_rP_fx_0$ are orthonormal.

Now we construct a unitary matrix $U = [u_1\ u_2\ u_3\ \cdots\ u_{4n}]$ with $u_1 = P_rP_fx_0$ and $u_2 = x_0$, and take unitary transformation to the Jacobian matrix F:

$$\begin{aligned} U^tFU &= -\beta U^tU - U^tx_0x_0^tD(x_0)AU - U^tx_0x_0^tAD(x_0)U \\ &\quad + U^tAD(x_0)U + U^t(I - x_0x_0^t)AD(x_0)\Delta(x_0)D(x_0)U \end{aligned} \tag{16}$$

The first term of right hand side of (16) is obviously equal to βI. The rest terms of the right hand side of (16) can be calculated as follows:

$$\begin{aligned} &-U^tx_0x_0^tD(x_0)AU = -\beta U^tx_0x_0^tU \\ &= -\beta \begin{bmatrix} x_0^tP_rP_fx_0 \\ x_0^tx_0 \\ u_3^tx_0 \\ \vdots \\ u_{4n}^tx_0 \end{bmatrix} \begin{bmatrix} x_0^tP_rP_fx_0 & x_0^tx_0 & x_0^tu_3 & \cdots & x_0^tu_{4n} \end{bmatrix} \\ &= -\beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -\beta & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & & \end{bmatrix} \end{aligned} \tag{17}$$

$$
\begin{aligned}
&-U^t x_0 x_0^t AD(x_0)U \\
&= -\begin{bmatrix} x_0^t P_f P_r^t x_0 \\ x_0^t x_0 \\ u_3^t x_0 \\ \vdots \\ u_{4n}^t x_0 \end{bmatrix} x_0^t AD(x_0) \begin{bmatrix} P_r P_f x_0 & x_0 & u_3 & \cdots & u_{4n} \end{bmatrix} \\
&= -\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \beta & x_0^t AD(x_0)u_3 & \cdots & x_0^t AD(x_0)u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -\beta & -x_0^t AD(x_0)u_3 & \cdots & -x_0^t AD(x_0)u_{4n} \\ 0 & 0 & & & \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & & \end{bmatrix} \qquad (18) \\
&U^t AD(x_0)U \\
&= \begin{bmatrix} x_0^t P_f P_r^t \\ x_0^t \\ u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} \begin{bmatrix} AD(x_0)P_f P_r^t x_0 & AD(x_0)x_0 & AD(x_0)u_3 & \cdots & AD(x_0)u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} x_0^t P_f P_r^t \\ x_0^t \\ u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} \begin{bmatrix} \beta P_r P_f x_0 & \beta x_0 & AD(x_0)u_3 & \cdots & AD(x_0)u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} \beta & 0 & x_0^t P_r P_f u_3 AD(x_0)u_3 & \cdots & x_0^t P_r P_f AD(x_0)u_{4n} \\ 0 & \beta & x_0^t AD(x_0)u_3 & \cdots & x_0^t AD(x_0)u_{4n} \\ 0 & 0 & & & \\ \vdots & \vdots & & A_1 & \\ 0 & 0 & & & \end{bmatrix} \qquad (19)
\end{aligned}
$$

where

$$
A_1 = \begin{bmatrix} u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} AD(x_0) \begin{bmatrix} u_3 & \cdots & u_{4n} \end{bmatrix} \qquad (20)
$$

Finally, the last term of the right hand side of (16) is:

$$U^t(I - x_0 x_0^t)AD(x_0)\Delta(x_0)D(x_0)U$$

$$= \begin{bmatrix} x_0^t P_f P_r^t \\ x_0^t \\ u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} (I - x_0 x_0^t) A D(x_0) \Delta(x_0) D(x_0) \begin{bmatrix} P_r P_f x_0 & x_0 & u_3 & \cdots & u_{4n} \end{bmatrix}$$

$$= \begin{bmatrix} x_0^t P_f P_r^t (I - x_0 x_0^t) A D(x_0) \\ x_0^t (I - x_0 x_0^t) A D(x_0) \\ u_3^t (I - x_0 x_0^t) A D(x_0) \\ \vdots \\ u_{4n}^t (I - x_0 x_0^t) A D(x_0) \end{bmatrix} \Delta(x_0) D(x_0) \begin{bmatrix} P_r P_f x_0 & x_0 & u_3 & \cdots & u_{4n} \end{bmatrix} \quad (21)$$

Now by using the following equations, which are proved in Appendix B, for $x \in R^{4n}$

$$x^t p_r^t D(x) x = 0 \quad (22)$$

$$x^t p_f D(x) x = 0 \quad (23)$$

$$x^t p_r^t D(x) P_r P_f x = 0 \quad (24)$$

$$x^t p_f D(x) P_r P_f x = 0 \quad (25)$$

together with (12), we have

$$\Delta(x_0) D(x_0) P_r P_f x_0 = \left(\frac{p_r x_0 x_0^t p_r^t}{s_r(x_0)} - \frac{p_f x_0 x_0^t p_f}{s_f(x_0)}\right) D(x_0) P_r P_f x_0 = 0$$

$$\Delta(x_0) D(x_0) x_0 = \left(\frac{p_r x_0 x_0^t p_r^t}{s_r(x_0)} - \frac{p_f x_0 x_0^t p_f}{s_f(x_0)}\right) D(x_0) x_0 = 0$$

thus, (21) becomes

$$U^t (I - x_0 x_0^t) A D(x_0) \Delta(x_0) D(x_0) U$$

$$= \begin{bmatrix} x_0^t P_f P_r^t A D(x_0) \\ 0 \\ u_3^t A D(x_0) \\ \vdots \\ u_{4n}^t A D(x_0) \end{bmatrix} \begin{bmatrix} 0 & 0 & \Delta(x_0) D(x_0) u_3 & \cdots & \Delta(x_0) D(x_0) u_{4n} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & x_0^t P_f P_r^t A D(x_0) \Delta(x_0) D(x_0) u_3 & \cdots & x_0^t P_r P_f A D(x_0) \Delta(x_0) D(x_0) u_{4n} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{bmatrix} \quad (26)$$

where

$$A_2 = \begin{bmatrix} u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} AD(x_0)\Delta(x_0)D(x_0) \begin{bmatrix} u_3 & \cdots & u_{4n} \end{bmatrix} \tag{27}$$

Now plugging (17), (18), (19) and (26) into (16) and emerging all the terms, then we decompose the Jacobian matrix F as

$$U^t F U = \begin{bmatrix} 0 & 0 & \times & & \cdots & \times \\ 0 & -2\beta & 0 & & \cdots & 0 \\ 0 & 0 & & & & \\ \vdots & \vdots & & -\beta I + A_1 + A_2 & & \\ 0 & 0 & & & & \end{bmatrix} \tag{28}$$

3.3 Stability analysis

(28) shows that the Jacobian matrix F has, at least, one eigenvalue equal to zero, and one equal to -2β. Based on the principle of the first approximation, we can the following result:

Lemma 1 *(i) If x_0 is a stable equilibrium point of (7) with $D(x)$ having a block structure $s = f = 1$, then $\beta \geq 0$ and all real parts of the eigenvalues of $-\beta I + A_1 + A_2$ are nonpositive. (ii) There is no any exponentially stable equilibrium point of (7) at all, for there is, at least, one zero eigenvalue of the the Jacobian matrix F.*

Remarks: (1) The first part of this lemma is what we need, as the lower bound of structured singular value must be nonnegative. (2) However, the nonnegative β and nonpositive real parts of the eigenvalues of $-\beta I + A_1 + A_2$ are just necessary conditions for x_0 to be a stable equilibrium point. (3) The second part of this lemma shows that the convergence of continuous algorithm is not exponential. (4) The result can be extended to the case where $s > 1$ and $f > 1$

The zero eigenvalue here is due to the structure of matrix A and matrix $D(x)$ no matter what a matrix M is. Now question is whether there exists any stable equilibrium point at all because of this zero eigenvalue of the Jacobian matrix. Numerical experiences show that the answer is certainly positive. However the prove of it is not so trivial just by using the principle of the first approximation, as in this critical case, the principle is inclusive. We use the following lemma, which consider all high order terms related to the zero eigenvalues, for our proving:

Lemma 2 *Consider a system*

$$\left.\begin{aligned} \dot{z} &= f_z(z, y) \\ \dot{y} &= B(y - y_0) + p(z, y) \end{aligned}\right\} \tag{29}$$

with an equilibrium point (z_0, y_0)*, and suppose that* $p(z, y_0) = 0$ *for all* z *near* z_0 *and*

$$\frac{\partial p}{\partial y}(z_0, y_0) = 0$$

if $\dot{z} = f_z(z, y_0)$ *has a stable equilibrium point at* $z = z_0$ *and the eigenvalues of* B *all have negative real part, then the system (29) has a stable equilibrium point at* (z_0, y_0)*.*

The prove of the results are based on the central manifold theory, see [5] and the references there in for details.

Now we make a linear coordinate transformation with unitary matrix U^t with U defined in the last subsection:

$$v = U^t x$$

then (7) becomes:

$$\frac{dv}{dt} = (I - vv^t)U^t AD(Uv)Uv \tag{30}$$

Corresponding to the equilibrium point x_0 of (7), the equilibrium point of (30) is: $v_0 = [\ 0\ 1\ 0\ \cdots\ 0\ ]^t$. Let $z = [\ z_1\ z_2\ ]^t = [\ e_1\ e_2\ ]^t v$, $y = [\ e_3\ \cdots e_{4n}\ ]^t v$, $z_0 = [\ 0\ 1\ ]^t$, $y_0 = [\ 0\ \cdots\ 0\ ]^t$, and $B = -\beta I + A_1 + A_2$, then (30) can be rearranged as:

$$\left.\begin{aligned} \dot{z} &= [\ e_1\ e_2\ ]^t (I - vv^t)U^t AD(Uv)Uv \\ \dot{y} &= By + [\ e_3\ \cdots\ e_{4n}\ ]^t (I - vv^t)UAD(Uv)Uv - By \end{aligned}\right\} \tag{31}$$

Comparing (31) with (29), we have

$$f_z(z, y) = [\ e_1\ e_2\ ]^t (I - vv^t)U^t AD(Uv)Uv \tag{32}$$

$$p(z, y) = [\ e_3\ \cdots\ e_{4n}\ ]^t (I - vv^t)UAD(Uv)Uv - By \tag{33}$$

By using the following equation, which is proved in Appendix B,

$$D(U[z\ y_0]^t) = D(z_1 P_f P_r x_0 + z_2 x_0) = D(x_0) \tag{34}$$

from (33), we have

$$\begin{aligned} &p(z, y_0) = p([\ z_1\ z_2\ ]^t, [\ 0\ \cdots\ 0\ ]^t) \\ &= [\ e_3\ \cdots\ e_{4n}\ ]^t (I - [\ z_1\ z_2\ 0\ \cdots\ 0\ ]^t [\ z_1\ z_2\ 0\ \cdots\ 0\ ])U^t AD(x_0)U[\ z_1\ z_2\ 0\ \cdots\ 0\ ] \end{aligned} \tag{35}$$

As

$$\begin{aligned} &[\ e_3\ \cdots\ e_{4n}\ ]^t (I - [\ z_1\ z_2\ 0\ \cdots\ 0\ ]^t [\ z_1\ z_2\ 0\ \cdots\ 0\ ])U^t \\ &= \begin{bmatrix} 0 & 0 & \\ \vdots & \vdots & I \\ 0 & 0 & \end{bmatrix} U^t = \begin{bmatrix} u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} \end{aligned}$$

and

$$AD(x_0)U[\ z_1\ z_2\ 0\ \cdots\ 0\ ] = AD(x_0)(z_1P_rP_fx_0 + z_2x_0) = \beta(z_1P_rP_fx_0 + z_2x_0)$$

(35) becomes

$$p(z, y_0) = \begin{bmatrix} u_3 \\ \vdots \\ u_{4n} \end{bmatrix} \beta(z_1P_rP_fx_0 + z_2x_0) = 0 \tag{36}$$

Thus we've proved $p(z, y_0) = 0$. Now we prove

$$\frac{\partial p}{\partial y}(z_0, y_0) = 0$$

Actually it is obvious, as from (33)

$$\begin{aligned}
&\left[\frac{\partial p_i(z,y)}{\partial y_j}\right]_{(z,y)=(z_0,y_0)} = \left[\frac{\partial[\ 0\ 0\ e_i^t\ ]U^t f(x)}{\partial y_j} - \frac{\partial e_i^t By}{\partial y_j}\right]_{(z,y)=(z_0,y_0)} \\
&= \left[[\ 0\ 0\ e_i^t\ ]U^t\frac{\partial f(x)}{\partial y_j}\right]_{(z,y)=(z_0,y_0)} - e_i^tBe_i \\
&= \left[[\ 0\ 0\ e_i^t\ ]U^t\sum_{k=1}^{4n}\frac{\partial f(x)}{\partial x_k}\frac{\partial x_k}{\partial y_j}\right]_{(z,y)=(z_0,y_0)} - e_i^tBe_i \\
&= \left[[\ 0\ 0\ e_i^t\ ]U^t\frac{\partial f(x)}{\partial x}U[\ 0\ 0\ e_j^t\ ]^t\right]_{x=x_0} - e_i^tBe_j \\
&= [\ 0\ 0\ e_i^t\ ]U^tFU[\ 0\ 0\ e_j^t\ ]^t - e_i^tBe_j = 0
\end{aligned}$$

Note e_i here is the ith standard base in $\mathbb{R}^{4n-2}$. Finally, consider $\dot{z} = f_z(z, y_0)$. From (33),

$$\begin{aligned}
\dot{z} &= f_z(z, y_0) = [\ e_1\ e_2\ ]^t(I - [\ z_1\ z_2\ 0\ \cdots\ 0\ ]^t[\ z_1\ z_2\ 0\ \cdots\ 0\ ])U^tAD(x_0)U[\ z_1\ z_2\ 0\ \cdots\ 0\ ]^t \\
&= \begin{bmatrix} (1-z_1^2)x_0^tP_fP_r^t - z_1z_2x_0^t \\ -z_1z_2x_0^tP_fP_r^t + (1-z_2^2)x_0^t \end{bmatrix} AD(x_0)(z_1P_rP_fx_0 + z_2x_0) \\
&= \begin{bmatrix} (1-z_1^2)x_0^tP_fP_r^t - z_1z_2x_0^t \\ -z_1z_2x_0^tP_fP_r^t + (1-z_2^2)x_0^t \end{bmatrix} \beta(z_1P_rP_fx_0 + z_2x_0) \\
&= \beta\begin{bmatrix} z_1(1-z_1^2-z_2^2) \\ z_1(1-z_1^2-z_2^2) \end{bmatrix}
\end{aligned} \tag{37}$$

If $\beta = 0$, then (37) is obviously stable. For $\beta > 0$, choose an energy function:

$$V(z) = (1 - z_1^2 - z_2^2)^2 + \frac{z_1^2}{z_2^2}$$

It is obviously locally positive definite in a neighborhood of $z_0 = [\ 0\ 1\ ]^t$. Computing $\dot{V}(z)$ along the trajectories of (37) gives:

$$\begin{aligned}
\dot{V}(x) &= (1 - z_1^2 - z_2^2)(-2z_1\dot{(z_1)} - 2z_2\dot{(z_2)}) + 2\frac{z_1}{z_2}(\frac{z_1\dot{z_1}}{z_2} - \frac{z_1^2\dot{z_2}}{z_2}) \\
&= -2\beta(1 - z_1^2 - z_2^2)(z_1^2 + z_2^2)(1 - z_1^2 - z_2^2) + 2\beta\frac{z_1}{z_2}(\frac{z_1^2}{z_2} - \frac{z_1^2}{z_2})(1 - z_1^2 - z_2^2) \\
&= -2\beta(1 - z_1^2 - z_2^2)^2(z_1^2 + z_2^2) \leq 0
\end{aligned}$$

thus $V(x)$ is indeed a Lyapunov function (37), so (37) is stable. From Lemma 2, the following lemma is proved.

Lemma 3 *If the real part of each eigenvalue of $-\beta I + A_1 + A_2$ are negative, and $\beta \leq 0$, then x_0 is a stable equilibrium point of (7).*

Remarks (1) x_0 is a stable but not asymptotically stable equilibrium point, for x_0 is not an isolated equilibrium point, actually any vector in the subspace of (x_0, $P_r P_f x_0$) is an equilibrium point. (2) The conditions in Lemma 3 is sufficient, not necessary, as there are cases where some of the real part of the eigenvalues of $-\beta I + A_1 + A2$ are zero , the result still holds.

Example 1 *Given a matrix M, generated by MATLAB random matrix generator:*

$$M = \begin{bmatrix} 0.0754 + 0.2058i & 0.3877 - 0.5169i & -0.3714 + 0.1587i \\ -0.1361 + 0.0755i & 0.5001 + 0.0231i & -0.1409 - 0.1691i \\ -0.1059 + 0.1669i & 0.3015 - 0.4662i & -0.0794 + 0.2153i \end{bmatrix}$$

Now we calculate the lower bound of $\mu(M)$, β, by solving (7). To do so, a MATLAB function, 'ode23', is used, which use automatic step Runge-Kutta-Fehlberg integration method. The desired accuracy of the solution is set to 10^6. Let the block structure be of $r = 1$ and $m = 2$. The initial vector of x is generated also by the MATLAB random matrix generator. The solution is shown in Fig.1. x is convergent to an equilibrium point and $x^t AD(x)x$ is convergent to $\beta = 1.0124$. As the structured singular value $\mu(M)$ is equal to its upper bound in the case the block number is less than 3, so we can obtain its value by calculating its upper bound. We use the MATLAB function 'mu' to calculate the upper bound. The result shows that $\mu(M) = 1.0124$, exactly equal to β. The Jacobian matrix F in this case

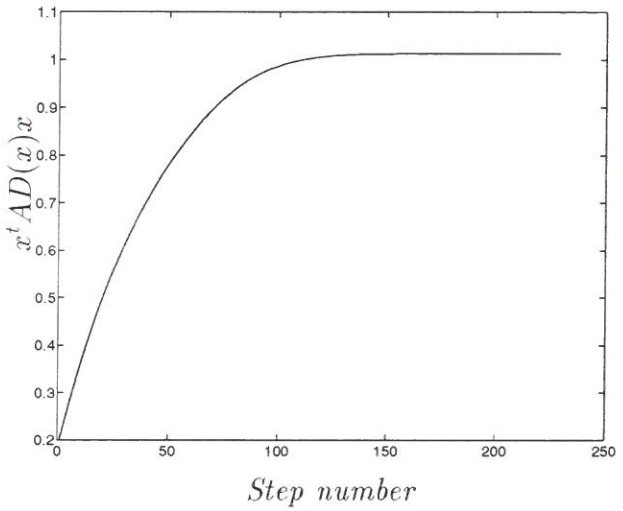


Figure 1: Continuous power iteration: $x^t AD(x)x$ is convergent to $\beta = \mu(M)$

has the eigenvalues:

$$
\begin{array}{l}
-2.0248 \\
0.0000 \\
-1.6287 \\
-0.5875 + 0.1059i \\
-0.5875 - 0.1059i \\
-1.2772 \\
-0.9866 + 0.1784i \\
-0.9866 - 0.1784i \\
-1.1574 \\
-0.9331 \\
-1.0834 \\
-0.9485
\end{array}
$$

The first one is equal to -2β, the second one is zero and the remained are all of negative real parts. We didn't find initial vector of x such that $x^t AD(x)x$ is convergent to other value.

4 Two extreme cases: spectral radius and maximum singular value

As it well known that the structured singular value is at some where between the spectral radius $\rho(M)$ and maximum singular value $\bar{\sigma}(M)$, which are special structured singular

values with $s = 1$ and $f = 0$, $s = 0$ and $f = 1$ respectively. One, of cause, can also use the continuous algorithm to calculate these two values or the lower bound of them. All results of stability analysis on equilibrium points in the last section is, of cause, still valid to these two special cases. However, in these two cases, the structure matrix $D(x)$ is very simple, we can completely characterization all stable equilibrium points.

4.1 Lower bound of spectral radius $\rho(M)$

For calculating spectral radius, $r = n$ and $m = 0$, thus $a = a_1$ and $w = w_1$ and

$$D(x) = \begin{bmatrix} 0 & d_r^r(x)I_n & 0 & -d_r^i(x)I_n \\ d_r^r(x)I_n & 0 & d_r^i(x)I_n & 0 \\ 0 & d_r^i(x)I_n & 0 & d_r^r(x)I_n \\ -d_r^i(x)I_n & 0 & d_r^r(x)I_n & 0 \end{bmatrix}$$

Correspondingly, $\Delta(x)$ in (12) becomes

$$\Delta(x) = \frac{P_r x x^t P_r^t}{s_r(x)}$$

Now we consider the Jacobian matrix in this case. Define a matrix

$$P_k = \begin{bmatrix} kI_n & 0 & 0 & 0 \\ 0 & -k^{-1}I_n & 0 & 0 \\ 0 & 0 & kI_n & 0 \\ 0 & 0 & 0 & -k^{-1}I_n \end{bmatrix} \text{ with } k = \frac{\|w_0\|}{\|a_0\|}$$

It is prove in Appendix B that the following equation holds

$$AD(x)P_k = P_k AD(x) \tag{38}$$

(38) and (15) show that $P_k x_0$ and $P_r P_f P_k x_0$ are both eigenvectors of matrix $AD(x_0)$ associated with eigenvalue β. Note that it is easy to verify that x_0, $P_r P_f x_0$, $P_k x_0$ and $P_r P_f P_k x_0$ are orthonormal. Now let $u_3 = P_k x_0$ and $u_4 = P_r P_f P_k x_0$, then (20) becomes:

$$\begin{aligned} A_1 &= \begin{bmatrix} u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} AD(x_0) \begin{bmatrix} u_3 & \cdots & u_{4n} \end{bmatrix} \\ &= \begin{bmatrix} \beta & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A_3 & \\ 0 & 0 & & & \end{bmatrix} \end{aligned} \tag{39}$$

where

$$A_3 = \begin{bmatrix} u_5^t \\ \vdots \\ u_{4n}^t \end{bmatrix} AD(x_0) \begin{bmatrix} u_5 & \cdots & u_{4n} \end{bmatrix}$$

and with the following equations prove in Appendix B:

$$x_0^t P_r^t D(x_0) u_3 = x_0^t P_r^t D(x_0) P_k x_0 = 0 \tag{40}$$

$$x_0^t P_r^t D(x_0) u_4 = x_0^t P_r^t D(x_0) P_r P_f P_k x_0 = \frac{-s_r(x_0)}{\|a_0\|\|w_0\|} \tag{41}$$

and the fact that $P_r x_0$ is a linear combination of $P_r P_f x_0$ and $P_r P_f P_k x_0$ so it is also an eigenvector of $AD(x_0)$ with eigenvalue of β, and that $x_0^t P_k P_f P_r^t P_r x_0 = 2\|a_0\|\|w_0\|$, (27) becomes:

$$\begin{aligned}
A_2 &= \begin{bmatrix} u_3^t \\ \vdots \\ u_{4n}^t \end{bmatrix} AD(x_0) \frac{P_r x_0 x_0^t P_r^t}{s_r(x_0)} D(x_0) \begin{bmatrix} u_3 & \cdots & u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} x_0^t P_k AD(x_0) P_r x_0 \\ x_0^t P_k P_f P_r^t AD(x_0) P_r x_0 \\ u_5 AD(x_0) P_r x_0 \\ \vdots \\ u_{4n} AD(x_0) P_r x_0 \end{bmatrix} \frac{x_0^t P_r^t D(x_0)}{s_r(x_0)} \begin{bmatrix} u_3 & u_4 & u_5 & \cdots & u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} \beta x_0^t P_k P_r x_0 \\ \beta x_0^t P_k P_f P_r^t P_r^t x_0 \\ \beta u_5 P_r x_0 \\ \vdots \\ \beta u_{4n} P_r x_0 \end{bmatrix} \frac{1}{s_r(x_0)} \begin{bmatrix} 0 & \frac{-s_r(x_0)}{\|a_0\|\|w_0\|} & x_0^t P_r^t D(x_0) u_5 & \cdots & x_0^t P_r^t D(x_0) u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \beta 2\|a\|\|w\| \\ \\ 0 \\ \vdots \\ 0 \end{bmatrix} \frac{1}{s_r(x_0)} \begin{bmatrix} 0 & \frac{-s_r(x_0)}{\|a\|\|w\|} & x_0^t P_r^t D(x_0) u_5 & \cdots & x_0^t P_r^t D(x_0) u_{4n} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -2\beta & \times & \cdots & \times \\ 0 & 0 & & & \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & & \end{bmatrix}
\end{aligned} \tag{42}$$

Now plugging (39) and (42) into (28), we have:

$$U^t F U = \begin{bmatrix} 0 & 0 & \times & \times & \times & \cdots & \times \\ 0 & -2\beta & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -2\beta & \times & \cdots & \times \\ 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & -\beta I + A_3 & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix} \tag{43}$$

From (19) and (39) it is easy to see that the union of the spectrum of A_3 and $(\beta,\ \beta,\ \beta,\ \beta)$ is equal to the spectrum of $AD(x_0)$. Note that from (1), $Md_r(x_0)^* I a_0 = \beta a_0$, this means that there is an eigenvalue of M, say λ, with $|\lambda| = \beta$ (recall that $\beta \geq 0$) and $\frac{\lambda}{|\lambda|} = d_r(x_0)^*$. In general, it is easy to see that if λ is an eigenvalue of M then $\lambda d_r(x_0)^*$ is an eigenvalue of $Md_r(x_0)^*$. And further, one can easily verify that if λ is an eigenvalue of M, then $AD(x_0)$ has eigenvalues $\lambda d_r(x_0)^*$, $\lambda d_r(x_0)^*$, $\lambda^* d_r(x_0)$ and $\lambda^* d_r(x_0)$. Thus if M has eigenvalue $\lambda_1, \cdots, \lambda_n$, and $\beta = |\lambda_i|$, $d_r(x_0) = \frac{\lambda_i}{|\lambda_i|}$, then $AD(x_0)$ has eigenvalues

$$\begin{matrix} \lambda_1 \frac{\lambda_i^*}{|\lambda_i|} & \lambda_1 \frac{\lambda_i^*}{|\lambda_i|} & \lambda_1^* \frac{\lambda_i}{|\lambda_i|} & \lambda_1^* \frac{\lambda_i}{|\lambda_i|} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{i-1} \frac{\lambda_i^*}{|\lambda_i|} & \lambda_{i-1} \frac{\lambda_i^*}{|\lambda_i|} & \lambda_{i_1}^* \frac{\lambda_i}{|\lambda_i|} & \lambda_{i_1}^* \frac{\lambda_i}{|\lambda_i|} \\ \beta, & \beta, & \beta, & \beta, \\ \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} & \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} & \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} \end{matrix}$$

Then from (43), the Jacobian matrix F has the eigenvalues

$$\begin{matrix} \lambda_1 \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_1 \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_1^* \frac{\lambda_i}{|\lambda_i|} - \beta & \lambda_1^* \frac{\lambda_i}{|\lambda_i|} - \beta \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{i-1} \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_{i-1} \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_{i_1}^* \frac{\lambda_i}{|\lambda_i|} - \beta & \lambda_{i_1}^* \frac{\lambda_i}{|\lambda_i|} - \beta \\ 0, & -2\beta, & 0, & -2\beta, \\ \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} - \beta & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} - \beta \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_{i+1} \frac{\lambda_i^*}{|\lambda_i|} - \beta & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} - \beta & \lambda_{i+1}^* \frac{\lambda_i}{|\lambda_i|} - \beta \end{matrix}$$

Hence from the principle of the first approximation, the following theorem is proved:

Theorem 1 *Suppose x_0 is a stable equilibrium point of (7) with $D(x)$ having a structure $s = 1$ and $f = 0$, then $\beta = |\lambda_i|$, where λ_i is one of λ_k, $k = 1, \cdots n$, the eigenvalues of M, and $\beta \geq \text{real}(\lambda_k \frac{\lambda_i^*}{|\lambda_i|})$ for all $k \neq i$.*

Remarks: (1) β is not necessarily equal to $\rho(M)$, the spectral radius of matrix M, in fact β can has different values equal to the magnitudes of different eigenvalues of M, as shown in Example 2 later on. In this sense, what we can calculated is a lower bound of the spectral radius. (2) Unfortunately we find that if $\beta = |\lambda_i| = \text{real}(\lambda_k \frac{\lambda_i^*}{|\lambda_i|})$ and $\text{imag}(\lambda_k \frac{\lambda_i^*}{|\lambda_i|}) \neq 0$ for a certain $k \neq i$, then limit cycling can happen for both x and β. We shall show it in Example 3.

Now the question is whether there exists any stable equilibrium point with $\beta = \rho(M)$. Suppose the eigenvalues of M are:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

and the multiplicity of λ_1 is 1, then

$$\text{real}(\lambda_k \frac{\lambda_1}{|\lambda_1|}) < |\lambda_1|, \; k \neq 1 \tag{44}$$

It can be proved as follows. It is obvious that if $|\lambda_k| < |\lambda_1|$, then (44) is held. So we now consider the case $|\lambda_k| = |\lambda_1|$. It is also clear that the left hand side of (44) is less or equal to the right hand side. Now suppose there is an eigenvalue, say λ_j $j \neq 1$, satisfying

$$\text{real}(\lambda_j \frac{\lambda_1}{|\lambda_1|}) = |\lambda_1| \tag{45}$$

Denote the real parts and the imaginary parts of λ_1 and λ_j by λ_{1r}, λ_{1i}, λ_{jr}, λ_{ji}, then (45) has the following form:

$$\lambda_{jr}^2 + \lambda_{ji}^2 = \lambda_{1r}^2 + \lambda_{1i}^2$$

$$\lambda_{jr}\lambda_{1r} + \lambda_{ji}\lambda_{1i} = \lambda_{1r}^2 + \lambda_{1i}^2$$

there is only one solution for this two equations: $\lambda_{jr} = \lambda_{1r}$ and $\lambda_{ji} = \lambda_{1i}$. As λ_1 has multiplicity 1, so j=1, that is contradictory to the assumption of $j \neq 1$. So (45) does not hold for $j \neq 1$, and therefore we proved (44). Then, by use of Lemma 2 and the fact that

$$D(z_1 x_0 + z_2 P_r P_f x_0 + z_3 P_k x_0 + z_4 P_r P_f P_k x_0) = D(x_0) \tag{46}$$

which is proved in Appendix B, along the same procedure for proving Lemma 3, the following result can be proved (we omit all details of the prove):

Theorem 2 *Suppose M has eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, and λ_1 has multiplicity 1, then the equilibrium point x_0 with $\beta = |\lambda_1|$ is a stable equilibrium point of (7) with $D(x)$ having a structure of $s = 1$ and $f = 0$.*

Remarks: The condition of 1 multiplicity for λ_1 can be relaxed actually. But the prove could be difficult as $f_z(z, y_0)$ is rather complicated in this case.

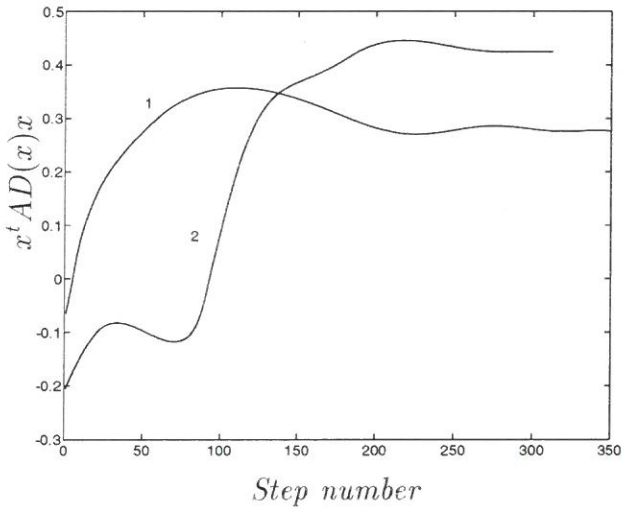


Figure 2: Continuous power iteration. Curve 1: $x^t AD(x)x$ is convergent to $\beta = |\lambda_1| = 0.2759$, the second largest eigenvalue of M. Curve 2: $x^t AD(x)x$ is convergent to $\beta = |\lambda_3| = 0.4242$, the spectral radius $\rho(M)$

Example 2 *Let M be the same as in the Example 1 and the block structure be of $r = 3$ and $f = 0$. The eigenvalues of M are:*

$$\lambda_1 = 0.2746 + 0.0269i \quad \lambda_2 = 0.1304 + 0.0030i \quad \lambda_3 = 0.0910 + 0.4143i$$

and $\rho(M) = |\lambda_3| = 0.4242$. From Theorem 2, $|\lambda_3|$ is certainly related to a stable equilibrium point. However, from Theorem 1, $|\lambda_1| = 0.2759$ is also related to another stable equilibrium point. Now we use the same method as in Example 2 to solve (7). Initial values are generated, again, by MATLAB random matrix generator. Fig. 2 shows the solutions. Curve 1 and 2 start with two different values of $x(0)^t AD(x(0))x(0)$ corresponding to two different initial values of $x(0)$ and end up with the values equal to $|\lambda_1|$ and $|\lambda_2|$ corresponding to two stable equilibrium points of (7). This example shows that the continuous power iteration algorithm is not guaranteed to converge to the spectral radius $\rho(M)$. In general, the continuous power iteration algorithm is not guaranteed to converge to the structured singular value $\mu(M)$ either.

Example 3 *Let M be given as*

$$M = \begin{bmatrix} -44.1104 - 20.3430i & 13.7763 - 31.6756i & 32.6037 + 19.1156i \\ -9.8071 - 37.6925i & 27.1522 - 7.5651i & 4.5312 + 30.0890i \\ -28.2306 - 36.0750i & 25.2184 - 20.1463i & 18.9582 + 29.9080i \end{bmatrix}$$

and the block structure be of $r = 3$ and $f = 0$. The eigenvalues of M are:

$$\lambda_1 = 1 + 2j \quad \lambda_2 = 1 \quad \lambda_3 = 0$$

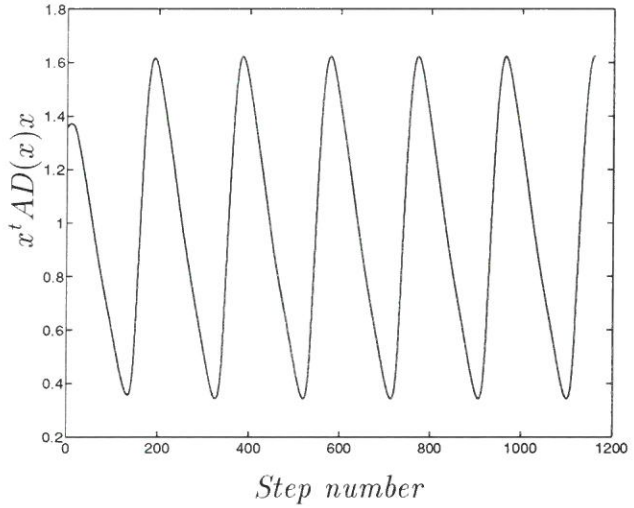


Figure 3: Continuous power iteration. limit cycling happens near to the neighborhood of the equilibrium point x_0 with $\beta = \lambda_2 = 1$

and $\rho(M) = |\lambda_1| = 2.2360$. *Note that* $|\lambda_2| = \text{real}(\lambda_1 \frac{\lambda_2}{|\lambda_2|})$ *and* $\text{imag}(\lambda_1 \frac{\lambda_2}{|\lambda_2|}) = 2 \neq 0$. *Though for most of the initial values* $x(0)$ *generated by random generator,* x *converge to an equilibrium point* x_0 *with* $\beta = \rho(M) = |\lambda_1| = 2.2360$, *we did find an initial point* $x(0)$ *such that both* x *and* β *converge to a limit cycling, see Fig. 3.*

4.2 Maximum Singular Value

Now we consider another extreme case of the structured singular value: the maximum singular value $\bar{\sigma}(M)$. In this case $D(x)$ has the structure of $r = 0$ and $m = n$, thus $a = a_2$, $w = w_2$ and

$$D(x) = \begin{bmatrix} d_f(x)I_n & 0 & 0 & 0 \\ 0 & d_f^{-1}(x)I_n & 0 & 0 \\ 0 & 0 & d_f(x)I_n & 0 \\ 0 & 0 & 0 & d_f^{-1}(x)I_n \end{bmatrix}$$

Correspondingly, $\Delta(x)$ in (12) becomes

$$\Delta(x) = -\frac{P_f x x^t P_f}{s_f(x)}$$

In this case, $D(x)$ along the trajectory of (7) is a constant matrix, for $\dot{D}(x) = 0$, which is proved in Appendix B. $D(x)$ are thus completely determined by the initial value of $x(0)$. Without loss of generality, we can choose $x(0)$ such that $D(x(0)) = I$. In this case, (7)

becomes:

$$\dot{x} = (1 - xx^t)Ax \tag{47}$$

Actually, if $D(x(0))$ is not equal to I, then one can always find a coordinate transformation:

$$y = \begin{bmatrix} \frac{1}{\|a(0)\|\Delta} I_n & & & \\ & \frac{1}{\|w(0)\|\Delta} I_n & & \\ & & \frac{1}{\|a(0)\|\Delta} I_n & \\ & & & \frac{1}{\|w(0)\|\Delta} I_n \end{bmatrix} x \text{ with } \Delta = \frac{1}{\sqrt{\frac{\|a\|^2}{\|a(0)\|^2} + \frac{\|w\|^2}{\|w(0)\|^2}}}$$

Then y will also satisfy (47). Now we consider general case of (47) where A is a arbitrary symmetric matrix, taking A defined in (3) as a special case. Suppose that x_0 is an equilibrium point of (47), then it is an eigenvector of A with eigenvalue $\beta = x_0^t A x_0$. As A is a symmetric matrix, there is an unitary matrix $U = [u_1\ u_2\ \cdots\ u_{4n}]$ with $u_1 = x_0$, such that

$$U^t A U = \text{diag}(\lambda_1,\ \lambda_2,\ \cdots, \lambda_{4n})$$

where λ_i: $i = 1,\ \cdots,\ 4n$ are the eigenvalues of A, and $\lambda_1 = \beta$. The Jacobian matrix F at equilibrium point x_0 is:

$$F = \left[-x_0^t A x_0 I - x_0 x_0^t A + (I - x_0 x_0^t)A\right]$$

Take a coordinate transformation to F with the unitary U then

$$\begin{aligned} U^t F U &= -\beta I - 2U^t x_0 x_0^t A U + U^t A U \\ &= -\beta I - 2\beta U^t x_0 x_0^t U + U^t A U \\ &= \begin{bmatrix} -2\beta & 0 & \cdots & 0 \\ 0 & \lambda_2 - \beta & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda_{4n} - \beta \end{bmatrix} \end{aligned} \tag{48}$$

From the principle of the first approximation, if x_0 is a stable equilibrium point, then

$$\beta \geq 0 \text{ and } \beta \geq \lambda_i,\ i = 2,\ \cdots,\ 4n \tag{49}$$

and obviously, there is only one β equal to the maximum eigenvalue of A which satisfies (49). If A, however, is negative definite, then there is no stable equilibrium point at all. Now we show that if A is nonnegative definite, then any equilibrium point in the eigenspace associated with maximum eigenvalue β is a global stable equilibrium point. Suppose the eigenvalues of A can be arranged as follows

$$\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} \geq \cdots \lambda_{4n}$$

and suppose $x(0) = \sum_{i=1}^{4n} a_i u_i$, a_i, $i = 1,\ \cdots,\ k$ are not all equal to zero and thus $\|e^{At}x(0)\|$ will not equal to zero for $t \in (0, \infty)$. It is then easy to check that[3]

$$x = \frac{e^{At}x(0)}{\|e^{At}x(0)\|}$$

is a solution of (47). Now it also is easy to see that when $t \to \infty$ x will converge to the eigenspace associated with eigenvalue β, as

$$\begin{aligned} \lim_{t\to\infty} x &= \lim_{t\to\infty} \frac{\sum_{i=1}^{4n} a_i e^{\lambda_i t} u_i}{\sqrt{\sum_{i=1}^{4n} a_i^2 e^{2\lambda_i t}}} \\ &= \lim_{t\to\infty} \frac{e^{\lambda_1 t}(\sum_{i=1}^{k} a_i u_i + \sum_{i=k+1}^{4n} a_i e^{(\lambda_i-\lambda_1)t} u_i)}{e^{\lambda_1 t}\sqrt{\sum_{i=1}^{k} a_i^2 + \sum_{i=k+1}^{4n} a_i^2 e^{2(\lambda_i-\lambda_1)t}}} \\ &= \frac{\sum_{i=1}^{k} a_i u_i}{\sqrt{\sum_{i=1}^{k} a_i^2}} \end{aligned} \tag{50}$$

Note that the global stability here is not in the exact sense, as we need that the initial value $x(0)$ is not orthogonal to the eigenspace associated to the maximum eigenvalue β and $\|x(0)\| = 1$. Now we consider the original problem where A is defined in (3). It is easy to check that if M has a singular value σ_i, then

$$\begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix}$$

will have eigenvalues σ_i and $-\sigma_i$, thus A will have eigenvalues σ_i, σ_i, $-\sigma_i$ and $-\sigma_i$. Thus the maximum eigenvalue of A is the maximum singular value $\bar{\sigma}(M)$. We can now summary all the results in this section in the following theorem:

Theorem 3 *Suppose $D(x)$ has the structure of $s = 0$ and $f = 1$ and M has the maximum eigenvalue $\bar{\sigma}(M)$. If the initial value $x(0)$ is not orthogonal to the eigenspace of $AD(x_0)$ associated the maximum eigenvalue $\bar{\sigma}(M)$ and $\|x(0)\| = 1$, then x in (7) will always convergent to the eigenspace of A associated the maximum eigenvalue $\bar{\sigma}(M)$, and $\beta = \bar{\sigma}(M)$.*

Remarks: (1) Numerically, the condition of $x(0)$ has component in the directions of the eigenspace of $AD(x_0)$ associated the maximum eigenvalue $\bar{\sigma}(M)$ is not important. Actually, the rounding errors sustained during solving (7) ensure that $x(t)$ $(t > 0)$ can have components in those directions. This can be shown in the following examples. (2) if A is positive definite, then the condition $\|x(0)\| = 1$ can be relaxed, as in this case, $\|x\|$ will always converge to 1.

Example 4 *Let again M be given as in Example 1 and the block structure be of $r = 0$ and $m = 3$. The singular value of M are*

$$\sigma_1 = 1.1731 \quad \sigma_2 = 0.2358 \quad \sigma_3 = 0.0569$$

Now we use the same way to solve (7) as in Example 1. Fig.4 shows the results. For curve 1, the initial state vector $x(0)$ is again generated by the random generator, an equilibrium point is reached with $\beta = \bar{\sigma}(M) = 1.1731$. For curve 2, the initial state vector x(0) is, this time, set to the value such that $x(0)$ is an eigenvector of $AD(x(0))$ associated with the eigenvalue of $AD(x(0))$: $\sigma_2 = 0.2358$. So $x(0)$ itself is an equilibrium point of (7). Actually we choose $x(0)$ such that $\|\dot{x}\|_{x=x(0)} < 10^{-15}$. However this $x(0)$ is not a stable equilibrium point. Thus curve 2 finally evolves to the $\beta = \bar{\sigma}(M) = 1.1731$ corresponding a stable equilibrium point.

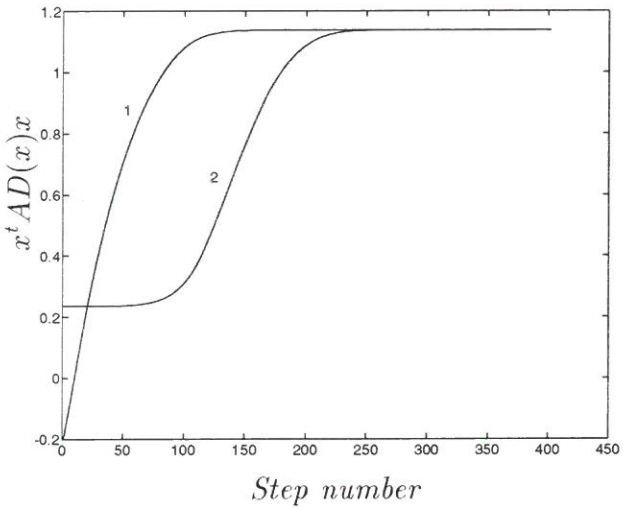


Figure 4: Continuous power iteration. Curve 1: $x^t AD(x)x$ is convergent to $\beta = \dot{\sigma}(M) = 1.1731$. Curve 2: x converges to an stable equilibrium point from an unstable equilibrium point and $x^t AD(x)x$ is convergent to $\beta = \bar{\sigma}(M) = 1.1731$ from 0.2358, the second largest singular value of M

5 Conclusion

In this paper, the stability analysis was made for the continuous algorithm of power iteration for the lower bound of structured singular value. Lemma 1 and Lemma 3 give a necessary condition and a sufficient condition for an equilibrium point to be a stable equilibrium point for general cases. Theorem 1 and Theorem 2 specializes these results to the extreme case for spectral radius calculation. Theorem 3 gives a sufficient condition (also necessary) for an equilibrium point to be a stable equilibrium point for another extreme case of maximum singular value calculation. For the case of spectral radius calculation, it is not guaranteed that spectral radius can be obtained as shown in Example 2. It is also found that the continuous algorithm can not be completely free of limit cycling for the case of spectral radius calculation. If this happens, one could simply try another initial point, since it is actually guaranteed from Theorem 2 that there is no limit cycling near to the equilibrium point associated with spectral radius. This properties should also hold for general cases of structured singular value cases. But it is still not proved yet.

Appendix

A The derivatives of $D(x)$

Matrix $D(x)$ is defined in (4). $D(x)$ is actually equal to a sum of two matrices:

$$D(x) = D_r(x) + D_f(x) \tag{51}$$

where

$$D_r(x) = \begin{bmatrix} 0 & 0 & d_r^r(x)I_r & 0 & 0 & 0 & -d_r^i(x)I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_r^r(x)I_r & 0 & 0 & 0 & d_r^i(x)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_r^i(x)I_r & 0 & 0 & 0 & d_r^r(x)I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -d_r^i(x)I_r & 0 & 0 & 0 & d_r^r(x)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_f(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_f(x)I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_f^{-1}(x)I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_f(x)I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_f^{-1}(x)I_m \end{bmatrix}$$

Obviously, $D_r(x)$ is constructed by scalar $d_r(x)$ and $D_f(x)$ is constructed by scalar $d_f(x)$. First, we calculate the derivative of $d_r(x)$. From (5),

$$\begin{aligned}
\frac{\partial d_r(x)}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\frac{w_1^* a_1}{\sqrt{w_1^* a_1 a_1^* w_1}}\right) \\
&= \frac{\frac{\partial w_1^*}{\partial x_j} a_1 + w_1^* \frac{\partial a_1}{\partial x_j}}{\sqrt{w_1^* a_1 a_1^* w_1}} - \frac{w_1^* a_1 \left(\frac{\partial w_1^*}{\partial x_j} a_1 a_1^* w_1 + w_1^* \frac{\partial a_1}{\partial x_j} a_1^* w_1 + w_1^* a_1 \frac{\partial a_1^*}{\partial x_j} w_1 + w_1^* a_1 a_1^* \frac{\partial w_1}{\partial x_j}\right)}{2\left(\sqrt{w_1^* a_1 a_1^* w_1}\right)^3} \\
&= \frac{w_1^* a_1 a_1^* w_1 \left(\frac{\partial w_1^*}{\partial x_j} a_1 + w_1^* \frac{\partial a_1}{\partial x_j}\right) - (w_1^* a_1)^2 \left(\frac{\partial a_1^*}{\partial x_j} w_1 + a_1^* \frac{\partial w_1}{\partial x_j}\right)}{2|w_1^* a_1|^3} \\
&= \frac{d_r(x) d_r(x)^* \left(\frac{\partial w_1^*}{\partial x_j} a_1 + w_1^* \frac{\partial a_1}{\partial x_j}\right) - (d_r(x))^2 \left(\frac{\partial a_1^*}{\partial x_j} w_1 + a_1^* \frac{\partial w_1}{\partial x_j}\right)}{2|w_1^* a_1|} \\
&= d_r(x) \left[\frac{d_r(x)^* \left(\frac{\partial w_1^*}{\partial x_j} a_1 + w_1^* \frac{\partial a_1}{\partial x_j}\right) - d_r(x) \left(\frac{\partial a_1^*}{\partial x_j} w_1 + a_1^* \frac{\partial w_1}{\partial x_j}\right)}{2|w_1^* a_1|}\right] \\
&= -j d_r(x) \operatorname{imag}\left[\frac{d_r(x) \left(\frac{\partial a_1^*}{\partial x_j} w_1 + a_1^* \frac{\partial w_1}{\partial x_j}\right)}{|w_1^* a_1|}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-jd_r(x)}{|w_1^* a_1|} \operatorname{imag}\left\{ \left(d_r^r(x) + jd_r^i(x)\right) \left[\left(\frac{\partial a_{1r}^t}{\partial x_j} - j\frac{\partial a_{1i}^t}{\partial x_j} \right) (w_{1r} + jw_{1i}) \right.\right. \\
&\quad \left.\left. + \left(a_{1r}^t - ja_{1i}^t\right) \left(\frac{\partial w_{1r}}{\partial x_j} + j\frac{\partial w_{1i}}{\partial x_j} \right) \right] \right\} \\
&= \frac{-jd_r(x)}{|w_1^* a_1|} \left(d_r^i(x)\frac{\partial a_{1r}^t}{\partial x_j} w_{1r} + d_r^i(x)\frac{\partial a_{1i}^t}{\partial x_j} w_{1i} + d_r^i(x) a_{1r}^t \frac{\partial w_{1r}}{\partial x_j} + d_r^i(x) a_{1i}^t \frac{\partial w_{1i}}{\partial x_j} \right. \\
&\quad \left. + d_r^r(x)\frac{\partial a_{1r}^t}{\partial x_j} w_{1i} - d_r^r(x)\frac{\partial a_{1i}^t}{\partial x_j} w_{1r} + d_r^r(x) a_{1r}^t \frac{\partial w_{1i}}{\partial x_j} - d_r^r(x) a_{1i}^t \frac{\partial w_{1r}}{\partial x_j} \right) \\
&= \frac{-jd_r(x)}{|w_1^* a_1|} \begin{bmatrix} a_{1r}^t & w_{1r}^t & a_{1i}^t & w_{1i}^t \end{bmatrix} \begin{bmatrix} 0 & d_r^i(x)I_r & 0 & d_r^r(x)I_r \\ d_r^i(x)I_r & 0 & -d_r^r(x)I_r & 0 \\ 0 & -d_r^r(x)I_r & 0 & d_r^i(x)I_r \\ d_r^r(x)I_r & 0 & d_r^i(x)I_r & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial a_{1r}}{x_j} \\ \frac{\partial w_{1r}}{x_j} \\ \frac{\partial a_{1i}}{x_j} \\ \frac{\partial w_{1i}}{x_j} \end{bmatrix} \\
&= \frac{-jd_r(x)}{|w_1^* a_1|} x^t \begin{bmatrix} 0 & 0 & d_r^i(x)I_r & 0 & 0 & 0 & d_r^i(x)I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_r^i(x)I_r & 0 & 0 & 0 & -d_r^r(x)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_r^r(x)I_r & 0 & 0 & 0 & d_r^i(x)I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_r^r(x)I_r & 0 & 0 & 0 & -d_r^i(x)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial x}{\partial x_j} \\
&= \frac{-jd}{|w_1^* a_1|} x^t p_r^t D(x) e_j \qquad (52)
\end{aligned}
$$

where matrix p_r is defined in (12). From (52),

$$\frac{\partial d_r^r(x)}{\partial x_j} = \operatorname{real}(\frac{\partial d_r(x)}{\partial x_j}) = d_r^i(x) x^t p_r^t D(x) e_j / s_r(x)$$

with $s_r = |w_1^* a_1|$ and

$$\frac{\partial d_r^i(x)}{\partial x_j} = \operatorname{imag}(\frac{\partial d_r(x)}{\partial x_j}) = -d_r^r(x) x^t p_r^t D(x) e_j / s_r(x)$$

Thus from (51) we have

$$
\begin{aligned}
\frac{\partial D_r(x)}{\partial x_j} &= \begin{bmatrix} 0 & 0 & d_r^i(x)I_r & 0 & 0 & 0 & d_r^i(x)I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_r^i(x)I_r & 0 & 0 & 0 & -d_r^r(x)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_r^r(x)I_r & 0 & 0 & 0 & d_r^i(x)I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_r^r(x)I_r & 0 & 0 & 0 & -d_r^i(x)I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x^t p_r^t D(x) e_j / s_r(x) \\
&= D(x) p_r x^t p_r^t D(x) e_j / s_r(x) \qquad (53)
\end{aligned}
$$

Now we calculate the partial derivative $d_f(x)$.

$$\frac{\partial d_f(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\|w_2\|}{\|a_2\|} = \frac{\|a_2\|}{2\|w_2\|}\frac{\partial}{\partial x_j}\left(\frac{w_{2r}^t w_{2r} + w_{2i}^t w_{2i}}{a_{2r}^t a_{2r} + a_{2i}^t a_{2i}}\right)$$

$$= d_f^{-1}(x)\left[\frac{w_{2r}^t \frac{\partial w_{2r}}{\partial x_j} + w_{2i}^t \frac{\partial w_{2i}}{\partial x_j}}{a_{2r}^t a_{2r} + a_{2i}^t a_{2i}} - \frac{(w_{2r}^t w_{2r} + w_{2i}^t w_{2i})\left(a_{2r}^t \frac{\partial a_{2r}}{\partial x_j} + a_{2i}^t \frac{\partial a_{2i}}{\partial x_j}\right)}{(a_{2r}^t a_{2r} + a_{2i}^t a_{2i})^2}\right]$$

$$= \frac{d_f^{-1}(x)}{\|a_2\|^4}\left[\|a_2\|^2\left(w_{2r}^t\frac{\partial w_{2r}}{\partial x_j} + w_{2i}^t\frac{\partial w_{2i}}{\partial x_j}\right) - \|w_2\|^2\left(a_{2r}^t\frac{\partial a_{2r}}{\partial x_j} + a_{2i}^t\frac{\partial a_{2i}}{\partial x_j}\right)\right]$$

$$= \frac{d_f^{-1}(x)}{\|a_2\|^4}\begin{bmatrix} -\|w_2\|^2 a_{2r}^t & \|a_2\|^2 w_{2r}^t & -\|w_2\|^2 a_{2i}^t & \|a_2\|^2 w_{2-}^t \end{bmatrix}\begin{bmatrix} \frac{\partial a_{2r}}{\partial x_j} \\ \frac{\partial w_{2r}}{\partial x_j} \\ \frac{\partial a_{2i}}{\partial x_j} \\ \frac{\partial w_{2i}}{\partial x_j} \end{bmatrix}$$

$$= \frac{-d_f(x)}{\|a_2\|\|w_2\|}\begin{bmatrix} a_{2r}^t & w_{2r}^t & a_{2i}^t & w_{2i}^t \end{bmatrix}\begin{bmatrix} d_f(x) & 0 & 0 & 0 \\ 0 & -d_f^{-1}(x) & 0 & 0 \\ 0 & 0 & d_f(x) & 0 \\ 0 & 0 & 0 & -d_f^{-1}(x) \end{bmatrix}\begin{bmatrix} \frac{\partial a_{2r}}{\partial x_j} \\ \frac{\partial w_{2r}}{\partial x_j} \\ \frac{\partial a_{2i}}{\partial x_j} \\ \frac{\partial w_{2i}}{\partial x_j} \end{bmatrix}$$

$$= \frac{-d_f(x)}{\|a_2\|\|w_2\|}x^t\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_f(x)I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_f^{-1}(x)I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_f(x)I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_f^{-1}(x)I_m \end{bmatrix}\frac{\partial x}{\partial x_j}$$

$$= -d_f(x)x^t p_f D e_j / s_f(x) \tag{54}$$

where matrix p_f is defined in (12) and $s_f(x) = \frac{1}{\|a_2\|\|w_2\|}$. It is easy to get the derivative of $d_f^{-1}(x)$:

$$\frac{\partial d_f^{-1}(x)}{\partial x_j} = -d_f^{-2}(x)\frac{\partial d_f(x)}{\partial x_j} = \frac{d_f^{-1}(x)}{\|a_2\|\|w_2\|}x^t p_f D e_j \tag{55}$$

Now, using (54) and (55), we have

$$\frac{\partial D_f(x)}{\partial x_j} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_f(x)I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_f^{-1}(x)I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_f(x)I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_f^{-1}(x)I_m \end{bmatrix} x^t p_f D(x) e_j / s_f(x)$$

$$= D(x)p_f x^t p_f D(x)e_j/s_f(x) \tag{56}$$

Finally, it is easy to get the partial derivative of $D(x)$ from (53) and (56):

$$\frac{\partial D(x)}{\partial x_j} = \frac{\partial D_r(x)}{\partial x_j} + \frac{\partial D_f(x)}{\partial x_j} = D(x)\left(p_r x^t p_r^t/s_r(x) - p_f x^t p_f/s_f(x)\right) D(x)e_j$$

and thus

$$\begin{aligned}\frac{\partial D(x)}{\partial x_j}x &= D(x)\left(p_r x^t p_r^t/s_r(x) - p_f x^t p_f/s_f(x)\right) D(x)e_j \\ &= D(x)\left(p_r x x^t p_r^t/s_r(x) - p_f x x^t p_f/s_f(x)\right) D(x)e_j\end{aligned}$$

This is just (11). The derivative of $D(x)$ with respect to time, $\frac{dD(x)}{dt}$, has the similar form with the partial derivative of $D(x)$ with respect to x_j, $\frac{\partial D(x)}{\partial x_j}$. Actually, we only need to replace e_j by $\dot{x}$ to get $\frac{dD(x)}{dt}$ from $\frac{\partial D(x)}{\partial x_j}$. That is

$$\frac{dD(x)}{dt} = D(x)\left(p_r x^t p_r^t/s_r(x) - p_f x^t p_f/s_f(x)\right) D(x)\dot{x}$$

In the case $r = 0$ and $m = n$, $a = a_2$, $w = w_2$ and $D(x) = D_f(x)$, then

$$\begin{aligned}\frac{dD(x)}{dt} &= -D_f(x)p_f x^t p_f D_f(x)\left(I - xx^t\right) AD_f(x)/s_r(x) \\ &= -D_f(x)p_f\left(x^t p_f D_f(x)D_f^{-1}(x)Ax - x^t p_f D_f(x)xx^t AD_f(x)x\right)/s_r x \\ &= -D_f(x)p_f\left(x^t p_f Ax - x^t p_f D_f(x)xx^t AD_f(x)x/s_r(x)\right) \qquad (57)\end{aligned}$$

One may easily check that

$$x^t p_f Ax = 0 \text{ and } x^t p_f D_f(x)x = 0$$

Thus (57) shows that $\frac{dD(x)}{dt} = 0$

B Prove of several equalities

B.1 Equation (15)

With P_r and P_f defined in (14), it is easy to check that:

$$D_r(x)P_r = -P_r D_r(x),\ \ D_r(x)P_f = -P_f D_r(x),\ \ D_f(x)P_f = P_f D_f(x)$$

$$D_f(x)P_r = P_r D_f(x) \text{ and } D_r(x)P_k = -P_k^{-1}D_r(x)$$

and

$$AP_r = -P_r A,\ \ AP_f = -P_f A \text{ and} AP_k = -P_k^{-1}A$$

Thus (15) can be proved as follows.

$$\begin{aligned} AD(x)P_rP_f &= A\left(D_r(x)+D_f(x)\right)P_rP_f \\ &= A\left(-P_rD_r(x)+P_rD_f(x)\right)P_f \\ &= -P_rA\left(-D_r(x)+D_f(x)\right)P_f \\ &= -P_rA\left(P_fD_r(x)+P_fD_f(x)\right) \\ &= P_rP_fA\left(D_r(x)+D_f(x)\right) \\ &= P_rP_fAD(x) \end{aligned}$$

Note that in (38) $D(x) = D_r(x)$ $(D_f(x) = 0)$. Then

$$AD(x)P_k = AD_r(x)P_k = A\left(-P_k^{-1}\right)D_r(x) = P_kAD(x)$$

B.2 Equations (22 - 25)

The following equations can be easily checked:

$$p_r^tD_f(x) = 0 \quad p_fD_r(x) = 0$$

Then for (22):

$$\begin{aligned} x^tp_r^tD(x)x &= x^tp_r^tD_r(x)x \\ &= -d_r^r(x)a_{1i}^tw_{1r} + d_r^i(x)a_{1i}^tw_{1i} + d_r^r(x)w_{1i}^ta_{1r} + d_r^i(x)w_{1i}^ta_{1i} \\ &\quad +d_r^i(x)a_{1r}^tw_{1r} + d_r^r(x)a_{1r}^tw_{1i} + d_r^i(x)w_{1r}^ta_{1r} - d_r^r(x)w_{1r}^ta_{1i} \\ &= -2d_r^r(x)\left(w_{1r}^ta_{1i} - w_{1i}^ta_{1r}\right) + 2d_r^i(x)\left(w_{1r}^ta_{1r} + w_{1i}^ta_{1i}\right) \\ &= 2\|w_1^*a_1\|\left(-d_r^r(x)d_r^i(x) + d_r^r(x)d_r^i(x)\right) = 0 \end{aligned}$$

For (23)

$$\begin{aligned} x^tp_fD(x)x &= x^tp_fD_f(x)x \\ &= \left(a_{2r}^ta_{2r} + a_{2i}^ta_{2i}\right)d_f(x) - \left(w_{2r}^tw_{2r} + w_{2i}^tw_{2i}\right)d_f(x)^{-1} \\ &= \|w_2\|\|a_2\| - \|w_2\|\|a_2\| = 0 \end{aligned}$$

For (24)

$$\begin{aligned} x^tp_r^tD(x)P_rP_fx &= x^tp_r^tD_r(x)P_rP_fx \\ &= d_r^r(x)a_{1i}^tw_{1i} + d_r^i(x)a_{1i}^tw_{1r} - d_r^r(x)w_{1i}^ta_{1i} + d_r^i(x)w_{1i}^ta_{1r} \\ &\quad -d_r^i(x)a_{1r}^tw_{1i} + d_r^r(x)a_{1r}^tw_{1r} - d_r^i(x)w_{1r}^ta_{1i} - d_r^r(x)w_{1r}^ta_{1r} \\ &= 0 \end{aligned}$$

For (25)

$$\begin{aligned} x^tp_fD(x)P_rP_fx &= x^tp_fD_f(x)P_rP_fx \\ &= \left(-a_{2r}^ta_{2i} + a_{2i}^ta_{2r}\right)d_f(x) - \left(-w_{2r}^tw_{2i} + w_{2i}^tw_{2r}\right)d_f(x)^{-1} \\ &= 0 \end{aligned}$$

B.3 Prove of (34) and (46)

Let $\tilde{x}$ denotes the linear combination of vector x and $P_r P_f x$, or $\tilde{x} = z_1 x + z_2 P_r P_f x$, and correspondingly

$$\begin{bmatrix} \tilde{a}_{1r} \\ \tilde{a}_{2r} \\ \tilde{w}_{1r} \\ \tilde{w}_{2r} \\ \tilde{a}_{1i} \\ \tilde{a}_{2i} \\ \tilde{w}_{1i} \\ \tilde{w}_{2i} \end{bmatrix} = \begin{bmatrix} z_1 a_{1r} - z_2 a_{1i} \\ z_1 a_{2r} - z_2 a_{2i} \\ z_1 w_{1r} - z_2 w_{1i} \\ z_1 w_{2r} - z_2 w_{2i} \\ z_1 a_{1i} + z_2 a_{1r} \\ z_1 a_{2i} + z_2 a_{2r} \\ z_1 w_{1i} + z_2 w_{1r} \\ z_1 w_{2i} + z_2 w_{2r} \end{bmatrix}$$

and also $\tilde{a}_1 = \tilde{a}_{1r} + j\tilde{a}_{1i}$, $\tilde{w}_1 = \tilde{w}_{1r} + j\tilde{w}_{1i}$, $\tilde{a}_2 = \tilde{a}_{2r} + j\tilde{a}_{2i}$, $\tilde{w}_2 = \tilde{w}_{2r} + j\tilde{w}_{2i}$. Now for proving (34), it is enough to prove that

$$d_r(\tilde{x}) = d_r(x),\ d_f(\tilde{x}) = d_f(x)$$

First we see the expression of $\tilde{w}_1^* \tilde{a}_1$:

$$\begin{aligned} \tilde{w}_1^* \tilde{a}_1 &= \left(z_1 w_{1r}^t - z_2 w_{1i}^t\right)\left(z_1 a_{1r} - z_2 a_{1i}\right) + \left(z_1 w_{1i}^t + z_2 w_{1r}^t\right)\left(z_1 a_{1i} + z_2 a_{1r}\right) \\ &\quad j\left[\left(z_1 w_{1r}^t - z_2 w_{1i}^t\right)\left(z_1 a_{1i} + z_2 a_{1r}\right) - \left(z_1 w_{1i}^t + z_2 w_{1r}^t\right)\left(z_1 a_{1r} - z_2 a_{1i}\right)\right] \\ &= \left(z_1^2 + z_2^2\right)\left(w_{1r}^t a_{1r} + w_{1i}^t a_{1i}\right) j\left(z_1^2 + z_2^2\right)\left(w_{1r}^t a_{1i} - w_{1i}^t a_{1i}\right) \\ &= \left(z_1^2 + z_2^2\right) w_1^* a_1 \end{aligned}$$

thus, from the definition of $d_r(x)$:

$$d_r(\tilde{x}) = \frac{\tilde{w}_1^* \tilde{a}_1}{|\tilde{w}_1^* \tilde{a}_1|} = \frac{(z_1^2 + z_2^2)\, w_1^* a_1}{|\,(z_1^2 + z_2^2) w_1^* a_1)} = \frac{w_1^* a_1}{|w_1^* a_1|} = d_r(x)$$

Next, we see the expression of $\|\tilde{w}_2\|$:

$$\begin{aligned} \|\tilde{w}_2\| &= \sqrt{\left(z_1 w_{2r}^t - z_2 w_{2i}^t\right)\left(z_1 w_{2r} - z_2 w_{2i}\right) + \left(z_1 w_{2i}^t + z_2 w_{2r}^t\right)\left(z_1 w_{2i} + z_2 w_{2r}\right)} \\ &= \sqrt{\left(z_1^2 + z_2^2\right)\left(w_{2r}^t w_{2r} + w_{2i}^t w_{2i}\right)} \\ &= \sqrt{(z_1^2 + z_2^2)}\|w_2\| \end{aligned}$$

$\|\tilde{a}_2\|$ has the similar expression of $\|\tilde{w}_2\|$. Then

$$d_f(\tilde{x}) = \frac{\tilde{w}_2}{\tilde{a}_2} = \frac{w_2}{a_2} = d_f(x)$$

Now we come to prove (46). Note that $D_f(x) = 0$ in this case. So with the same way above, we have to prove that

$$d_r(z_1 x + z_2 P_r P_f x + z_3 P_k x + z_4 P_r P_f P_k x) = d_r(x) \tag{58}$$

From (34), proved in the beginning of this section, the left hand side of (58) is equivalent to

$$d_r\left(z_3(z_1x + z_2P_kx) + z_4P_rP_f(z_1x + z_2P_kx)\right) = d_r(z_1x + z_2P_kx)$$

Thus we only need to prove

$$d_r(z_1x + z_2P_kx) = d_r(x) \tag{59}$$

As now $a = a_1$ and $w = w_1$, then

$$z_1x + z_2P_kx = \begin{bmatrix} z_1a_r + z_2ka_r \\ z_1w_r - z_2k^{-1}w_r \\ z_1a_i + z_2ka_i \\ z_1w_i - z_2k^{-1}w_i \end{bmatrix} = \begin{bmatrix} (z_1 + z_2k)a_r \\ (z_1 - z_2k^{-1})w_r \\ (z_1 + z_2k)a_i \\ (z_1 - z_2k^{-1})w_i \end{bmatrix}$$

Let now $\tilde{a} = (z_1+z_2k)(a_r+ja_i) = (z_1+z_2k)a$ and $\tilde{w} = (z_1-z_2k^{-1})(w_r+jw_i) = (z_1+z_2k^{-1})w$, then

$$\tilde{w}^*\tilde{a} = (z_1 + z_2k)w^*(z_1 + z_2k^{-1})a = (z_1 + z_2k)(z_1 + z_2k^{-1})w^*a$$

and then,

$$\begin{aligned} d_f(z_1x + z_2P_kx) &= \frac{\tilde{w}^*\tilde{a}}{|\tilde{w}^*\tilde{a}|} \\ &= \frac{(z_1 + z_2k)(z_1 - z_2k^{-1})w^*a}{|(z_1 + z_2k)(z_1 - z_2k^{-1})||w^*a|} \\ &= \frac{(z_1 + z_2k)(z_1 - z_2k^{-1})}{|(z_1 + z_2k)(z_1 - z_2k^{-1})|}d_r(x) \end{aligned}$$

It is easy to see that near to equilibrium point where $z_1 = 1$ and $z_2 = 0$ ($x = x_0$), $(z_1 + z_2k)(z_1 - z_2k^{-1}) > 0$, and thus

$$\frac{(z_1 + z_2k)(z_1 - z_2k^{-1})}{|(z_1 + z_2k)(z_1 - z_2k^{-1}|} = 1$$

and then (59) follows.

B.4 Prove of (40) and (41)

Recall that now $D_f(x) = 0$ and $a = a_1$ and $w = w_1$. For (40)

$$x_0^tP_r^tD(x_0)P_kx_0 = x_0^tP_r^tD(x_0)\begin{bmatrix} ka_{r0} \\ -k^{-1}w_{r0} \\ ka_{i0} \\ -k^{-1}w_{i0} \end{bmatrix}$$

$$= \begin{bmatrix} d_r^r(x_0)w_{i0}^t + d_r^i(x_0)w_{r0}^t & -d_r^r(x_0)a_{i0}^t + d_r^i(x_0)a_{r0}^t \end{bmatrix}$$

$$
\begin{aligned}
& d_r^i(x_0)w_{i0}^t - d_r^r(x_0)w_{r0}^t \quad d_r^i(x_0)a_{i0}^t + d_r^i(x_0)a_{r0}^t \,] \begin{bmatrix} ka_{r0} \\ -k^{-1}w_{r0} \\ ka_{i0} \\ -k^{-1}w_{i0} \end{bmatrix} \\
&= d_r^r(x_0)\left(k^{-1}-k\right)\left(w_{r0}^t a_{i0} - w_{i0}^t a_{r0}\right) + d_r^i(x_0)\left(-k^{-1}+k\right)\left(w_{r0}^t a_{r0} + w_{i0}^t a_{i0}\right) \\
&= \left(k-k^{-1}\right)\left(d_r^r(x_0)d_r^i(x_0) - d_r^r(x_0)d_r^i(x_0)\right)s_r(x_0) = 0
\end{aligned}
$$

and similarly for (41)

$$
\begin{aligned}
& x_0^t P_r^t D(x_0) P_r P_f P_k x_0 = x_0^t P_r^t D(x_0) \begin{bmatrix} -ka_{i0} \\ k^{-1}w_{i0} \\ ka_{r0} \\ -k^{-1}w_{r0} \end{bmatrix} \\
&= [\, d_r^r(x_0)w_{i0}^t + d_r^i(x_0)w_{r0}^t \quad - d_r^r(x_0)a_{i0}^t + d_r^i(x_0)a_{r0}^t \\
&\qquad d_r^i(x_0)w_{i0}^t - d_r^r(x_0)w_{r0}^t \quad d_r^i(x_0)a_{i0}^t + d_r^i(x_0)a_{r0}^t \,] \begin{bmatrix} -ka_{-i0} \\ k^{-1}w_{i0} \\ ka_{r0} \\ -k^{-1}w_{r0} \end{bmatrix} \\
&= -d_r^r(x_0)\left(k^{-1}+k\right)\left(w_{r0}^t a_{r0} + w_{i0}^t a_{i0}\right) - d_r^i(x_0)\left(k^{-1}+k\right)\left(w_{r0}^t a_{i0} - w_{i0}^t a_{r0}\right) \\
&= \left(k+k^{-1}\right)\left(d_r^r(x_0)^2 + d_r^r(x_0)^2\right)s_r(x_0) \\
= \; & -(k+k^{-1})s_r(x_0)
\end{aligned}
$$

as

$$
k + k^{-1} = \frac{\|w_0\|}{\|a_0\|} + \frac{\|a_0\|}{\|w_0\|} = \frac{\|w_0\|^2 + \|a_0\|^2}{\|w_0\|\|a_0\|} = \frac{1}{\|w_0\|\|a_0\|}
$$

then from, (41) follows.

References

[1] Andy Pachard, Michael K.H.Fan and John Doyle, "A power method for the structured singular value", *Proc. 27th Conf. Decision Contr.*, Austin, Texas, Dec. 1988, pp. 2132-2137.

[2] Yi Cheng, De Moor B., "Continuous power iteration for computing low bound of μ", Internal report(93-56I), Electrical Department, Catholic University of Leuven, Belgium, 1993. (submitted to ACC 94 Conference)

[3] T. Nanda, "Differential equations and the QR algorithm", *SIAM J. Numer. Anal.*, Vol. 22, No.2, April, 1985. pp.311-321.

[4] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd Edition, Prentice Hall, 1993.

[5] A. Isidori, *Nonlinear Control Systems-An Introduction*, 2nd Edition, Springer-Verlag, 1990.