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Lur'e Systems with Multilayer Perceptron and Recurrent Neural Networks: Absolute Stability and Dissipativity

J. A. K. Suykens, J. Vandewalle, B. De Moor

***Abstract*— Sufficient conditions for absolute stability and dissipativity of continuous-time recurrent neural networks with two hidden layers are presented. In the autonomous case this is related to a Lur'e system with multilayer perceptron nonlinearity. Such models are obtained after parameterizing general nonlinear models and controllers by a multilayer perceptron with one hidden layer and representing the control scheme in standard plant form. The conditions are expressed as matrix inequalities and can be employed for nonlinear H_∞ control and imposing closed-loop stability in dynamic backpropagation.**

***Index Terms*—Lur'e systems, Lur'e–Postnikov Lyapunov function, matrix inequalities, multilayer recurrent neural networks, nonlinear H_∞ control.**

I. Introduction

In this paper we investigate a class of nonlinear models and controllers that are parameterized by multilayer perceptrons. As a result recurrent neural networks [6], [28] with two hidden layers are obtained as closed-loop system equations. It is well known that multilayer perceptrons with one or more hidden layers are universal approximators in the sense that they are able to approximate any static continuous nonlinear function arbitrarily well on a compact interval [4], [10], [14]. In this sense "generality" is preserved after parameterizing nonlinear systems by means of multilayer perceptrons. On the other hand, the layered structure and the fact that the two-hidden layer recurrent neural networks contain sector type nonlinearities can be exploited in order to derive matrix inequalities [5] as sufficient conditions for global asymptotic stability. Observability, controllability, and identifiability issues for a class of recurrent neural networks have been studied in [1], [2], and [18]. In this paper an absolute stability criterion will be derived based on a Lur'e–Postnikov Lyapunov function.

In the area of nonlinear H_∞ control, the problem of extending results from linear H_∞ control to input affine and general nonlinear systems received considerable interest. Solutions have been presented for the state and output feedback case, in terms of the solution to Hamilton–Jacobi–Isaac equations [3], [11], [12], [22], [23], [27]. The notion of dissipativity as proposed by Willems [26] and later developed by Hill and Moylan for nonlinear systems [7], [8], plays an important role in this context. In this paper we investigate dissipativity of the nonautonomous two-hidden layer recurrent neural networks. This is done with respect to a supply rate of quadratic form (including the cases of passivity and finite L_2 gain) and a storage function of

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The authors are with the Department of Electrical Engineering, Katholieke Universiteit Leuven, ESAT-SISTA, Kardinaal Mercierlaan 94, B-3001 Leuven (Heverlee), Belgium.

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quadratic form plus integral terms. The condition is also expressed as a matrix inequality.

The derived matrix inequalities for absolute stability and dissipativity can be employed for controller synthesis. Nonlinear H_∞ control for recurrent neural networks involves then solving a constrained nonlinear optimization problem which takes into account the matrix inequality. It also offers a way of improving Narendra's dynamic backpropagation procedure [15]. Dynamic backpropagation basically means that the controller is trained by optimizing on one or a set of specific reference inputs. However, nothing is guaranteed then concerning closed-loop stability or generalization of the controller toward other reference inputs, which do not belong to the training set. The matrix inequalities can be used in order to impose global asymptotic stability of the closed-loop scheme or guarantee a certain disturbance attenuation level for the control scheme in standard plant form. A similar approach has been taken for discrete-time multilayer recurrent neural networks in NL_q theory, with applications to stabilization and control of systems that possess one or multiple equilibria, are periodic, quasiperiodic, or chaotic [19].

This paper is organized as follows. In Section II we discuss the parameterization of nonlinear models and controllers by a multilayer perceptron. The closed-loop systems can be represented as a two-hidden layer recurrent neural network, which is explained in Section III. In Section IV a sufficient condition for absolute stability of this form is derived. In Section V a condition for dissipativity is given. In Section VI it is discussed how to employ the resulting matrix inequalities for nonlinear H_∞ control and modified dynamic backpropagation.

II. A Parameterization of Models and Controllers by Neural Nets

For a given nonlinear plant, let us consider nonlinear models of the form

$$\begin{aligned} \dot{x} &= A^{\mathcal{M}}x + B^{\mathcal{M}}u + f(x,u) + K^{\mathcal{M}}\epsilon \\ y &= C^{\mathcal{M}}x + \epsilon \end{aligned} \tag{1}$$

with $f(\cdot,\cdot)$:$\mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ a continuous nonlinear mapping and $f(0,0) = 0$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^l$, and state vector $x \in \mathbb{R}^n$. The matrix $K^{\mathcal{M}} \in \mathbb{R}^{n\times l}$ denotes a Kalman gain, in order to model process noise. Parameterizing $f(\cdot,\cdot)$ with a one-hidden layer multilayer perceptron and zero bias terms gives

$$\mathcal{M}: \quad \begin{cases} \dot{x} = & A^{\mathcal{M}}x + B^{\mathcal{M}}u + W_{AB}^{\mathcal{M}}\sigma(V_A^{\mathcal{M}}x + V_B^{\mathcal{M}}u) \\ & \quad + K^{\mathcal{M}}\epsilon \\ y = & C^{\mathcal{M}}x + \epsilon \end{cases} \tag{2}$$

with interconnection matrices $W_{AB}^{\mathcal{M}} \in \mathbb{R}^{n\times n_{h_x}}$, $V_A^{\mathcal{M}} \in \mathbb{R}^{n_{h_x}\times n}$, $V_B^{\mathcal{M}} \in \mathbb{R}^{n_{h_x}\times m}$. $\sigma(\cdot)$ denotes the activation function of the neural network (typically $\tanh(\cdot)$ nonlinearity), which is applied componentwise to the elements of a vector and $\sigma(0) = 0$. The model $\mathcal{M}$ can be considered as the continuous-time version of neural state-space models, proposed in [20]. In case $W_{AB}^{\mathcal{M}} = 0$, $V_A^{\mathcal{M}} = 0$, $V_B^{\mathcal{M}} = 0$, (2) corresponds to a Kalman filter with steady-state Kalman gain $K^{\mathcal{M}}$. The model parameters of (2) might be identified using a prediction error algorithm. The gradient of the cost function can be computed using a sensitivity method (known as Narendra's dynamic backpropagation in the field of neural networks [15]).

Now, let us consider in connection to the model $\mathcal{M}$ a nonlinear state feedback controller $\mathcal{C}_1$, a linear dynamic output feedback controller with saturation $\mathcal{C}_2$, or a nonlinear output feedback controller $\mathcal{C}_3$

$$\begin{aligned} \mathcal{C}_1\text{:} \quad & u = K^{\mathcal{C}}x + W^{\mathcal{C}}\sigma(V^{\mathcal{C}}x) \\ \mathcal{C}_2\text{:} \quad & \begin{cases} \dot{z} &= E^{\mathcal{C}}z + F^{\mathcal{C}}y + F_2^{\mathcal{C}}\ d \\ u &= \sigma(G^{\mathcal{C}}z + H^{\mathcal{C}}y + H_2^{\mathcal{C}}\ d) \end{cases} \\ \mathcal{C}_3\text{:} \quad & \begin{cases} \dot{z} &= E^{\mathcal{C}}z + F^{\mathcal{C}}y + F_2^{\mathcal{C}}d + W_{EF}^{\mathcal{C}} \\ & \quad \sigma(V_E^{\mathcal{C}}z + V_F^{\mathcal{C}}y + V_{F_2}^{\mathcal{C}}\ d) \\ u &= W_{GH}^{\mathcal{C}}\sigma(V_G^{\mathcal{C}}z + V_H^{\mathcal{C}}y + V_{H_2}^{\mathcal{C}}\ d) \end{cases} \end{aligned} \tag{3}$$

with controller state $z \in \mathbb{R}^{n_z}$ and reference input $d \in \mathbb{R}^l$. The matrices are of dimension $K^{\mathcal{C}} \in \mathbb{R}^{m\times n}, W^{\mathcal{C}} \in \mathbb{R}^{m\times n_{h_u}}$, $V^{\mathcal{C}} \in \mathbb{R}^{n_{h_u}\times n}, E^{\mathcal{C}} \in \mathbb{R}^{n_z\times n_z}$, $F^{\mathcal{C}} \in \mathbb{R}^{n_z\times l}, F_2^{\mathcal{C}} \in \mathbb{R}^{n_z\times l}$, $G^{\mathcal{C}} \in \mathbb{R}^{m\times n_z}, H^{\mathcal{C}} \in \mathbb{R}^{m\times l}$, $H_2^{\mathcal{C}} \in \mathbb{R}^{m\times l}, W_{EF}^{\mathcal{C}} \in \mathbb{R}^{n_z\times n_{h_z}}$, $V_E^{\mathcal{C}} \in \mathbb{R}^{n_{h_z}\times n_z}, V_F^{\mathcal{C}} \in \mathbb{R}^{n_{h_z}\times l}$, $V_{F_2}^{\mathcal{C}} \in \mathbb{R}^{n_{h_z}\times l}$, $W_{GH}^{\mathcal{C}} \in \mathbb{R}^{m\times n_{h_u}}, V_G^{\mathcal{C}} \in \mathbb{R}^{n_{h_u}\times n_z}, V_H^{\mathcal{C}} \in \mathbb{R}^{n_{h_u}\times l}, V_{H_2}^{\mathcal{C}} \in \mathbb{R}^{n_{h_u}\times l}$ with n_{h_u}, n_{h_z} the number of hidden neurons in the hidden layers.

It is straightforward to observe that the closed-loop systems for the model $\mathcal{M}$, connected to one of the controllers $\mathcal{C}_i$ $(i = 1,2,3)$, can be written in the form

$$\dot{p} = Ap + B\sigma(Np + H\sigma(Cp + D_2w) + D_1w) + D_0w \tag{4}$$

with state vector $p = [x; z]$ and exogenous input $w = [\epsilon; d]$. One obtains

- $[\mathcal{M} - \mathcal{C}_1]$:

$$\begin{aligned} A &= A^{\mathcal{M}} + B^{\mathcal{M}}K^{\mathcal{C}}, \qquad B = [W_{AB}^{\mathcal{M}}B^{\mathcal{M}}W^{\mathcal{C}}] \\ N &= \begin{bmatrix} V_A^{\mathcal{M}} + V_B^{\mathcal{M}}K^{\mathcal{C}} \\ V^{\mathcal{C}} \end{bmatrix}, \qquad H = \begin{bmatrix} V_B^{\mathcal{M}}W^{\mathcal{C}} \\ 0 \end{bmatrix}, \qquad C = V^{\mathcal{C}} \end{aligned} \tag{5}$$

- $[\mathcal{M} - \mathcal{C}_2]$:

$$\begin{aligned} A &= \begin{bmatrix} A^{\mathcal{M}} & 0 \\ F^{\mathcal{C}}C^{\mathcal{M}} & E^{\mathcal{C}} \end{bmatrix}, \qquad B = \begin{bmatrix} W_{AB}^{\mathcal{M}} & B^{\mathcal{M}} \\ 0 & 0 \end{bmatrix} \\ N &= \begin{bmatrix} V_A^{\mathcal{M}} & 0 \\ H^{\mathcal{C}}C^{\mathcal{M}} & G^{\mathcal{C}} \end{bmatrix}, \qquad H = \begin{bmatrix} V_B^{\mathcal{M}} & 0 \\ 0 & 0 \end{bmatrix} \\ C &= \begin{bmatrix} H^{\mathcal{C}}C^{\mathcal{M}} & G^{\mathcal{C}} \\ 0 & 0 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} H^{\mathcal{C}} & H_2^{\mathcal{C}} \\ 0 & 0 \end{bmatrix} \\ D_1 &= \begin{bmatrix} 0 & 0 \\ H^{\mathcal{C}} & H_2^{\mathcal{C}} \end{bmatrix}, \qquad D_0 = \begin{bmatrix} K^{\mathcal{M}} & 0 \\ F^{\mathcal{C}} & F_2^{\mathcal{C}} \end{bmatrix} \end{aligned} \tag{6}$$

- $[\mathcal{M} - \mathcal{C}_3]$:

$$\begin{aligned} A &= \begin{bmatrix} A^{\mathcal{M}} & 0 \\ F^{\mathcal{C}}C^{\mathcal{M}} & E^{\mathcal{C}} \end{bmatrix}, \qquad B = \begin{bmatrix} W_{AB}^{\mathcal{M}} & B^{\mathcal{M}}W_{GH}^{\mathcal{C}} & 0 \\ 0 & 0 & W_{EF}^{\mathcal{C}} \end{bmatrix} \\ N &= \begin{bmatrix} V_A^{\mathcal{M}} & 0 \\ V_H^{\mathcal{C}}C^{\mathcal{M}} & V_G^{\mathcal{C}} \\ V_F^{\mathcal{C}}C^{\mathcal{M}} & V_E^{\mathcal{C}} \end{bmatrix}, \qquad H = \begin{bmatrix} V_B^{\mathcal{M}}W_{GH}^{\mathcal{C}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ C &= \begin{bmatrix} V_H^{\mathcal{C}}C^{\mathcal{M}} & V_G^{\mathcal{C}} \\ V_F^{\mathcal{C}}C^{\mathcal{M}} & V_E^{\mathcal{C}} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} V_H^{\mathcal{C}} & V_{H_2}^{\mathcal{C}} \\ V_F^{\mathcal{C}} & V_{F_2}^{\mathcal{C}} \end{bmatrix} \\ D_1 &= \begin{bmatrix} 0 & 0 \\ V_H^{\mathcal{C}} & V_{H_2}^{\mathcal{C}} \\ V_F^{\mathcal{C}} & V_{F_2}^{\mathcal{C}} \end{bmatrix}, \qquad D_0 = \begin{bmatrix} K^{\mathcal{M}} & 0 \\ F^{\mathcal{C}} & F_2{}^{\mathcal{C}} \end{bmatrix}. \end{aligned} \tag{7}$$

III. Two-Hidden Layer Recurrent Neural Nets and Lur'e Systems

Because the closed-loop systems for (2), (3) can be represented as a recurrent neural network with two hidden layers, we will study absolute stability and dissipativity of the following form:

$$\begin{aligned} \dot{p} &= Ap + B\sigma_1(Np + H\sigma_2(Cp + D_2w) + D_1w) + D_0w \\ e &= Ep + F\sigma_3(Mp + J\sigma_4(Gp + L_2w) + L_1w) + L_0w \end{aligned} \tag{8}$$

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with external input $w \in \mathbb{R}^{n_w}$, output $e \in \mathbb{R}^{n_e}$, and state vector $p \in \mathbb{R}^{n_p}$. For the static nonlinearities $\sigma_i(\cdot)$ with number of hidden neurons n_{h_i} we assume the sector conditions $[0, k_i]$ $(i = 1, 2, 3, 4,$ respectively). The interconnection matrices are of appropriate dimension. Note that when the closed-loop systems are written in standard plant form, w and e in (8) correspond to the exogenous input and regulated output, respectively. The discrete-time version of multilayer recurrent neural networks has been studied in the context of NL_q theory [19].

Let us consider the autonomous case of (8) (zero external input w) and denote the state vector by $x \in \mathbb{R}^n$

$$\dot{x} = Ax + B\sigma_1(Nx + H\sigma_2(Cx)). \tag{9}$$

For $N = 0$ the two-hidden layer recurrent neural network (9) can be represented as the Lur'e system:

$$\begin{aligned} \dot{x} &= Ax + B_1\mu \\ \nu &= C_1 x \\ \mu &= \varphi(\nu) = B_2\sigma_1(H\sigma_2(C_2\nu)) \end{aligned} \tag{10}$$

with $B_1B_2 = B, C_2C_1 = C$ and $B_1 \in \mathbb{R}^{n \times m}, B_2 \in \mathbb{R}^{m \times n_{h_1}}, C_2 \in \mathbb{R}^{n_{h_2} \times l}, C_1 \in \mathbb{R}^{l \times n}$. The nonlinearity $\varphi(\cdot)$ is a multilayer perceptron with two hidden layers, zero bias terms and activation functions $\sigma_1(\cdot)$, $\sigma_2(\cdot)$. The interconnection matrices of the output layer and hidden layers are B_2, H, C_2, respectively. The Lur'e representation (10) consists of the linear dynamic system $[A, B_1, C_1]$ interconnected by feedback to $\varphi(\cdot)$, which in general does not satisfy a sector condition. However, $\varphi(\cdot)$ is composed of units with activation functions that do satisfy a sector condition. This enables us to represent (10) as the Lur'e system

$$\begin{aligned} \dot{x} &= A_*x + B_*s \\ r &= C_*x + D_*s \\ \mu &= \sigma_1(z) \\ v &= \sigma_2(\nu) \end{aligned} \tag{11}$$

with $s = [\mu; v]$, $r = [z; \nu]$, $z = Nx + Hv$, $v = \sigma_2(Cx)$, and

$$A_* = A, B_* = [B \quad 0], \qquad C_* = \begin{bmatrix} N \\ C \end{bmatrix}, \qquad D_* = \begin{bmatrix} 0 & H \\ 0 & 0 \end{bmatrix}$$

and $\sigma_1(\cdot)$, $\sigma_2(\cdot)$ belonging to sector $[0, k]$. Absolute stability criteria for such Lur'e systems are readily available in the literature (see, e.g., [13], [16], and [25]), such as the circle and Popov criterion which are related to a quadratic Lyapunov function and a Lur'e–Postnikov Lyapunov function, respectively. However, the nonzero D_* matrix in the Lur'e representation complicates the analysis. In the next section we will derive a new criterion which is directly based on the form (9).

IV. Absolute Stability Criterion

In this section we derive a sufficient condition for global asymptotic stability of the form (9). The following lemma is based on the Lur'e–Postnikov Lyapunov function:

$$\begin{aligned} V(x) = x^TPx &+ \sum_{i=1}^{n_{h_1}} 2\omega_i \cdot \int_0^{z_i} \sigma_{1_i}(\rho)k_1 \, d\rho \\ &+ \sum_{j=1}^{n_{h_2}} 2\gamma_j \cdot \int_0^{\nu_j} \sigma_{2_j}(\rho)k_2 \, d\rho \end{aligned} \tag{12}$$

with $P = P^T > 0$, $\omega_i > 0$, $\gamma_j > 0$ and $z = Nx + H\sigma_2(\nu)$, $\nu = Cx$. This Lyapunov function is positive everywhere with $V(0) = 0$ and radially unbounded.

Lemma 1: Let $\Gamma = \mathrm{diag}\{\gamma_j\}$, $\Omega = \mathrm{diag}\{\omega_i\}$, $T = \mathrm{diag}\{\tau_j\}$, $\Phi = \mathrm{diag}\{\phi_i\}$ be diagonal matrices with $\gamma_j > 0$, $\omega_i > 0$, $\tau_j > 0$, $\phi_i > 0$ for $i = 1, \cdots, n_{h_1}$ and $j = 1, \cdots, n_{h_2}$ and consider an arbitrary constant $\alpha > 0$. Assume the following condition on the slope of $\sigma_2(\cdot)$:

$$0 \le \frac{d\sigma_{2_j}}{d\nu_j} \le k_3, k_3 \ge 1, \qquad \forall j \tag{13}$$

in addition to the sector conditions $\sigma_1(\cdot) \in [0, k_1]$ and $\sigma_2(\cdot) \in [0, k_2]$. Furthermore, let Y be shown in the equation at the bottom of the page, and

$$Z = \begin{bmatrix} A^TC^T & 0 \\ \Omega H & B^TC^T & 0 \\ 0 & 0 & \sqrt{\alpha}I \end{bmatrix}$$

where Z is assumed to be a full rank matrix. Then, if there exist a $P = P^T > 0, \alpha$ and diagonal matrices Γ, Ω, T, Φ such that the matrix inequality

$$Y + k_1k_3ZZ^T < 0 \tag{14}$$

is satisfied, and (9) is globally asymptotically stable (or absolutely stable in the large) with the origin as a unique equilibrium point.

Proof: From the sector conditions on $\sigma_1(\cdot)$, one has the inequalities [25]: $\sigma_{1_i}(z_i)[\sigma_{1_i}(z_i) - k_1n_i^Tx - k_1h_i^T\sigma_2(\nu)] \le 0, \forall x \in \mathbb{R}^n$ and $\sigma_2(\nu_j)[\sigma_{2_j}(\nu_j) - k_2c_j^Tx] \le 0, \forall x \in \mathbb{R}^n$, for all i, j where n_i^T, h_i^T, c_j^T denote the ith row of the matrices N, H and the jth row of the matrix C, respectively. By taking the time derivative of the Lyapunov function (12) and applying the $\mathcal{S}$-procedure [5] one obtains

$$\begin{aligned} \dot{V} \le\ & [Ax + B\sigma_1(z)]^TPx + x^TP[Ax + B\sigma_1(z)] \\ & + 2k_2\sigma_2(\nu)^T\Gamma C[Ax + B\sigma_1(z)] \\ & + 2k_1\sigma_1(z)^T\Omega[NAx + NB\sigma_1(z) \\ & + H\frac{d\sigma_2(\nu)}{d\nu}C(Ax + B\sigma_1(z))] \\ & - 2\sigma_2(\nu)^TT[\sigma_2(\nu) - k_2Cx] - 2\sigma_1(z)^T\Phi \\ & \cdot [\sigma_1(z) - k_1Nx - k_1H\sigma_2(\nu)]. \end{aligned}$$

Defining $\xi = [x; \sigma_1(z); \sigma_2(\nu)]$, this is written as the quadratic form $\xi^T(Y + k_1Z\Lambda Z^T)\xi < 0$ with [9]

$$\Lambda = \begin{bmatrix} 0 & \frac{d\sigma_2}{dy} & 0 \\ \frac{d\sigma_2}{dy} & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \le k_3I.$$

Multiplication from the left and right with the full rank matrix Z yields $Z\Lambda Z^T \le k_3ZZ^T$. The inequality has to hold for all nonzero ξ, which gives (14). □

V. Dissipativity

In this section we analyze input–output (I/O) properties of (8). Therefore, we associate to (8) a supply rate of the form [7], [8]

$$s(w, e) = [w^T e^T]\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}\begin{bmatrix} w \\ e \end{bmatrix} \tag{15}$$

$$Y = Y^T = \begin{bmatrix} A^TP + PA & PB + k_1A^TN^T\Omega + k_1N^T\Phi & k_2A^TC^T\Gamma + k_2C^TT \\ B^TP + k_1\Omega NA + k_1\Phi N & -2\Phi + k_1\Omega NB + k_1B^TN^T\Omega & k_2B^TC^T\Gamma + k_1\Phi H \\ k_2\Gamma CA + k_2TC & k_2\Gamma CB + k_1H^T\Phi & -2T - \alpha I \end{bmatrix}$$

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with Q_{11}, Q_{22} symmetric matrices. The following types of supply rates are of special importance: $s(w,e) = 2w^T e$ (passivity) and $s(w,e) = \zeta^2 w^T w - e^T e$ (finite L_2-gain ζ).

System (8) with supply rate (15) is called dissipative [7], [8] if for all locally square integrable w and all $t_f \geq 0$, one has $\int_0^{tf} s(w(t),e(t))\, dt \geq 0$ with $p(0) = 0$ and $s(w,e)$ evaluated along the trajectory of (8). System (8) is dissipative, then with respect to this supply rate if there exists a storage function $\phi(p)$: $\mathbb{R}^n \mapsto \mathbb{R}$ satisfying $\phi(p) > 0$ for $p \neq 0$ and $\phi(0) = 0$ and $\dot{\phi}(p) \leq s(w,e), \forall w \in \mathbb{R}^{n_w}, \forall e \in \mathbb{R}^{n_e}$ [7], [8].

Here we investigate a storage function which consists of a quadratic form plus integral terms

$$\phi(p) = p^T P p + \sum_{i=1}^{n_{h_1}} 2\omega_i \cdot \int_0^{z_i} \sigma_{1_i}(\rho) k_1 \, d\rho + \sum_{j=1}^{n_{h_2}} 2\gamma_j \cdot \int_0^{\nu_j} \sigma_{2_j}(\rho) k_2 \, d\rho \tag{16}$$

with $P = P^T > 0, \omega_i > 0, \gamma_j > 0$ and $z = Np + H\sigma_2(\nu)$, $\nu = Cp$. Note the resemblance with the Lur'e–Postnikov Lyapunov function for the autonomous case.

In order to derive the next lemma, we have to assume $D = D_0, D_i = 0\ (i = 1,2), L = L_0, L_i = 0\ (i = 1,2)$ and $F = 0$ in (8), giving

$$\begin{aligned} \dot{p} &= Ap + B\sigma_1(Np + H\sigma_2(Cp)) + Dw \\ e &= Ep + Lw. \end{aligned} \tag{17}$$

This assumption is not needed for the special case of a quadratic storage function.

Lemma 2: Let $\Gamma = \mathrm{diag}\{\gamma_j\}$, $\Omega = \mathrm{diag}\{\omega_i\}$, $T = \mathrm{diag}\{\tau_j\}$, $\Phi = \mathrm{diag}\{\phi_i\}$ be diagonal matrices with $\gamma_j > 0$, $\omega_i > 0$, $\tau_j > 0$, $\phi_i > 0$ for $i = 1, \cdots, n_{h_1}$, and $j = 1, \cdots, n_{h_2}$ and consider an arbitrary constant $\alpha > 0$. Assume the following condition on the slope of $\sigma_2(\cdot)$:

$$0 \leq \frac{d\sigma_{2_j}}{d\nu_j} \leq k_3, k_3 \geq 1, \qquad \forall j \tag{18}$$

in addition to the sector conditions $\sigma_1(\cdot) \in [0, k_1]$, $\sigma_2(\cdot) \in [0, k_2]$. Furthermore, let Y be shown in (19), at the bottom of the page, and

$$Z = \begin{bmatrix} A^T C^T & 0 & 0 & 0 \\ B^T C^T & \Omega H & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{\alpha} I & \frac{1}{2}\sqrt{\alpha} I \\ D^T C^T & 0 & 0 & 0 \end{bmatrix}$$

where Z is assumed to be a full rank matrix and $Y_{44} = -Q_{11} - L^T E^T Q_{22} E L - Q_{12} L - L^T Q_{12}^T$. Then, if there exist a $P = P^T > 0$, α and diagonal matrices Γ, Ω, T, Φ such that the matrix inequality

$$Y + k_1 k_3 Z Z^T < 0 \tag{20}$$

is satisfied, system (17) is dissipative with respect to supply rate (15) and storage function (16).

Proof: The outline of the proof is similar to the proof of Lemma 1. We investigate under what condition $\dot{\phi} - s(w,e) \leq 0$ holds. Using the $\mathcal{S}$-procedure [5] and the inequalities from the sector conditions of the nonlinearities, checking dissipativity yields:

$$\dot{\phi} - s(w,e) \leq \xi^T (Y + k_1 Z \Lambda Z^T) \xi < 0$$

with $\xi = [p; \sigma_1(z); \sigma_2(\nu); w]$ and

$$\Lambda = \begin{bmatrix} 0 & \frac{d\sigma_2}{dz_2} & 0 & \\ \frac{d\sigma_2}{dz_2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \leq k_3 I.$$

Multiplication from the left and right with the full rank matrix Z gives $Z\Lambda Z^T \leq k_3 Z Z^T$. The quadratic form has to be negative for all nonzero ξ, which is satisfied if (20) holds. □

Remark: The notions of dissipativity with finite L_2 gain and storage function are the same as in the context of nonlinear H_∞ control. However, the derived condition here is only sufficient, due to the storage function which is not a general positive definite function but takes a similar form as a Lur'e–Postnikov Lyapunov function. Hence in this sense, loss of dissipativity has a similar meaning as loss of absolute stability for the Lur'e–Postnikov Lyapunov function in Lemma 1 when the matrix inequality is not satisfied.

VI. Nonlinear H_∞ Control for Recurrent Neural Networks and Modified Dynamic Backpropagation

Considering a supply rate with finite L_2-gain, the condition of Lemma 2 can be employed for nonlinear H_∞ control in order to design one of the controllers (3), based on the recurrent neural network model (2). Additional linear filters could be taken into account for the control scheme in standard plant form. The nonlinear H_∞ optimal control problem is formulated then as

$$\min_{\theta_c, P, \Omega, \Gamma, T, \Phi, \alpha} \ \zeta \quad \text{s.t.} \quad Y + k_1 k_3 Z Z^T < 0 \tag{21}$$

where θ_c denotes the controller parameter vector (related to $\mathcal{C}_1, \mathcal{C}_2$, or $\mathcal{C}_3$) and the matrix inequality from Lemma 2 is taken into account for $Q_{11} = \zeta^2 I$, $Q_{22} = -I, Q_{12} = 0$ in (15). One seeks for the minimal disturbance attenuation level ζ such that the matrix inequality is satisfied. The resulting nonlinear optimization problem is nonconvex and possibly nondifferentiable (when the two largest eigenvalues of the matrix inequality coincide) [17].

The matrix inequality from Lemma 1 can be used to impose global asymptotic stability of the closed-loop scheme for Narendra's dynamic backpropagation [15]. Controller design using dynamic backpropagation means that the neural controller is trained by optimizing on one or a set of specific reference inputs. This may lead to instabilities of the control scheme. One can overcome this problem by the modified dynamic backpropagation scheme

$$\begin{aligned} \min_{\theta_c, P, \Omega, \Gamma, T, \Phi, \alpha} \ & J_{\mathrm{track}}(\theta_c) \\ &= \int_0^{t_f} [d(t) - \hat{y}(t;\theta)]^T \, [d(t) - \hat{y}(t;\theta)] \, dt \\ & \text{s.t.} \quad Y + k_1 k_3 Z Z^T < 0 \end{aligned} \tag{22}$$

$$Y = Y^T = \begin{bmatrix} A^T P + PA - E^T Q_{22} E & PB + k_1 A^T N^T \Omega + k_1 N^T \Phi & k_2 A^T C^T \Gamma + k_2 C^T T & PD - E^T (Q_{12}^T + Q_{22} L) \\ \cdot & -2\Phi + k_1 \Omega N B + k_1 B^T N^T \Omega & k_2 B^T C^T \Gamma + k_1 H^T \Phi & k_1 \Omega N D \\ \cdot & \cdot & -2T - \alpha I & k_2 \Gamma C D \\ \cdot & \cdot & \cdot & Y_{44} \end{bmatrix} \tag{19}$$

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where $\hat{y}(t;\theta)$ is the output of the neural control scheme, $J_{\mathrm{track}}(\theta_c)$ is a cost function for the tracking error, defined on a given specific reference input $d(t)$ and t_f is a finite time horizon. The matrix inequality constraint can be related to Lemma 1 as well as to Lemma 2, respectively, for imposing global asymptotic stability or I/O stability with a fixed disturbance attenuation level ζ^*. Such methods have been successfully applied for the discrete-time recurrent neural networks using NL_q theory in [19].

VII. Conclusion

In this paper absolute stability and dissipativity of continuous-time recurrent neural networks with two hidden layers have been studied. These types of models occur when one considers nonlinear models and controllers that are parameterized by multilayer perceptrons with one hidden layer. For the autonomous case a classical Lur'e system representation and Lur'e system with multilayer perceptron nonlinearity is given. Sufficient conditions for absolute stability and dissipativity have been derived from a Lur'e–Postnikov Lyapunov function and a storage function of the same form. The criteria are expressed as matrix inequalities. They can be employed in order to impose closed-loop stability in Narendra's dynamic backpropagation procedure and for nonlinear H_∞ control.

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Noninteracting Control via Static Measurement Feedback for Nonlinear Systems with Relative Degree

S. Battilotti

***Abstract*— In this paper the authors give a necessary and sufficient geometric condition for achieving noninteraction via static measurement feedback for nonlinear systems with vector relative degree. Their analysis relies on the theory of connections and as a result gives systematic procedures for constructing a decoupling feedback law.**

***Index Terms*—Measurement feedback, noninteracting control.**

I. The Class of Systems and Control Laws

Let us consider the affine nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j(x)u_j \\ y_i &= h_i(x), \qquad i = 1, \cdots, m, \\ z &= k(x) \end{aligned} \tag{1}$$

where $x \in \mathcal{M}$, a smooth (Hausdorff) manifold, the u_i's are input functions from a suitable function space (e.g., measurable $\mathbb{R}$-valued functions defined on closed intervals of the form $[0, T]$), the y_i's are the $\mathbb{R}$-valued output functions and $z \in \mathbb{R}^s$ is the vector of

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The author is with the Dipartimento di Informatica e Sistemistica, 00184 Roma, Italy.

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