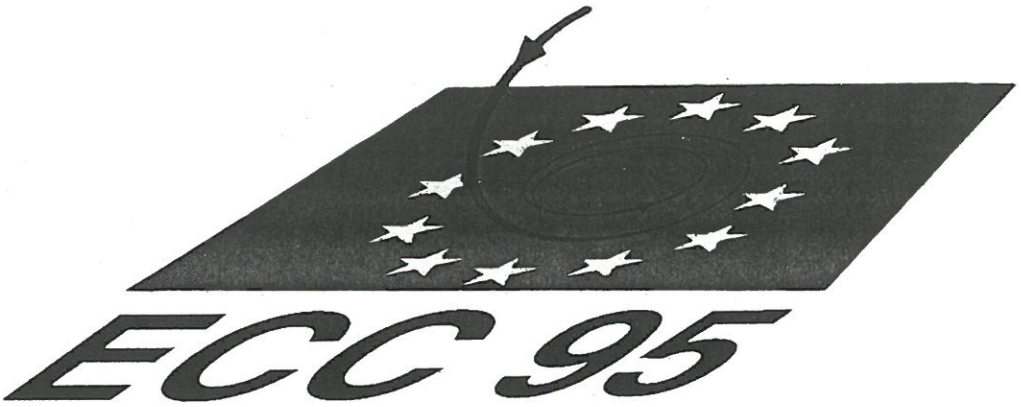
94-62



Proceedings of the third

European Control Conference

ECC 95

September 5-8, 1995
Roma, Italy

Under the auspices of the EUCA

A. Isidori, S. Bittanti, E. Mosca
A. De Luca, M.D. Di Benedetto, G. Oriolo
Editors

Volume 3 of 4
Part two

Proceedings of 3rd
European Control Conference
Rome, Italy, September 1995

STABILITY CRITERIA FOR NEURAL CONTROL SYSTEMS *

J.A.K. Suykens, B. De Moor and J. Vandewalle
Katholieke Universiteit Leuven
Department of Electrical Engineering, ESAT-SISTA
Kardinaal Mercierlaan 94
B-3001 Leuven (Heverlee), Belgium
Tel: 32/16/32 11 11 Fax: 32/16/32 19 86
E-mail: Johan.Suykens@esat.kuleuven.ac.be

Keywords: Neural control, recurrent neural networks, NL_q systems, global asymptotic stability, dissipativity, L_2-gain, Linear Matrix Inequalities (LMIs), (modified) dynamic backpropagation

Abstract

A model based framework for neural control design is proposed with nonlinear state space models and nonlinear dynamic output feedback controllers, both parametrized by multilayer perceptrons. Process noise and measurement noise can be taken into account in the framework. Sufficient conditions for global asymptotic stability and I/O stability (L_2-gain analysis) for such neural control systems are presented. The results are closely related to H_∞ control theory and μ theory. The criteria can be expressed as Linear Matrix Inequalities. Narendra's dynamic backpropagation procedure for tracking a specific reference input is modified in order to guarantee closed loop stability. A crucial point in the theory is the introduction of the so-called NL_q system form, a concept which unifies many existing models and problems in recurrent neural networks, systems and control theory.

1 Introduction

Due to the fact that multilayer perceptrons are universal approximators [4][6], they are powerful achitectures in order to parametrize models and controllers. Furthermore they have a parallel structure and their weights can be learned and adapted on- or off-line from I/O data. Although there exist a lot of examples where neural nets are successful in solving complicated nonlinear optimal control problems (such as e.g. backing up a truck [11], controlling a space robot [13] or swinging up an inverted or double inverted pendulum [16][17]), general stability criteria for neural control systems are still lacking. The aim of this paper is precisely to present a model-based neural control framework (model-based like Nguyen & Widrow's emulator approach [11]) with stability criteria. Starting from I/O measurements, recurrent neural state space models are identified using dynamic backpropagation and neural state space controllers are designed based upon the identified models. Using state augmentation the neural control schemes are transformed into a so-called standard NL_q system form, for which the criteria are derived.

This paper is organized as follows. In Section 2 the model-based framework is proposed with neural state space models and neural state space controllers. In Section 3 it is stated that all neural state space control problems considered within the present framework are special cases of NL_q systems. In Section 4 sufficient conditions for global asymptotic stability and conditions for dissipativity with finite L_2-gain are presented for these NL_qs. In Section 5 this finally leads to a modification of Narendra's dynamic backpropagation procedure in order to ensure closed loop stability.

2 Neural state space models and controllers

Given a nonlinear plant, supposed to be of the form

$$\begin{cases} x_{k+1} &= f_0(x_k, u_k) + \varphi_k \\ y_k &= g_0(x_k, u_k) + \psi_k \end{cases} \quad (1)$$

with f_0 and g_0 continuous nonlinear mappings, input vector $u_k \in \mathbb{R}^m$, output vector $y_k \in \mathbb{R}^l$, state vector $x_k \in \mathbb{R}^n$ and $\varphi_k \in \mathbb{R}^n$, $\psi_k \in \mathbb{R}^l$ respectively process noise and measurement noise. The noise processes are assumed to be

*This research work was carried out at the ESAT laboratory and the Interdisciplinary Center of Neural Networks ICNN of the Katholieke Universiteit Leuven, in the framework of a Concerted Action Project of the Flemish Community, entitled *Applicable Neural Networks* and within the framework of the Belgian Programme on Interuniversity Poles of Attraction (IUAP-17 and IUAP-50), initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture.

zero mean white Gaussian with covariance matrices

$$E\{\begin{bmatrix} \varphi_k \\ \psi_k \end{bmatrix} [\varphi_s^T \ \psi_s^T]\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{ks}.$$

Predictors in innovations form for (1) were proposed in [18]. The models are of the form

$$\begin{cases} \hat{x}_{k+1} &= f(\hat{x}_k, u_k, \epsilon_k; \theta_m) \\ y_k &= g(\hat{x}_k, u_k, \epsilon_k; \theta_m) \end{cases} \quad (2)$$

parametrized by a parameter vector θ_m. ϵ_k is the prediction error (white noise innovations input). Within the present framework dynamic output feedback controllers of the form

$$\begin{cases} z_{k+1} &= h(z_k, y_k, d_k; \theta_c) \\ u_k &= s(z_k, y_k, d_k; \theta_c) \end{cases} \quad (3)$$

are considered, parametrized by the vector θ_c. The state of the controller is $z_k \in \mathbb{R}^{n_z}$ and the reference input $d_k \in \mathbb{R}^l$. Parametrizations by multilayer perceptrons for $f(.)$ and $g(.)$ lead then to neural state space models

$$\mathcal{M}_0(\theta_m): \begin{cases} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K\epsilon_k \\ y_k &= C\hat{x}_k + Du_k + \epsilon_k \end{cases}$$

$$\mathcal{M}_1(\theta_m): \begin{cases} \hat{x}_{k+1} &= W_{AB}\sigma(V_A\hat{x}_k + V_Bu_k + \beta_{AB}) + K\epsilon_k \\ y_k &= C\hat{x}_k + Du_k + \epsilon_k \end{cases}$$

$$\mathcal{M}_2(\theta_m): \begin{cases} \hat{x}_{k+1} &= W_{AB}\sigma(V_A\hat{x}_k + V_Bu_k + \beta_{AB}) + K\epsilon_k \\ y_k &= W_{CD}\sigma(V_C\hat{x}_k + V_Du_k + \beta_{CD}) + \epsilon_k \end{cases} \quad (4)$$

Here W_* and V_* denote the interconnection matrices, with compatible dimensions and β_* are bias vectors. As activation function $\sigma(.)$ we take the hyperbolic tangent function tanh(.), which is applied elementwise (diagonal nonlinearity). The linear model in innovations form $\mathcal{M}_0$ is the *Kalman filter* (see [8]). Linear mappings are considered as neural nets without hidden layer and with a linear activation function. The matrix K is the Kalman gain. Parametrizations by multilayer feedforward neural networks make sense because of their universal approximation ability (see e.g. [4],[6]). The models $\mathcal{M}_i$ are recurrent. Given a training set of I/O data, prediction error learning algorithms can be applied to identify the models and are discussed in [18]. The gradient of the cost function is computed using Narendra's sensitivity model approach in dynamic backpropagation [19].

For the controller, we consider the following parametrizations for $h(.)$ and $s(.)$ by linear maps and multilayer perceptrons

$$\mathcal{C}_0(\theta_c): \begin{cases} z_{k+1} &= Ez_k + Fy_k + F_2d_k \\ u_k &= Gz_k + Hy_k + H_2d_k \end{cases}$$

$$\mathcal{C}_1(\theta_c): \begin{cases} z_{k+1} &= Ez_k + Fy_k + F_2d_k \\ u_k &= \sigma(Gz_k + Hy_k + H_2d_k) \end{cases}$$

$$\mathcal{C}_2(\theta_c): \begin{cases} z_{k+1} &= W_{EF}\sigma(V_Ez_k + V_Fy_k + V_{F_2}d_k + \beta_{EF}) \\ u_k &= W_{GH}\sigma(V_Gz_k + V_Hy_k + V_{H_2}d_k + \beta_{GH}) \end{cases}$$

$$\mathcal{C}_3(\theta_c): \begin{cases} z_{k+1} &= W_{EF}\sigma(V_Ez_k + V_Fy_k + V_{F_2}d_k + \beta_{EF}) \\ u_k &= \sigma(W_{GH}\sigma(V_Gz_k + V_Hy_k + V_{H_2}d_k + \beta_{GH})). \end{cases} \quad (5)$$

Again W_*, V_* denote interconnection matrices and β_* bias vectors. We take tanh(.) for $\sigma(.)$. The controller $\mathcal{C}_0$ is a classical linear dynamic output feedback controller. The reason why we consider linear models and linear controllers in this paper on neural control is to make a comparison possible in the sequel between the linear and the nonlinear case and to show in what sense the results in this paper can be considered as extensions towards linear H_∞ control and μ control. The control scheme for this emulator approach is presented in Fig.1. Like in modern control theory we will consider a standard plant configuration (see [1]). This standard plant is a reorganized scheme with exogenous input w_k (consisting of the reference input and disturbance signals and a constant due to the bias terms of the neural nets), regulated output e_k (consisting of $d_k - \hat{y}_k$ and possibly also other variables of interest), sensed output y_k and actuator input u_k. The *augmented plant* $\mathcal{S}_i$ has inputs w_k, u_k and output e_k, y_k (see Fig.1).

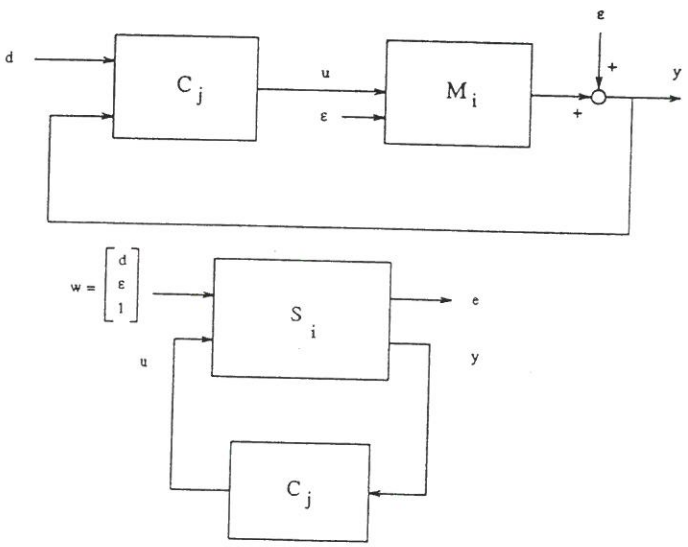


Figure 1: *Neural control scheme and standard plant.*

The following neural state space control problems Ξ_j^i in the Table correspond to one of the models $\mathcal{M}_i$ $(i = 0, 1, 2)$ connected to one of the controllers $\mathcal{C}_j$ $(j = 0, 1, 2, 3)$.

	$\mathcal{C}_0$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{M}_0$	Ξ_0^0	Ξ_1^0	-	-
$\mathcal{M}_1$	Ξ_0^1	Ξ_1^1	Ξ_2^1	Ξ_3^1
$\mathcal{M}_2$	Ξ_0^2	Ξ_1^2	Ξ_2^2	Ξ_3^2

3 NL_q systems

An essential concept in the derivation of stability criteria for the family of neural control problems Ξ_j^i is the so-called NL_q system (introduced in [19])

$$\begin{cases} x_{k+1} &= \Gamma_1(V_1\Gamma_2(V_2 ... \Gamma_q(V_q x_k + B_q v_k) + \\ & \quad ... + B_2 v_k) + B_1 v_k) \\ y_k &= \Lambda_1(W_1\Lambda_2(W_2 ... \Lambda_q(W_q x_k + D_q v_k) + \\ & \quad ... + D_2 v_k) + D_1 v_k) \end{cases} \quad (6)$$

where Γ_i, Λ_i $(i = 1, ..., q)$ are diagonal matrices with diagonal elements $\gamma_j(x_k, v_k)$, $\lambda_j(x_k, v_k) \in [0, 1]$, depending continuously on the state x_k and the input v_k (with respect to the standard plant, the state, input and output become p_k, w_k, e_k instead of x_k, v_k, y_k respectively). The matrices V_*, B_*, W_*, D_* are constant and have compatible dimensions. The term 'NL_q' refers to the alternating sequence of nonlinear and linear operators (q 'layers') acting on the input arguments x_k, v_k.

It can be shown then that all $\mathcal{M}_i$'s and Ξ_j^i's are special cases of NL_qs. For a proof we refer to [19]. Essential elements in the proof are that the nonlinearity $\sigma(.)$ belongs to sector $[0, 1]$ and the fact that for systems like e.g. the Hopfield network (with synchronous updating) $x_{k+1} = \sigma(Wx_k)$ can be written as $x_{k+1} = \Gamma(x_k)Wx_k$ with $\Gamma = \text{diag}\{\gamma_i\}$ and $\gamma_i = \sigma(w_i^T x_k)/(w_i^T x_k)$, which was shown before in [15]. Furthermore state augmentation is applied in order to bring the neural control systems in the NL_q form. We will illustrate this here for the problem $\boxed{\Xi_2^2}$: after defining additional state variables $\xi_k = \sigma(V_C \hat{x}_k)$ and $\eta_k = \sigma(V_G z_k)$ and supposing that $V_D = 0$, $V_H = 0$, $V_{H_2} = 0$, $\beta_{CD} = 0$, $\beta_{GH} = 0$ it is straightforward calculation to show that the state equation for the closed loop system can be written as

$$\begin{cases} \hat{x}_{k+1} &= W_{AB}\sigma(V_A\hat{x}_k + V_B W_G \eta_k + \beta_{AB}) + K\epsilon_k \\ z_{k+1} &= W_{EF}\sigma(V_E z_k + V_F W_C \xi_k + V_F \epsilon_k + V_{F_2} d_k + \\ & \quad \beta_{EF}) \\ \xi_{k+1} &= \sigma(V_C W_{AB}\sigma(V_A\hat{x}_k + V_B W_G \eta_k + \beta_{AB}) + \\ & \quad V_C K \epsilon_k) \\ \eta_{k+1} &= \sigma(V_G W_{EF}\sigma(V_E z_k + V_F W_{CD}\xi_k + V_F \epsilon_k + \\ & \quad V_{F_2} d_k + \beta_{EF})) \end{cases}$$

which is an NL_2 system

$$p_{k+1} = \Gamma_1(V_1\Gamma_2(V_2 p_k + B_2 w_k) + B_1 w_k)$$

with state $p_k = [\hat{x}_k; z_k; \xi_k; \eta_k]$ and exogenous input $w_k = [d_k; \epsilon_k; 1]$. Many other systems arising in neural networks, systems and control can be represented as NL_qs (see [19][20][21][22][23]). Among the examples of NL_qs are neural state space control problems, the Lur'e problem, Linear Fractional Transformations with real diagonal Δ block arising in robust control theory and μ theory, (multilayer) Hopfield nets, (generalized) CNNs, locally recurrent globally feedforward networks and others.

4 Stability criteria for NL_qs

4.1 Autonomous case

The following condition is sufficient for global asymptotic stability of an autonomous NL_q system (with unique equilibrium point $p = 0$).

Theorem 1 [Diagonal scaling]. A sufficient condition for global asymptotic stability of the autonomous NL_q

$$p_{k+1} = (\prod_{i=1}^{q} \Gamma_i(p_k)V_i)\, p_k$$

is to find diagonal matrices D_i such that

$$\|D_{tot} V_{tot} D_{tot}^{-1}\|_2^q = \beta_D < 1 \quad (7)$$

where

$$V_{tot} = \begin{bmatrix} 0 & V_2 & & & 0 \\ & 0 & V_3 & & \\ & & \ddots & & \\ & & & 0 & V_q \\ V_1 & & & & 0 \end{bmatrix}$$

- $V_i \in \mathbb{R}^{n_{h_i} \times n_{h_{i+1}}}$, $n_{h_1} = n_{h_{q+1}} = n_p$
- $D_{tot} = \text{diag}\{D_2, D_3, ..., D_q, D_1\}$, $D_i \in \mathbb{R}^{n_{h_i} \times n_{h_i}}$ diagonal matrices with nonzero diagonal elements.

The proof is based on the Lyapunov function $V(p) = \|D_1 p\|_2$ which is radially unbounded (hence the equilibrium point $p = 0$ is unique) and makes use of the properties of induced norms (see [19]). Finding such a matrix D_{tot} corresponds to solving a convex optimization problem [2]. The condition (7) is closely related to work in robust control theory (e.g. [7][12]). We should stress that the condition (7) is only sufficient.

Remarks

- Sharper criteria related to diagonal dominance instead of diagonal scaling are discussed in [19]. In that case 'full' instead of diagonal matrices D_i are allowed. The condition is of the form $c(P_{tot})\,\|P_{tot} V_{tot} P_{tot}^{-1}\|_2^q < 1$ where P_{tot} is a blockdiagonal matrix and $c(P_{tot})$ is a correction factor which is larger than 1 and depends on the degree of diagonal dominance of the matrix $P_{tot}^T P_{tot}$.
- Theorem 1 is only meaningful for models which are in itself stable. Otherwise no feasible point to (7) will be found. In the other case one may impose local stability for the linearized system at $p = 0$ (like in [17]) and minimize the correction factor $c(P_{tot})$ from the previous remark (see [19]).

4.2 I/O properties

An I/O equivalent NL_q representation for the standard plant, needed to formulate the next Theorem, is

$$\begin{bmatrix} p_{k+1} \\ e_k^{ext} \end{bmatrix} = (\prod_{i=1}^{q} \Omega_i(p_k, w_k) R_i) \begin{bmatrix} p_k \\ w_k \end{bmatrix} \quad (8)$$

with $\Omega_i = \text{diag}\{\Gamma_{i,e}, \Lambda_{i,e}, 0\}$ $(i = 1, ..., q)$ and $R_i = \text{blockdiag}\{M_i, N_i, 0\}$, $R_q = [M_q; N_q; 0]$ $(i = 1, ..., q-1)$ where $\Gamma_{1,e} = \Gamma_1$, $\Gamma_{i,e} = \text{diag}\{\Gamma_i, I\}$, $M_1 = [V_1\ B_1]$, $M_i = [V_i\ B_i; 0\ I]$, $\Lambda_{1,e} = \Lambda_1$, $\Lambda_{i,e} = \text{diag}\{\Lambda_i, I\}$, $N_1 = [W_1\ D_1]$, $N_i = [W_i\ D_i; 0\ I]$ $(i = 2, ..., q)$. Furthermore e_k^{ext} corresponds to e_k augmented with a number of zero elements in order to make $\prod_{i=1}^{q} R_i$ square. Note that $\|\Omega_i\| \leq 1$ because $\|\Gamma_i\| \leq 1$ and $\|\Lambda_i\| \leq 1$ (1,2 or ∞ norm). The following Theorem holds then

Theorem 2 [l_2 theory - Diagonal scaling]. Given the representation (8), if there exist matrices D_i such that

$$\|D_{tot} R_{tot} D_{tot}^{-1}\|_2^q = \beta_D < 1, \quad (9)$$

then the following holds

1. For a finite time horizon N:

$$\sum_{k=0}^{N-1} \|e_k\|_2^2 \leq \beta_D^2 \sum_{k=0}^{N-1} \|w_k\|_2^2 + r_0^2 \quad (10)$$

with $r_0 = \|D_1 p_0\|_2$.

2. For $N \to \infty$: There exist constants c_1, c_2 such that

$$c_2(1 - \beta_D^2)\|p\|_2^2 + \|e\|_2^2 \leq \beta_D^2 \|w\|_2^2 + c_1 \|p_0\|_2^2 \quad (11)$$

provided that $\{w_k\}_{k=0}^{\infty} \in l_2$.

Here

- $$R_{tot} = \begin{bmatrix} 0 & R_2 & & & 0 \\ & 0 & R_3 & & \\ & & \ddots & & \\ & & & 0 & R_q \\ R_1 & & & & 0 \end{bmatrix}$$
- $R_i \in \mathbb{R}^{n_{r_i} \times n_{r_{i+1}}}$, $n_{r_1} = n_{r_{q+1}} = n_p + n_w$
- $D_{tot} = \text{diag}\{D_2, D_3, ..., D_q, D_{S_1}\}$, $D_{S_1} = \text{diag}\{D_1, I_{n_w}\}$, $D_1 \in \mathbb{R}^{n_p \times n_p}$, $D_i \in \mathbb{R}^{n_{r_i} \times n_{r_i}}$ diagonal matrices with nonzero diagonal elements.

A proof is given in [19] and is closely related to the proof given by Packard & Doyle [12] for the state space upper bound test in μ theory. It can also be shown that the latter is a special case of the NL_q theorems under certain conditions (real diagonal uncertainty block). Like in the autonomous case conditions for the diagonal dominance case can also be derived here. Furthermore perturbed NL_qs (with parametric uncertainties on the interconnection matrices) can be considered: conditions for robust performance are derived in [19].

Remarks

- For the linear control system Ξ_0^0 with Kalman filter $\mathcal{M}_0$ and linear dynamic output feedback controller $\mathcal{C}_0$ the closed loop system is an NL_q with $q = 1$ and $\Omega_1 = I$ in (8). Theorem 2 reduces then to the H_∞ control problem with linear dynamic output feedback (see e.g. Packard & Doyle [12]).
- There is a close relationship between the internal stability criteria (autonomous case) and the property of finite L_2-gain in Theorem 2. This was already stated e.g. in the work of Hill & Moylan [5] and van der Schaft [24] and becomes clear through the concept of dissipativity. A dynamic system with input w_k and output e_k and state vector p_k is called *dissipative* if there exists a nonnegative function $V(p) : \mathbb{R}^{n_p} \to \mathbb{R}$ with $V(0) = 0$, called the *storage function*, such that $\forall w \in \mathbb{R}^{n_w}$ and $\forall k \geq 0$:

$$V(p_{k+1}) - V(p_k) \leq W(e_k, w_k)$$

where $W(e_k, w_k)$ is called the *supply rate* ([3]). The NL_q system is dissipative under the condition of Theorem 2, with storage function $V(p) = \|D_1 p\|_2^2$, supply rate $W(e_k, w_k) = \beta_D^2 \|w_k\|_2^2 - \|e_k\|_2^2$ and finite L_2-gain $\beta_D < 1$.

5 Control design

5.1 Tracking a specific reference input: dynamic backpropagation

Within the classical framework of neural control, learning to track a given specific reference input is possible using e.g. the dynamic backpropagation procedure of Narendra & Parthasarathy [9]. The sensitivity method can be used to compute the gradient of the following cost function

$$\min_{\theta_c} J(\theta_c) = \frac{1}{2N} \sum_{k=1}^{N} \{\|d_k - \hat{y}_k(\theta_c)]\|_2^2 + \lambda \|u_k(\theta_c)\|_2^2\} \quad (12)$$

where d_k is the specific reference input and $\hat{y}_k$, u_k are the output of the neural state space model and the controller respectively. θ_c is the parameter vector of the controller. Stability of the closed loop system is not guaranteed in this case and moreover it is not clear how the learned controller will generalize towards other reference inputs. In Theorem 2 on the other hand no specific inputs to the NL_q system are considered, but inputs w_k belonging to a class of functions (l_2 in this case). The idea is now precisely to combine the dynamic backpropagation paradigm with conditions from NL_q theory.

5.2 Analysis problem

By the analysis problem we mean checking stability of a given learned controller. With respect to Theorem 1 and 2 this means, given a fixed matrix Z_{tot} in

$$\|D_{tot} Z_{tot} D_{tot}^{-1}\|_2^q < 1 \tag{13}$$

with $Z_{tot} = V_{tot}$ (autonomous case) or $Z_{tot} = R_{tot}$ (I/O case) and D_{tot}=diag$\{D_2, D_3, ..., D_q, D_{S_1}\}$, $D_{S_1} = D_1$ (autonomous case) or D_{S_1} =diag$\{D_1, I\}$ (I/O case), find diagonal matrices D_i such that (13) holds. This means solving the LMI problem

$$Z_{tot}^T D_{tot}^2 Z_{tot} < D_{tot}^2 \tag{14}$$

which means $\lambda_{max}(Z_{tot}^T D_{tot}^2 Z_{tot} - D_{tot}^2) < 0$. LMIs (Linear Matrix Inequalities) are expressions of the form $M(x) < 0$ with $M = M^T$ with unknown $x \in \mathbb{R}^n$ and $M(x) = M_0 + \sum_{i=1}^m x_i M_i$ with $M_0 = M_0^T$, $M_i = M_i^T$ and play an important role in systems and control problems. An overview on this subject is given in the book [2]. Finding a feasible point to LMIs is a convex problem for which efficient polynomial-time algorithms exist [10].

5.3 Synthesis problem

For the controller synthesis problem, two type of problems can be considered

1. *Type 1 problem:*
   For the diagonal scaling case this corresponds to a feasibility problem in θ_c and D_{tot}

$$Z_{tot}(\theta_c)^T D_{tot}^2 Z_{tot}(\theta_c) < D_{tot}^2. \tag{15}$$

2. *Type 2 problem:*
   A modified dynamic BP procedure

$$\min_{\theta_c, D_i} J(\theta_c) \quad \text{such that} \quad M(\theta_c, D_i) < 0 \tag{16}$$

   in order to impose closed loop stability. Here $M(\theta_c, D_i) = Z_{tot}(\theta_c)^T D_{tot}^2 Z_{tot}(\theta_c) - D_{tot}^2$ and $J(\theta_c)$ corresponds to (12).

Methods for solving such optimization problems, involving singular value constraints are discussed by Polak & Wardi in [14]. For the Type 2 problem one basically needs to compute the gradient of the object function and the constraint. For the constraint one has a problem of the form

$$\min_{x \in \mathbb{R}^n} \lambda_{max}[Q(x)], \qquad Q = Q^T > 0 \tag{17}$$

where λ_{max} is the maximal eigenvalue of Q. $Q > 0$ denotes the matrix inequality. Some properties related to $\lambda_{max}(x)$ are that this function is Lipschitz continuous and differentiable whenever $\lambda_1(x) \neq \lambda_2(x)$. In that case the gradient is equal to

$$[\nabla \lambda_{max}[Q(x)]]_i = u_1(x)^T \frac{\partial Q(x)}{\partial x_i} u_1(x) \tag{18}$$

where $u_1(x)$ is the unit eigenvector corresponding to the largest eigenvalue $\lambda_1(x)$ of $Q(x)$. At the points where $\lambda_{max}(x)$ is not differentiable a generalized gradient must be considered

$$\partial \lambda_{max}[Q(x)] = co\{v \in \mathbb{C}^n \mid v_i = (Uz)^* \tfrac{\partial Q(x)}{\partial x_i}(Uz), \; z \in \mathbb{C}^{k(x)}, \|z\| = 1\} \tag{19}$$

where *co* denotes the convex hull of the set in $\{.\}$ and $k(x)$ is such that $\lambda_1(x) = \lambda_2(x) = ... = \lambda_{k(x)}(x) > \lambda_{k(x)+1}(x)$. Also note that the use of a generalized gradient doesn't lead necessarily to a descent direction like in differentiable problems.

One can exploit the fact that the Type 1 or 2 problem contains a *convex subproblem*. For a fixed value of θ_c, say θ_c^0, the 'inner' feasibility problem

$$M(\theta_c^0, D_i) < 0 \tag{20}$$

is convex. The 'outer' optimization problem is then

$$\min_{\theta_c} J(\theta_c) \quad \text{such that} \quad \lambda_{max}[M(\theta_c)] < 0. \tag{21}$$

The gradient of $J(\theta_c)$ follows from dynamic backpropagation and the gradient for the constraint at a given point θ_c^0 is given by

$$[\nabla \lambda_{max}[M(\theta_c^0)]]_i = u_1(\theta_c^0)^T \frac{\partial M(\theta_c^0, D_i^{(*)})}{\partial \theta_{c_i}^0} u_1(\theta_c^0) \tag{22}$$

with $u_1(\theta_c^0)$ the unit eigenvector corresponding to the largest eigenvalue $\lambda_1(\theta_c^0)$ of $M(\theta_c^0, D_i^{(*)})$ and $D_i^{(*)}$ the solution to the convex feasibility problem (20).

6 Conclusion

In this paper we derived sufficient conditions for global asymptotic stability and I/O stability with finite L_2-gain of neural control systems. Neural network models and controllers are in state space form. In order to derive the criteria, closed loop systems were formulated in standard plant form and transformed into so-called NL_q systems by means of state augmentation. Theorems of NL_q theory are closely related to H_∞ control and μ-theory (in fact an extension towards 'layered' systems, like recurrent neural networks). Criteria can be formulated as LMIs. Narendra's dynamic backpropagation procedure can be used in order to identify the neural state space models from I/O data and to let the neural state space controller track a specific reference input. Instead of specific reference inputs, one considers in NL_q theory inputs belonging to the class of functions l_2. This finally leads to a modification of Narendra's backpropagation algorithm in order to ensure desirable properties for the closed loop system such as internal and I/O stability. Because of space limitations the interested reader is referred to [19][22] for examples. Up

till now principles from NL_q theory have shown to be useful e.g. for stabilizing and controlling complex behaviour (including chaos) and controlling nonlinear distortion in electrodynamic loudspeakers (see [22]).

Hence a complete understanding of the behaviour of NL_q systems (such as necessary and sufficient conditions for global asymptotic stability; asymptotic stability with multiple equilibria; parametrization of all stabilizing controllers for NL_qs) would be important, could have an immediate impact on the theory of recurrent neural networks and systems and control theory and may lead to a unifying framework for analysis and design of complex nonlinear systems.

References

[1] Boyd S., Barratt C. (1991). *Linear controller design, limits of performance*, Prentice-Hall.

[2] Boyd S., El Ghaoui L., Feron E., Balakrishnan V. (1994). *Linear matrix inequalities in system and control theory*, SIAM (Studies in Applied Mathematics), Vol.15.

[3] Byrnes C.I., Lin W. (1994). Losslessness, feedback equivalence and the global stabilization of discrete-time nonlinear systems, *IEEE Transactions on Automatic Control*, Vol.39, No.1, pp.83-98.

[4] Funahashi K-I. (1989). On the Approximate Realization of continuous Mappings by Neural Networks, *Neural Networks*, Vol.2, pp.183-192.

[5] Hill D.J., Moylan P.J. (1980). Connections between finite-gain and asymptotic stability, *IEEE Transactions on Automatic Control*, Vol.AC-25, No.5, pp.931-936.

[6] Hornik K., Stinchcombe M., White H. (1989). Multilayer feedforward networks are universal approximators, *Neural Networks*, Vol.2, pp.359-366.

[7] Kaszkurewicz E., Bhaya A. (1993). Robust stability and diagonal Liapunov functions, *SIAM Journal on Matrix Analysis and Applications*, Vol.14, No.2, pp.508-520.

[8] Ljung L. (1979). Asymptotic behavior of the extended Kalman filter as a parameter estimator for linear systems, *IEEE Transactions on Automatic Control*, Vol.AC-24, No.1, pp.36-50.

[9] Narendra K.S., Parthasarathy K. (1991). Gradient methods for the optimization of dynamical systems containing neural networks, *IEEE Transactions on Neural Networks*, Vol.2, No.2, pp.252-262.

[10] Nesterov Y., Nemirovskii A. (1994). *Interior point polynomial algorithms in convex programming*, SIAM (Studies in Applied Mathematics), Vol.13.

[11] Nguyen D., Widrow B. (1990). Neural networks for self-learning control systems, *IEEE Control Systems Magazine*, 10(3), pp.18-23.

[12] Packard A., Doyle J. (1993). The complex structured singular value, *Automatica*, Vol.29, No.1, pp.71-109.

[13] Parisini T., Zoppoli R. (1994). Neural networks for feedback feedforward nonlinear control systems, *IEEE Transactions on Neural Networks*, Vol.5, No.3, pp.436-449.

[14] Polak E., Wardi Y. (1982). Nondifferentiable optimization algorithm for designing control systems having singular value inequalities, *Automatica*, Vol.18, No. 3, pp.267-283.

[15] Suykens J.A.K., De Moor B., Vandewalle J. (1993). Neural network models as linear systems with bounded uncertainty, applicable to robust controller design, *NOLTA 93 International Symposium on Nonlinear Theory and its Applications, Honolulu, Hawaii*, Vol.2, pp.419-422.

[16] Suykens J.A.K., De Moor B., Vandewalle J. (1993). Stabilizing neural controllers: a case study for swinging up a double inverted pendulum, *NOLTA 93 International Symposium on Nonlinear Theory and its Applications, Honolulu, Hawaii*, Vol.2, pp.411-414.

[17] Suykens J.A.K., De Moor B., Vandewalle J. (1994). Static and dynamic stabilizing neural controllers, applicable to transition between equilibrium points, *Neural Networks*, Vol.7, No.5, pp.819-831.

[18] Suykens J.A.K., De Moor B., Vandewalle J. (1994). Nonlinear system identification using neural state space models, applicable to robust control design, *International Journal of Control*, to appear.

[19] Suykens J.A.K., De Moor B., Vandewalle J. (1994). NL_q theory: a neural control framework with global asymptotic stability criteria, *K.U. Leuven, Department of Electrical Engineering, ESAT-SISTA, Technical report 94-83I*, submitted for publication.

[20] Suykens J.A.K., De Moor B., Vandewalle J. (1995). NL_q Theory: Unifications in the Theory of Neural Networks, Systems and Control, *European Symposium on Artificial Neural Networks (ESANN '95), Brussels Belgium*, April 1995.

[21] Suykens J.A.K., Vandewalle J. (1995). Generalized Cellular Neural Networks represented in the NL_q framework, *IEEE International Symposium on Circuits and Systems (ISCAS 95), Seattle, Washington, USA*, May 1995.

[22] Suykens J.A.K. (1995). *Artificial neural networks for modelling and control of nonlinear systems*, PhD Thesis, K.U.Leuven, Belgium, Dep. Electr. Eng., May 1995.

[23] Suykens J.A.K., Vandewalle J. (1995). NL_q theory: a unifying framework for analysis, design and applications of complex nonlinear systems, *NDES'95 - International Workshop on 'Nonlinear dynamics in electronic systems'*, Dublin Ireland, July 1995.

[24] van der Schaft A.J. (1992). L_2-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control, *IEEE Transactions on Automatic Control*, Vol.AC-37, pp.770-784.