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THE EXTENDED LINEAR COMPLEMENTARITY PROBLEM AND ITS APPLICATIONS IN THE MAX-PLUS ALGEBRA*

BART DE SCHUTTER AND BART DE MOOR†

Abstract. In this paper we give a survey of our research on the Extended Linear Complementarity Problem (ELCP). First we discuss the link between the ELCP and other generalizations of the Linear Complementarity Problem, and we present an algorithm to find all the solutions of an ELCP. Next we introduce the max-plus algebra and show how it can be used to model a certain class of discrete event systems. Finally we show that the ELCP can be used to solve many important problems in the max-plus algebra.

Key words. Linear complementarity problem, generalized linear complementarity problem, discrete event systems, max-plus algebra.

1. Introduction. In this survey paper we present the main results of our research on the Extended Linear Complementarity Problem (ELCP) and on the applications of the ELCP in the max-plus algebra.

The ELCP is an extension of the Linear Complementarity Problem (LCP). Many other generalizations of the LCP are special cases of the ELCP and in a way the ELCP can be considered as the "most general linear" generalization of the LCP. We present an algorithm to find all solutions of an ELCP. This algorithm yields a description of the complete solution set of an ELCP by finite points, generators for the extreme rays and a basis for the linear subspace associated with the largest affine subspace of the solution set. In that way it provides an insight in the geometrical structure of the solution set of the ELCP and related problems.

The formulation of the ELCP arose from our work in the study of discrete event systems, examples of which are flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication networks and logistic systems. We introduce the max-plus algebra (which has maximization and addition as basic operations) and we show by an example how a certain class of discrete event systems can be modeled by a state space model that is "linear" in the max-plus algebra. Next we indicate how the ELCP can be used to solve a system of multivariate max-algebraic polynomial equalities and inequalities. This allows us to solve many other problems in the max-plus algebra such as computing max-algebraic matrix factorizations, performing state space transformations for max-linear time-invariant discrete event systems, computing state space realizations of the impulse response of a max-linear time-invariant discrete event system and so on.

1.1. Notations and definitions. If a is a vector then a_i represents the ith component of a. If A is an m by n matrix then the entry on the ith row and the jth column is denoted by a_{ij}. We use $A_{\cdot,j}$ to represent the jth column of A. The n by n identity matrix is denoted by I_n and the m by n zero matrix by $O_{m\times n}$.

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We assume that the reader is familiar with the basic concepts of linear complementarity problems, polyhedra and linear inequalities (See e.g. Chapters 1 and 2 of [6] for an introduction to these subjects).

If $\mathcal{P}$ is a polyhedron defined by $\mathcal{P} = \{x \mid Ax \geq b\}$, then the lineality space of $\mathcal{P}$ is denoted by $\mathcal{L}(\mathcal{P})$: $\mathcal{L}(\mathcal{P}) = \{x \mid Ax = 0\}$. In this paper points of $\mathcal{L}(\mathcal{P})$ are called *central generators* of $\mathcal{P}$. We say that a set of central generators is minimal and complete if it is a basis of $\mathcal{L}(\mathcal{P})$. A minimal face of $\mathcal{P}$ is a face that does not contain any other face of $\mathcal{P}$.

Consider a polyhedral cone $\mathcal{K}$ defined by $\mathcal{K} = \{x \mid Ax \geq 0\}$. Let t be the dimension of $\mathcal{L}(\mathcal{K})$. A face of $\mathcal{K}$ of dimension $t+1$ is called a *minimal proper face*. If G is a minimal proper face of $\mathcal{K}$ and if $e \in G$ with $e \notin \mathcal{L}(\mathcal{K})$, then any arbitrary point u of G can be represented as

$$u \;=\; \sum_{c_k \in \mathcal{C}} \lambda_k c_k \;+\; \kappa e \qquad \text{with } \lambda_k \in \mathbb{R} \text{ for all } k \text{ and } \kappa \geq 0 \;\;,$$

where $\mathcal{C}$ is a minimal complete set of central generators of $\mathcal{K}$. We call e an *extreme generator* corresponding to G. If $\mathcal{C}$ is a minimal complete set of central generators of $\mathcal{K}$ and if $\mathcal{E}$ is a minimal complete set of extreme generators of $\mathcal{K}$, i.e. a set obtained by selecting exactly one point of each minimal proper face of $\mathcal{K}$, then any arbitrary point u of $\mathcal{K}$ can be represented uniquely as

$$u \;=\; \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geq 0 \text{ for all } k\,. \tag{1}$$

One of the possible formulations of the Linear Complementarity Problem (LCP) is the following:

Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find $w,\, z \in \mathbb{R}^n$ such that

$$\begin{aligned} w, z &\geq 0 \\ w &= q + Mz \\ z^T w &= 0 \;\;, \end{aligned}$$

or show that no such w and z exist.

2. The Extended Linear Complementarity Problem. In this section we introduce the Extended Linear Complementarity Problem (ELCP). We discuss the connection between the ELCP and various generalizations of the LCP. We present an algorithm to find all the solutions of an ELCP and we give a geometric characterization of the solution set of a general ELCP.

2.1. Problem formulation. Consider the following problem:

Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_1, ϕ_2, ..., ϕ_m of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i \;=\; 0 \tag{2}$$

subject to

$$Ax \geq c \tag{3}$$

$$Bx = d \;\;, \tag{4}$$

or show that no such x exists.

We call this problem the Extended Linear Complementarity Problem since it is an extension of the Linear Complementarity Problem: if we set $x = \left[\begin{array}{cc} w^T & z^T \end{array}\right]^T$, $A = I_{2n}$, $B = [\, I_n \; -M \,]$, $c = O_{2n\times 1}$, $d = q$ and $\phi_j = \{j, j+n\}$ for $j = 1, 2, \ldots, n$ in the formulation of the ELCP, we get an LCP. So the LCP can be considered as a particular case of the ELCP.

Since (2) corresponds to the complementarity condition $z^T w = 0$ of the LCP, we call (2) the *complementarity condition* of the ELCP. One possible interpretation of this condition is the following. Since $Ax \geq c$, all the terms in (2) are nonnegative. Hence, condition (2) is equivalent to

$$\prod_{i\in\phi_j} (Ax - c)_i \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, m\,.$$

So we could say that each set ϕ_j corresponds to a group of inequalities of $Ax \geq c$ and that in each group at least one inequality should hold with equality, i.e. the corresponding residue should be equal to 0:

$$\forall j \in \{1, 2, \ldots, m\} : \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \;.$$

2.2. The ELCP and other extensions of the LCP. It is easy to verify that the following extensions of the LCP are also special cases of the ELCP (See [10, 11] for proofs):

- the Vertical LCP of Cottle and Dantzig [5]:
  Let $M \in \mathbb{R}^{m\times n}$ with $m \geq n$ and let $q \in \mathbb{R}^m$. Suppose that M and q are partitioned as follows:

$$M = \left[\begin{array}{c} M_1 \\ M_2 \\ \vdots \\ M_n \end{array}\right] \quad \text{and} \quad q = \left[\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_n \end{array}\right]$$

  with $M_i \in \mathbb{R}^{m_i\times n}$ and $q_i \in \mathbb{R}^{m_i}$ for $i = 1, 2, \ldots, n$ and with $\displaystyle\sum_{i=1}^{n} m_i = m$.
  Now find $z \in \mathbb{R}^n$ such that

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z_i \prod_{j=1}^{m_i} (q_i + M_i z)_j &= 0 \qquad \text{for } i = 1, 2, \ldots, n\,; \end{aligned}$$

- the Horizontal LCP:
  Given M, $N \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, find non-trivial w, $z \in \mathbb{R}^n$ such that

$$\begin{aligned} w, z &\geq 0 \\ Mz + Nw &= q \\ z^T w &= 0 \;; \end{aligned}$$

- the Generalized LCP of De Moor [9]:
  Given $Z \in \mathbb{R}^{p\times n}$ and m subsets ϕ_1, ϕ_2, ..., ϕ_m of $\{1,2,\ldots,p\}$, find a non-trivial $u \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i\in\phi_j} u_i \;=\; 0$$

  subject to $u \geq 0$ and $Zu = 0$;
- the Extended Generalized Order LCP of Gowda and Sznajder [21]:
  Given $k+1$ matrices B_0, B_1, ..., $B_k \in \mathbb{R}^{n\times n}$ and $k+1$ vectors b_0, b_1, ..., $b_k \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that

$$(B_0x + b_0) \wedge (B_1x + b_1) \wedge \ldots \wedge (B_kx + b_k) \;=\; 0$$

  where $\wedge$ is the entrywise minimum: if x, $y \in \mathbb{R}^n$ then $x \wedge y \in \mathbb{R}^n$ and $(x \wedge y)_i = \min(x_i, y_i)$ for $i = 1, 2, \ldots, n$;
- the Extended LCP of Mangasarian and Pang [20, 22]:
  Given M, $N \in \mathbb{R}^{m\times n}$ and a polyhedral set $\mathcal{P} \subseteq \mathbb{R}^m$, find x, $y \in \mathbb{R}^n$ such that

$$\begin{aligned} x, y &\geq 0 \\ Mx - Ny &\in \mathcal{P} \\ x^T y &= 0 \;\; ; \end{aligned}$$

- the Generalized LCP of Ye [27]:
  Given A, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times k}$ and $q \in \mathbb{R}^m$, find x, $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^k$ such that

$$\begin{aligned} x, y, z &\geq 0 \\ Ax + By + Cz &= q \\ x^T y &= 0 \;\; ; \end{aligned}$$

- the mixed LCP [6]:
  Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, find $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$\begin{aligned} v &\geq 0 \\ a + Au + Cv &= 0 \\ b + Du + Bv &\geq 0 \\ v^T(b + Du + Bv) &= 0 \;\; ; \end{aligned}$$

- the Extended HLCP of Sznajder and Gowda [26]:
  Given $k+1$ matrices $C_0, C_1, \ldots, C_k \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$ and $k-1$ vectors d_1, d_2, ..., $d_{k-1} \in \mathbb{R}^n$ with positive components, find x_0, x_1, ..., $x_k \in \mathbb{R}^n$ such that

$$C_0x_0 = q + \sum_{j=1}^{k} C_jx_j$$

$$
\begin{aligned}
x_0, x_1, \ldots, x_k &\geq 0 \\
d_j - x_j &\geq 0 \qquad \text{for } j = 1, 2, \ldots, k-1 \\
x_0^T x_1 &= 0 \\
(d_j - x_j)^T x_{j+1} &= 0 \qquad \text{for } j = 1, 2, \ldots, k-1\,;
\end{aligned}
$$

- the generalization of the LCP alluded to in [16]:
  Given n positive integers m_1, m_2, ..., m_n, n matrices A_1, A_2, ..., A_n with $A_i \in \mathbb{R}^{p \times m_i}$ for all i and a vector $b \in \mathbb{R}^p$, find x_1, x_2, ..., x_n with $x_i \in \mathbb{R}^{m_i}$ for all i such that

$$
\begin{aligned}
\sum_{i=1}^{n} \prod_{j=1}^{m_i} (x_i)_j &= 0 \\
\sum_{i=1}^{n} A_i x_i &\leq b \\
x_i &\geq 0 \qquad \text{for } i = 1, 2, \ldots, n\,.
\end{aligned}
$$

So we could say that the ELCP can be considered as a unifying framework for the LCP and its various generalizations. The underlying geometrical explanation for the fact that all these generalizations of the LCP are particular cases of the ELCP is that they all have a solution set that consists of the union of faces of a polyhedron, and that the union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP [10, 11]:

PROPOSITION 2.1. *In general the solution set of an ELCP consists of the union of faces of a polyhedron.*

PROPOSITION 2.2. *The union of any arbitrary set $\mathcal{F}$ of faces of an arbitrary polyhedron $\mathcal{P}$ can be described by an ELCP.*

So with every union of faces of a polyhedron there corresponds an ELCP and vice versa. Therefore, we claim that ELCP can be considered as the most general linear extension of the LCP.

2.3. The homogeneous ELCP. In order to solve the ELCP we make it homogeneous: we introduce a real number $\alpha \geq 0$ and we define

$$
u = \begin{bmatrix} x \\ \alpha \end{bmatrix}, \; P = \begin{bmatrix} A & -c \\ O_{1\times n} & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} B & -d \end{bmatrix}.
$$

Then we get a *homogeneous ELCP* of the following form:

Given $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{q \times n}$ and ϕ_1, ϕ_2, ..., $\phi_m \subseteq \{1, 2, \ldots, p\}$, find a non-trivial $u \in \mathbb{R}^n$ such that

$$
\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pu)_i \; = \; 0 \tag{5}
$$

subject to

$$
Pu \geq 0 \tag{6}
$$

$$
Qu = 0\,. \tag{7}
$$

Note that this problem always has the trivial solution $u = O_{n\times 1}$.

In the next section we present an algorithm to solve the homogeneous ELCP. We also indicate how the solutions of a general ELCP can be retrieved from the solutions of the corresponding homogeneous ELCP.

2.4. The ELCP algorithm. A homogeneous ELCP can be considered as a system of homogeneous linear equalities and inequalities subject to a complementarity condition. The solution set of the system (6)−(7) is a polyhedral cone $\mathcal{K}$. Let $\mathcal{C}$ be a minimal complete set of central generators of $\mathcal{K}$ and let $\mathcal{E}$ be a minimal complete set of extreme generators of $\mathcal{K}$. Any point u of $\mathcal{K}$ can be represented uniquely as a combination of the form (1). Later on we shall see that any solution u of the homogeneous ELCP can be written as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geq 0 \text{ for all } k \tag{8}$$

with $\mathcal{E}_s \subseteq \mathcal{E}$ (See Theorem 2.4).

To compute $\mathcal{C}$ and $\mathcal{E}$ we use an algorithm that is an adaptation and extension of the double description method of Motzkin [23]. In the kth step of the algorithm the *partial complementarity condition* is defined as follows:

$$\prod_{i \in \phi_j} (Pu)_i = 0 \qquad \text{for all } j \text{ such that } \phi_j \subseteq \{1, 2, \ldots, k\} \; .$$

If there are no sets ϕ_j such that $\phi_j \subseteq \{1, 2, \ldots, k\}$ then the partial complementarity condition is satisfied by definition. For $k \geq p$ the partial complementarity condition coincides with the full complementarity condition (5). Note that central generators of $\mathcal{K}$ always satisfy the (partial) complementarity condition. During the iteration we already remove extreme generators that do not satisfy the partial complementarity condition since combinations of the form (8) that contain such generators cannot yield solutions of the homogeneous ELCP [10, 11]. This leads to the following algorithm to determine the central and the extreme generators of the solution set of the homogeneous ELCP defined by (5)−(7):

Algorithm 1 : Computation of the central and the extreme generators.
Initialization:

- $\mathcal{C}_0 := \{c_i \mid c_i = (I_n)_{.,i} \text{ for } i = 1, 2, \ldots, n\}$
- $\mathcal{E}_0 := \emptyset$

Iteration:

for $k := 1, 2, \ldots, p+q$ **do**

- Calculate the intersection of the current polyhedral cone (described by $\mathcal{C}_{k-1}$ and $\mathcal{E}_{k-1}$) with the half-space or hyperplane determined by the kth inequality or equality of the system (6)−(7). This yields a new polyhedral cone described by a minimal complete set of central generators $\mathcal{C}_k$ and a minimal complete set of extreme generators $\mathcal{E}_k$.
- Remove the extreme generators that do not satisfy the partial complementarity condition.

Result: $\mathcal{C} := \mathcal{C}_{p+q}$ and $\mathcal{E} := \mathcal{E}_{p+q}$

The complementarity condition (5) requires that in each group of inequalities of $Pu \geq 0$ that corresponds to some set ϕ_j at least one inequality should hold with equality. As a consequence, the complementarity condition is satisfied either by all the points of $\mathcal{K}$ (if all the ϕ_j's are empty) or only by points that lie on the border of $\mathcal{K}$ (if at least one of the ϕ_j's is not empty). Since a linear combination of the central generators always satisfies the complementarity condition and since we have only rejected extreme generators that cannot yield solutions of the ELCP, any arbitrary

solution of the homogeneous ELCP can be represented by (8) for some $\mathcal{E}_s \subseteq \mathcal{E}$. However, not every nonnegative combination of the extreme generators satisfies the complementarity condition. So in general we cannot always take $\mathcal{E}_s = \mathcal{E}$. Therefore we introduce the concept of cross-complementarity:

DEFINITION 2.3 (Cross-complementarity). *Let $\mathcal{E}$ be a set of extreme generators of an homogeneous ELCP. A subset $\mathcal{E}_s$ of $\mathcal{E}$ is called cross-complementary if every combination of the form*

$$u = \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \kappa_k \geq 0 \text{ for all } k$$

satisfies the complementarity condition.

The *cross-complementarity graph* $\mathcal{G}_c$ that corresponds to a set of extreme generators $\mathcal{E}$ is defined as follows. We have one vertex v_k for each extreme generator $e_k \in \mathcal{E}$. There is an edge between two different vertices v_k and v_l if the corresponding extreme generators e_k and e_l are cross-complementary. Now we can determine the set Γ of the maximal cross-complementary subsets of $\mathcal{E}$.

Algorithm 2 : Determination of the maximal cross-complementary subsets of extreme generators.
Initialization:

- $\Gamma := \emptyset$
- Construct the cross-complementarity graph $\mathcal{G}_c$ that corresponds to $\mathcal{E}$.
- Select a vertex v_1 of $\mathcal{G}_c$ and let $\mathcal{S} := \{e_1\}$.

Depth-first search in $\mathcal{G}_c$:

- Select a new vertex v^{new} that is connected by an edge with all the vertices that correspond to elements of $\mathcal{S}$. Add the corresponding extreme generator e^{new} to the test set: $\mathcal{S}^{\text{new}} := \mathcal{S} \cup \{e^{\text{new}}\}$.
- **if** $\mathcal{S}^{\text{new}}$ is cross-complementary
  then Let $\mathcal{S} := \mathcal{S}^{\text{new}}$. Select a new vertex and add it to the test set.
  else If $\mathcal{S}$ is not a subset of one of the sets that are already in Γ, then add $\mathcal{S}$ to Γ: $\Gamma := \Gamma \cup \{\mathcal{S}\}$. Go back to the last point where a choice has been made.
- Continue until all possible choices have been considered.

Result: Γ

Now we can characterize the solution set of the homogeneous ELCP:

THEOREM 2.4. *Let $\mathcal{C}$ be a minimal complete set of central generators of the solution set of a homogeneous ELCP, let $\mathcal{E}$ be a minimal complete set of extreme generators of the solution set of the homogeneous ELCP and let Γ be the set of the maximal cross-complementary subsets of $\mathcal{E}$. Then u is a solution of the homogeneous ELCP if and only if there exists a set $\mathcal{E}_s \in \Gamma$ such that*

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geq 0 \text{ for all } k.$$

The solutions of the original ELCP can be retrieved from the solutions of the homogeneous ELCP as follows. Any solution u of the homogeneous ELCP has the following form: $u = \begin{bmatrix} x_u^T & \alpha_u \end{bmatrix}^T$ with $\alpha_u \geq 0$. Now we normalize the central and the extreme generators such that their α component is either 0 or 1. For any central generator c we have $\alpha_c = 0$. For an extreme generator e there are two possibilities:

either $\alpha_e = 0$ or $\alpha_e > 0$. If $\alpha_e = 0$, we leave e as it is. If $\alpha_e > 0$, we divide each component of e by α_e such that the α component of e becomes 1. Note that the new e will still be a solution of the homogeneous ELCP. Define

$$\mathcal{X}^{\rm cen} = \{x_c \,|\, c \in \mathcal{C}\},\ \ \mathcal{X}^{\rm ext} = \{x_e \,|\, e \in \mathcal{E}, \alpha_e = 0\} \text{ and } \mathcal{X}^{\rm fin} = \{x_e \,|\, e \in \mathcal{E}, \alpha_e = 1\}\ .$$

If $\mathcal{X}^{\rm fin}$ is empty then the original ELCP has no solution. For each $\mathcal{E}_s \in \Gamma$ we construct the corresponding sets $\mathcal{X}_s^{\rm ext} \subseteq \mathcal{X}^{\rm ext}$ and $\mathcal{X}_s^{\rm fin} \subseteq \mathcal{X}^{\rm fin}$. All the ordered pairs $\left(\mathcal{X}_s^{\rm ext}, \mathcal{X}_s^{\rm fin}\right)$ for which $\mathcal{X}_s^{\rm fin}$ is not empty, are put in a set Λ. Finally, we remove all the generators $x_k^{\rm e} \in \mathcal{X}^{\rm ext}$ that do not appear in one of the ordered pairs $\left(\mathcal{X}_s^{\rm ext}, \mathcal{X}_s^{\rm fin}\right) \in \Lambda$.

Let $\mathcal{P}$ be the polyhedron defined by the system of linear equalities and inequalities of the original ELCP. There exists a pointed polyhedron $\mathcal{P}_{\rm red}$ with $\mathcal{P} = \mathcal{P}_{\rm red} + \mathcal{L}(\mathcal{P})$ such that the elements of $\mathcal{X}^{\rm ext}$ are extreme generators of $\mathcal{P}_{\rm red}$ and such that the elements of $\mathcal{X}^{\rm fin}$ are finite points of $\mathcal{P}_{\rm red}$. We can give the following geometrical interpretation to the sets $\mathcal{X}^{\rm cen}$, $\mathcal{X}^{\rm ext}$, $\mathcal{X}^{\rm fin}$ and Λ:

- $\mathcal{X}^{\rm cen}$ is a basis for the lineality space of $\mathcal{P}$. We call $\mathcal{X}^{\rm cen}$ a minimal complete set of *central generators* of (the solution set of) the ELCP;
- $\mathcal{X}^{\rm ext}$ is a set of extreme generators of $\mathcal{P}_{\rm red}$ that satisfy the complementarity condition. We say that $\mathcal{X}^{\rm ext}$ is a minimal complete set of *extreme generators* of (the solution set of) the ELCP;
- $\mathcal{X}^{\rm fin}$ is a set of finite vertices of $\mathcal{P}_{\rm red}$ that satisfy the complementarity condition. We call $\mathcal{X}^{\rm fin}$ a minimal complete set of *finite points* of (the solution set of) the ELCP;
- Λ is the set of *ordered pairs of maximal cross-complementary subsets* of $\mathcal{X}^{\rm ext}$ and $\mathcal{X}^{\rm fin}$. Each ordered pair $\left(\mathcal{X}_s^{\rm ext}, \mathcal{X}_s^{\rm fin}\right) \in \Lambda$ determines a face $\mathcal{F}_s$ of $\mathcal{P}_{\rm red}$ that belongs to the solution set of the ELCP: $\mathcal{X}_s^{\rm ext}$ is a minimal complete set of extreme generators of $\mathcal{F}_s$ and $\mathcal{X}_s^{\rm fin}$ is the set of the finite vertices of $\mathcal{F}_s$.

Now we can characterize the solution set of the general ELCP:

THEOREM 2.5. *Let $\mathcal{X}^{\rm cen}$ be a minimal complete set of central generators of a general ELCP, let $\mathcal{X}^{\rm ext}$ be a minimal complete set of extreme generators of the ELCP, let $\mathcal{X}^{\rm fin}$ be a minimal complete set of finite points of the ELCP and let Λ be the set of ordered pairs of maximal cross-complementary subsets of $\mathcal{X}^{\rm ext}$ and $\mathcal{X}^{\rm fin}$. Then x is a solution of the ELCP if and only if there exists an ordered pair $\left(\mathcal{X}_s^{\rm ext}, \mathcal{X}_s^{\rm fin}\right) \in \Lambda$ such that*

$$x \ = \sum_{x_k^{\rm c} \in \mathcal{X}^{\rm cen}} \lambda_k x_k^{\rm c} + \sum_{x_k^{\rm e} \in \mathcal{X}_s^{\rm ext}} \kappa_k x_k^{\rm e} + \sum_{x_k^{\rm f} \in \mathcal{X}_s^{\rm fin}} \mu_k x_k^{\rm f}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geq 0$, $\mu_k \geq 0$ for all k and $\sum_k \mu_k = 1$.

More information on the method used to find all solutions of a system of linear inequalities can be found in [23]. For a more detailed and precise description of the algorithms the interested reader is referred to [10], where also a worked example can be found, and to [11].

2.5. The computational complexity of the ELCP. Our ELCP algorithm yields the entire solution set of the ELCP. This leads to large execution times and storage space requirements especially if the number of variables and equations is large. Therefore, it might be interesting to develop algorithms that only find one solution. However, the following theorem shows that the ELCP is intrinsically a computationally hard problem:

THEOREM 2.6. *In general the ELCP with rational data is an NP-hard problem.*

Proof. See [10, 11]. □

So the ELCP can probably not be solved in polynomial time (unless the class P would coincide with the class NP). The interested reader is referred to [17] for an extensive treatment of NP-completeness.

3. A link between the max-plus algebra and the ELCP. The formulation of the ELCP arose from our research on discrete event systems, examples of which are flexible manufacturing systems, traffic networks and telecommunications networks. Normally the behavior of discrete event systems is highly nonlinear. However, when the order of the events is known or fixed, some of these systems can be described by a description that is "linear" in the max-plus algebra [1]. In this section we first give a short introduction to the max-plus algebra. Next we show by an example how we can obtain a max-algebraic model for certain discrete event systems. Finally we discuss the connection between the ELCP and systems of multivariate max-algebraic polynomial equalities and inequalities.

3.1. The Max-Plus Algebra. The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y = \max(x, y) \tag{9}$$

$$x \otimes y = x + y \tag{10}$$

with $x, y \in \mathbb{R} \cup \{-\infty\}$. The reason for choosing these symbols is that many results from conventional linear algebra can be translated to the max-plus algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. The structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*. The neutral element for $\oplus$ in $\mathbb{R}_{\max}$ is denoted by ε. So $\varepsilon = -\infty$. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. Let $x, r \in \mathbb{R}$. The rth max-algebraic power of x is defined as follows:

$$x^{\otimes^r} = r \cdot x \ . \tag{11}$$

We have $\varepsilon^{\otimes^r} = \varepsilon$ if $r \geq 0$. If $r < 0$ then $\varepsilon^{\otimes^r}$ is not defined. If $a_1, a_2, \ldots, a_n \in \mathbb{R}_\varepsilon$ then

$$\bigoplus_{i=1}^{n} a_i \ = \ a_1 \oplus a_2 \oplus \ldots \oplus a_n \quad \text{and} \quad \bigotimes_{i=1}^{n} a_i \ = \ a_1 \otimes a_2 \otimes \ldots \otimes a_n \ .$$

The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then

$$(A \oplus B)_{ij} \ = \ a_{ij} \oplus b_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \, .$$

If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} \ = \ \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \, .$$

Let $k \in \mathbb{N}$. The kth max-algebraic power of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined recursively as follows:

$$A^{\otimes^k} = A^{\otimes^{k-1}} \otimes A \qquad \text{if } k > 0 \, ,$$
$$A^{\otimes^0} = E_n \ ,$$

where E_n is the n by n max-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i, and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The m by n zero matrix in the max-plus algebra is denoted by $\mathcal{E}_{m\times n}$: $(\mathcal{E}_{m\times n})_{ij} = \varepsilon$ for all i, j.

For more information on the max-plus algebra the interested reader is referred to [1, 8].

3.2. Max-Algebraic Modeling of a Class of Discrete Event Systems. Now we show by a simple example that certain classes of discrete event systems can be modeled using the max-plus algebra.

EXAMPLE 3.1.**: A simple production system**

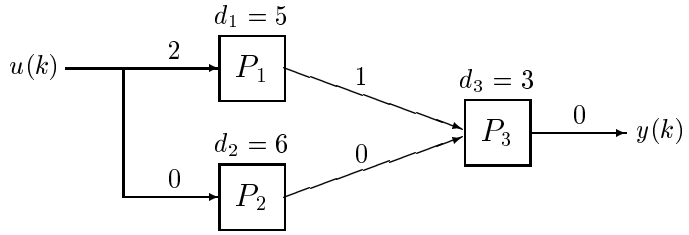


FIG. 1. *A simple production system.*

Consider the system of Figure 1. This production system consists of 3 processing units: P_1, P_2 and P_3. Raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly takes place. The processing times for P_1, P_2 and P_3 are respectively $d_1 = 5$, $d_2 = 6$ and $d_3 = 3$ time units. We assume that it takes 2 time units for the raw material to get from the input source to P_1 and that it takes 1 time unit for the finished product of processing unit P_1 to reach P_3. The other transportation times are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. Define:

- $u(k)$: time instant at which raw material is fed to the system for the $(k+1)$st time,
- $x_i(k)$: time instant at which the ith processing unit starts working for the kth time,
- $y(k)$: time instant at which the kth finished product leaves the system.

A processing unit can only start working on a new product if it has finished processing the previous one. If we assume that each processing unit starts working as soon as all parts are available, we find the following evolution equations for the system:

$$\begin{aligned}
x_1(k+1) &= \max\,(\,x_1(k)+5,\; u(k)+2\,)\\
x_2(k+1) &= \max\,(\,x_2(k)+6,\; u(k)\,)\\
x_3(k+1) &= \max\,(\,x_1(k+1)+5+1,\; x_2(k+1)+6,\; x_3(k)+3\,)\\
&= \max\,(\,x_1(k)+11,\; x_2(k)+12,\; x_3(k)+3,\; u(k)+8\,)\\
y(k) &= x_3(k)+3\ .
\end{aligned}$$

If we rewrite these equations in max-algebraic matrix notation, we obtain

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix} \otimes x(k) \;\oplus\; \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \otimes u(k)\\
y(k) &= \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix} \otimes x(k)\ ,
\end{aligned}$$

where $x(k) = [\ x_1(k)\ \ x_2(k)\ \ x_3(k)\ ]^T$. $\diamond$

The result of Example 3.1 can be generalized: if we limit ourselves to time-invariant deterministic discrete event systems in which the sequence of the events and the duration of the activities are fixed or can be determined in advance (such as repetitive production processes), then the behavior of a system with m inputs and l outputs can be described by equations of the form

$$x(k+1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \tag{12}$$

$$y(k) = C \otimes x(k) \tag{13}$$

where $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$ with an initial condition $x(0) = x_0$. We call (12)−(13) an *nth order state space model*. The vector x represents the state, u is the input vector and y is the output vector of the system. Note that the description (12)−(13) closely resembles the conventional state space description

$$\begin{aligned} x(k+1) &= Ax(k) \ + \ Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

for discrete linear time-invariant systems. Because of this analogy we call discrete event systems that can be described by a state space model of the form (12)−(13) *max-linear time-invariant* discrete event systems. This analogy is also one of the reasons why the symbols $\oplus$ and $\otimes$ are used to represent the basic operations of the max-plus algebra.

More information on max-algebraic modeling and analysis of discrete event systems can be found in [1, 2, 3, 4, 7, 24] and the references cited therein.

3.3. Systems of multivariate max-algebraic polynomial equalities and inequalities. Consider the following problem:

Given $p_1 + p_2$ positive integers $m_1, m_2, \ldots, m_{p_1+p_2}$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, 2, \ldots, p_1 + p_2$; $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, 2, \ldots, p_1\,, \tag{14}$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} \leq b_k \qquad \text{for } k = p_1+1, p_1+2, \ldots, p_1+p_2\,, \tag{15}$$

or show that no such x exists.

We call (14)−(15) a *system of multivariate max-algebraic polynomial equalities and inequalities*. Note that the exponents may be negative or real. In the next section we shall see that many important max-algebraic problems can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities. In [15] we have proved the following theorem:

THEOREM 3.2. *A system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an Extended Linear Complementarity Problem.*

We shall illustrate this by an example:

EXAMPLE 3.3.

Consider the following system of multivariate max-algebraic polynomial equalities and

inequalities:

$$4 \otimes {x_1}^{\otimes -2} \otimes x_2 \otimes {x_3}^{\otimes 4} \otimes {x_5}^{\otimes 2} \;\oplus\; 3 \otimes {x_3}^{\otimes -3} \otimes {x_4}^{\otimes -1} \;\oplus$$

$$(-3) \otimes {x_1}^{\otimes -2} \otimes {x_2}^{\otimes 2} \otimes {x_4}^{\otimes 2} \otimes {x_5}^{\otimes 2} = 1 \tag{16}$$

$$x_1^{\otimes -3} \otimes x_2 \otimes {x_3}^{\otimes -2} \otimes {x_5}^{\otimes 3} \;\oplus\; (-1) \otimes x_3 \otimes {x_4}^{\otimes 3} = -3 \tag{17}$$

$$7 \otimes {x_1}^{\otimes -3} \otimes x_2 \otimes {x_4}^{\otimes -2} \otimes {x_5}^{\otimes 3} \leq 5 \tag{18}$$

with $x \in \mathbb{R}^5$.
Using definitions (10) and (11) we find that the first term of (16) is equivalent to

$$4 - 2x_1 + x_2 + 4x_3 + 2x_5 \;\; .$$

The other terms of (16) can be rewritten in a similar way. Each term has to be smaller than 1 and at least one of them has to be equal to 1. So we get a group of three inequalities in which at least one inequality should hold with equality. If we also take (17) and (18) into account, we get the following ELCP:

Given

$$A = \begin{bmatrix} 2 & -1 & -4 & 0 & -2 \\ 0 & 0 & 3 & 1 & 0 \\ 2 & -2 & 0 & -2 & -2 \\ 3 & -1 & 2 & 0 & -3 \\ 0 & 0 & -1 & -3 & 0 \\ 3 & -1 & 0 & 2 & -3 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 3 \\ 2 \\ -4 \\ 3 \\ 2 \\ 2 \end{bmatrix} ,$$

find $x \in \mathbb{R}^5$ such that

$$(Ax - c)_1 \, (Ax - c)_2 \, (Ax - c)_3 \;+\; (Ax - c)_4 \, (Ax - c)_5 = 0$$

subject to $Ax \geq c$.
The ELCP algorithm yields the extreme generators and the finite points of Table 1 and the pairs of maximal cross-complementary sets of Table 2. There are no central generators. Any solution of the system (16)−(18) can now be expressed as

$$x = \lambda_1 x_1^{\mathrm{c}} + \sum_{x_k^{\mathrm{e}} \in \mathcal{X}_s^{\mathrm{ext}}} \kappa_k x_k^{\mathrm{e}} + \sum_{x_k^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{fin}}} \mu_k x_k^{\mathrm{f}}$$

for $s = 1$, 2 or 3 with $\lambda_1 \in \mathbb{R}$, $\kappa_k \geq 0$, $\mu_k \geq 0$ for all k and $\sum_k \mu_k = 1$. $\diamond$

4. Applications. In this section we discuss some important max-algebraic problems that can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities. These problems can also be reformulated as an ELCP and they can thus be solved using the ELCP algorithm. In general their solution set consists of the union of faces of a polyhedron.

4.1. Matrix factorizations. Consider the following problem:

Given a matrix $A \in \mathbb{R}_\varepsilon^{m \times n}$ and a positive integer p, find $B \in \mathbb{R}_\varepsilon^{m \times p}$ and $C \in \mathbb{R}_\varepsilon^{p \times n}$ such that

$$A = B \otimes C \;\; ,$$

or show that no such factorization exists.

TABLE 1
The generators and the finite points of the ELCP of Example 3.3.

	$\mathcal{X}^{\mathrm{cen}}$	$\mathcal{X}^{\mathrm{ext}}$				$\mathcal{X}^{\mathrm{fin}}$	
	x_1^{c}	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_1^{f}	x_2^{f}
x_1	1	0	0	0	0	0	0
x_2	0	-32	-3	1	14	13	-13
x_3	0	3	0	0	3	1	1
x_4	0	-1	0	0	-1	-1	-1
x_5	1	10	1	-1	-13	-10	3

TABLE 2
The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\mathrm{ext}}$ and $\mathcal{X}^{\mathrm{fin}}$ of Example 3.3.

s	$\mathcal{X}_s^{\mathrm{ext}}$	$\mathcal{X}_s^{\mathrm{fin}}$
1	$\{x_1^{\mathrm{e}}, x_4^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_2^{\mathrm{f}}\}$
2	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_2^{\mathrm{f}}\}$
3	$\{x_3^{\mathrm{e}}, x_4^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}\}$

So we have to find b_{ik} and c_{kj} for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$ such that

$$\bigoplus_{k=1}^{p} b_{ik} \otimes c_{kj} = a_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \ ,$$

and this can clearly be considered as a system of multivariate max-algebraic polynomial equations in b_{ik} and c_{kj}.

This technique can easily be extended to the factorization of A as the product of three or more matrices of a specified size. It is also possible to impose a certain structure on the composing matrices (e.g. triangular, diagonal, Hessenberg, ...).

4.2. Transformation of state space models. Consider a discrete event system that can be described by a state space model of the form

$$x(k+1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \tag{19}$$

$$y(k) = C \otimes x(k) \tag{20}$$

with initial condition $x(0)$.

Suppose that we want to find another state space model with system matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and with initial condition $\tilde{x}(0)$ that describes the same input-output behavior as the original state space model. This can be done as follows. If we can find a common factor L of A and C with $A = \hat{A} \otimes L$ and $C = \hat{C} \otimes L$ and if we define

$$\tilde{A} = L \otimes \hat{A}, \ \ \tilde{B} = L \otimes B, \ \ \tilde{C} = \hat{C} \ \text{ and } \tilde{x}(0) = L \otimes x(0),$$

then the state space model

$$\tilde{x}(k+1) = \tilde{A} \otimes \tilde{x}(k) \ \oplus \ \tilde{B} \otimes u(k)$$
$$y(k) = \tilde{C} \otimes \tilde{x}(k)$$

with initial condition $\tilde{x}(0)$ has the same input-output behavior as the system (19)–(20) with initial condition $x(0)$ [14, 15].

If M is a common factor of A, B and $x(0)$ such that $A = M \otimes \hat{A}$, $B = M \otimes \hat{B}$ and $x(0) = M \otimes \hat{x}(0)$, then the state space model with

$$\tilde{A} = \hat{A} \otimes M, \quad \tilde{B} = \hat{B}, \quad \tilde{C} = C \otimes M \text{ and } \tilde{x}(0) = \hat{x}(0)$$

also has the same input-output behavior as the original system.

So to obtain another state space realization of the given system, we try to find a factorization

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L \qquad \text{or} \qquad \begin{bmatrix} A & B & x(0) \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} & \hat{B} & \hat{x}(0) \end{bmatrix} .$$

As has been shown in the previous subsection these matrix factorizations can be considered as systems of multivariate max-algebraic polynomial equalities.

4.3. State space realization. Consider a discrete event system that can be described by an nth order state space model of the form (19)–(20) with $A \in \mathbb{R}_{\varepsilon}^{n \times n}$, $B \in \mathbb{R}_{\varepsilon}^{n \times m}$ and $C \in \mathbb{R}_{\varepsilon}^{l \times n}$.

If we apply a unit impulse: $e(k) = 0$ if $k = 0$, and $e(k) = \varepsilon$ if $k \neq 0$, to the ith input of the system and if $x(0) = \mathcal{E}_{n \times 1}$, we get $y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{\cdot,i}$ for $k = 1, 2, \ldots$ as the output of the system. This output is called the impulse response due to an impulse at the ith input. Note that $y(k)$ corresponds to the ith column of the matrix $G_{k-1} = C \otimes A^{\otimes^{k-1}} \otimes B$ for $k = 1, 2, 3, \ldots$. Therefore, the sequence $\{G_k\}_{k=0}^{\infty}$ is called the *impulse response* of the system. The G_k's are called the impulse response matrices or *Markov parameters*.

Suppose that A, B and C are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the discrete event system is max-linear and time-invariant and that there is no noise present). How can we construct A, B and C from the G_k's? The triple (A, B, C) will be called a *state space realization* of the given impulse response. If we make the dimension of A minimal, we have a minimal state space realization.

We assume that the system can be described by an rth order state space model (see e.g. [18, 19] for methods to determine lower and upper bounds for the minimal system order). For sake of simplicity we only consider the partial realization problem: i.e. we look for a realization that only fits the first, say, N Markov parameters. State space realizations of the entire impulse response can be constructed by solving the partial realization problem for several values of N [10]. So now we try to find $A \in \mathbb{R}_{\varepsilon}^{r \times r}$, $B \in \mathbb{R}_{\varepsilon}^{r \times m}$ and $C \in \mathbb{R}_{\varepsilon}^{l \times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = G_k , \qquad \text{for } k = 0, 1, 2, \ldots, N-1 .$$

If we write out these equations, we get

$$\bigoplus_{p=1}^{r} c_{ip} \otimes b_{pj} = (G_0)_{ij} \tag{21}$$

for $i = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, m$, and

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} c_{ip} \otimes (A^{\otimes^k})_{pq} \otimes b_{qj} = (G_k)_{ij}$$

for $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N-1$. Since

$$(A^{\otimes^k})_{pq} = \bigoplus_{i_1=1}^{r} \bigoplus_{i_2=1}^{r} \cdots \bigoplus_{i_{k-1}=1}^{r} a_{pi_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} q} \ ,$$

this can be rewritten as

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} \bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} {a_{uv}}^{\otimes^{\gamma_{kpqsuv}}} \otimes b_{qj} = (G_k)_{ij} \tag{22}$$

for $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N-1$, where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. Clearly, (21)–(22) is a system of multivariate max-algebraic polynomial equations with the entries of A, B and C as unknowns.

For more information on this method to construct state space realizations and for some worked examples the interested reader is referred to [10, 12, 13, 14].

4.4. Max-max problems. Now we consider systems of max-algebraic equations that also have multivariate max-algebraic polynomials (instead of constants) on the right-hand side. Since in general there do not exist inverse elements w.r.t. $\oplus$ in $\mathbb{R}_{\max}$, we cannot simply transfer terms from the right-hand side to the left-hand side as we would do in conventional algebra. However, these problems can also be solved using the ELCP approach.

Consider the following problem:

Given integers $m_k, p_k \in \mathbb{N}_0$ for $k = 1, 2, \ldots, q$ and real numbers a_{ki}, b_{kij}, c_{kl} and d_{klj} for $k = 1, 2, \ldots, q$; $i = 1, 2, \ldots, m_k$; $j = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, p_k$, find $x \in \mathbb{R}^n$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{b_{kij}}} = \bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{d_{klj}}} \tag{23}$$

for $k = 1, 2, \ldots, q$.

If we define q dummy variables t_1, t_2, ..., t_q such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{b_{kij}}} = t_k \qquad \text{for } k = 1, 2, \ldots q \,,$$

then problem (23) is equivalent to

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{b_{kij}}} \otimes {t_k}^{\otimes^{-1}} = 0 \qquad \text{for } k = 1, 2, \ldots, q \,,$$

$$\bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{d_{klj}}} \otimes {t_k}^{\otimes^{-1}} = 0 \qquad \text{for } k = 1, 2, \ldots, q \,.$$

This is a system of multivariate max-algebraic polynomial equalities that can be transformed into an ELCP.

4.5. Mixed max-min problems. We can also use the ELCP to solve mixed max-min problems. First we introduce the $\oplus'$ operation: $x \oplus' y = \min(x, y)$ with $x, y \in \mathbb{R} \cup \{-\infty, +\infty\}$. This leads to the extended structure $(\mathbb{R} \cup \{-\infty, +\infty\}, \oplus, \oplus', \otimes)$ which is called the max-min-plus algebra. More information about the max-min-plus algebra can be found in [8, 25]. If $a_1, a_2, \ldots, a_n \in \mathbb{R} \cup \{-\infty, +\infty\}$ then

$$\bigoplus_{i=1}^{n}{}' a_i = a_1 \oplus' a_2 \oplus' \ldots \oplus' a_n \ .$$

Now we consider the following problem:

Given integers $m_k, m_{kl_1} \in \mathbb{N}_0$ for $k = 1, 2, \ldots, m$ and $l_1 = 1, 2, \ldots, m_k$ and real numbers $a_{kl_1l_2}$, b_k and $c_{kl_1l_2j}$ for $k = 1, 2, \ldots, m$; $l_1 = 1, 2, \ldots, m_k$; $l_2 = 1, 2, \ldots, m_{kl_1}$ and $j = 1, 2, \ldots, n$, find a vector $x \in \mathbb{R}^n$ that satisfies

$$\bigoplus_{l_1=1}^{m_k}{}' \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes c_{kl_1l_2j}} = b_k \qquad \text{for } k = 1, 2, \ldots, m \, .$$

Using a technique that is similar to the one used in Section 4.4 this problem can also be transformed into an ELCP.

We can also use this technique to transform systems of combined max-min equations of the following form into an ELCP:

$$\bigoplus_{l_1}{}' \bigoplus_{l_2} \bigoplus_{l_3}{}' \ldots \bigoplus_{l_q} a_{kl_1l_2\ldots l_q} \otimes \bigotimes_{j=1}^{n} x_i^{\otimes c_{kl_1l_2\ldots l_qj}} = b_k \, , \qquad \text{for } k = 1, \ldots, m \ ,$$

or analogous equations but with $\bigoplus$ replaced by $\bigoplus'$ and vice versa or when some of the equalities are replaced by inequalities.

Furthermore, we can also use this technique to solve systems of form (23) but with some of the $\bigoplus$-summations replaced by $\bigoplus'$-summations, and/or with some of the equalities replaced by inequalities.

4.6. Problems in the symmetrized max-plus algebra. We can also use the ELCP to solve problems in the symmetrized max-plus algebra such as constructing matrices with a given max-algebraic characteristic polynomial, determining max-algebraic singular value decompositions and QR decompositions of a matrix, solving systems of "linear equations" in the symmetrized max-plus algebra and so on. However, it would lead us too far to go further into this matter.

For more information on this subject (and on the other problems that were discussed in this section) the reader is referred to [10, 15].

5. Conclusions and Future Research. In this paper we have introduced the Extended Linear Complementarity Problem (ELCP) and sketched an algorithm to find all its solutions. Since this algorithm yields all solutions, it provides a geometrical insight in the solution set of an ELCP and other problems that can be reduced to an ELCP. However, we are not always interested in obtaining all solutions of an ELCP. Therefore, our further research efforts will concentrate on algorithms that yield only one solution of an ELCP. Although we have shown that in general the ELCP is NP-hard, it may be interesting to determine which subclasses of the ELCP can be solved with a polynomial time algorithm.

We have demonstrated that the ELCP can be used to solve a system of multivariate max-algebraic polynomial equalities and inequalities. This allows us to solve many

max-algebraic problems. Furthermore, we can also use the ELCP to solve problems in the max-min-plus algebra and the symmetrized max-plus algebra. Therefore, the ELCP is a powerful mathematical tool for solving max-algebraic problems. It would be interesting to make a more thorough study of the class of problems that can be reduced to solving a system of multivariate max-algebraic polynomial equalities and inequalities. Every instance of this class can then be reformulated as an ELCP.

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