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FRACTALS BY NUMBERS

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Abstract

We introduce an extension of an earlier defined simple, number-based matrix substitution system for obtaining fractal matrices, by considering cyclic substitutions. The elements of the resulting matrices are related to representations of their addresses in a mixed number base. The Hutchinson operator for the limit form of a geometrical representation of the fractal matrix is derived. It is shown that the class of fractal limit sets obtainable from cyclic substitutions does not extend the class obtainable from the simple substitutions.

1. INTRODUCTION

A fractal matrix (fractrix) has been defined earlier[1–4] as the matrix with integer elements which results in het limit from the recursion:

$$E_{m+1} = E_m \oplus E_1 \,. \tag{1}$$

E_1 is the *generator-matrix*: it is an *integer valued* matrix, generally of dimension p and size $\boldsymbol{\sigma} \triangleq (\sigma_1, \sigma_2, \ldots, \sigma_p)$, where σ_i is the matrix-size in the ith direction. Also, E_1 has at least one element that is 0. The operation $\oplus$ is the so-called *bigsum*-operation: let A and B be two matrices of the same dimension p and let $e_A(\mathbf{c})$ be the element of A located at position $\mathbf{c} = (c_1, c_2, \ldots, c_p)$, $c_i \in \{0, 1, \ldots, \sigma_i - 1\}$; then $e_A(\mathbf{c}) \oplus B$ is the matrix resulting from

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adding $e_A(\mathbf{c})$ to all elements of B, and $C = A \oplus B$ is the matrix obtained by replacing each element $e_A(\mathbf{c})$ in A by $e_A(\mathbf{c}) \oplus B$.

In this contribution, we expand Eq. (1) into the *q-cyclic* matrix recursion:

$$E_{ql+i} = E_{ql+i-1} \oplus M_i \,, \tag{2}$$

where $q \in \mathbb{N}_0$ is fixed; $l = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, q$; and where the generator-matrices $M_1, M_2, \ldots, M_q$ have the same dimension p and M_i has size $\boldsymbol{\sigma}_{M_i} = (\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots, \sigma_p^{(i)})$. Recursion Eq. (2) starts with $E_0 = [0]$, the matrix with unique 0-element. Observe that when all M_i are taken identical, then the cyclic recursion Eq. (2) reduces to the *simple* recursion shown in Eq. (1). Fractrices generated by Eq. (2) will be called *cyclic*.

In what follows, we will look at the fractal limit of certain geometrical representations of E_m for this cyclic recursion when $m \to \infty$; this is an extension of what was done before[3] for certain limit forms of Eq. (1). At first sight, we expect that the cyclic recursion will extend the class of limiting fractals that can be obtained by the simple recursion, but it will become clear that this is not the case.

As was shown in previous papers,[1,3] the elements in E_m resulting from the simple recursion could be explained in terms of *base-σ_i (or σ_i-ary, or σ_i-adic*[3]) representations of their coordinates. In Sec. 3, we will show how the E_m-elements in a cyclic recursion are connected with number representations in a mixed basis. These "mixary" and "cyclary" numbers as they will be called, will be introduced in Sec. 2. Section 4 will consider the geometrical fractal limit sets associated with $E_m(m \to \infty)$; the associated Hutchinson operators will be discussed in Sec. 5.

2. MIXARY AND CYCLARY NUMBER REPRESENTATIONS

Consider the real-line interval $[0, 1]$, and divide it into p_1 ($p_1 \in \mathbb{N}\backslash\{0, 1\}$) equal subintervals (see Fig. 1). Number each interval thus obtained in natural order from 0 to $p_1 - 1$, starting from the left. Select the interval with number a_1. In a second step, subdivide this selected interval into p_2 equal parts, which are again numbered in natural order from 0 to $p_2 - 1$, and pick out the interval with number a_2. Then the string $a_1 a_2$ completely specify the selected interval: it can be considered as the *address* of the selected interval. In a 3rd

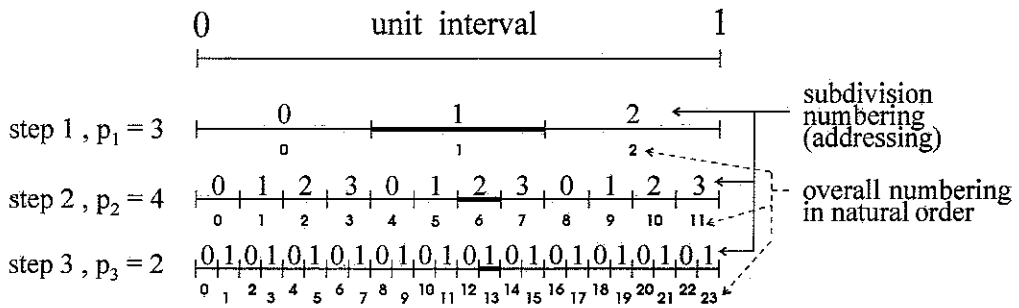


Fig. 1 Subdividing $[0, 1]$ in 3-steps, producing $D_{\langle 3,4,2\rangle}[0, 1]$. Subinterval number 13 in $D_{\langle 3,4,2\rangle}[0, 1]$ has address 121, which is the $\langle 3, 4, 2\rangle$-mixary representation of 13. The leftmost point of this interval has coordinate $0.1_3 2_4 1_2$ in $\langle 3, 4, 2\rangle$-mixary representation.

step, we once again partition this interval into p_3 equal parts, and number these from 0 to $p_3 - 1$. Select subinterval with number a_3: its address is then the string $a_1a_2a_3$. Such a subdividing, numbering and addressing procedure can be continued indefinitely. In general, after m steps, we will have divided the $[0,1]$-interval into $\pi_m \triangleq \Pi_{i=1}^{m} p_i$ equal intervals, each of which has a uniquely defined address formed by a connected string of m digits represented by:

$$a_1 a_2 __ a_m \,, \tag{3}$$

with $a_i \in \mathcal{Z}_{p_i} \triangleq \{0, 1, 2, \ldots, p_i - 1\}$, $p_i \in \mathbb{N}\backslash\{0, 1\}$.

Such an *m-level subdivision* of $[0,1]$ will be denoted by $D_{s_m}[0, 1]$, where $s_m \triangleq \langle p_1, p_2, \ldots, p_m\rangle$ is the *subdividing sequence.*

The π_m subintervals resulting after the mth dividing step can also be *numbered* in a natural order from 0 to $\pi_m - 1$. The subinterval with number n in $D_{s_m}[0, 1]$, has as address the string:

$$a_1^{(m)} a_2^{(m)} __ a_m^{(m)} \,,$$

where

$$a_m^{(m)} = n \bmod p_m$$

$$\text{and, for } 1 \le i < m : a_i^{(m)} = \lfloor \ldots \lfloor \lfloor n/p_m \rfloor / p_{m-1} \rfloor \ldots / p_{i+1} \rfloor \bmod p_i \,. \tag{4}$$

Conversely, the address $a_1a_2 __ a_m$ indicates interval number n in $D_{s_m}[0, 1]$, where:

$$n = a_1 \times p_2p_3 \ldots p_m + a_2 \times p_3p_4 \ldots p_m \ldots + a_{m-1} \times p_m + a_m \,. \tag{5}$$

As an example, in the $D_{\langle 3,7,4\rangle}[0, 1]$-division, interval number 81 has address 261. Interval number 81 in the $D_{\langle 3,7,4,2,5\rangle}[0, 1]$-division, has address 02001. For convenience, we will write:

$$\begin{aligned} 81 &= (261)_{\langle 3,7,4\rangle} \text{ or } 2_3 6_7 1_4 \\ \text{and} \quad 81 &= (02001)_{\langle 3,7,4,2,5\rangle} \text{ or } 0_3 2_7 0_4 0_2 1_5 \,. \end{aligned}$$

The rightmost expressions have the advantage that each digit bears the index of the corresponding subdivision.

Observe that when $p_i = s$ for all i, the address of interval number n in D_{s_m} is given by the common s-ary representation of n. For example, interval #19 in the 3-level division $D_{\langle 3,3,3\rangle}[0, 1]$ has address $(201)_{\langle 3,3,3\rangle}$; while in the 5-level division $D_{\langle 3,3,3,3,3\rangle}[0, 1]$, it has address $(00201)_{\langle 3,3,3,3,3\rangle}$. Both addresses correspond to the 3-ary representation of 19 written as a 3-digit and 5-digit string respectively.

The correspondence between n and its related address as given by Eqs. (4) and (5) can be interpreted as a natural extension of the common s-ary number representation, where the number-base now consists of a *mixture* of numbers $p_1, p_2, p_3, \ldots$. Therefore, the address of cell $\#n$ in $D_{s_m}[0, 1]$ will be called the *s_m-mixary* (or *$\langle p_1, p_2, \ldots, p_m\rangle$-mixary*) representation of n. (Number representations with a mixed basis were used in some ancient civilizations, like the Babylonian and the Mayan.[5])

Now consider the leftmost point of subinterval $\#n$ in $D_{s_m}[0, 1]$. Its coordinate in $[0, 1]$ is given by the rational number:

$$\frac{n}{\pi_m} \overset{(5)}{=} \frac{a_1}{p_1} + \frac{a_2}{p_1p_2} + \frac{a_3}{p_1p_2p_3} + \cdots \frac{a_m}{p_1p_2 \ldots p_m} \,, \tag{6}$$

where $a_1 a_2 __ a_m$ is the $\langle p_1 p_2, \ldots, p_m \rangle$-mixary representation of n. In extension to what is used in the common s-ary representation of numbers in the interval $[0, 1]$, we will write:

$$\frac{n}{\pi_m} = 0.a_1 a_2 __ a_m \triangleq \sum_{i=1}^{m} \frac{a_i}{\pi_i}. \tag{7}$$

For example, interval #81 in $D_{\langle 3,7,4,2,5 \rangle}[0, 1]$ starts at the point with coordinate 0.02001 in $\langle 3, 7, 4, 2, 5 \rangle$-mixary representation. We prefer to write this as $0.0_3 2_7 0_4 0_2 1_5$.

In the limit, when $m \to \infty$ with $s_\infty = \langle p_1, p_2, p_3, \ldots \rangle$, the subintervals in $D_{s_m}[0, 1]$ collapse into mere points with as coordinates:

$$0.a_1 a_2 a_3 \ldots \triangleq \lim_{m \to \infty} \sum_{i=1}^{m} \frac{a_i}{\pi_i}, \tag{8}$$

where the right-side limit is in general a real number (not necessarily rational), and the left-side expression its s_∞-mixary representation.

In what follows, we will particularly be interested in cases where the dividing sequence s_∞ is a *q-periodic* sequence $\langle p_1, p_2, \ldots, p_q, p_1, p_2, \ldots, p_q, p_1, p_2 \ldots \rangle$, which will be represented by $\langle \overline{p_1, p_2, \ldots, p_q} \rangle$. The corresponding subsequence for finite m will be represented by $s_m = \langle \overline{p_1, p_2, \ldots, p_q}; m \rangle$. For example, with $s_\infty = \langle \overline{2, 3, 5} \rangle$, we have $s_1 = \langle \overline{2, 3, 5}; 1 \rangle = \langle 2 \rangle$, $s_5 = \langle \overline{2, 3, 5}; 5 \rangle = \langle 2, 3, 5, 2, 3 \rangle$, etcetera. Mixary number representations based on a q-periodic (cyclic) dividing sequence $\langle \overline{p_1, p_2, \ldots, p_q} \rangle$, will be called $\langle \overline{p_1, p_2, \ldots, p_q} \rangle$*-cyclary.*

Observe that there are real numbers with two different cyclary representations, just as it may be the case for s-ary representations. For example, $\frac{19}{30} = 0.1_3 1_2 3_5 \overline{2_3 1_2 4_5} = 0.1_3 1_2 4_5 \overline{0_3 0_2 0_5}$, where the overbarred digit-string is repeated endlessly.

3. NUMBER-THEORETIC INTERPRETATION AND GRAPHICAL REPRESENTATION OF CYCLIC FRACTRICES

We now return to the cyclic recursion Eq. (2), while restricting ourselves to matrices of dimension $p = 2$ (at most), and to the 2-cyclic case (extensions to the other cases being rather trivial). So, we consider:

$$\begin{aligned} E_{2l+1} &= E_{2l} \oplus A \\ E_{2l+2} &= E_{2l+1} \oplus B, \end{aligned} \tag{9}$$

where A and B are two-dimensional integer matrices of size $\boldsymbol{\sigma}_A = (r_A, c_A)$ and $\boldsymbol{\sigma}_B = (r_B, c_B)$ respectively (r and c referring to rows and columns respectively). $e_A(i, j)$, with $0 \le i \le r_A - 1$ and $0 \le j \le c_A - 1$ denotes the element at row i and column j in A.

The cyclic recursion Eq. (9) generates the matrix sequence $E_1 = A$, $E_2 = A \oplus B$, $E_3 = A \oplus B \oplus A$, $E_4 = A \oplus B \oplus A \oplus B$, etcetera. E_m has size

$$\boldsymbol{\sigma}_m = (\pi_{m,r}, \pi_{m,c}) = (\underbrace{r_A r_B r_A r_B \ldots r_*}_{m \text{ factors}}, \underbrace{c_A c_B c_A c_B \ldots c_*}_{m \text{ factors}}),$$

where the *-index of the rightmost factors is either "A" or "B", dependent on the value of m (here: A when m is odd, else B). Rows and columns in E_m are numbered in natural order from 0 to $(\pi_{m,r} - 1)$ and from 0 to $(\pi_{m,c} - 1)$ respectively.

Here follows a property which is an extension of the fundamental decomposition property for simple fractrices (Property 2.3[1]).

Property. *Let $e_m(r, c)$ denote the element at row r and column c in E_m. Let $r_1 r_2 __ r_m$ be the $\langle \overline{r_A, r_B}; m\rangle$-cyclary representation of r and $c_1 c_2 __ c_m$ the $\langle \overline{c_A, c_B}; m\rangle$-cyclary representation of c. Then*

$$e_m(r, c) = e_A(r_1, c_1) + e_B(r_2, c_2) + e_A(r_3, c_3) + e_B(r_4, c_4) + \cdots + e_*(r_m, c_m)\,. \qquad (10)$$

$(r_1 r_2 __ r_m, c_1 c_2 __ c_m)$ will be called the $\langle \overline{\sigma_A, \sigma_B}; m\rangle$-cyclary representation of (r, c). $e_m(r, c)$ as given by Eq. (10) is the weight of this representation, and will sometimes be indicated by $w(r_1 r_2 __ r_m, c_1 c_2 __ c_m)$.

Example. Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, so $\sigma_A = (r_A, c_A) = (2, 3)$ and $\sigma_B = (r_B, c_B) = (2, 1)$ (observe that we interprete B to be 2-dimensional, although it is strictly 1-dimensional: such extensions of the dimensionality can always be made by giving size 1 to the directions in which the matrix does not actually extend). Then $e_5(23, 25) = e_A(0, 2) + e_B(1, 0) + e_A(1, 2) + e_B(2, 0) + e_A(1, 1) = -2$, because

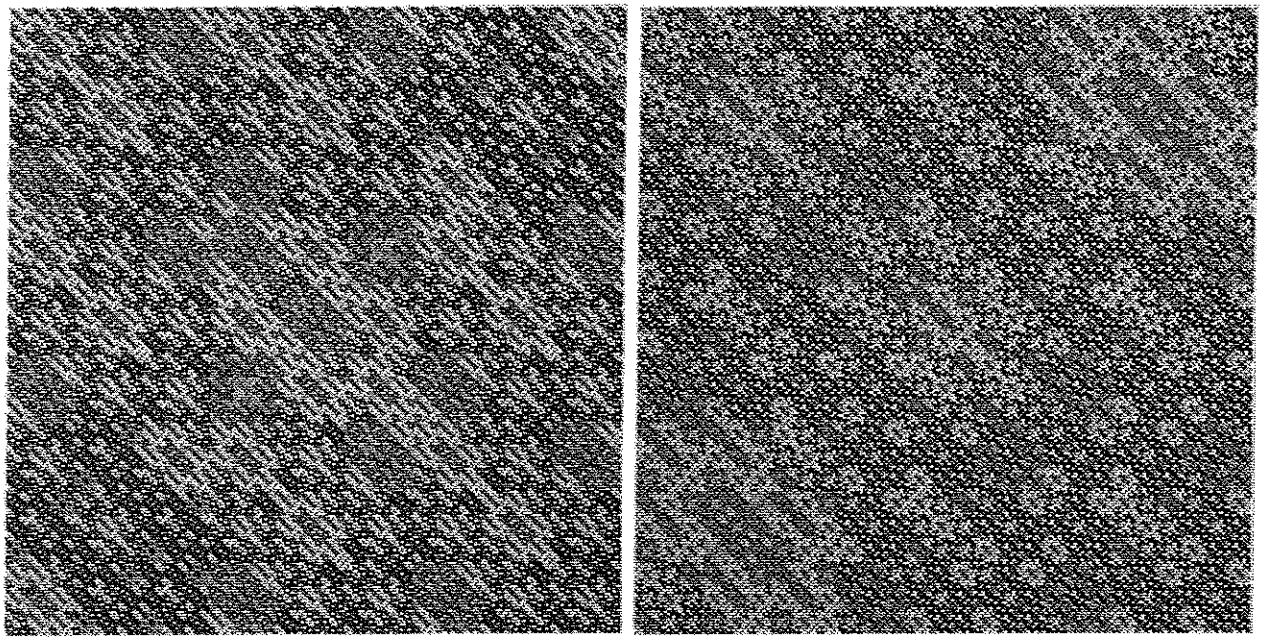
$$23 \text{ has } \langle \overline{2, 3}; 5\rangle\text{-cyclary representation } 0_2 1_3 1_2 2_3 1_2$$

$$25 \text{ has } \langle \overline{3, 1}; 5\rangle\text{-cyclary representation } 2_3 0_1 2_3 0_1 1_3\,.$$

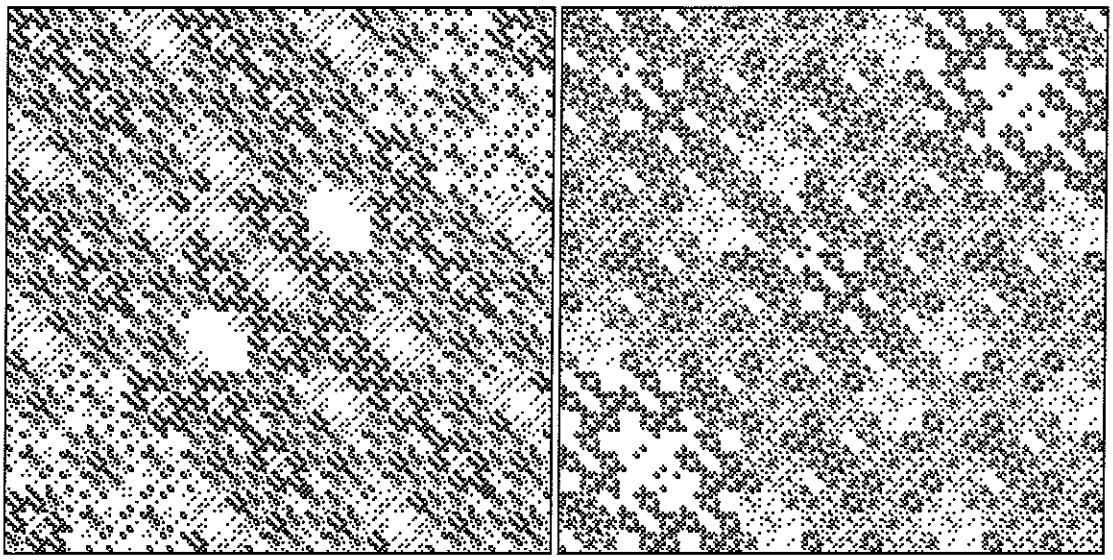
We will now describe how to associate a geometrical object with E_m that is confined to the unit square $[0, 1]^2$ (the point $(0, 0)$ being the top-left point). For that purpose, divide $[0, 1]^2$ in equally sized cells by subdividing it in vertical (row-)direction according to the $\langle \overline{r_A, r_B}; m\rangle$-dividing sequence, and in horizontal (column-)direction according to the $\langle \overline{c_A, c_B}; m\rangle$-dividing sequence. If we put in each cell at location (r, c) its corresponding weight as given by Eq. (10), we obtain the matrix E_m. But we can also attribute a color to each point of the half-open cell $[\frac{r}{\pi_{m,r}}, \frac{r+1}{\pi_{m,r}}) \times [\frac{c}{\pi_{m,c}}, \frac{c+1}{\pi_{m,c}})$, in 1-1 correspondence with the number associated with that cell. In that way, we obtain a colored geometrical object $G\hat{\mathcal{L}}(E_m)$ confined to the unit square. $G\hat{\mathcal{L}}_k(E_m)$ will represent the subset of all points of $G\hat{\mathcal{L}}(E_m)$ that have the color associated with weight k (or: that map into k): it is called the *k-indicator*[3] set of E_m. A few examples of $G\hat{\mathcal{L}}(E_m)$ and $G\hat{\mathcal{L}}_k(E_m)$ are given in Fig. 2. $G\mathcal{L}_k(E_m)$ is the *restricted* k-indicator set: it is the set of top-left points of the half-open intervals belonging to $G\hat{\mathcal{L}}_k(E_m)$, i.e., the set of points with coordinate $(\frac{r}{\pi_{m,r}}, \frac{c}{\pi_{m,c}})$ whose $\langle \overline{\sigma_A, \sigma_B}; m\rangle$-cyclary representation has weight k as given by Eq. (10):

$$G\mathcal{L}_k(E_m) = \{(0.r_1 r_2 __ r_m, 0.c_1 c_2 __ c_m) | e_A(r_1, c_1) + e_B(r_2, c_2) + \cdots + e_*(r_m, c_m) = k\}\,.$$

In what follows, we will consider the related limit sets.



(a) (b)



(c) (d)

Fig. 2 Graphical representation $G\hat{\mathcal{L}}(E_5)$ for a 2-cyclic fractal matrix. **(a)** $E_5 = A \oplus B \oplus A \oplus B \oplus A$; **(b)** $E_5 = B \oplus A \oplus B \oplus A \oplus B$. $\left(A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right.$; the color code ranges from left to right from -5 to $9\Big)$. **(c)** and **(d)**: the corresponding 1-indicator sets $G\hat{\mathcal{L}}_1(E_5)$ of (a) and (b) respectively.

4. LIMIT SETS

For simplicity, we shall now restrict to one-dimensional matrices (the higher dimensional cases are notationally and graphically a bit more involved, but conceptually very much the

same). So, let the generators be the integer matrices:

$$A = [e_A(0)\, e_A(1) \cdots e_A(\alpha-1)] \quad \text{and} \quad B = [e_B(0)\, e_B(1) \cdots e_B(\beta-1)]\,, \tag{11}$$

with at least one of the elements in both A and B equal to 0. E_m, as given by Eq. (9) has then $\pi_m = \alpha^{\frac{m}{2}}\beta^{\frac{m}{2}}$ elements when m is even, and $\pi_m = \alpha^{\frac{m+1}{2}}\beta^{\frac{m-1}{2}}$ elements when m is odd. According to Eq. (10), element number n in E_m has value:

$$e_m(n) = e_A(c_1) + e_B(c_2) + e_A(c_3) + e_B(c_4) + \cdots e_*(c_m)\,, \tag{12}$$

where $c_1c_2 __ c_m$ is the $\langle\overline{\alpha,\,\beta};\, m\rangle$-cyclary representation of n (also to be denoted, when convenient, by $(c_1)_\alpha(c_2)_\beta __ (c_m)_*$, with $(c_i)_\alpha \in \mathcal{Z}_\alpha \triangleq \{0,\, 1, \ldots,\, \alpha-1\}$ and $(c_i)_\beta \in \mathcal{Z}_\beta$). Then,

$$G\mathcal{L}_k(E_m) = \{0.(c_1)_\alpha(c_2)_\beta __ (c_m)_* | e_A(c_1) + e_B(c_2) \cdots + e_*(c_m) = k\}\,. \tag{13}$$

When $m \to \infty$, the *straightforward extension* of this set is:

$$\mathcal{F}_k^{\overline{A,B}} \triangleq \{0.(c_1)_\alpha(c_2)_\beta(c_3)_\alpha \cdots | \lim_{m\to\infty}[e_A(c_1) + e_B(c_2) \cdots + e_*(c_m)] = k\}\,. \tag{14}$$

Notation. Let $0.(c_1)_\alpha(c_2)_\beta(c_3)_\alpha \ldots$ be a real $\langle\overline{\alpha,\,\beta}\rangle$-cyclary number. Then $0.c\rfloor_i^{\overline{\alpha,\,\beta}}$ denotes the number $0.(c_1)_\alpha(c_2)_\beta __ (c_i)_*$. The symbol $\tilde{q}_{m_i}^{(i)}$ denotes a string $q_1^{(i)}q_2^{(i)} __ q_{m_i}^{(i)}$ of m_i digits, where i indicates an ith item. Then $0.c\rfloor_i^{\overline{\alpha,\,\beta}}\tilde{q}_{m_i}^{(i)}$ is the $\langle\overline{\alpha,\,\beta};\, i+m_i\rangle$-cyclary number $0.(c_1)_\alpha(c_2)_\beta(c_3)_\alpha __ (c_i)_* q_1^{(i)}q_2^{(i)} __ q_{m_i}^{(i)}$. Hereby, $q_{m_i}^{(i)}$ should continue the $(\alpha,\,\beta)$-alternation of $0.c\rfloor_i^{\overline{\alpha,\,\beta}}$ in the proper way. The *weight* of this concatenated number is then:

$$w(0.c\rfloor_i^{\overline{\alpha,\,\beta}}\tilde{q}_{m_i}^{(i)}) = e_A(c_1) + e_B(c_2) \cdots + e_*(c_i) + w(\tilde{q}_{m_i}^{(i)})\,,$$

$$\text{with} \quad w(\tilde{q}_{m_i}^{(i)}) = e_{\overline{*}}(q_1^{(i)}) + e_*(q_2^{(i)}) + e_{\overline{*}}(q_3^{(i)}) + \cdots + e_{*'}(q_{m_i}^{(i)})\,,$$

where $\overline{*} = A$ when $* = B$, and $\overline{*} = B$ when $* = A$ ($*'$, just like $*$, is either A or B, in proper termination of the sequence involved).

Following the arguments in an earlier paper,[3] $\mathcal{F}_k^{\overline{A,B}}$ is *not* the complete set of points that in the limit map into k. The complete limit set is:

$$\mathcal{G}_k^{\overline{A,B}} \triangleq \{0.(c_1)_\alpha(c_2)_\beta(c_3)_\alpha \ldots | \text{ there is a sequence } \langle 0.c\rfloor_i^{\overline{\alpha,\,\beta}}\tilde{q}_{m_i}^{(i)}\rangle_{i\in\mathbb{N}_0}$$
$$\text{whose weight-sequence } \langle w(0.c\rfloor_i^{\overline{\alpha,\,\beta}}\tilde{q}_{m_i}^{(i)})\rangle_{i\in\mathbb{N}_0} \text{ converges to } k\}\,. \tag{15}$$

Observe that $\mathcal{F}_k^{\overline{A,B}}$ as given by Eq. (14) satisfies $\mathcal{F}_k^{\overline{A,B}} \subseteq \mathcal{G}_k^{\overline{A,B}}$, $\mathcal{F}_k^{\overline{A,B}}$ also being the set of points satisfying Eq. (15) under the restriction that all $\tilde{q}_{m_i}^{(i)}$ are empty strings.

5. HUTCHINSON OPERATORS FOR $\mathcal{F}_k^{\overline{A,B}}$ AND $\mathcal{G}_k^{\overline{A,B}}$

Let $\mathcal{P}^{\overline{\alpha,\,\beta}}$ be a set of $\langle\overline{\alpha,\,\beta}\rangle$-cyclary real numbers (points) in $[0,1]$, and let $x = 0.(c_1)_\alpha(c_2)_\beta(c_3)_\alpha \ldots \in \mathcal{P}^{\overline{\alpha,\,\beta}}$. Then $0.b_\beta x$ represents the $\langle\overline{\beta,\,\alpha}\rangle$-cyclary number $0.b_\beta(c_1)_\alpha(c_2)_\beta(c_3)_\alpha \ldots$ (notice the reversal of α and β), and

$$0.b_\beta\mathcal{P}^{\overline{\alpha,\,\beta}} \triangleq \{0.b_\beta x | x \in \mathcal{P}^{\overline{\alpha,\,\beta}}\}\,. \tag{16}$$

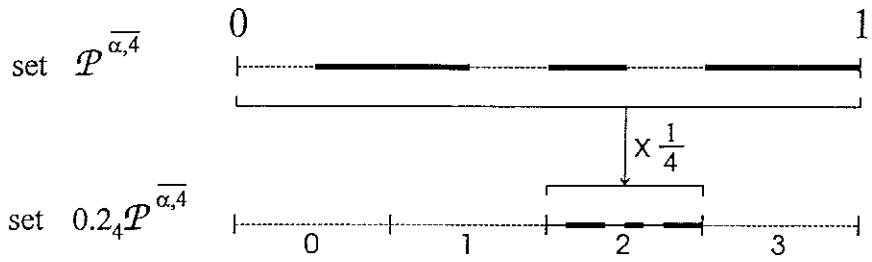


Fig. 3 Relationship between the point sets $\mathcal{P}^{\overline{\alpha,\beta}}$ and $0.b_\beta\mathcal{P}^{\overline{\alpha,\beta}} = \frac{b}{\beta} + \frac{1}{\beta}\mathcal{P}^{\overline{\alpha,\beta}}$ (here $\beta = 4$, $b = 2$). The last is an affine contractive mapping of the first.

This set consists of a reduced copy of $\mathcal{P}^{\overline{\alpha,\beta}}$ (reduction factor β), which is placed in subinterval number b_β of the β-division of $[0,1]$ (see Fig. 3). It can also be represented by:

$$0.b_\beta\mathcal{P}^{\overline{\alpha,\beta}} = \frac{b}{\beta} + \frac{1}{\beta}\mathcal{P}^{\overline{\alpha,\beta}}. \tag{17}$$

The numbers (points) $0.b_\beta x$ are $\langle\overline{\beta,\ \alpha}\rangle$-cyclary numbers, and they are to be associated with the matrix recursion of Eq. (9), but with A and B swapped. $\mathcal{F}_k^{\overline{B,A}}$ and $\mathcal{G}_k^{\overline{B,A}}$ are the limiting indicator sets as defined by Eqs. (14) and (15), but now for the A, B-switched case.

Now, let $Q_k^{\overline{A,B}}$ be either $\mathcal{F}_k^{\overline{A,B}}$ or $\mathcal{G}_k^{\overline{A,B}}$ (but fixed in the context), then all points of $0.b_\beta Q_k^{\overline{A,B}}$ map into $e_B(b) + k$, or equivalently: $0.b_\beta Q_{k-e_B(b)}^{\overline{A,B}}$ maps into k. Taking this into account gives the following *joint Hutchinson operator* for $Q_k^{\overline{A,B}}$ and $Q_k^{\overline{B,A}}$ (in extension to what was found in earlier[3] for the simple recursion):

$$\begin{cases} Q_k^{\overline{A,B}} = \bigcup_{a=0}^{\alpha-1} 0.aQ_{k-e_A(a)}^{\overline{B,A}} \\ Q_k^{\overline{B,A}} = \bigcup_{b=0}^{\beta-1} 0.bQ_{k-e_B(b)}^{\overline{A,B}}, \end{cases} \tag{18}$$

where $Q_l^{\overline{A,B}} = \phi$ and $Q_l^{\overline{B,A}} = \phi$ (ϕ being the empty set) whenever l does not belong to the range of integers generated in the E_m-recursion Eq. (9).

Example. Let $A = [0\ 1\ 0]$ and $B = [1\ 0\ 2\ 1]$. The range of the elements in E_m when $m \to \infty$ is $\mathbb{N}$. So $Q_k^{\overline{A,B}} = Q_k^{\overline{B,A}} = \phi$ for $k < 0$, while these sets are nonempty for all $k \geq 0$. The associated Hutchinson operator as given by Eq. (18) becomes:

$$\forall k \in \mathbb{N}: \begin{cases} Q_k^{\overline{A,B}} = 0.0_3Q_k^{\overline{B,A}} \cup 0.1_3Q_{k-1}^{\overline{B,A}} \cup 0.2_3Q_k^{\overline{B,A}} \\ Q_k^{\overline{B,A}} = 0.0_4Q_{k-1}^{\overline{A,B}} \cup 0.1_4Q_k^{\overline{A,B}} \cup 0.2_4Q_{k-2}^{\overline{A,B}} \cup 0.3_4Q_{k-1}^{\overline{A,B}} \end{cases}. \tag{19}$$

A graphical representation as a multiple reduction copy machine[6] is given in Fig. 4(a).

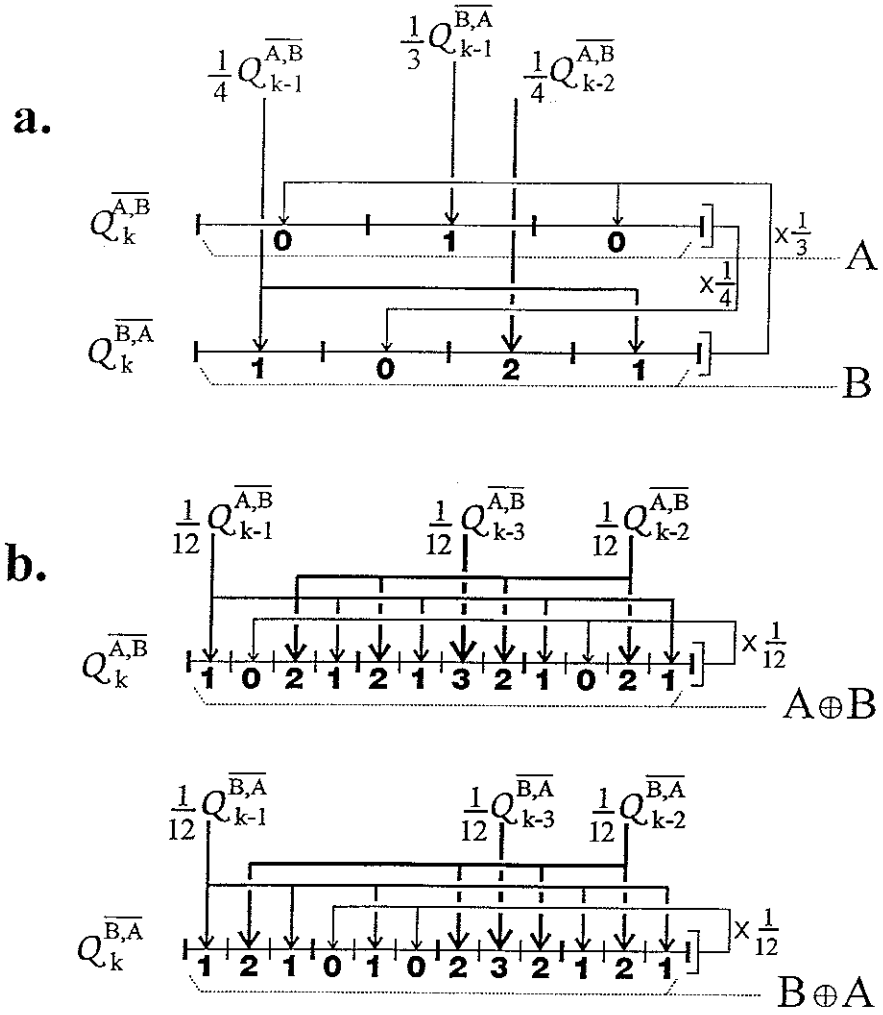


Fig. 4 (a) Graphical representation of the collage relationship between the $Q_k^{\overline{A,B}}$ and $Q_k^{\overline{B,A}}$-sets for $A = [0\,1\,0]$ and $B = [1\,0\,2\,1]$. (b) Decoupling of $Q_k^{\overline{A,B}}$ and $Q_k^{\overline{B,A}}$.

As each $Q_k^{\overline{A,B}}$ contains copies of $Q_j^{\overline{A,B}}$ and $Q_j^{\overline{B,A}}$ for all $j < k$, this produces an hierarchical chain of Q_k-sets of ever increasing complexity as k increases. What is the nature of these sets?

By definition [Eq. (14)], $\mathcal{F}_0^{\overline{A,B}}$ is the set of all points in $[0,1]$ whose $\langle\overline{3,4}\rangle$-cyclary representation has *either* a 0 *or* 2 $\in \mathcal{Z}_3$ at all *odd* positions and a 1 $\in \mathcal{Z}_4$ at all *even* positions. This is denoted by $\mathcal{F}_0^{\overline{A,B}} = \left\{0.\binom{0}{2}_3 1_4 \binom{0}{2}_3 1_4 \binom{0}{2}_3 \ldots\right\}$. In a similar way, $\mathcal{F}_0^{\overline{B,A}} = \left\{0.1_4 \binom{0}{2}_3 1_4 \binom{0}{2}_3 \ldots\right\}$. Both $\mathcal{F}_0^{\overline{A,B}}$ and $\mathcal{F}_0^{\overline{B,A}}$ are *perfect Cantor sets.* As the matrix elements are nonnegative, it follows from the definition Eq. (15), that $\mathcal{G}_0^{\overline{A,B}} = \mathcal{F}_0^{\overline{A,B}}$ and $\mathcal{G}_0^{\overline{B,A}} = \mathcal{F}_0^{\overline{B,A}}$.

$\mathcal{F}_1^{\overline{A,B}}$ is clearly the set of all points whose $\langle\overline{3,4}\rangle$-cyclary representation has 0 *or* 2 at all odd positions and a 1 at all even positions, except for *just one* arbitrary position where it should be a 0 *or* 3 when this position is even, and a 1 when this position is odd. A similar characterization can be found for $\mathcal{F}_1^{\overline{B,A}}$.

What about $\mathcal{G}_1^{\overline{A,B}}$? It certainly contains $\mathcal{F}_1^{\overline{A,B}}$, but also all limits of sequences $\left\langle 0.1_4\tilde{q}_{m_1}^{(1)}, 0.1_4\binom{0}{2}_3\tilde{q}_{m_2}^{(2)}, 0.1_4\binom{0}{2}_3 1_4\tilde{q}_{m_3}^{(3)}, 0.1_4\binom{0}{2}_3 1_4\binom{0}{2}_3\tilde{q}_{m_4}^{(4)}\ldots\right\rangle$ for which $w(\tilde{q}_{m_i}^{(i)}) = 1$, i.e., all points of the form $0.1_4\binom{0}{2}_3 1_4\binom{0}{2}_3\ldots$. Now this is just the set $\mathcal{F}_0^{\overline{A,B}}$ obtained above, and so $\mathcal{F}_0^{\overline{A,B}} \cup \mathcal{F}_1^{\overline{A,B}} \subseteq \mathcal{G}_1^{\overline{A,B}}$. Actually, it is possible to use the results from an earlier paper[3] to prove that $\mathcal{G}_1^{\overline{A,B}} = \mathcal{F}_0^{\overline{A,B}} \cup \mathcal{F}_1^{\overline{A,B}}$ (a similar relationship holds for the $\overline{B,A}$-case), by considering what follows.

The combination of both equations in Eq. (18) produces:

$$\begin{aligned} Q_k^{\overline{A,B}} &= \bigcup_{b=0}^{\beta-1}\bigcup_{a=0}^{\alpha-1} 0.a_\alpha b_\beta Q_{k-e_A(a)-e_B(b)}^{\overline{A,B}} \\ Q_k^{\overline{B,A}} &= \bigcup_{b=0}^{\beta-1}\bigcup_{a=0}^{\alpha-1} 0.b_\beta a_\alpha Q_{k-e_A(a)-e_B(b)}^{\overline{B,A}}, \end{aligned} \tag{21}$$

where $0.a_\alpha b_\beta \mathcal{P}^{\overline{A,B}}$ places a $\frac{1}{\alpha\beta}$-copy of $\mathcal{P}^{\overline{A,B}}$ in the subinterval with address $a_\alpha b_\beta$ in the $\langle\alpha,\ \beta\rangle$-subdivision of $[0,1]$ (and $0.b_\beta a_\alpha \mathcal{P}^{\overline{B,A}}$ places a $\frac{1}{\alpha\beta}$-copy of $\mathcal{P}^{\overline{B,A}}$ in the subinterval with address $b_\beta a_\alpha$ in the $\langle\beta,\ \alpha\rangle$-subdivision of $[0,1]$). This is illustrated for the same example as above in Fig. 4(b).

Observe that both equations in Eq. (20) are decoupled, although $Q_k^{\overline{A,B}}$ and $Q_k^{\overline{B,A}}$ are not independent. This equation makes it clear that $Q_k^{\overline{A,B}}$ (or $Q_k^{\overline{B,A}}$) are also the limit forms obtained from the simple recursion:

$$E'_{n+1} = E'_n \oplus E',$$

where the generator E'_1 is now given by $A \oplus B$ (or $B \oplus A$, respectively). Step n in this last recursion corresponds to step $2n$ in Eq. (9). This means that, as far as the limits $\mathcal{F}_k^{\overline{A,B}}$ and $\mathcal{G}_k^{\overline{A,B}}$ are concerned, nothing new can emerge from these 2-cyclic recursions as compared with what can be obtained from the simple recursive schemes. It should be clear now that this is also the case for general q-cyclic recursions. It also means that the whole theory concerning the limit forms (and corresponding iterated function systems, and fractal dimensions) as developed earlier,[3] can be applied to the case of q-cyclic recursions by taking the generator to be $M_1 \oplus M_2 \oplus \cdots \oplus M_q$ [referring to Eq. (2)]. More in particular, we recall that:

(a) $\lim_{m\to\infty} G\hat{\mathcal{L}}_k(E_m) = \lim_{m\to\infty} G\mathcal{L}_k(E_m) = \mathcal{G}_k^{\overline{M_1,M_2,\ldots,M_q}}$ = closure of $\mathcal{F}_k^{\overline{M_1,M_2,\ldots,M_q}}$,

(b) $\mathcal{G}_k^{\overline{M_1,M_2,\ldots,M_q}} = \bigcup_{j\le k} \mathcal{F}_j^{\overline{M_1,M_2,\ldots,M_q}}$ when all M_i are nonnegative. When there is only one 0 in $M_1 \oplus M_2 \oplus \cdots \oplus M_q$, the $\mathcal{F}_k$ and $\mathcal{G}_k$ form so called *hierarchical Zeno-sets*, with fractal dimension 0. With more than one 0, the resulting $\mathcal{F}_k$ and $\mathcal{G}_k$ are *hierarchical Cantor sets*, all with the same fractal dimension (the unit interval). When one of the M_i has elements of both signs, then $\mathcal{G}_k^{\overline{M_1,M_2,\ldots,M_q}} = [0,\ 1]$ (the unit interval).

(c) $\mathcal{G}_k$ is also the attractor of the hierarchical iterated function system based on the corresponding Hutchinson operator.

In analogy with the 2-cyclic Eq. (18), the q-cyclic case will also show an interdependency of all $Q_k^{\overline{C(M_1,M_2,\ldots,M_q)}}$, where $\overline{C(M_1,M_2,\ldots,M_q)}$ denotes any cyclic permutation of $M_1,\ M_2,\ldots,\ M_q$.

It also follows that there is a true challenge left in characterizing the $\mathcal{F}_k$ and $\mathcal{G}_k$-sets for nonperiodic extensions of the matrix recursion Eq. (2). In the example above, this would mean that the alternation of A and B occurs in a nonperiodic way. Are the resulting limit sets conceivable? Clearly, the associated Hutchinson-operators and iterated function systems are not.

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