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SHIFT-ADD CORRELATION PATTERNS OF ORBITS OF ONE-DIMENSIONAL LINEAR CELLULAR AUTOMATA

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We consider the two-dimensional state-time evolution pattern (orbit) of a one-dimensional linear cellular automaton (CA) defined over a finite commutative ring $\mathcal{R}$, and correlate this to shifted versions of itself by adding the original pattern and its shifted versions. The resulting pattern is called the *shift-add (correlation) pattern*. A *dual* counterpart of this pattern is also introduced: it forms a so-called *pseudo*-CA orbit. We show that the set of shift-add patterns for different shifts interrelate like a pseudo-CA whose states are two-dimensional patterns over $\mathcal{R}$. The set of dual shift-add patterns forms likewise a regular CA. Complementary ways of viewing the shift-add correlation space are also presented. Finally, the nature of shift-add correlation patterns for the newly defined class of pseudo-CA is investigated.

1. Introduction

Shift-add correlation patterns (SAP's) were introduced in [Barbé *et al.*, 1995], in order to detect hidden structures in coarse-graining invariant orbits of binary cellular automata (CA) with a linear rule in $\mathbb{Z}_2$ (the field of integers modulo 2). For that $\mathbb{Z}_2$-case, the SAP coincides with the *shift-identity pattern* that indicates where the CA's orbit and a translated version of it display identical values [Barbé, 1995]. This paper generalizes the SAP-concept for linear CA defined over a finite commutative ring $\mathcal{R}$ with multiplicative unity. $\mathcal{R}^*$ denotes the set of nonzero elements in $\mathcal{R}$. [As finite (Galois) fields form a subset of these rings, everything that follows also holds for linear CA over finite fields GF(p^n) (p prime). When $n = 1$, these fields become the sets of integers modulo p, with addition and multiplication modulo p; for example, the $\mathbb{Z}_2$-case mentioned above forms such a field].

After a few notational conventions and a review of some basic relationships for linear $\mathcal{R}$-CA, this section defines the *complete shift-add correlation function* and several of its derivatives: SAP's, dual SAP's, SAP-signature (all related to a given original CA orbit). Also the concept of a *pseudo*-CA and its regular companion is introduced. Section 2 takes a closer look into the nature of the shift-add correlation patterns and their duals. Section 3 uncovers a similarity between the dual SAP's and the original CA orbit. Section 4 presents a few alternatives for moving around in the four-dimensional complete shift-add function space. Section 5 is about SAP's of SAP's.

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Preliminaries, Definitions

1. Notational convention. A p-dimensional sequence over a set M is a function s: $A_1 \times A_2 \times \cdots \times A_p \to M$, where each set A_i is either N or $\mathbb{Z}$ (respectively the set of nonnegative integers and the set of integers). $s(a_1, a_2, \ldots, a_p)$ will denote the value of the sequence for the specific element $(a_1, a_2, \ldots, a_p) \in A_1 \times A_2 \times \cdots \times A_p$. The whole sequence, which is commonly denoted by $(s(a_1, a_2, \ldots, a_p))_{(a_1, a_2, \ldots, a_p) \in A_1 \times A_2 \times \cdots \times A_p}$ will also be represented by the shorter notation $\bar{s}(a_1, a_2, \ldots, a_p)$ (the domain over which the a_i range is given by the context). A restricted version of the sequence, where one or more of the variables are kept fixed, is represented by writing these fixed variables as subscripts, while the others remain between the brackets. For example, with a_1 and a_2 fixed, we write $s_{a_1, a_2}(a_3, a_4, \ldots, a_p)$. The corresponding $(p-2)$-dimensional sequence is then represented by $\bar{s}_{a_1, a_2}(a_3, a_4, \ldots, a_p)$.
2. The orbit of a one-dimensional linear CA over $\mathcal{R}$ is a two-dimensional sequence $\bar{a}(r, c) \equiv (a(r, c))_{(r,c) \in \mathbb{N} \times \mathbb{Z}}$ produced by the recursion

$$a(r, c) = \sum_{j \in R_1} \rho(1, j) a(r-1, c-j), \quad (1)$$

with R_1 a finite subset of $\mathbb{Z}$, and fixed coefficients $\rho(1, j) \in \mathcal{R}^*$ for all $j \in R_1$. We use the common matrix convention for the indices: r refers to rows, c refers to columns. The rowwise recursion Eq. (1) starts with a so-called *initial state configuration* $(a(0, c))_{c \in \mathbb{Z}}$, which is a one-dimensional bidirectional infinite sequence in $\mathcal{R}$. The set of coefficients $\rho(1, j)$, $j \in R_1$, fixes the *local evolution rule* of the CA. The rule can also be written as a polynomial $R(x)$ over $\mathcal{R}$ [Martin *et al.*, 1984; Barbé, 1995]:

$$R(x) = \sum_{j \in R_1} \rho(1, j) x^j \in \mathcal{R}[x]. \quad (2)$$

The summation in Eq. (1) and Eq. (2) could be taken over all $j \in \mathbb{Z}$ provided that $\rho(1, j) = 0$ for all $j \notin R_1$. Equation (1) can be generalized into

$$a(r, c) = \sum_{j \in R_k} \rho(k, j) a(r-k, c-j) \quad (0 \le k \le r), \quad (3)$$

where $\rho(k, j)$ is the (k, j)th element in the CA-rule orbit generated from the particular initial state sequence $(\rho(0, c))_{c \in \mathbb{Z}}$, for which $\rho(0, 0) = 1$ and $\rho(0, c) = 0$ when $c \neq 0$ [Barbé, 1995], and where

$$R_v \stackrel{\text{def}}{=} \{j | \rho(v, j) \neq 0\} \quad (\text{for all } v \in \mathbb{N}). \quad (4)$$

As $(\rho(r, c))_{(r,c) \in \mathbb{N} \times \mathbb{Z}}$ is an orbit itself, it also satisfies Eqs. (1) and (3), and so:

$$\rho(r, c) = \sum_{j \in R_1} \rho(1, j) \rho(r-1, c-j)$$

and, more generally

$$\rho(r, c) = \sum_{j \in R_k} \rho(k, j) \rho(r-k, c-j) \quad (0 \le k \le r). \quad (5)$$

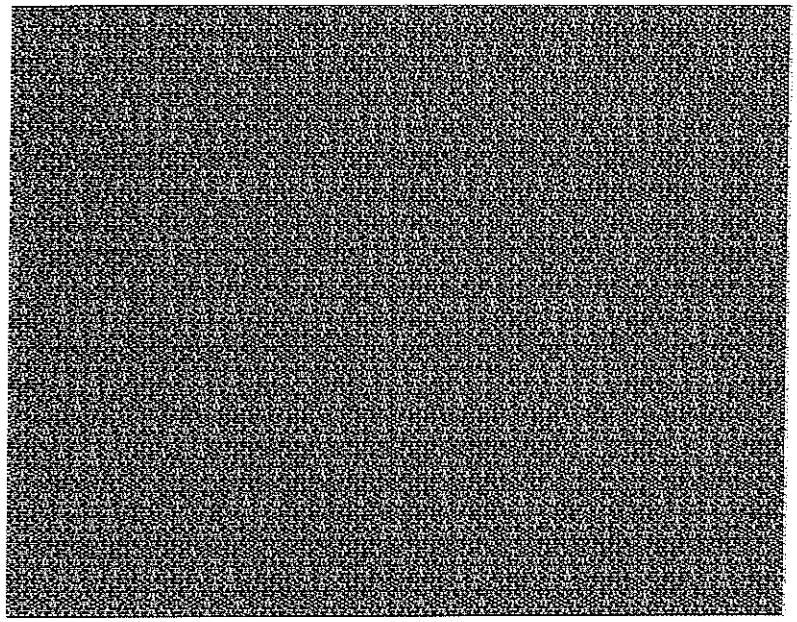
A straightforward visualization for the two-dimensional CA orbit is obtained by associating with each $a(r, c)$-value in the orbit, a colored pixel at the (r, c)th pixel position in some display coordinate system, whereby the color is in 1–1 correspondence with the $a(r, c)$-value (r refers to the row, and c to the column).This visualization is the *orbit pattern*. An example of a CA orbit pattern is given in Fig. 1(a). Figure 1(b) shows the corresponding *rule orbit pattern*. In view of the limited size of the display, only a fragment of the orbit pattern can be shown. These figures, and all those that follow, concern a CA over the finite field GF(4), with CA-rule $R(x) = 2x^{-1} + 2 + 2x \in \text{GF}(4)[x]$ and with a quasi-periodic initial state sequence s_{CTM} which is an extended and coded form of the Thue-Morse sequence [see the Appendix for a description of this sequence and of the operations in GF(4)].
3. Fix an $\bar{a}(r, c)$-orbit (by fixing an initial state configuration and a CA-rule), and also fix $\alpha, \beta \in \mathcal{R}^*$. Consider the variables $v \in \mathbb{N}$, $h \in \mathbb{Z}$, $r \in \mathbb{N}$, $c \in \mathbb{Z}$ and define the *complete shift-add correlation function*

$$s(v, h, r, c) \stackrel{\text{def}}{=} \alpha a(r, c) + \beta a(r+v, c+h). \quad (6)$$

The four-dimensional function $s(v, h, r, c)$ will be looked upon in different ways, by fixing some of the variables.

- When (v, h) in Eq. (6) are considered fixed, we obtain the restricted two-dimensional sequence $\bar{s}_{v,h}(r, c)$, which is called the (v, h)-*shift-add correlation pattern* [(v, h)-SAP] of the underlying $\bar{a}$-orbit. Its visualization, as indicated in Fig. 2, is actually the pattern that is obtained by taking the original $\bar{a}(r, c)$-orbit,

(a)



(b)

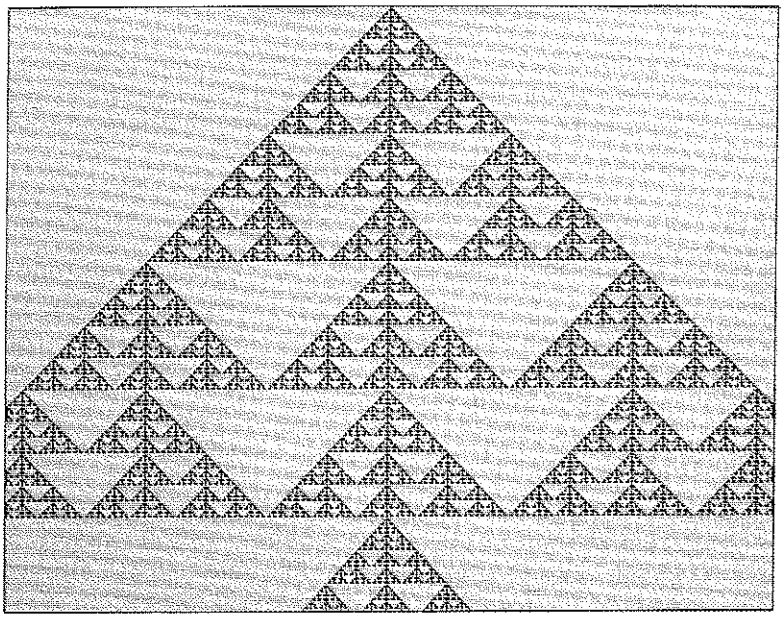


Fig. 1. (a) An orbit pattern $\bar{a}(r, c)$ over GF(4) [partial view: $0 \leq r \leq 160$, $-100 \leq c \leq 100$; the top row (initial state configuration) corresponds to $r = 0$]. The CA-rule is $R(x) = 2x^{-1} + 2 + 2x$ and the initial state sequence is the coded Thue-Morse sequence. The pixel color code for the cell-states is 0 = yellow, 1 = blue, 2 = black, 3 = red. (b) The rule-orbit pattern corresponding with the CA-rule $R(x) = 2x^{-1} + 2 + 2x \in \mathrm{GF}(4)[x]$.

copying it and shifting this copy vertically over v rows upwards, and horizontally over h columns to the left and then adding pixelwise the original pattern and its shifted version in the region of overlap, after having weighted them respectively with the numbers α and β. Special cases are the so-called *basic* h-SAP's with no vertical shift: $\bar{s}_{0,h}(r, c)$.

- $\bar{s}_h(c) \stackrel{\text{def}}{=} \bar{s}_{0,h}(0, c)$ represents the *top row of the basic* h-SAP. When h is also considered as variable, we get the function $\bar{s}(h, c)$ whose associated two-dimensional pattern is obtained by stacking the top rows of all basic h-SAP's on top of each other in the proper order. This pattern is called the SAP-*signature*. It lies at the basis of constructing the SAP's and their

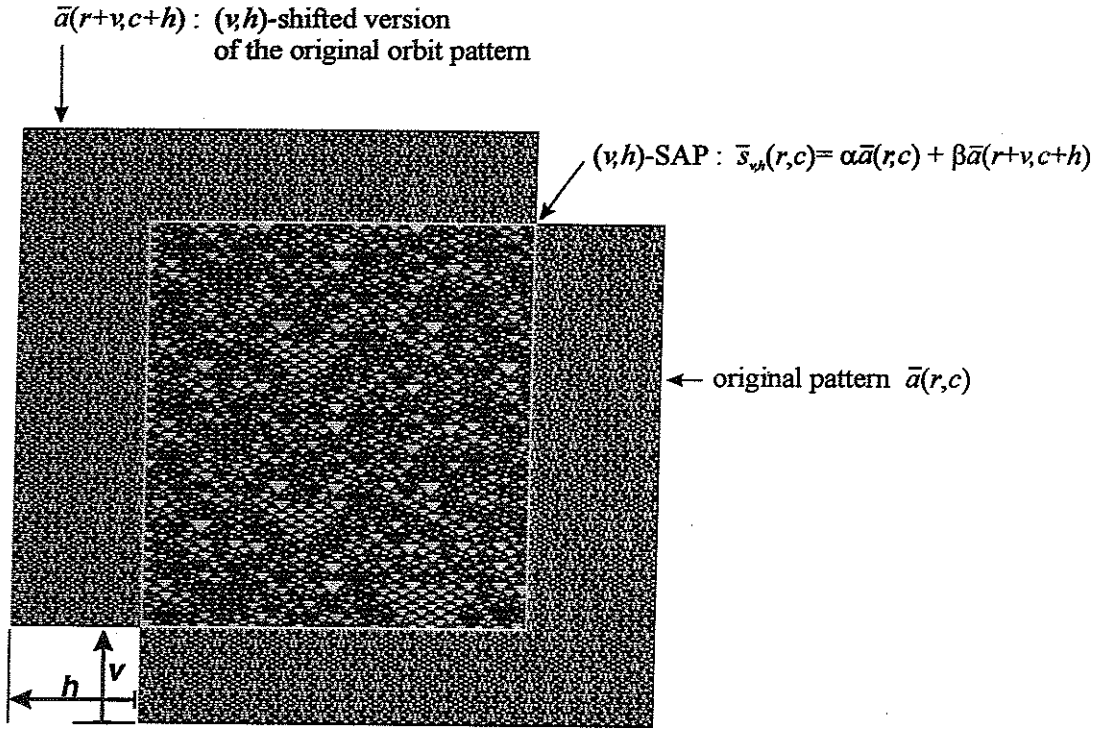


Fig. 2. The (v, h)-shift-add pattern (SAP) $\bar{s}_{v,h}(r, c) \stackrel{\text{def}}{=} \alpha\bar{a}(r, c)+\beta\bar{a}(r+v, c+h)$ of the orbit in Fig. 1(a) [$(v, h) = (32, 48)$, $\alpha = 1$, $\beta = 3$], is found in the overlap area of the original pattern and its (v, h)-shifted version. The display area for the original pattern is: $0 \leq r \leq 160 + v$, $-80 \leq c \leq 80 + h$. Display area for the overlapping SAP: $0 \leq r \leq 160$, $-80 \leq c \leq 80$.

duals. The SAP-signature for our example is given in Fig. 3(a).

- The (r, c)-*dual shift-add* (*correlation*) *pattern* [(r, c)-DSAP] corresponds to the function given by Eq. (6), but restricted to a fixed (r, c). So it will be represented by $\bar{s}^*_{r,c}(v, h)$. The *-superscript is an extra indication for the duality. Its associated two-dimensional pattern shows how a pixel at *fixed* position (r, c) in the (v, h)-SAP changes its colour (value) when the (v, h)-shift is varied. Figure 4(a) shows an example. Special cases are the *basic* c-DSAP's for which $r = 0$: they are denoted by $\bar{s}^*_{0,c}(v, h)$.
- There are two interesting visualizations of the four-dimensional complete shift-add correlation function [Eq. (6)] as a hierarchical superposition of two two-dimensional arrays:
  - The first one (see Fig. 5) consists of a (v, h)-indexed cellular matrix structure, where each (v, h)-cell displays the corresponding (v, h)-SAP. It will be denoted by $\bar{\mathcal{S}}(v, h)$, and so $\mathcal{S}(v, h) = \bar{s}_{v,h}(r, c)$. The top row patterns of $\bar{\mathcal{S}}(v, h)$ are then just the basic h-SAP's $\bar{s}_{0,h}(r, c)$.
  - The second one (see Fig. 6) is the dual representation $\bar{\mathcal{S}}^*(r, c)$ in which the (r, c)-labeled cells contain the corresponding (r, c)-DSAP's $\bar{s}^*_{r,c}(v, h)$. The top row patterns in $\bar{\mathcal{S}}^*(r, c)$ are just the basic c-DSAP's $\bar{s}^*_{0,c}(v, h)$.

$\bar{\mathcal{S}}(v, h)$ and $\bar{\mathcal{S}}^*(r, c)$ also form a *dual pair* in the sense that a pattern $\bar{s}_{v_o,h_o}(r, c)$ in $\bar{\mathcal{S}}(v, h)$ can be obtained by extracting from each $\bar{s}^*_{r,c}(v, h)$-pattern in $\bar{\mathcal{S}}^*(r, c)$ the pixel located at the fixed position (v_o, h_o), and by joining these extracted pixels together in their proper (r, c)-order. In the same way, pattern $\bar{s}^*_{r_o,c_o}(v, h)$ in $\bar{\mathcal{S}}^*(r, c)$ can be obtained by extracting from each pattern in $\bar{\mathcal{S}}(v, h)$ the pixel located at the fixed position (r_0, c_0), and by joining these pixels together in their proper (v, h)-order.

4. We will further need the function

$$\theta(v) \stackrel{\text{def}}{=} 1 - \sum_{j \in R_v} \rho(v, j) = 1 - [R(1)]^v \quad (v \in \mathbb{N}) \tag{7}$$

(all operations in $\mathcal{R}$), where $R(1)$ is the value obtained by replacing x in the polynomial form

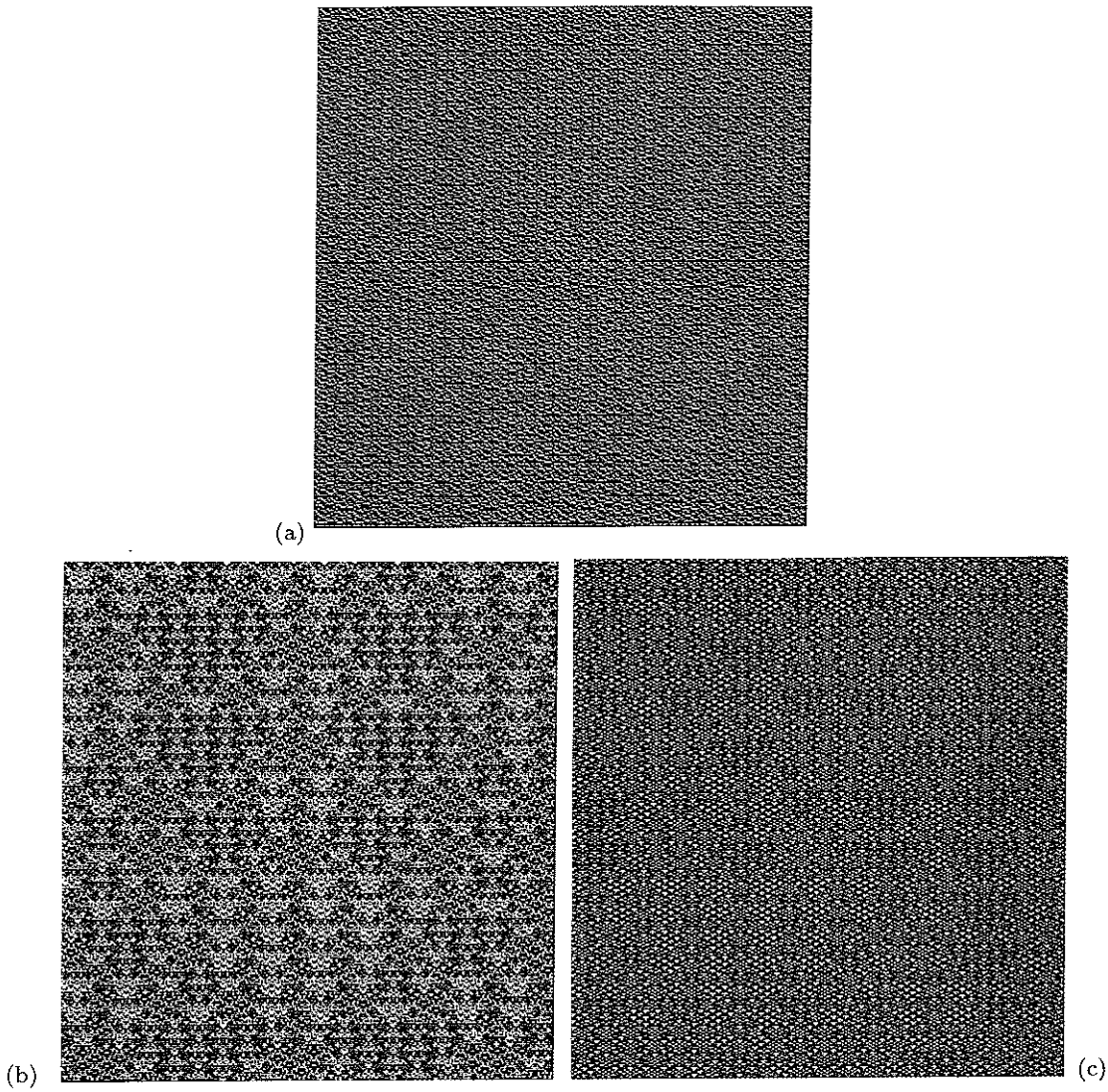


Fig. 3. (a) The SAP-signature $\bar{s}(h, c) = \alpha a(r, c) + \beta a(0, c + h)$ ($\alpha = 1$, $\beta = 3$). Display window: $-80 \leq h \leq 80$ (along the vertical axis, increasing downwards); $-80 \leq c \leq 80$ (horizontal axis). (b) The basic h-SAP $\bar{s}_h(r, c)$ ($h = 11$, $0 \leq r \leq 160$, $-80 \leq c \leq 80$). It is the CA orbit pattern with as initial state sequence the sequence on the horizontal row $h(= 11)$ in the SAP-signature. (c) The regular companion $\bar{\eta}^*_{0,c}(v, h)$ of the basic c-DSAP $\bar{s}^*_c(v, h)$ (dual shift-add pattern, $c = 49$, $0 \leq v \leq 160$, $-80 \leq h \leq 80$). It is the CA orbit pattern with as initial state configuration the sequence on the vertical column $c(= 49)$ in the SAP-signature.

$R(x)$ of the CA-rule by 1. The last equality in Eq. (7) follows from the fact that the sequence $\bar{\rho}_v(j)$ has associated generating polynomial $[R(x)]^v$ (see [Barbé, 1995; Martin *et al.*, 1984]). It follows that for all $0 \leq m \leq k$ (m, $k \in \mathbb{N}$) :

$$\theta(k) = \theta(m) + \theta(k - m) - \theta(m)\theta(k - m). \quad (8)$$

Notice that, because of the finitude of $\mathcal{R}$, $(\theta(k))_{k\in\mathbb{N}}$ is necessarily an eventually periodic sequence with a minimal period smaller than the number of elements in $\mathcal{R}$. Actually, when $\mathcal{R}$ is taken to be the finite field GF(p^n), $(\theta(k))_{k\in\mathbb{N}}$ is a periodic sequence with a minimal period that divides $p^n - 1$ [as a matter of fact, when $R(1) \neq 0$, the period is the so-called *order* of $R(1)$ in GF(p^n): it divides $p^n - 1$ (see, for example [Mc. Eliece, 1987]). When $R(1) = 0$ or 1, the period of $\theta(v)$ is clearly 1].

5. A (one-dimensional) *pseudo-cellular automaton* with rule $R(x) = \sum_{j\in R_1} \rho(1, j)x^j$ is a CA-like structure that evolves according to the recursion

$$\forall r \geq 1 :$$

$$q(r, c) = \sum_{j\in R_1} \rho(1, j)q(r - 1, c - j) + \gamma\theta(1), \quad (9)$$

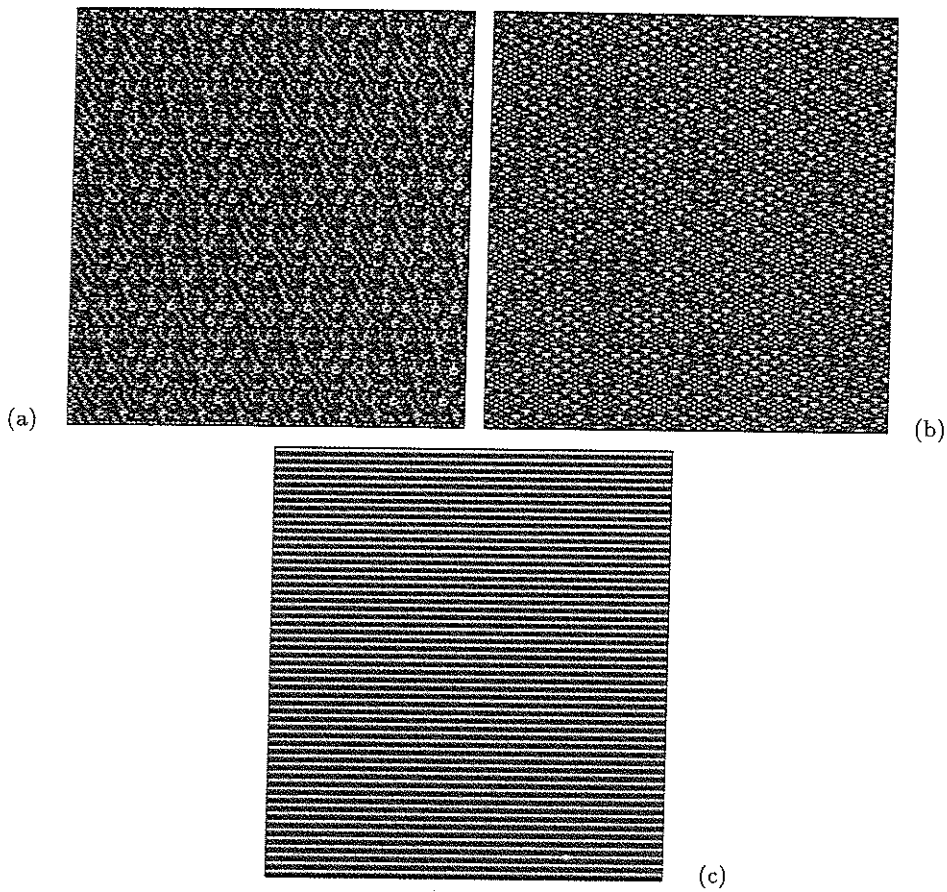


Fig. 4. (a) The (r, c)-DSAP (dual shift-add pattern) $\bar{s}_{r,c}(v, h) = \alpha a(r, c) + \beta a(r + v, c + h)$, for $[(r, c) = (23, 33), \alpha = 1, \beta = 3]$. Display area: $0 \leq v \leq 160$ (vertical axis), $-80 \leq h \leq 80$ (horizontal axis). (b) The regular companion of the DSAP in (a). It is the CA orbit with the same initial state sequence (top row) of the DSAP displayed in (a). (c) The rowwise constant and periodic pattern $\alpha\theta(v)a_{r,c}$, $[\alpha = 1, a_{r,c} = a_{23,33} = 2; \theta(v) = 0$ for $v \equiv 0(\mathrm{mod}\,3)$, $\theta(v) = 3$ for $v \equiv 1(\mathrm{mod}\,3)$, $\theta(v) = 2$ for $v \equiv 2(\mathrm{mod}\,3)]$. The pattern in (a) is the pixelwise addition (in GF(4)) of the pattern in (b) and of the pattern in (c): it is a pseudo-CA orbit pattern (Corollary 2.2).

starting with some initial state configuration $\bar{q}(0, c)$, whereby γ is a nonzero constant from $\mathcal{R}$, and where $\theta(1) = 1 - \sum_{j \in R_1} \rho(1, j) \neq 0$. (If $\gamma = 0$ and/or $\theta(1) = 0$, then Eq. (9) would reduce to the regular CA-relationship.)

Property 1.1. *For the orbit of a pseudo-CA as defined by Eq. (9), it holds that for all $(r, c) \in \mathbb{N} \times \mathbb{Z}$ and $0 \leq k \leq r$:*

$$q(r, c) = \sum_{j \in R_k} \rho(k, j) q(r-k, c-j) + \gamma\theta(k), \quad (10)$$

where $\theta(k)$ satisfies Eq. (7).

Proof. By induction: Eq. (9), together with Eq. (10) for some $1 \leq k \leq r$, imply that

$$q(r, c) = \sum_{j \in R_{k+1}} \rho(k+1, j) q(r-k-1, c-j) + \gamma\theta(k+1). \quad (11)$$

Indeed, using Eq. (9), Eq. (10) can be written as

$$q(r, c) = \sum_{j \in R_k} \rho(k, j) \left[\sum_{l \in R_1} \rho(1, l) q(r-k-1, c-j-l) + \gamma\theta(1) \right] + \gamma\theta(k)$$

$$= \gamma \left[\theta(1) \sum_{j \in R_k} \rho(k, j) + \theta(k) \right] + \sum_{j \in R_k} \sum_{l \in R_1} \rho(k, j) \times \rho(1, l) q(r-k-1, c-j-l).$$

The first term in this expression equals $\gamma[\theta(1) \times (1 - \theta(k)) + \theta(k)] \overset{(8)}{=} \theta(k+1)$, while the second term can be transformed, by putting $z = j + l$, into

$$\sum_{z \in R_{k+1}} \left[\sum_{l \in R_1} \rho(k, z-l) \rho(1, l) \right] q(r-k-1, c-z) \overset{(5)}{=} \sum_{z \in R_{k+1}} \rho(k+1, z) q(r-k-1, c-z).$$

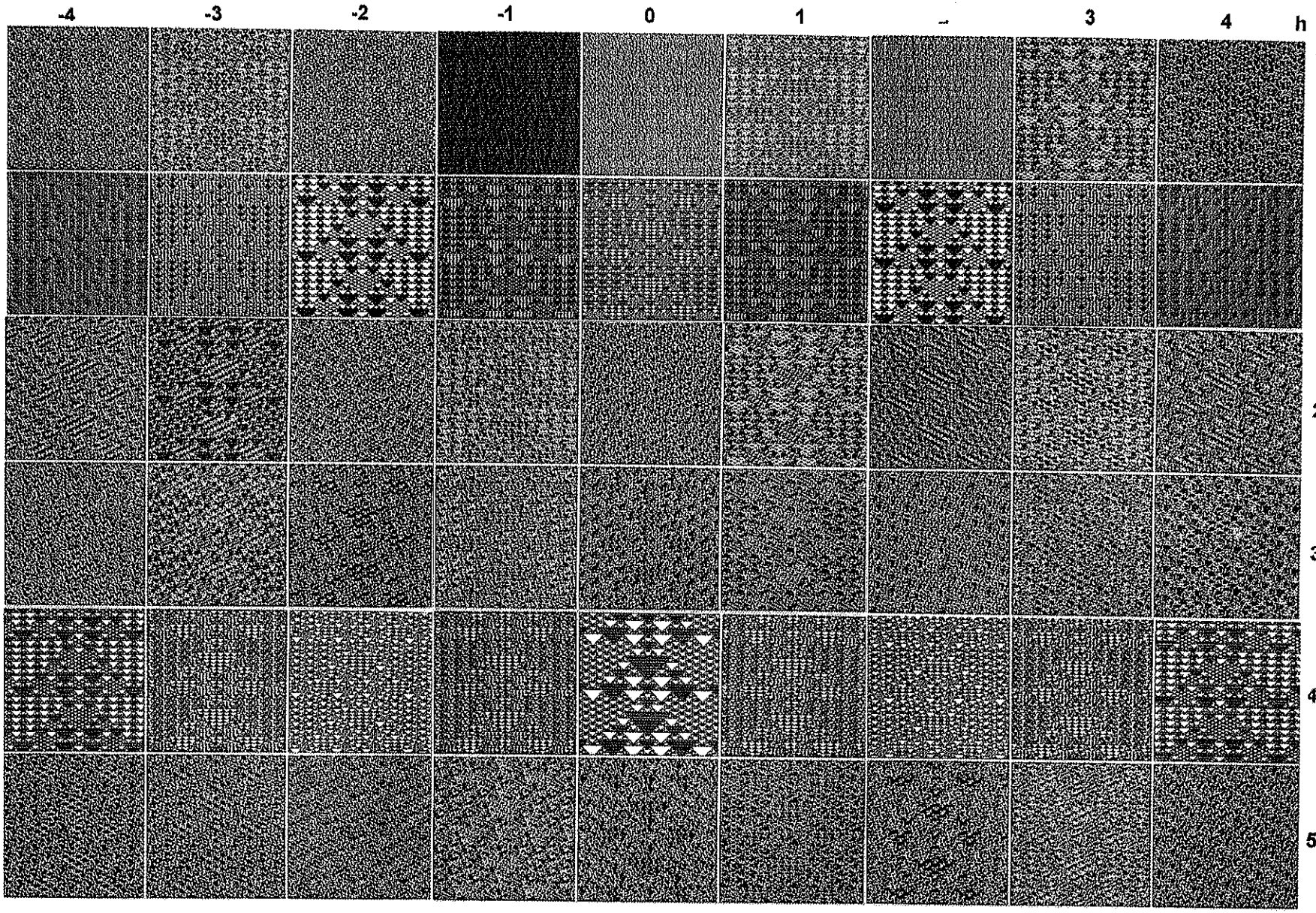


Fig. 5. The array $\bar{S}(v, h)$ $(0 \leq v \leq 5, -4 \leq h \leq 4)$. Each (v, h)-cell in this array contains the (v, h)-SAP $\bar{s}_{v,h}(r, c)$ (displayed for $0 \leq r \leq 160$, $-80 \leq c \leq 80$). When the pattern in a cell is considered as the state of the cell, the patterns in this $\bar{S}(v, h)$-array constitute among themselves the orbit of a *pseudo* cellular automaton (Property 2.2 + Corollary 2.1): each pattern is the (pixelwise) sum of its corresponding regular companion in the $\bar{N}(v, h)$-array (shown in Fig. 7), and of a proper multiple of the original orbit pattern (shown in Fig. 8). Each pattern in $\bar{S}(v, h)$ is itself a CA orbit pattern (Property 2.1.A). The top row patterns in $\bar{S}(v, h)$ are the *basic* h-SAP's [cf. Fig. 3(b)].

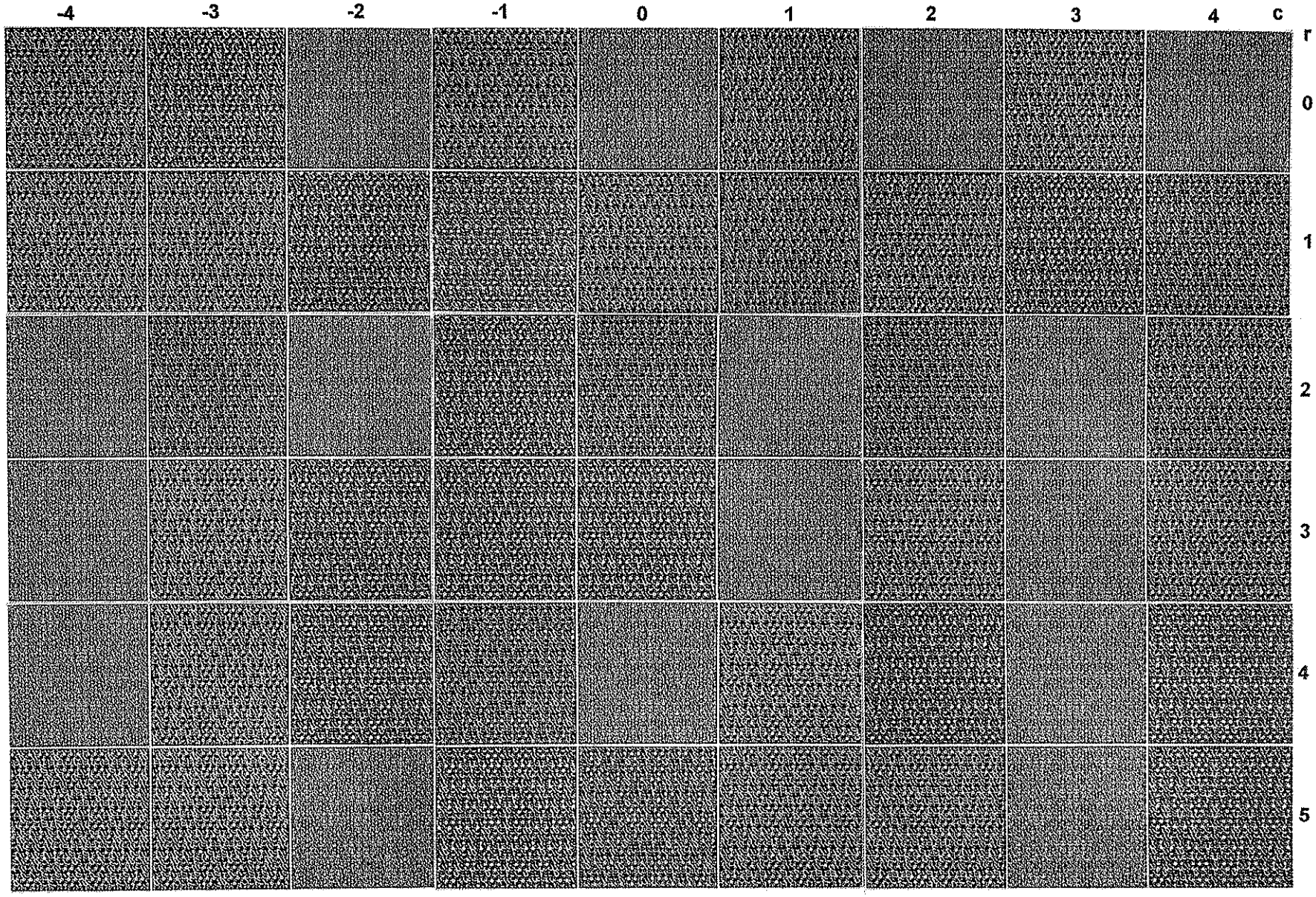


Fig. 6. The array $\bar{\mathcal{S}}^*(r, c)$ $(0 \leq r \leq 5, -4 \leq c \leq 4)$. Each (r, c)-cell in this array contains the (r, c)-dual SAP $\bar{s}^*_{r,c}(v, h)$ (displayed for $0 \leq v \leq 160$, $-80 \leq h \leq 80$). When the pattern in a cell is considered as the state of the cell, the patterns in this $\bar{\mathcal{S}}^*(r, c)$-array constitute among themselves the orbit of a cellular automaton (Main Property 2.4): for the given example with rule $R(x) = 2(x^{-1} + 1 + x)$, each pattern $\bar{s}^*_{r,c}(v, h)$ equals $2[\bar{s}_{r-1,c-1}(v, h) + \bar{s}_{r-1,c}(v, h) + \bar{s}_{r-1,c+1}(v, h)]$ $(r \geq 1$; pixelwise addition and multiplication in GF(4)). Each pattern is itself a *pseudo* CA-orbit (as illustrated in Fig. 4). The top row patterns are the basic c-DSAP's [cf. Fig. 3(c)].

Together, these two terms produce Eq. (11). Then start the induction with $k = 1$ in Eq. (10) [this expression then becomes equal to Eq. (9), and so is true by definition]. ■

6. The *regular companion* of a pseudo-CA is the ordinary CA that has the *same rule* and the *same initial configuration* as the pseudo-CA. Formally, let $\eta(r, c)$ be the orbit of the regular companion of the pseudo-CA defined in Eq. (9), then for all $(r, c) \in \mathbb{N} \times \mathbb{Z}$, $0 \leq k \leq r$:

$$\eta(r, c) = \sum_{j \in R_k} \rho(k, j)\eta(r-k, c-j)\,, \qquad (12)$$

with $\eta(0, c) \equiv q(0, c)$.

If multiplication of a pattern (sequence) with a scalar is defined to be a pixelwise (elementwise) multiplication with that scalar; and addition of two patterns (sequences) is defined to be pixelwise (elementwise) addition, we can formulate

Property 1.2. *The orbit of a pseudo-CA can be obtained from its regular companion's orbit by adding the rowwise constant pattern $\gamma\bar{\theta}(r)$ to it:*

$$\bar{q}(r, c) = \bar{\eta}(r, c) + \gamma\bar{\theta}(r)\,. \qquad (13)$$

Proof. From Eq. (10), it follows that

$$q(r, c) = \sum_{j \in R_r} \rho(r, j)q(0, c-j) + \gamma\theta(r)\,.$$

Compare this with Eq. (12), taken with $k = r$, to conclude that the summation term equals $\eta(r, c)$, and so $q(r, c) = \eta(r, c) + \gamma\theta(r)$. As this equality holds for all $(r, c) \in \mathbb{N} \times \mathbb{Z}$, Eq. (13) follows. ■

2. The nature of SAP's and DSAP's and their interrelationship

We will now discuss the structure of the $\bar{s}_{v,h}(r, c)$- and $\bar{s}^*_{r,c}(v.h)$-patterns, and of the complete shift-add correlation function representations $\bar{S}(v, h)$ and $\bar{S}^*(r, c)$. Before stating the main properties, we introduce two lemmas.

Lemma 2.1. *For all $(v, h, r, c) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{Z}$ and all $0 \leq k \leq r$:*

$$s(v, h, r, c) = \sum_{j \in R_k} \rho(k, j)s(v, h, r-k, c-j)\,. \quad (14)$$

Proof

$$s(v, h, r, c) \overset{(6)}{=} \alpha a(r, c) + \beta a(r+v, c+h)$$
$$\overset{(3)}{=} \sum_{j \in R_k} \rho(k, j)[\alpha a(r-k, c-j) + \beta a(r+v-k, c+h-j)]\,.$$

Equation (14) then follows by using Eq. (6) again. ■

Lemma 2.2. *For all $(v, h, r, c) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{Z}$ and all k, $0 \leq k \leq v$:*

$$s(v, h, r, c) = \sum_{j \in R_k} \rho(k, j)s(v-k, h-j, r, c) + \alpha\theta(k)a(r, c)\,. \qquad (15)$$

Proof

$$s(v, h, r, c) \overset{(6)}{=} \alpha a(r, c) + \beta a(r+v, c+h)$$
$$\overset{(3)}{=} \alpha a(r, c) + \beta \sum_{j \in R_k} \rho(k, j)a(r+v-k, c+h-j)\,. \qquad (16)$$

Now, using Eq. (7), we can write

$$a(r, c) = \theta(k)a(r, c) + \sum_{j \in R_k} \rho(k, j)a(r, c)\,.$$

Introduce this into Eq. (16), combine the terms involving the summation, and use Eq. (6) again to obtain Eq. (15). ■

Observe a similarity between Eqs. (3–10) concerning the CA- and pseudo-CA orbit structure on the one hand, and Eqs. (14–15) on the other hand. This leads to the next six Properties about the structure of the SAP's and DSAP's and their interrelationship.

Property 2.1. *CA-nature of the SAP-elements in $\bar{S}(v, h)$.*

A. *The (v, h)-SAP $\bar{s}_{v,h}(r, c)$ is the orbit pattern of a CA operating under the same rule $R(x)$ as the original CA with $\bar{s}_{v,h}(c)$ as initial state configuration. Formally: for all $(r, c) \in \mathbb{N} \times \mathbb{Z}$ and all $k \in \mathbb{N}$, $0 \leq k \leq r$:*

$$s_{v,h}(r, c) = \sum_{j \in R_k} \rho(k, j)s_{v,h}(r-k, c-j)\,. \qquad (17)$$

B. *The basic h-SAP is the CA orbit pattern whose initial state configuration is the corresponding h-row in the SAP-signature. Formally:*

$$s_{0,h}(r,\, c) = \sum_{j \in R_r} \rho(r,\, j) s_h(c - j)\,. \qquad (18)$$

Proof. Apply Lemma 2.1 under the assumption of a fixed $(v,\, h)$ to obtain Eq. (17), and fixed $(v,\, h) = (0,\, h)$ to obtain Eq. (18). The corresponding initial configurations are found for $r = k = 1$. ■

A basic h-SAP, generated according to Property 2.1.B, is shown in Fig. 3(b).

Property 2.2. *The pseudo-CA nature of* $\bar{\mathcal{S}}(v, h)$.

The $(v,\, h)$*-SAP's in* $\bar{\mathcal{S}}(v,\, h)$ *form the states of a pseudo-CA orbit over the* $(v,\, h)$*-variables, with the same rule* $R(x)$ *as for the original* $\bar{a}$*-orbit, and with initial state configuration the sequence of basic h-SAP's. Formally:*

$$\mathcal{S}(v,\, h) = \sum_{j \in R_k} \rho(k,\, j) \mathcal{S}(v - k,\, h - j) + \alpha\theta(k)\mathcal{A}\,, \qquad (19)$$

where $\mathcal{A}$ *is used for consistency of notation (aesthetic reasons) as a replacement for the orbit-notation* $\bar{a}(r,\, c)$.

Proof. This follows from Lemma 2.2 by considering $(v,\, h)$ fixed:

$$s_{v,h}(r,\, c) = \sum_{j \in R_k} \rho(k,\, j) s_{v-k,h-j}(r,\, c) + \alpha\theta(k) a(r,\, c)\,.$$

As this holds for all $(r,\, c) \in \mathbb{N} \times \mathbb{Z}$, it holds pixelwise for all patterns involved, and so:

$$\bar{s}_{v,h}(r,\, c) = \sum_{j \in R_k} \rho(k,\, j) \bar{s}_{v-k,h-j}(r,\, c) + \alpha\theta(k)\bar{a}(r,\, c)\,. \qquad (20)$$

Equation (19) reformulates Eq. (20) by using the convention $\mathcal{S}(v,\, h) \stackrel{\text{def}}{=} \bar{s}_{v,h}(r,\, c)$. The initial configuration is obtained for $v = k = 1$. ■

Definition 2.1. $\bar{\mathcal{N}}(v,\, h)$, $(v,\, h) \in \mathbb{N} \times \mathbb{Z}$, is the regular companion of the pseudo-CA orbit $\bar{\mathcal{S}}(v,\, h)$, i.e. $\bar{\mathcal{N}}(v,\, h)$ is the array whose cell located at $(v,\, h)$ contains the pattern $\bar{\eta}_{v,h}(r,\, c)$ obtained through the regular CA-rule

$$\bar{\eta}_{v,h}(r,\, c) = \sum_{j \in R_k} \rho(k,\, j) \bar{\eta}_{v-k,h-j}(r,\, c) \qquad (21)$$

from the initial state configuration $(\bar{\eta}_h(r,\, c))_{h \in \mathbb{Z}} \equiv (\bar{s}_h(r,\, c))_{h \in \mathbb{Z}}$ [i.e. the sequence of top row patterns in the $\bar{\mathcal{S}}(v,\, h)$-array].

Part of the $\bar{\mathcal{N}}(v,\, h)$-array for our example is shown in Fig. 7.

As a direct consequence of this definition and of Eq. (20), we can formulate the following corollary.

Corollary 2.1. *The* $\bar{\mathcal{S}}(v,\, h)$*-array (which is a pseudo-CA orbit) can be obtained as the pattern-wise sum of its regular companion* $\bar{\mathcal{N}}(v,\, h)$ *and the rowwise constant pattern-array* $\alpha\theta(v)\mathcal{A}$ *of multiples of the original orbit:*

$$\bar{\mathcal{S}}(v,\, h) = \bar{\mathcal{N}}(v,\, h) + \alpha\theta(v)\mathcal{A}\,. \qquad (22)$$

The different elements of $\alpha\theta(v)\mathcal{A}$ for our example are shown in Fig. 8.

Property 2.3. *The CA-nature of the elements* $\bar{\eta}_{v,h}(r,\, c)$ *in* $\bar{\mathcal{N}}(v,\, h)$.

The $\bar{\eta}_{v,h}(r,\, c)$*-patterns are orbit patterns of a regular CA operating under the same rule* $R(x)$ *as the original CA, and with the initial state configuration* $\bar{\eta}_{v,h}(0,\, c) \equiv \bar{s}_{v,h}(0,\, c) - \alpha\theta_v \bar{a}(0,\, c)$. *Formally:*

$$\eta_{v,h}(r,\, c) = \sum_{j \in R_k} \rho(k,\, j) \eta_{v,h}(r - k,\, c - j)\,. \qquad (23)$$

Proof. From Eq. (22), it follows that for all $(r,\, c) \in \mathbb{N} \times \mathbb{Z}$:

$$s_{v,h}(r,\, c) = \eta_{v,h}(r,\, c) + \alpha\theta_v a(r,\, c)\,, \qquad (24)$$

or:

$$\begin{aligned} \eta_{v,h}(r,\, c) &= s_{v,h}(r,\, c) - \alpha\theta_v a(r,\, c) \\ &\stackrel{(17)}{=} \sum_{j \in R_k} \rho(k,\, j) s_{v,h}(r - k,\, c - j) \\ &\quad - \alpha\theta_v a(r,\, c) \\ &\stackrel{(24)}{=} \sum_{j \in R_k} \rho(k,\, j)[\eta_{v,h}(r - k,\, c - j) \\ &\quad + \alpha\theta_v a(r - k,\, c - j)] - \alpha\theta_v a(r,\, c)\,. \end{aligned}$$

Now invoke Eq. (3) to obtain Eq. (23). ■

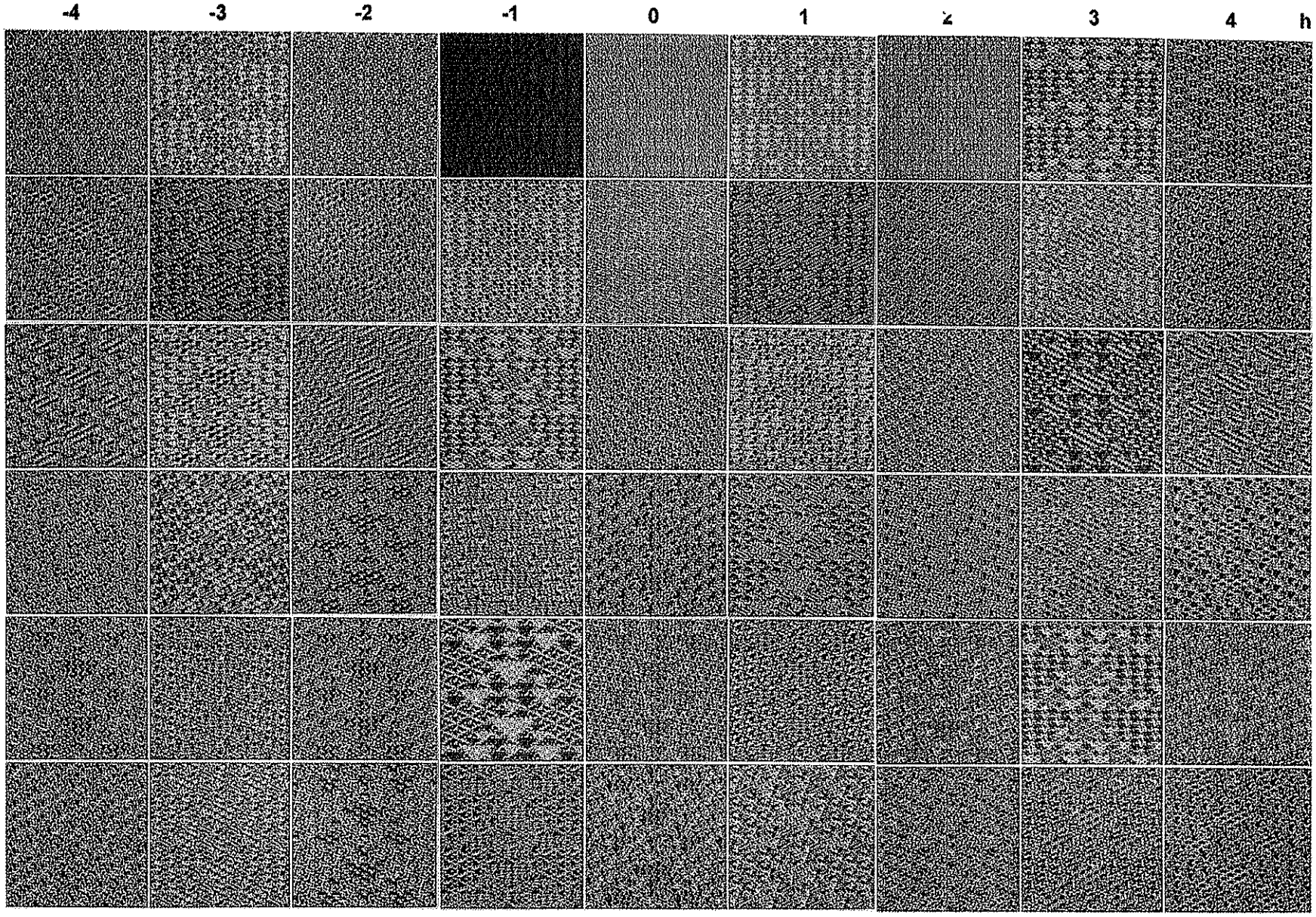


Fig. 7. The regular companion array $\bar{\mathcal{N}}(v, h)$, of the $\bar{\mathcal{S}}(v, h)$-array in Fig. 5 ($0 \leq v \leq 5$, $-4 \leq h \leq 4$). Each (v, h)-cell in this array contains the pattern $\bar{\eta}_{v,h}(r, c)$, which is a regular CA orbit pattern (Property 2.3). When the pattern in a cell is considered as the state of the cell, the patterns in this $\bar{\mathcal{N}}(v, h)$-array constitute among themselves the orbit of a regular CA (by definition): for the given example with rule $R(x) = 2(x^{-1}+1+x)$, each pattern $\bar{\eta}_{v,h}(r, c)$ equals $2[\bar{\eta}_{v-1,h-1}(r, c)+\bar{\eta}_{v-1,h}(r, c)+\bar{\eta}_{v-1,h+1}(r, c)]$ ($r \geq 1$; pixelwise addition and multiplication in GF(4)). The top row patterns are the same as the ones in the $\bar{\mathcal{S}}(v, h)$-array (Fig. 5).

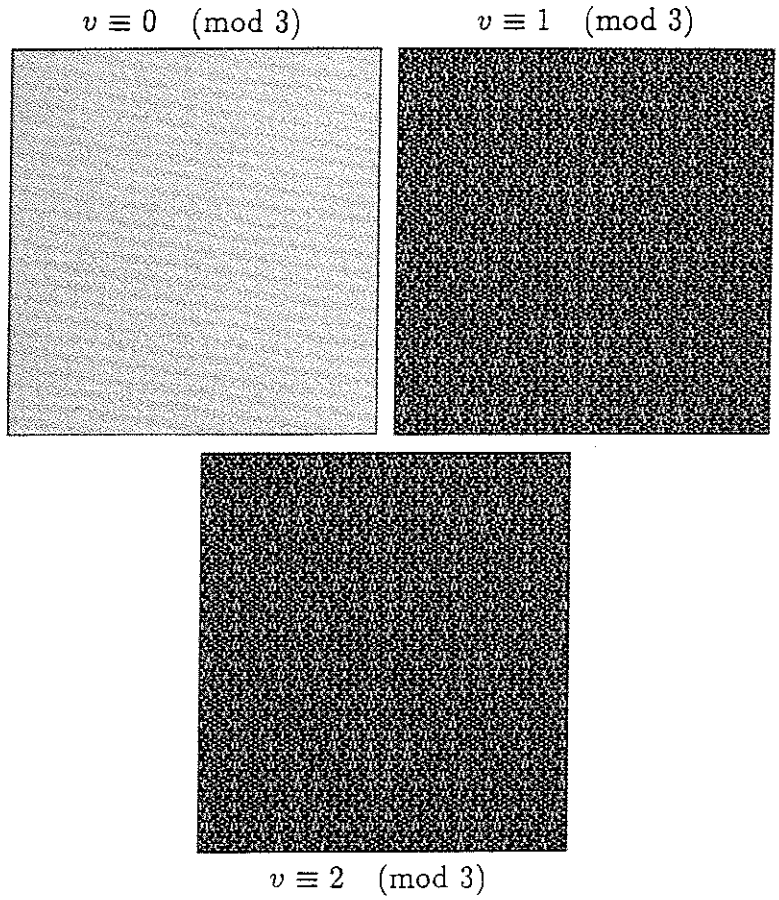


Fig. 8. The patterns $\alpha\theta(v)\mathcal{A}$, [i.e. the $\alpha \times \theta(v)$-multiple of the original orbit pattern $\mathcal{A} \equiv \bar{a}(r, c)$], (display area: $0 \leq r \leq 160$, $-80 \leq c \leq 80$). For the given example: $\alpha = 1$, $\theta(v) = 0$ for $v \equiv 0 (\mathrm{mod}\, 3)$, $\theta(v) = 3$ for $v \equiv 1 (\mathrm{mod}\, 3)$, $\theta(v) = 2$ for $v \equiv 2 (\mathrm{mod}\, 3)$. Adding $\alpha\theta(v)\mathcal{A}$ to any $\mathcal{N}$ (v, h)-pattern in Fig. 7 produces the corresponding $\mathcal{S}$ (v, h)-pattern in Fig. 5. (Corollary 2.1).

Property 2.4. *The pseudo-CA nature of the DSAP-elements in $\bar{\mathcal{S}}^*(r, c)$.*

A. *The (r, c)-DSAP $\bar{s}^*_{r,c}(v, h)$ is the orbit pattern of a pseudo-CA operating under the same rule $R(x)$ as the original CA, and with $\bar{s}^*_{0,c}(v, h)$ as initial state configuration. Formally:*

$$s^*_{r,c}(v, h) = \sum_{j \in R_k} \rho(k, j) s^*_{r,c}(v - k, h - j) + \alpha\theta(k) a_{r,c}\,. \tag{25}$$

B. *The basic c-DSAP is the pseudo-CA orbit pattern whose initial state configuration is the corresponding c-column in the SAP-signature. Formally:*

$$s^*_c(v, h) = \sum_{j \in R_v} \rho(v, j) s^*_c(h - j) + \alpha\theta(v) a_c\,. \tag{26}$$

Proof. Apply Lemma 2.2 under the assumption of a fixed (r, c) to obtain Eq. (25), and fixed $(r, c) = (0, c)$ to obtain Eq. (26). The corresponding initial configurations are found for $v = k = 1$. ■

Definition 2.2.

- $\bar{\eta}^*_{r,c}(v, h)$ is the regular companion of $\bar{s}^*_{r,c}(v, h)$, i.e. the pattern obtained through

$$\eta^*_{r,c}(v, h) = \sum_{j \in R_k} \rho(k, j) \eta^*_{r,c}(v - k, h - j)\,, \tag{27}$$

  from the initial state configuration $\bar{\eta}^*_{r,c}(h) \equiv \bar{s}^*_{r,c}(h)$.
- $\bar{\mathcal{N}}^*(r, c)$ is the array of these regular DSAP-companion patterns $\bar{\eta}^*_{r,c}(v, h)$.

Figure 9 shows part of the $\bar{\mathcal{N}}^*(r, c)$-array for our example.

Remarks. The regular companion of a basic c-DSAP is the regular CA orbit pattern [with rule

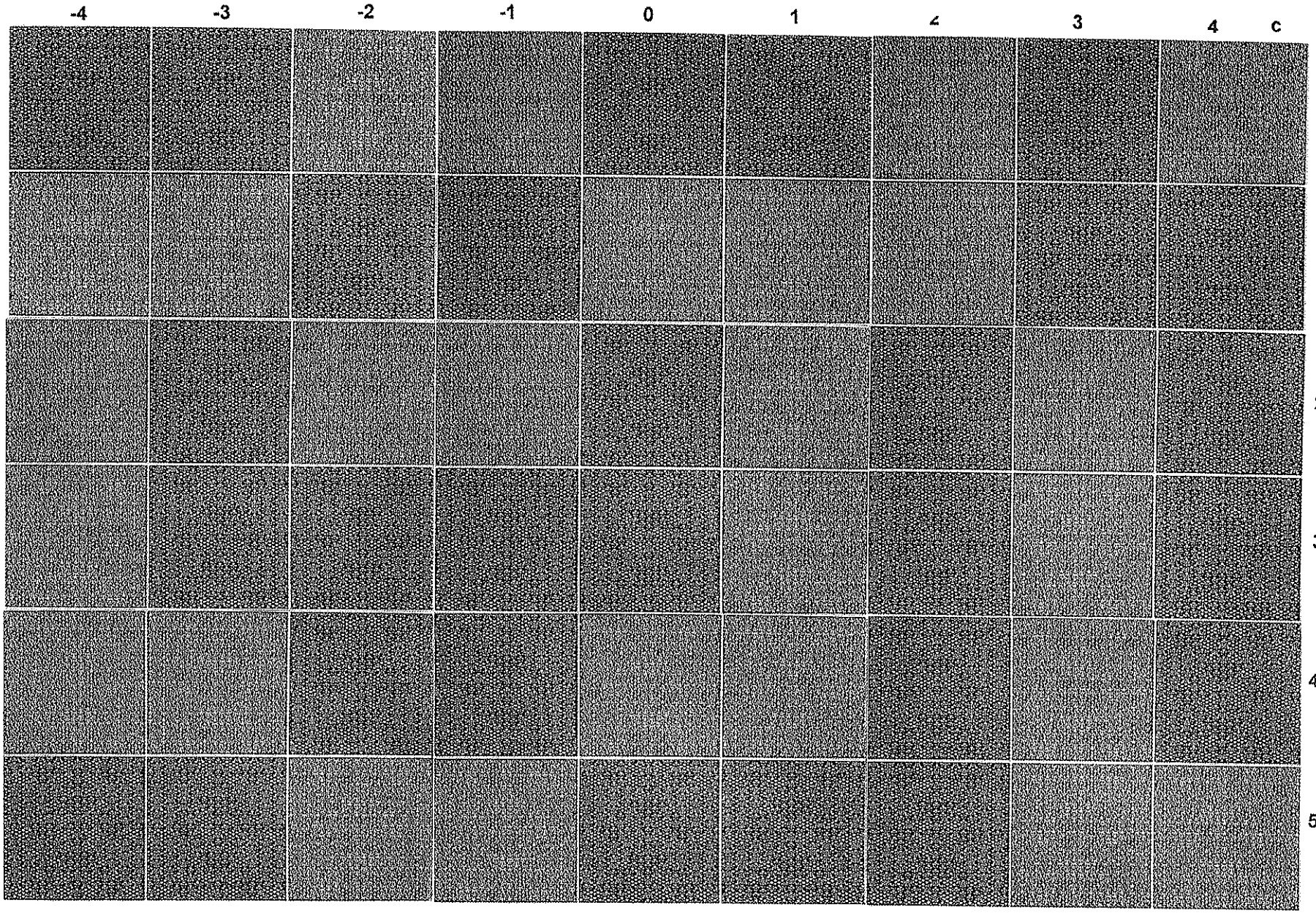


Fig. 9. The array $\tilde{\mathcal{N}}^*(r, c)$ of regular companions $\bar{\eta}^*_{r,c}(v, h)$ of the (r, c)-DSAP's in Fig. 6 ($0 \leq r \leq 5$, $-4 \leq c \leq 4$). Each (r, c)-cell contains the regular CA orbit pat $\bar{\eta}^*_{r,c}(v, h)$ (by definition). These patterns constitute among themselves the orbit of a regular CA (Property 2.6): for the given example with rule $R(x) = 2(x^{-1} + 1 +$ each pattern $\bar{\eta}^*_{r,c}(v, h)$ equals $2[\bar{\eta}_{r-1,c-1}(v, h) + \bar{\eta}^*_{r-1,c}(v, h) + \bar{\eta}^*_{r-1,c+1}(v, h)]$. The top row patterns $\bar{\eta}^*_{0,c}(v, h)$ are the CA orbits generated from the vertical sequence the SAP-signature (i.e. the regular companions of the basic c-DSAP's).

$R(x)$] whose initial state configuration is the corresponding c-column of the SAP-signature [$\bar{\eta}^*_{0,c}(h) \equiv \bar{s}^*_c(h)$ in Eq. (27)]. This is illustrated in Fig. 3(c).

As a consequence of this definition and of Eq. (25), we can formulate

Corollary 2.2. *The (r, c)-DSAP is the sum of its regular companion and the rowwise constant pattern $\alpha\bar{\theta}(v)a_{r,c}$:*

$$\bar{s}^*_{r,c}(v, h) = \bar{\eta}^*_{r,c}(v, h) + \alpha\bar{\theta}(v)a_{r,c}\,. \qquad (28)$$

This corollary is illustrated in Fig. 4.

Property 2.5. *The CA-nature of $\bar{\mathcal{S}}^*(r, c)$*

The (r, c)-DSAP's in $\bar{\mathcal{S}}^*(r, c)$ form the states of a CA orbit over the (r, c)-variables, with the rule $R(x)$ as for the original $\bar{a}$-orbit, and with initial configuration the sequence of basic c-DSAP's $(\bar{s}^*_{0,c}(v, h))_{c\in\mathbb{Z}}$. Formally:

$$\mathcal{S}^*(r, c) = \sum_{j\in R_k} \rho(k, j)\mathcal{S}^*(r-k, c-j)\,. \qquad (29)$$

Proof. This follows from Lemma 2.1 by considering the dual version of the function with (r, c) fixed:

$$s^*_{r,c}(v, h) = \sum_{j\in R_k} \rho(k, j)s^*_{r-k,c-j}(v, h)\,.$$

As this holds for all $(v, h) \in \mathbb{N}\times\mathbb{Z}$, it holds pixelwise for all patterns involved, and so:

$$\bar{s}^*_{r,c}(v, h) = \sum_{j\in R_k} \rho(k, j)\bar{s}^*_{r-k,c-j}(v, h)\,.$$

Applying $\mathcal{S}^*(r, c) \stackrel{\text{def}}{=} \bar{s}^*_{r,c}(v, h)$ produces Eq. (29). ■

Property 2.6. *The CA-nature of the $\bar{\mathcal{N}}^*(r, c)$-array.*

The $\bar{\eta}^*_{r,c}(v, h)$-patterns in the cells of $\bar{\mathcal{N}}^*(r, c)$ form the states of a CA orbit over the (r, c)-variables, with the same rule $R(x)$ as for the original $\bar{a}$-orbit, and with initial state configuration $(\bar{\mathcal{N}}^*(0, c))_{c\in\mathbb{Z}} = (\bar{\eta}^*_{0,c}(v, h))_{c\in\mathbb{Z}}$ [i.e. the sequence of regular companions of the basic c-DSAP's]. Formally:

$$\mathcal{N}^*(r, c) = \sum_{j\in R_k} \rho(k, j)\mathcal{N}^*(r-k, c-j)\,. \qquad (30)$$

Proof

$$\begin{aligned}\eta^*_{r,c}(v, h) &\stackrel{(28)}{=} s^*_{r,c}(v, h) - \alpha\theta(v)a_{r,c}\\ &\stackrel{(29)}{=} \sum_{j\in R_k} \rho(k, j)s^*_{r-k,c-j}(v, h) - \alpha\theta(v)a_{r,c}\\ &\stackrel{(28)}{=} \sum_{j\in R_k} \rho(k, j)\eta^*_{r-k,c-j}(v, h)\\ &\quad + \alpha\theta(v)\Big[\sum_{j\in R_k} \rho(k, j)a_{r-k,c-j} - a_{r,c}\Big]\,.\end{aligned}$$

Applying Eq. (3) shows that the second term equals 0, and so we obtain

$$\eta^*_{r,c}(v, h) = \sum_{j\in R_k} \rho(k, j)\eta^*_{r-k,c-j}(v, h)\,.$$

This holds for all $(v, h) \in \mathbb{N}\times\mathbb{Z}$. Use the convention $\mathcal{N}^*(r, c) = \eta^*_{r,c}(v, h)$ to obtain Eq. (30). ■

Observe that the $\bar{\mathcal{N}}(v, h)$-array and the $\bar{\mathcal{N}}^*(r, c)$-array form a dual pair, in the same way as $\bar{\mathcal{S}}(v, h)$ and $\bar{\mathcal{S}}^*(r, c)$ do: they are just different hierarchical representations of the same function in the four variables r, c, v, h. This follows directly from comparison of their definitions [Eq. (21) and Eq. (27)]. It also means that the pattern $\bar{\eta}_{v_o,h_o}(r, c)$ in $\bar{\mathcal{N}}(v, h)$ can be obtained by extracting the pixels located at the fixed position (v_o, h_o) in all patterns $\bar{\eta}^*_{r,c}(v, h)$ in $\bar{\mathcal{N}}^*(r, c)$. In a similar way, any pattern in $\bar{\mathcal{N}}^*(r, c)$ can be recovered from extracting the proper pixels in $\bar{\mathcal{N}}(v, h)$. The relationship between the $\bar{\mathcal{S}}$-, $\bar{\mathcal{S}}^*$-, $\bar{\mathcal{N}}$-, $\bar{\mathcal{N}}^*$-arrays and their patterns is summarized in Fig. 14(a).

3. The similarity between the DSAP's $\bar{s}^*_{v,h}(r, c)$ and the original $\bar{a}(r, c)$-orbit pattern

Definition 3.1. Two sequences (patterns) $((a(r, c))_{(r,c)\in\mathbb{N}\times\mathbb{Z}}$ and $((b(r, c))_{(r,c)\in\mathbb{N}\times\mathbb{Z}}$ are *numerically similar* when

$$\forall(r, c) \in \mathbb{N}\times\mathbb{Z}: \quad a(r, c) = \lambda b(r, c) + \nu \qquad (31)$$

for some invertible constant $\lambda \neq 0$ (i.e. a so-called *unit* in $\mathcal{R}$), and some constant ν (everything in $\mathcal{R}$!).

Observe that Eq. (31) also implies that $b(r, c) = \lambda^{-1}a(r, c) - \lambda^{-1}\nu$, so the similarity is guaranteed to be symmetric. Numeric similarity also implies *symbolic* similarity, in the sense that the $\bar{a}$-pattern and the $\bar{b}$-pattern can be transformed into each other through a 1–1 symbolic substitution. The underlying symbols may be just the coloured pixels used in the visualization (numeric and symbolic similarity were introduced earlier in [Barbé, 1992]).

The next property shows that there is a rather direct relationship between the (r_0, c_0)-DSAP and a *restricted and shifted version* $\bar{a}(v + r_0, h + c_0) \stackrel{\text{def}}{=} (a(v + r_0, h + c_0))_{(v,h)\in\mathbb{N}\times\mathbb{Z}}$ of the original orbit pattern. $\bar{a}(v + r_0, h + c_0)$ is obtained from the original orbit pattern $\bar{a}(v, h)$, by deleting the first r_0 rows (row 0 to row $r_0 - 1$), and by shifting the remaining 'decapitated' pattern over c_0 columns to the left (when $c_0 < 0$, it is actually a shift over $|c_0|$ columns to the right).

Property 3.1. *When β is an invertible element of R, the restricted and shifted pattern $\bar{a}(v + r_0, h + c_0)$ is numerically (and symbolically) similar to the (r_0, c_0)-DSAP $\bar{s}^*_{r_0,c_0}(v, h)$, as given by*

$$\bar{a}(v+r_0, h+c_0) = \beta^{-1}[\bar{s}^*_{r_0,c_0}(v, h) - \alpha a_{r_0,c_0}] \qquad (32)$$

or

$$\bar{s}^*_{r_0,c_0}(v, h) = \beta\bar{a}(v + r_0, h + c_0) + \alpha a_{r_0,c_0}\,. \qquad (33)$$

Proof. This is a trivial consequence of the definition of the $s(v, h, r, c)$-function [Eq. (6)], where (r, c) has been fixed to (r_0, c_0). ■

Remarks. When the original orbit pattern $\bar{a}(r, c)$ displays an overall uniform texture (meaning that the pattern $\bar{a}(r, c)$ has the same texture as its shifted versions $\bar{a}(r + v, c + h)$ [for all $(v, h) \in \mathbb{N} \times \mathbb{Z}$], as is the case in our example), Eq. (30) implies that the $\bar{\mathcal{S}}^*(r, c)$-array is symbolically similar to the original orbit $\bar{a}(r, c)$ itself, in the sense that there is a 1–1 mapping between the states in the $\bar{a}(r, c)$-orbit cells and the textures in the $\bar{\mathcal{S}}^*(r, c)$-cells. Hereby, the textures act as a kind of symbolic state. This can be clearly observed in Fig. 6, where there are four different textures, corresponding to, and arranged in the same way, as the numbers 0, 1, 2, 3 in the original orbit for our particular example. For easy comparison, part of this orbit is shown below over the same (r, c)-range as the $\bar{\mathcal{S}}^*(r, c)$-array in Fig. 6 $(0 \le r \le 5, -4 \le c \le 4)$:

3	1	0	2	3	1	0	3	0
3	3	1	2	0	3	3	1	2
0	2	0	1	2	0	2	0	2
0	3	1	1	1	0	3	0	3
0	3	1	2	0	3	1	0	1
3	3	0	1	2	3	3	0	1

Property 3.1 and Corollary 2.2 can be combined to obtain Corollary 3.1.

Corollary 3.1. *When β is an invertible element of $\mathcal{R}$, the restricted and shifted orbit pattern $\bar{a}(v + r_0, h + c_0)$ is a weighted sum of the regular companion $\bar{\eta}^*_{r_0,c_0}(v, h)$ of the (r_0, c_0)-DSAP and of a rowwise constant and periodic pattern:*

$$\bar{a}(v+r_0, h+c_0) = \beta^{-1}\{\bar{\eta}^*_{r_0,c_0}(v, h) - \alpha[R(1)]^v a_{r_0,c_0}\}\,.$$

Remarks. Property 3.1 and Corollary 3.1 have a special meaning for $r_0 = 0$, as they produce a relationship between the *complete* original $\bar{a}$-orbit pattern and the basic c_0-DSAP's $\bar{s}^*_{0,c_0}(v, h)$ or their regular companions $\bar{\eta}^*_{0,c_0}(v, h)$:

$$\bar{a}(v, h + c_0) = \beta^{-1}[\bar{s}^*_{0,c_0}(v, h) - \alpha a_{0,c_0}]$$

$$\bar{a}(v, h + c_0) = \beta^{-1}\{\bar{\eta}^*_{0,c_0}(v, h) - \alpha[R(1)]^v a_{0,c_0}\}\,.$$

4. Surfing in the other directions of the $s(v, h, r, c)$-space

The SAP-array $\bar{\mathcal{S}}(v, h)$ and its dual $\bar{\mathcal{S}}^*(r, c)$, are two possible representations of the four-dimensional sequence $\bar{s}(v, h, r, c)$ defined by the complete shift-add correlation function [Eq. (6)]. Both provide an additional view on the complexity of a given original CA orbit by revealing underlying hidden structures in it. The $\bar{\mathcal{S}}(v, h)$-array may be interpreted as a way of "surfing" through this four-dimensional space by moving through the two-dimensional discrete coordinate space (v, h) and thereby contemplating the emerging two-dimensional $\bar{s}_{v,h}(r, c)$-patterns in (r, c)-coordinate space. $\bar{\mathcal{S}}^*(r, c)$ does so in a dual fashion. There are still four other complementary possibilities of travelling around in this *bi*-two-dimensional space. These can be grouped in two

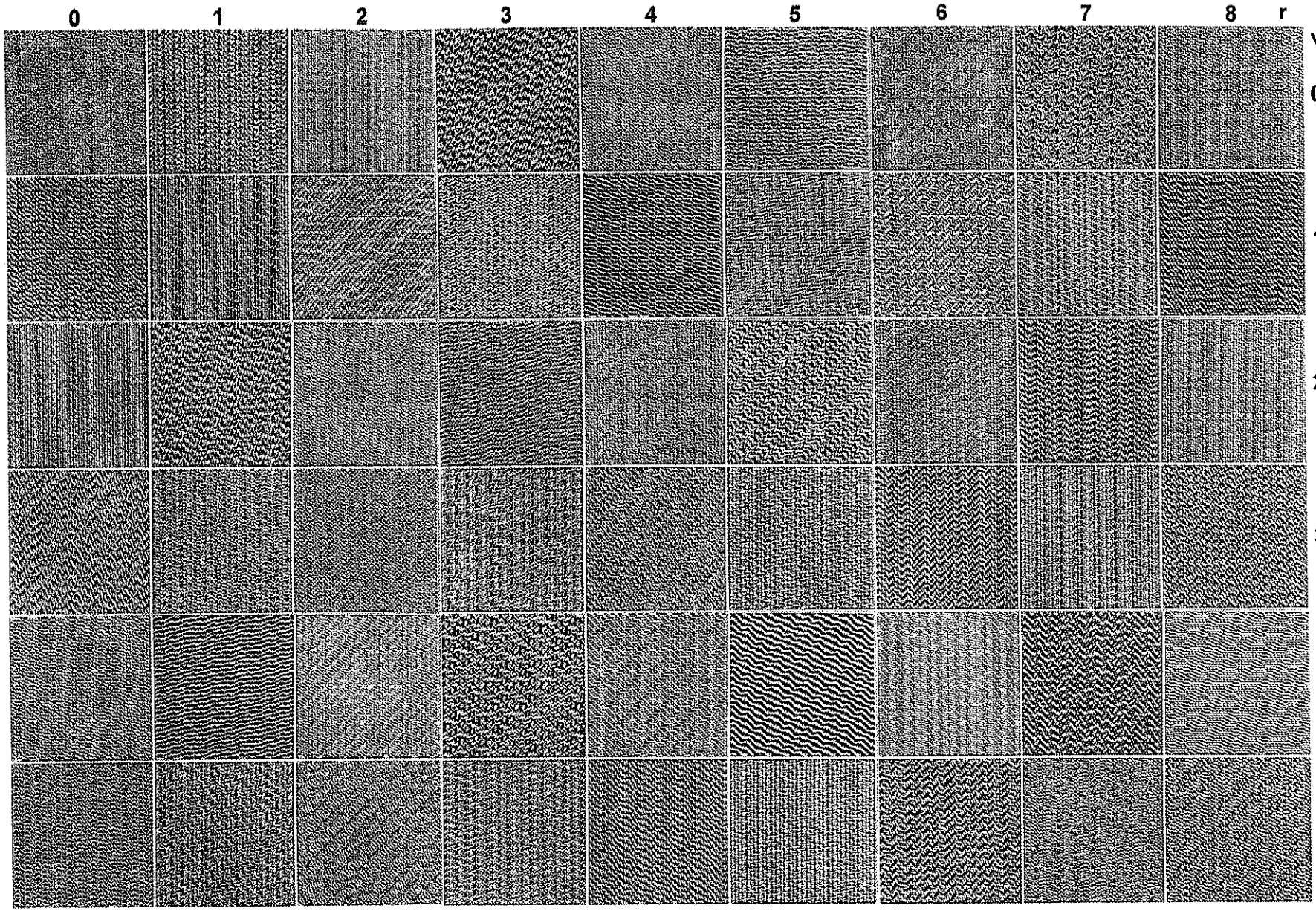


Fig. 10. The $\bar{T}(v, r)$-array of $\bar{s}_{v,r}(h, c)$-patterns. [$0 \leq v \leq 5$, $0 \leq r \leq 8$; $-80 \leq h \leq 80$ (vertical direction, increasing h-value downwards), $-80 \leq c \leq 80$ (horizontal direction)]. The top-left pattern $\bar{s}_{0,0}(h, c)$ is actually the SAP-signature.

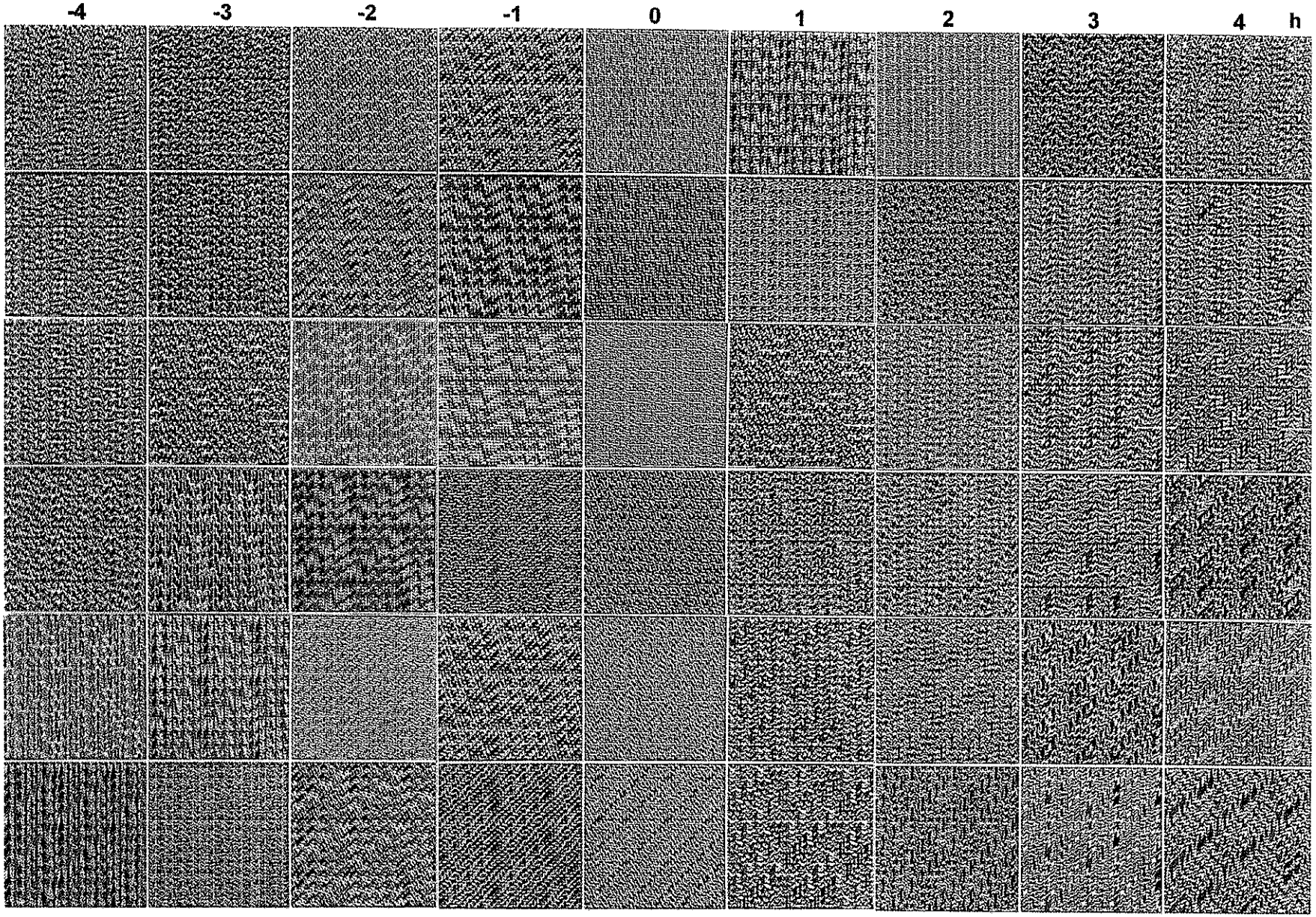


Fig. 11. The $\bar{T}^*(h, c)$-array of $\bar{s}^*_{h,c}(v, r)$-patterns. [$-2 \leq h \leq 3$, $-4 \leq c \leq 4$; $0 \leq v \leq 160$ (vertical direction), $0 \leq r \leq 160$ (horizontal direction)].

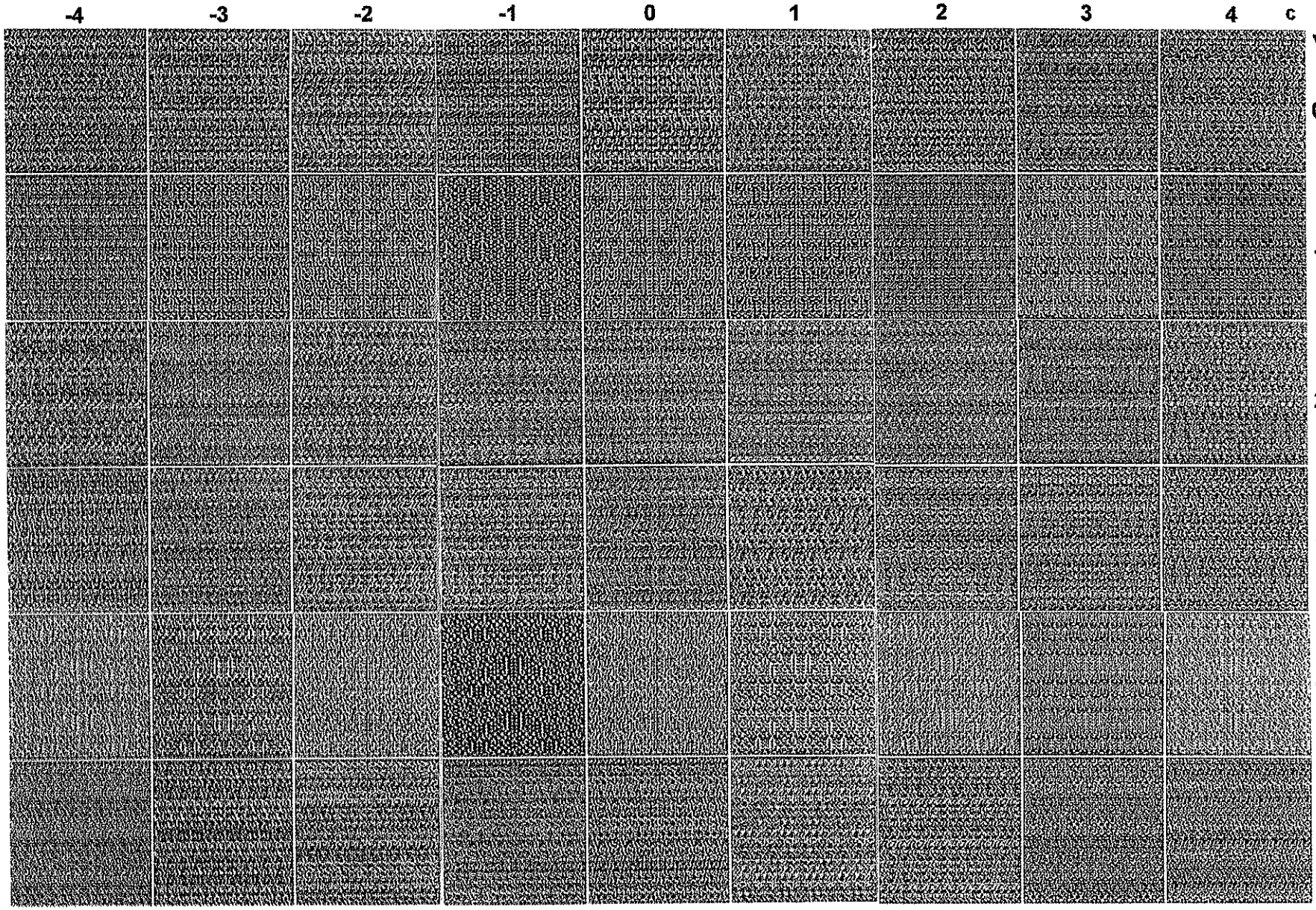


Fig. 12. The $\bar{\mathcal{U}}(v, c)$-array of $\bar{s}_{v,c}(r, h)$-patterns. [$0 \leq v \leq 5$, $-4 \leq c \leq 4$; $0 \leq r \leq 160$ (vertical direction), $-80 \leq h \leq 80$ (horizontal direction)].

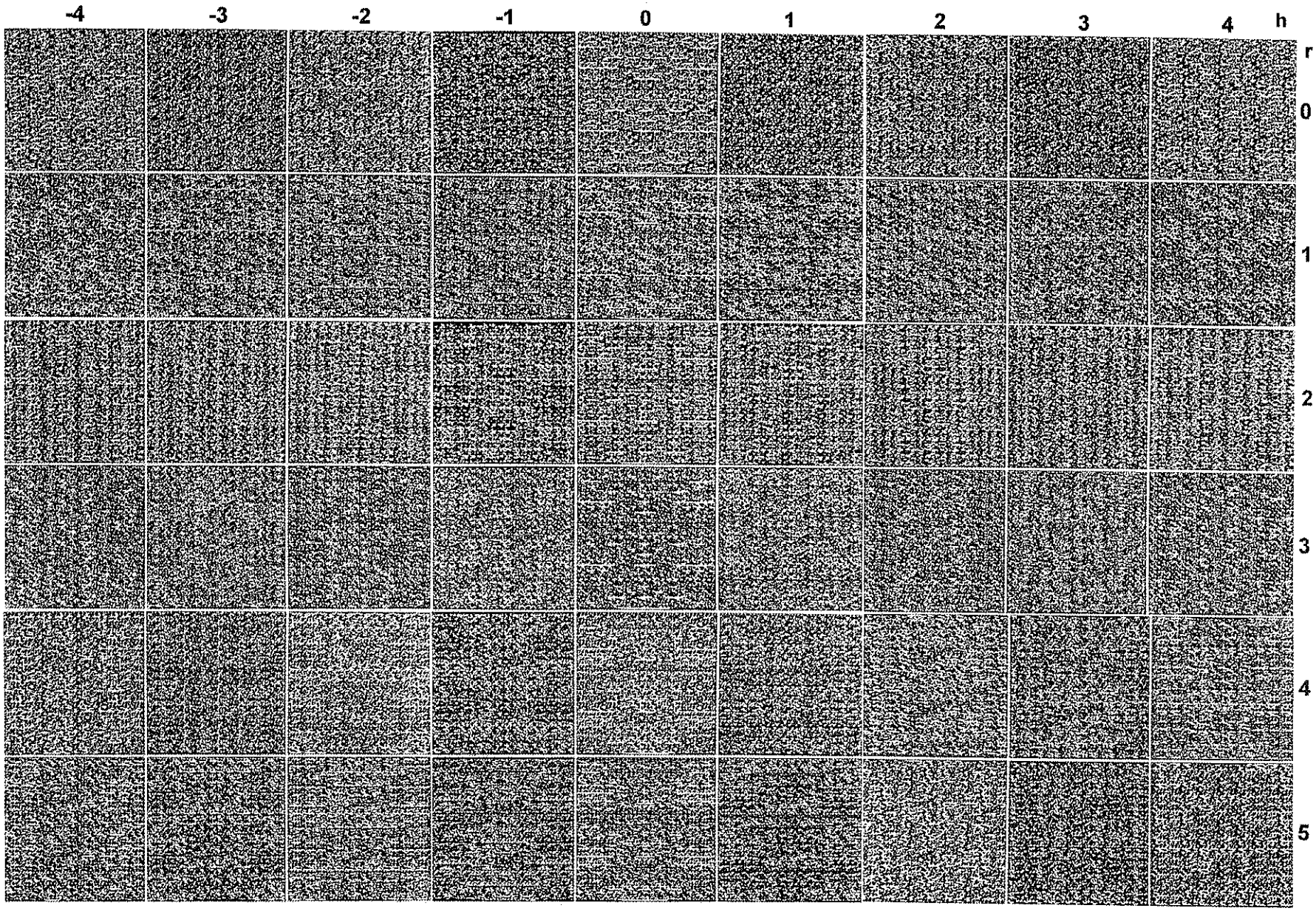


Fig. 13. The $\mathcal{U}^*(r, h)$-array of $\bar{s}^*_{r,h}(v, c)$-patterns. [$0 \leq r \leq 5$, $-4 \leq h \leq 4$; $0 \leq v \leq 160$ (vertical direction), $-80 \leq c \leq 80$ (horizontal direction)].

dual pairs. We define here these possibilities, and relate the new patterns to the SAP- and DSAP-arrays:

- the $\bar{T}(v, r)$-array of $\bar{s}_{v,r}(h, c)$-patterns.

  (recall that $\bar{s}_{v,r}(h, c)$ is a restricted version of the complete shift-add correlation function $s(v, h, r, c) = \alpha a(r, c) + \beta a(r + v, c + h)$, with (v, r) considered fixed and with (h, c) as actual variables.)

  The $\bar{s}_{v,r}(h, c)$-pattern, which is an element in this $\bar{T}(h, c)$-array, can also be considered as the vertical stacking of the r-rows which are extracted from the patterns on row v in the SAP-array $\bar{\mathcal{S}}(v, h)$.

  Part of the $\bar{T}(v, r)$-array for our particular example is shown in Fig. 10.
- the $\bar{T}^*(h, c)$-array of $\bar{s}^*_{h,c}(v, r)$-patterns.

  It is the dual of the $\bar{T}(v, r)$-array, and its elements are the $\bar{s}^*_{h,c}(v, r)$-patterns (the * just being an extra indication of the dual character).

  The $\bar{s}^*_{h,c}(v, r)$-pattern is the vertical stacking of the c-columns which are extracted from the patterns on column h in the SAP-array $\bar{\mathcal{S}}(v, h)$ (after having rotated these c-columns in horizontal position first).

  Part of the $\bar{T}^*(h, c)$-array is shown in Fig. 11.
- the $\bar{U}(v, c)$-array of $\bar{s}_{v,c}(r, h)$-patterns.

  The $\bar{s}_{v,c}(r, h)$-pattern is the vertical stacking of the v-rows extracted from the patterns on column c in the DSAP-array $\bar{\mathcal{S}}^*(r, c)$.

  Part of the $\bar{U}(v, c)$-array is shown in Fig. 12.
- the $\bar{U}^*(r, h)$-array of $\bar{s}^*_{r,h}(v, c)$-patterns.

  This is the dual of $\bar{U}(v, c)$.

  The $\bar{s}^*_{r,h}(v, c)$-pattern is the vertical stacking of the h-columns which are extracted from the patterns on row r in the DSAP-array $\bar{\mathcal{S}}^*(r, c)$ (after having rotated the h-columns in horizontal position first).

  Part of the $\bar{U}^*(r, h)$-array is shown in Fig. 13.

Notice that, contrary to the case for the SAP- and DSAP-arrays, there is no *explicit* relationship (CA- or pseudo-CA-like, or something of the same kind) among the patterns themselves in these $\bar{T}$-, $\bar{T}^*$-, $\bar{U}$-, $\bar{U}^*$-arrays. Neither is this the case for the pixels inside a pattern.

5. SAP's and DSAP's of SAP's and DSAP's

Let $\bar{p}(x, y)$ represent any of the patterns in the $\bar{\mathcal{S}}$-, $\bar{\mathcal{S}}^*$-, $\bar{\mathcal{N}}$-, $\bar{\mathcal{N}}^*$-, $\bar{T}$-, $\bar{T}^*$-, $\bar{U}$-, $\bar{U}^*$-arrays, whereby (x, y) has to be substituted by the proper variable pair from (r, c, v, h). It is then possible to construct a complete shift-add correlation function ${}^{(\bar{p})}s(x, y, u, w)$ for this pattern:

$$ {}^{(\bar{p})}s(x, y, u, w) = \alpha' p(x, y) + \beta' p(x + u, y + w) . $$

This can be taken as the basis for new $\bar{\mathcal{S}}$-, $\bar{T}$-, $\bar{U}$-like arrays, to be represented as ${}^{(\bar{p})}\bar{\mathcal{S}}$, ${}^{(\bar{p})}\bar{T}$, ${}^{(\bar{p})}\bar{U}$ (and starred duals). Then each pattern $\bar{p}'$ in one of these new arrays can be at the origin of a new set of arrays ${}^{(\bar{p}')}\bar{\mathcal{S}}$, ${}^{(\bar{p}')}\bar{T}$, ${}^{(\bar{p}')}\bar{U}$ (and starred duals). This procedure for constructing new pattern-arrays can be continued indefinitely. These patterns actually visualize *higher-order* shift-add correlations of the original orbit pattern. For example, the (v', h')-SAP of the (v, h)-SAP of the original orbit pattern can be explicitized as follows. Let

$$ \bar{p}(r, c) = \bar{s}_{v,h}(r, c) \quad [= (v, h)\text{−SAP of } a(r, c)] , $$

then

$$ \begin{aligned} {}^{(\bar{p})}s_{v',h'}(r, c) &= \alpha' s_{v,h}(r, c) + \beta' s_{v,h}(r + v', c + h') \\ &= \alpha'[\alpha a(r, c) + \beta a(r + v, c + h)] \\ &\quad + \beta'[\alpha a(r + v', c + h') \\ &\quad + \beta a(r + v + v', c + h + h')] , \end{aligned} $$

$[(v, h, v', h')$ fixed, (r, c) variable$]$.

Among the chains of correlation patterns of increasing order that can be obtained in the above-mentioned way, the chains formed by taking SAP's of SAP's are particularly interesting: all patterns and associated $\bar{\mathcal{S}}$-, $\bar{\mathcal{S}}^*$-, $\bar{\mathcal{N}}$-, $\bar{\mathcal{N}}^*$-arrays in these chains, always form a CA- or a pseudo-CA orbit (recall that for the patterns, the states are the elements of $\mathcal{R}$; for the arrays, the states are CA- or pseudo-CA orbit patterns over $\mathcal{R}$ themselves). This is clear from the results in Sec. 2: an SAP was defined for an original CA orbit pattern (with a given rule $R(x)$); but, according to Property 2.1.A, this SAP itself is a new CA orbit pattern (governed by the same rule $R(x)$). As such, it can be the basis of a new SAP, which will also be a CA orbit pattern (with the same rule). And so we can continue without limits. Each new step generates associated $\bar{\mathcal{S}}$-, $\bar{\mathcal{S}}^*$-, $\bar{\mathcal{N}}$-, $\bar{\mathcal{N}}^*$-arrays and related patterns. The patterns in the $\bar{\mathcal{N}}$-, $\bar{\mathcal{N}}^*$-arrays are also CA-patterns for which an SAP can be produced. So, any original CA orbit pattern may produce different chains of CA-patterns, by continuously taking SAP's from an SAP or from its associated regular companions

$\bar{\eta}$ and $\bar{\eta}^*$ [actually, $\bar{\eta}^*$ is produced in a DSAP-step, as is illustrated in Fig. 14(a)].

Observe that in Fig. 14(a), there is one kind of pattern that has a pseudo-CA nature (pCA): the dual SAP, $\bar{s}^*_{v,h}(r, c)$. What happens when we construct the SAP or DSAP of this last pattern? The fascinating thing is that this will again produce pseudo-CA orbits (and associated regular companion orbits). This is a direct consequence of the next property.

Property 5.1. *Let $\bar{q}(x, y)$ be a pseudo-CA orbit, satisfying*

$$q(x, y) = \sum_{j \in R_k} \rho(k, j) q(x - k, y - j) + \gamma\theta(k) .$$

Define its complete shift-add correlation function:

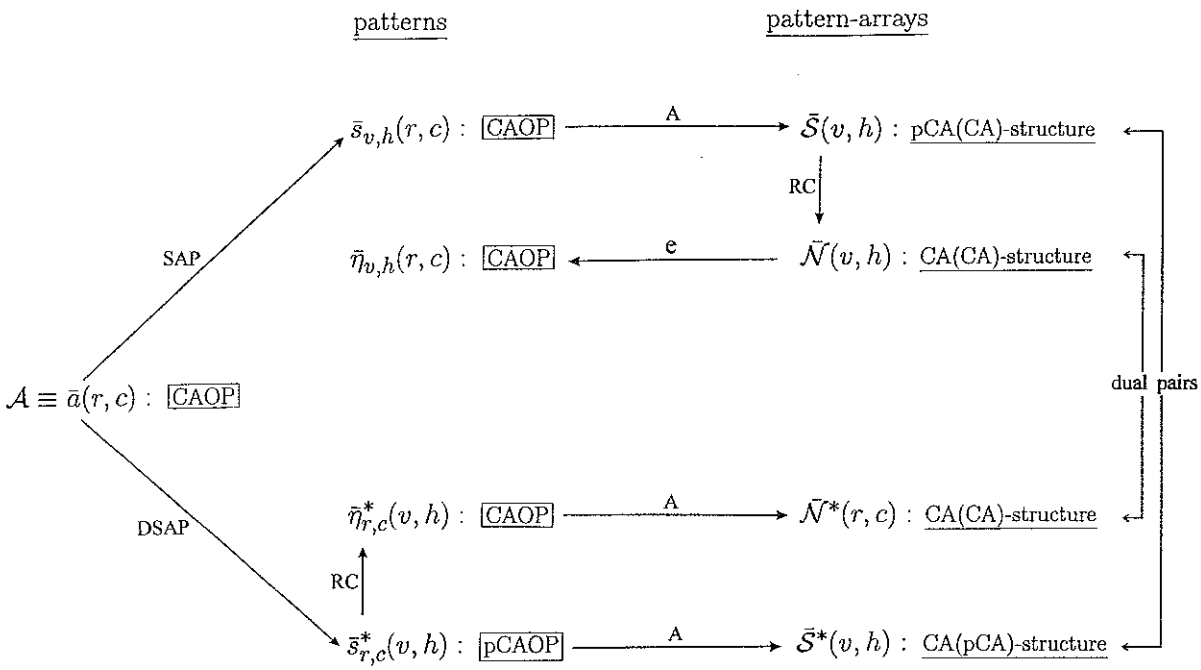
$$f(x, y, u, w) \stackrel{\text{def}}{=} \alpha' q(x, y) + \beta' q(x + u, y + w) , \tag{34}$$

and its associated (u, w)-SAP's: $\bar{f}_{u,w}(x, y)$; by considering (u, w) fixed in Eq. (34). Define also the dual (x, y)-DSAP: $\bar{f}^*_{x,y}(u, w)$; by fixing (x, y) in Eq. (34).

Then:

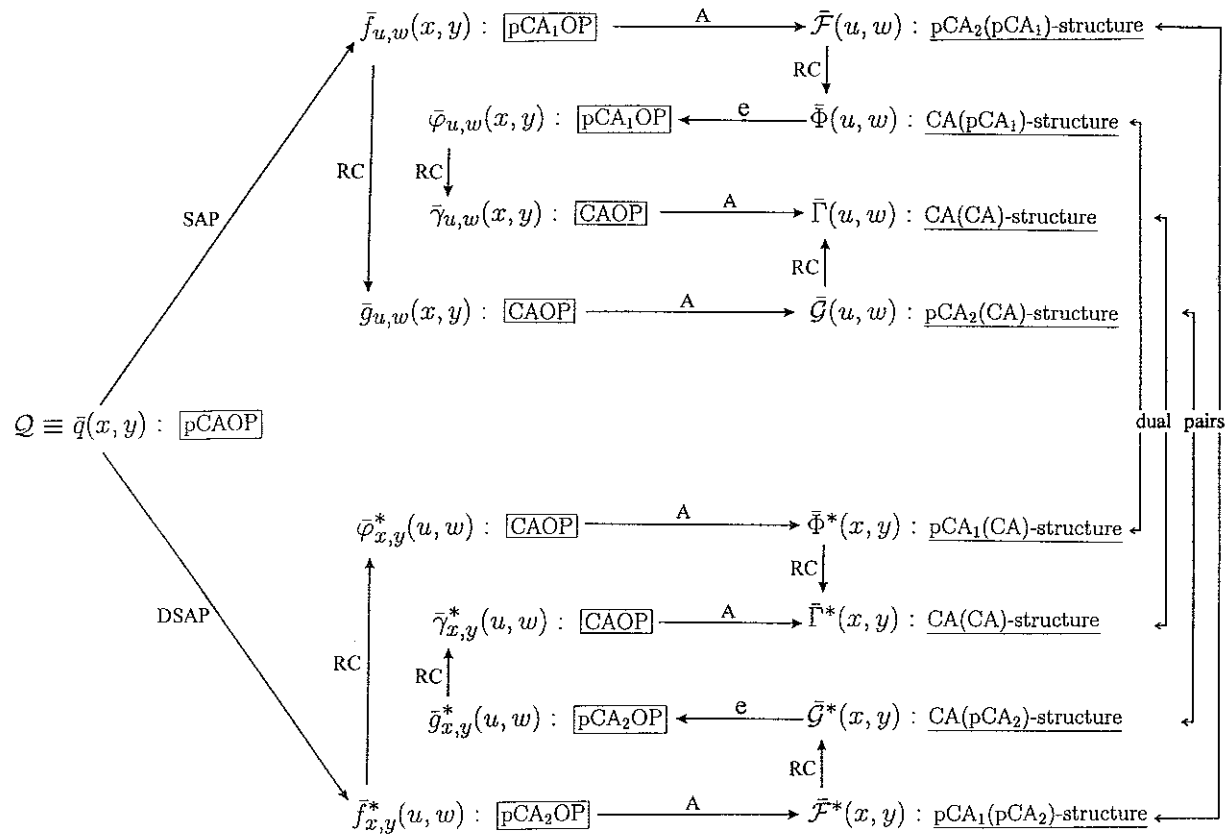
A. *Both $\bar{f}_{u,w}(x, y)$ and $\bar{f}^*_{x,y}(u, w)$ are pseudo CA orbits, satisfying:*

$$f_{u,w}(x, y) = \sum_{j \in R_k} \rho(k, j) f_{u,w}(x - k, y - j) + \gamma(\alpha' + \beta')\theta(k) ,$$



(a)

Fig. 14. (a) Relationship between different patterns and pattern-arrays that can be obtained from constructing the SAP's and the DSAP's of a CA orbit pattern. (b) Relationship between the different patterns and arrays that can be obtained from constructing the SAP's and DSAP's of a pseudo-CA orbit pattern. Higher order shift-add correlation patterns and corresponding arrays can be obtained by applying scheme (a) to any CA orbit pattern appearing in (a) or (b), or by applying scheme (b) to any pCA orbit pattern in (a) or (b). Notations: CAOP = CA orbit pattern; pCA$_i$OP = pseudo-CA orbit pattern [the index i, when present, indicates the specific form: $i = 1$ corresponds to the "pseudo"-term $\gamma(\alpha' + \beta')\theta(k)$, $i = 2$ corresponds to the "pseudo"-term $(\alpha' \mathcal{Q} + \beta'\gamma)\theta(k)$]. For the arrays, pCA$_i$(CA)-structure means that the structure is pseudo-CA-like on the array-level (corresponding to the specific value of the index i), while there is a regular CA-structure on the pattern-level. $\bar{x}_{a,b}(c, d) \xrightarrow{\text{A}} \bar{X}(a, b)$ refers to the construction of the array $\bar{X}(a, b)$ with elements $\bar{x}_{a,b}(c, d)$. $\bar{X}(a, b) \xrightarrow{\text{e}} \bar{x}_{a,b}(c, d)$: consider the general element $\bar{x}_{a,b}(c, d)$ from the array $\bar{X}(a, b)$. $\bar{X} \xrightarrow{\text{RC}} \bar{Y}$ indicates the construction of the regular companion $\bar{Y}$ of $\bar{X}$ (i.e. on the level of a pattern when $\bar{X}$ is a pattern, on the level of an array when $\bar{X}$ is an array; $\bar{Y}$ is always of the same nature as $\bar{X}$).



(b)

Fig. 14. (*Continued*)

$$f^*_{x,y}(u,w) = \sum_{j\in R_k} \rho(k,j) f_{x,y}(u-k, w-j) + (\alpha' q_{x,y} + \beta'\gamma)\theta(k)\,.$$

B. *The corresponding SAP-array* $\bar{\mathcal{F}}(u,w)$ *[with elements* $\mathcal{F}(u,w) = \bar{f}_{u,w}(x,y)$*] and DSAP-array* $\bar{\mathcal{F}}^*(x,y)$ *[with elements* $\mathcal{F}^*(x,y) = \bar{f}^*_{x,y}(u,w)$*] also have a pseudo-CA structure (with patterns as states). Formally:*

$$\mathcal{F}(u,w) = \sum_{j\in R_k} \mathcal{F}(u-k, w-j) + (\alpha'\mathcal{Q} + \beta'\gamma)\theta(k)\,,$$

where $\mathcal{Q} = \bar{q}(x,y)$.

$$\mathcal{F}^*(x,y) = \sum_{j\in R_k} \mathcal{F}^*(x-k, y-j) + \gamma(\alpha'+\beta')\theta(k)\,.$$

Proof. The proof runs along the same lines as those for Properties 2.1.A and 2.4.A (for part A.), and Properties 2.2 and 2.5 (for part B.). Where reference is made to Lemmas 2.1 and 2.2, these should be replaced by their equivalent ones for the complete shift-add correlation function of the pseudo-CA:

$$f(x,y,u,w) = \sum_{j\in R_k} \rho(k,j) f(x-k, y-j, u, w) + \gamma(\alpha'+\beta')\theta(k)\,,$$

$$f(x,y,u,w) = \sum_{j\in R_k} \rho(k,j) f(x,y,u-k,w-j) + [\alpha' q(x,y) + \beta'\gamma]\theta(k)\,.$$

(to be proven in the same way as the original Lemmas). ■

The SAP's and DSAP's of a pseudo-CA orbit pattern themselves being pseudo-CA orbit patterns, we can construct the regular companions of the last ones; and the associated arrays of regular companions. Also, as the $\bar{\mathcal{F}}$- and $\bar{\mathcal{F}}^*$-arrays exhibit a pseudo-CA structure, we can construct the regular companion arrays. As a matter of fact, as the $\bar{\mathcal{F}}$- and $\bar{\mathcal{F}}^*$-arrays have a two-level pseudo-CA structure, the situation has become a bit more complicated than in the case where we had a regular CA orbit as starting point, and where the associated pattern-arrays had pseudo-CA structure on

one level only (and regular CA-structure on the remaining level). The patterns in the regular companion array of $\bar{\mathcal{F}}$, i.e. in $\bar{\Phi}$, remain pseudo-CA-like (it is only on the array-level that $\bar{\Phi}$ has a regular CA-structure), and so we can construct the regular companions of the patterns in $\bar{\Phi}$. These regular companions can then again be put in an array $\bar{\Gamma}$, which is now CA-like on both levels.

Figure 14(b) summarizes the relationship between all possible patterns and arrays that can be obtained from the SAP's and DSAP's of a pseudo-CA orbit pattern. (We skip the proofs, which are as elementary as the ones in Sec. 2. Just keep in mind that a regular companion is the regular CA with the same rule and initial state configuration as the pseudo-CA of which it is the companion.)

Combining the schemes of Fig. 14(a) and Fig. 14(b) shows how an unlimited number of unlimited chains of CA- and pseudo-CA orbit patterns can be constructed from just one given original CA: each CA orbit pattern (CAOP) that appears as output in one of the diagrams shown in Fig. 14(a) and 14(b) can be considered as a new input for the scheme in Fig. 14(a). Also, each pseudo-CA orbit pattern (pCAOP) appearing as output of the diagrams in Fig. 14(a) and 14(b), can be taken as a new input for the scheme in Fig. 14(b).

6. Conclusion

We have started with the array of shift-add correlation patterns associated with a given orbit pattern of a linear CA over a finite commutative ring. As shown in earlier papers, the patterns in this array may possibly uncover hidden structures in that original CA-pattern. We have shown that this array can be considered as a pseudo-CA orbit pattern, where the states of this pseudo-CA are orbit patterns of a common regular CA. The underlying CA-rule for the regular as well as for the pseudo-CA orbits is the same as the one for the original orbit.

Then a dual version of this SAP-array has been introduced: the array of dual shift-add correlation patterns. Here the elements of the array are pseudo-CA orbit patterns, whereas the patterns among themselves form a regular CA orbit.

For the above-mentioned CA orbits, we have defined the corresponding regular companions. This produced a dual pair of pattern arrays $\bar{\mathcal{N}}(v, h)$ and $\bar{\mathcal{N}}^*(r, c)$. Each of these arrays can be considered as a two-level nesting of regular CA orbits. The top-row patterns in these regular companion arrays are the regular CA orbits generated respectively by the horizontal and vertical rows in the SAP-signature. This SAP-signature itself is an ordered representation of all shift-add correlation sequences of the initial state configuration of the original orbit.

The SAP's and DSAP's, together with two additional dual pairs, all provide complementary views on the complete shift-add correlation function. The examples that were shown in this paper give only a small glimpse of the huge diversity of patterns that hereby emerges. And these patterns are all related to a single given original CA orbit. Each new orbit produces an additional infinitude of new SAP-related patterns.

It has also been shown that one can produce SAP's and DSAP's of SAP's and DSAP's in an unlimited way. An interesting question here is whether a particular sequence of consecutive SAP's may become ultimately periodic.

A final note: readers who are able to "stereogaze" could try this on the individual patterns shown here. In most cases, a three-dimensional abstract structure will emerge.

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Appendix A

Addition and multiplication in GF(4)

The addition and multiplication tables of the numbers 0, 1, 2, 3 in GF(4) are (see, for example, Mc. Eliece [1987]):

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Appendix B

The coded Thue-Morse sequence s_{CTM}

The Thue-Morse sequence $(s_{TM}(n))_{n\in\mathbb{N}}$ is the binary sequence defined by

$$\begin{aligned} s_{TM}(n) &= 0 \text{ if the number of 1's in the binary representation of } n \text{ is even} \\ &= 1 \text{ if the number of 1's in the binary representation of } n \text{ is odd}. \end{aligned}$$

(see, for example, Schroeder [1990]).

First, we will *extend* this sequence over $\mathbb{Z}$, by creating the sequence $(s_{ETM}(n))_{n\in\mathbb{Z}}$ obtained by reflecting the Thue-Morse sequence around $n = 0$ in a complementwise way:

$$\begin{array}{lll} \text{for } n \in \mathbb{N}: & s_{ETM}(n) = s_{TM}(n) & \\ \text{for } n < 0: & s_{ETM}(n) = 0 & \text{when } s_{TM}(|n| - 1) = 1 \\ & = 1 & \text{when } s_{TM}(|n| - 1) = 0. \end{array}$$

Then define the *coded* extended Thue-Morse sequence $(s_{CTM}(n))_{n\in\mathbb{Z}}$, as follows:

$$s_{CTM}(n) = 2s_{ETM}(2n+1) + s_{ETM}(2n+2).$$

This produces a sequence consisting of the numbers 0, 1, 2, 3, and codes the extended Thue-Morse sequence by partitioning it in consecutive strings of length two, and by replacing each binary 2-string thus obtained by its 4-adic equivalent. A few elements of the Thue-Morse sequence and of its extended and coded equivalent are shown here:

n	=	⋯	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	⋯
$s_{ETM}(n)$	=	⋯	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	1	⋯
coding		⋯		**3**		**1**		**0**		**2**		**3**		**1**		**0**		**3**		⋯
$s_{CTM}(n)$	=	⋯	0	2	3	0	**3**	1	**0**	**2**	**3**	1	**0**	**3**	0	2	3	1	0	⋯