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Linear System Identification, Structured Total Least Squares and the Riemannian SVD*

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Abstract

Many linear estimation problems in signal processing and system identification applications, lead to a structured total least squares formulation and a so-called Riemannian singular value decomposition. Examples that we give in this paper include noisy realization of impulse and step responses, (errors-in-variables) identification of linear systems, ARMA model parameter estimation and FIR filter multichannel blind identification. We also provide a maximum likelihood interpretation of the Riemannian SVD and discuss several extensions, such as including linear constraints or dealing with missing observations. We also point out some possible relations with so-called matrix completion and interpolation problems.

1 Total Least Squares

The *total linear least squares problem* for a given matrix $A \in \mathcal{R}^{p\times q}$, which is of full column rank (we assume that $p \geq q$), is to find a rank deficient least squares approximation $B \in \mathcal{R}^{p\times q}$:

$$(1) \qquad \min_{\substack{B \in \mathcal{R}^{p\times q} \\ y \in \mathcal{R}^q}} \|A - B\|_F^2 \text{ subject to } \begin{cases} By = 0\,, \\ y^T y = 1\,. \end{cases}$$

The two constraints ensure that the approximation matrix B is rank deficient. It is well known that the solution can be found from the singular value decomposition (SVD) of the matrix A, by removing from its dyadic decomposition, the term corresponding to the smallest singular value [8], [13]. As will be demonstrated in this paper, there are many instances in which the objective function $\|A - B\|_F^2 = \sum_{i=1}^{p}\sum_{j=1}^{q}(a_{ij} - b_{ij})^2$ must be extended with 'elementwise' nonnegative weights $w_{ij} \geq 0$, and, in addition, the approximating rank deficient matrix B is required to have some structure, such as for instance a Hankel or Toeplitz structure. In these cases we have the

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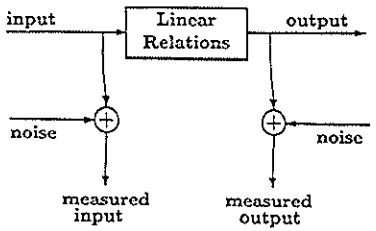

FIG. 1. *Total least squares configuration: 'Exact' noise-free outputs are generated by sending an 'exact' noise-free input 'through' a linear system. The inputs and outputs are however not known exactly, but are corrupted by additive noise (e.g. measurement noise). The total least squares modeling problem boils down to modifying the given observed data, such that the 'modified output' is generated 'through' a linear model by the 'modified input'.*

Structured/Weighted Total Least Squares problem (STLS):

$$(2) \qquad \min_{\substack{B \in \mathcal{R}^{p\times q} \\ y \in \mathcal{R}^q}} \sum_{i=1}^{p}\sum_{j=1}^{q}(a_{ij} - b_{ij})^2 w_{ij} \text{ subject to } \begin{cases} By = 0\,, \\ y^T y = 1\,, \\ B \text{ structured}\,. \end{cases}$$

Again, the rank deficiency is ensured by the constraints.

In this paper, we will confine ourselves to so-called *affine matrix structures*, i.e. structured matrices B that can be written as an affine (linear) combination of a given set $\{B_k; k = 0, \ldots, s\}$ of $s+1$ basis matrices as $B = B_0 + B_1 b_1 + \ldots + B_s b_s$. Here, the real coefficients $b_k, k = 1, \ldots, s$ 'parametrize' the structured matrix B. Examples of such affinely (linearly when $B_0 = 0$) structured matrices are (centro-, per-, skew-, ...) symmetric matrices, (block-) Hankel, (block-) Toeplitz, (block-) circulant, Brownian, Hankel + Toeplitz matrices, matrices with a certain zero structure (sparsity pattern), etc ....

Two major key observations form the underlying motivation for the results we present here:

- Matrices that are simultaneously affinely structured and rank deficient, frequently occur in many signal processing and system identification applications. In these applications, the combination of rank deficiency and structure implies that the underlying data are 'generated' by a model that is linear-in-the-parameters. However, in practical experimental circumstances, observations are always corrupted by additive (measurement) noise, and therefore, the data matrices are never rank deficient. Therefore it seems like a good idea[1] to try to modify the observations (as little as possible), so that the structured matrices that contain the modified data, are rank deficient. When we modify the data in a least squares sense, we call the problem a *structured total least squares* (STLS) problem.

[1] It actually is!

- A second observation lies in the insight that the solutions to STLS problems can be described by a *nonlinear generalized singular value decomposition* (derived in [2], [3]), which we have called the Riemannian SVD in [6]:

> Riemannian SVD
>
> Let A be the (structured full rank) 'observed' data matrix. Then the solution to the STLS problem (2) follows from the singular triplet (u, τ, v) corresponding to the smallest singular value τ in:
>
> $$\begin{aligned} A\,v &= D_v\,u\,\tau\,, & u^T D_v\,u &= 1\ (= v^T D_u\,v \text{ always})\,, \\ A^T u &= D_u\,v\,\tau\,, & v^T v &= 1\,, \end{aligned} \tag{3}$$
>
> Here $u \in \mathcal{R}^p$ and $v \in \mathcal{R}^q$ are a left, resp. right singular vector and $\tau \in \mathcal{R}$ is the singular value. D_u and D_v are symmetric positive definite (or nonnegative definite) matrices, the elements of which are quadratic functions of the components of u, resp. v. Their precise structure depends on the matrix structure and on the weights in the objective function.

One needs to find the minimal singular value τ because it can be shown that it is exactly equal to the objective function in (2). The 'modified' data matrix B can be reconstructed in general as a modification of the 'observed' data matrix A, a modification[2] that is multilinear in the elements of the singular vectors u and v and the singular value τ (for details and algorithms, we refer to [2], [3], [6]).

We have kept the discussion in this paper rather informal, do not give proofs nor derivations. This is more a paper of ideas, less one of rigor and proofs. For the same reason, we have included an exhaustive list of references, which however can be found in the papers mentioned at the end in the bibliography. In Section 2, we briefly discuss the so-called noisy realization problem and we give a maximum likelihood interpretation. Several extensions are discussed, such as starting from step response measurements instead of impulse responses, or adding linear constraints. We also discuss how to include weights, two extreme versions of which are cases where some of the data are known exactly, or some of the data are missing. We also point out some connections between STLS problems and so-called matrix completion and interpolation problems. Linear dynamic system identification (errors-in-variables) is the subject of Section 3, with ARMA model parameter estimation being discussed in Section 4. As a special application, we mention multichannel blind finite impulse response identification in Section 5. The conclusions can be found in Section 6.

2 Noisy Realization

Consider a linear discrete-time time-invariant dynamic system with one input $t_k \in \mathcal{R}$, one output $z_k \in \mathcal{R}$ and the state $x_k \in \mathcal{R}^n$ of the form

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d t_k\,, \\ z_k &= C_d x_k + D_d t_k\,, \end{aligned} \tag{4}$$

[2] In the unstructured case (1), the matrix B is obtained as a rank one modification of A as $B = A - u\tau v^T$.

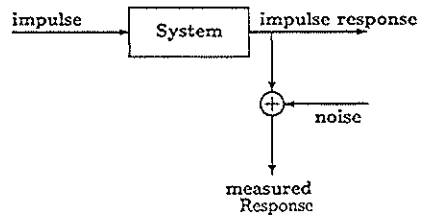

FIG. 2. *Impulse response experiment: An impulse is sent through a linear system with zero initial state. The impulse response is observed, corrupted by additive measurement noise. Actually, there are many engineering applications in which signals can be modeled as the sum of a certain number of (un-)damped sinusoids, which is always the impulse response of a linear system. Examples include modal and vibrational analysis in mechanical engineering, harmonic retrieval and spectral estimation, biomedical signal processing from nuclear magnetic resonance measurements, etc...*

and A_d, B_d, C_d, D_d are vectors/matrices of appropriate dimensions. The impulse response $h_k, k \geq 0$ for this single-input single-output (SISO) system is obtained by applying an impulse input of the form $u_k = (1, 0, 0, 0, ...)$, starting from an initial state $x_0 = 0$:

$$h_0 = D_d\,,\ h_k = C_d A_d^{k-1} B_d\,,\ k \geq 1\,. \tag{5}$$

It can be shown that for generic linear systems the impulse response consists of a sum of damped sinusoids, which makes that this class of models is very useful in many applications. The *realization problem* is to find from the impulse response samples $h_k, k \geq 0$, the system matrices A_d, B_d, C_d and D_d. It is called the *noisy realization* problem when the observations are corrupted by (additive) noise. When a step input of the form $u_k = (1, 1, 1, 1, 1, 1, ...)\ ,\ k \geq 0$, is applied starting from $x_0 = 0$, we obtain the so-called step response, which is

$$s_0 = D_d\,,\ s_k = D_d + \sum_{i=1}^{k} C_d A_d^{i-1} B_d\,,\ k \geq 1. \tag{6}$$

We treat (noisy) *step response realization* in Section 2.2. For general input sequences (i.e. no impulse nor step), one can observe the outputs and then try to find the matrices A_d, B_d, C_d and D_d, directly from inputs and outputs. This is called *system identification*. When inputs and outputs are corrupted by noise, we have the *errors-in-variables system identification* problem, which is the subject of Section 3. When the input sequence is an unobserved white noise sequence, the model (4) is called an ARMA model, the identification of which is treated in Section 4.

2.1 Problem and Solution

Consider the problem described in Figure 2, which is called the noisy realization problem. Let $h_i \in \mathcal{R}, i = 1, \ldots, N$ be the impulse response of a linear SISO system of order n as in (4). If there is no noise on the observations, then it is a well known result of linear system theory (see e.g. [9]) that the Hankel matrix $H \in \mathcal{R}^{p \times q}$ (with $N = p + q - 1$)[3] constructed

[3] In this paper, we will always assume that $p > q$

with these data is rank deficient if p and q are sufficiently large. As a matter of fact:

$$\text{rank}(H) = n \text{ when } \min(p,q) > n \,. \tag{7}$$

In a practical situation, the data are never known exactly, but they are corrupted by noise. Hence, the Hankel matrix A with the observed impulse response samples $a_i, i = 1,\ldots,p+q-1$ will be of full column rank q. Therefore, it is meaningful to formulate the following structured total least squares problem:

$$\min_{\substack{b_i \in \mathcal{R} \\ y \in \mathcal{R}^q}} \sum_{i=1}^{p+q-1} (a_i - b_i)^2 \text{ subject to } \begin{cases} B\,y = 0\,, \\ y^T y = 1\,, \\ B \text{ Hankel}\,, \end{cases} \tag{8}$$

where $B \in \mathcal{R}^{p\times q}$ is the rank deficient Hankel matrix with the 'modified' data b_i.
The rank deficiency of B is guaranteed by the existence of a non-trivial (i.e. not identically 0, as imposed by $y^T y = 1$) vector y in its null space.
The vector y contains the coefficients of the so-called characteristic polynomial of the linear model, of which the b_i are the impulse response samples. The roots of that characteristic polynomial determine the eigenfrequencies and dampings of the dynamic 'modes' of the system. Under general conditions of observability and controllability, they are also the eigenvalues of the matrix A_d in (4).
The order n of the linear approximating model with impulse response b_i can be controlled by the user, as we are free to pick q (with corresponding $p = N + 1 - q$). From (7) it then follows that the order of the linear model will be (at most) $n = q - 1$[4].

As is claimed in our general statement in the introduction, the solution can be obtained from the singular triplet (u,τ,v) corresponding to the smallest singular value in (3) (for a derivation we refer to [3]), where the weighting matrices $D_u \in \mathcal{R}^{q\times q}$ and $D_v \in \mathcal{R}^{p\times p}$ are banded symmetric Toeplitz positive definite matrix functions, the elements of which are quadratic in the components of u, resp. v:

$$D_u = T_u\, T_u^T\,,\; D_v = T_v\, T_v^T\,, \tag{9}$$

where $T_u \in \mathcal{R}^{q\times(p+q-1)}$ is a banded Toeplitz matrix, generated with the components of u as

$$T_u = \begin{pmatrix} u_1 & u_2 & \ldots & \ldots & u_{p-1} & u_p & 0 & \ldots & 0 \\ 0 & u_1 & u_2 & \ldots & u_{p-2} & u_{p-1} & u_p & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & u_1 & u_2 & \ldots & u_{p-2} & u_{p-1} & u_p \end{pmatrix} \tag{10}$$

and $T_v \in \mathcal{R}^{p\times(p+q-1)}$ is constructed similarly from the components of v.

2.2 A Maximum Likelihood Interpretation of D_u and D_v

Let us explain an easy 'trick' to find the matrix D_v in (9) for the Hankel case in the Riemannian SVD (3). Actually, this trick works for all possible affine matrix structures that one might consider and therefore, allows to find the weighting matrices D_u and D_v for any affine matrix structure that one would like to impose in the TLS problem (2).

[4] It can be shown that the order actually is $q - 1$.

As we will see below, this 'heuristic' derivation to find the precise structure of the STLS weighting matrices D_u and D_v can also be invoked when there are given weights in the objective function (including the special cases of *exact noiseless* data or *missing* data).
Because of the noise corrupting the data, the Hankel matrix A with 'observed' impulse response samples, is of the form

$$A = H + E = H + \begin{pmatrix} \epsilon_1 & \epsilon_2 & \ldots & \epsilon_q \\ \epsilon_2 & \epsilon_3 & \ldots & \epsilon_{q+1} \\ \epsilon_3 & \epsilon_4 & \ldots & \epsilon_{q+2} \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon_p & \epsilon_{p+1} & \ldots & \epsilon_{p+q-1} \end{pmatrix},$$

where the Hankel matrix H contains the 'exact', uncorrupted data and the Hankel matrix E contains the noise. If the order of the linear system generating the data in H is $q-1$, then, from (7), we know that H is rank deficient and therefore that there exists a non-trivial vector v in its null space: $Hv = 0$. Then we have

$$Av = Hv + Ev = Ev = e\,, \tag{11}$$

where e can be considered as an 'equation error' (it's a vector of residuals, that would have been 0 if A would have been equal to H (i.e. no noise)). Because the vector e is a linear vector function in the elements of E and also linear in the elements of v, we can rewrite it as

$$e = T_v\,\epsilon\,, \tag{12}$$

where T_v is now a matrix constructed with the components of v (it has the same structure as the matrix in (10)) and ϵ contains the elements $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{p+q-1})$.
If the noise sequence ϵ, that corrupts the data, is white, normal, zero mean with covariance matrix $\mathbf{E}(\epsilon\epsilon^T) = \sigma^2 I_{p+q-1}$, where σ^2 is the unknown variance, then the maximum likelihood estimate of the elements of v follows from minimizing over v the weighted two-norm of the residual vector:

$$\min_v\; e^T W^{-1} e\,,$$

where $W = \mathbf{E}(ee^T)$ is the covariance matrix of the equation error, which is given by

$$W = \mathbf{E}(ee^T) = T_v\,[\mathbf{E}(\epsilon\epsilon^T)]\,T_v^T = \sigma^2 (T_v T_v^T)\,.$$

Hence, the maximum likelihood solution under the noise assumptions as specified above, follows from the minimum over $v \in \mathcal{R}^q$, of the objective function

$$v^T A^T (T_v\, T_v^T)^{-1} A v\,, \tag{13}$$

which is exactly the expression obtained in [1]. We also refer to [10] for more details and equivalences with other methods and approaches.
The matrix D_v required for the Riemannian SVD (3) is exactly equal to

$$D_v = T_v\, T_v^T\,,$$

from which it can be concluded that D_v is equal (up to within an unknown scalar) to the covariance matrix of the equation error defined in (11).
It is also straightforward to derive expression (13) from the Riemannian SVD (3) when

assuming the invertibility of D_v (which is not the case for all affine matrix structures), one eliminates u, to find that

$$\tau^2 = v^T A^T D_v^{-1} A v \, .$$

Using the same 'heuristic' derivation with $A^T u$, one can also derive the exact expression for D_u, which is precisely (up to within an unknown scalar) the covariance matrix of the equation error $A^T u$.

2.3 Step Response Realization

It is sometimes more convenient to measure the step response (6) of a linear system than the impulse response (5). The relation between these two sequences is straightforward:

$$h_0 = s_0 \, , \; h_k = s_k - s_{k-1} \, , \; k \geq 1 \, .$$

Therefore, it follows from (7) that, for a linear system of order $q-1$, the Hankel matrix

$$S = \begin{pmatrix} s_1 - s_0 & s_2 - s_1 & \dots & s_q - s_{q-1} \\ s_2 - s_1 & s_3 - s_2 & \dots & s_{q+1} - s_q \\ s_3 - s_2 & s_4 - s_3 & \dots & s_{q+2} - s_{q+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_p - s_{p-1} & s_{p+1} - s_p & \dots & s_{p+q-1} - s_{p+q-2} \end{pmatrix} , \tag{14}$$

will be of rank $q-1$.
In the case that the step response samples are not observed exactly, but corrupted by additive noise, it makes sense to define a structured total least squares problem, which is similar to the one in (8) but where now B is required to be rank deficient and the difference of two Hankel matrices as the matrix S in (14).

We can find the matrices D_u and D_v for this case, using the heuristic trick as explained in the previous subsection. Illustrated for the case $p = 4, q = 3$, we have

$$c = Av = Ev = T_v \, c = \begin{pmatrix} -v_1 & v_1 - v_2 & v_2 - v_3 & v_3 & 0 & 0 & 0 \\ 0 & -v_1 & v_1 - v_2 & v_2 - v_3 & v_3 & 0 & 0 \\ 0 & 0 & -v_1 & v_1 - v_2 & v_2 - v_3 & v_3 & 0 \\ 0 & 0 & 0 & -v_1 & v_1 - v_2 & v_2 - v_3 & v_3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} . \tag{15}$$

The matrix D_v is given by $D_v = T_v T_v^T$. The construction of the matrix D_u is similar, starting from $A^T u$.

2.4 Adding Linear Constraints to y

In many applications, there is some a priori information on the poles of the system (which, under general conditions, are the eigenvalues of A_q in (4)). In some cases, this a priori information can be formulated as a set of linear constraints[5] on y :

[5] Here we only briefly treat the case of homogeneous linear constraints of the form $Ly = 0$. The case of non-homogeneous constraint of the form $Ly = a$ for a given matrix L and given vector a, can be treated as an homogeneous one of the form $(L \;\; a)z = 0$, with an additional linear constraint on z. We omit the details here.

Known poles: If $\alpha \pm j\beta$ is a known complex pole pair of the linear system, then it is easily shown that this corresponds to a pair of linear constraints on the vector y of the form

$$Ly = 0 \text{ where } L = \begin{pmatrix} 1 & \rho\cos\phi & \rho^2\cos(2\phi) & \dots \\ 0 & \rho\sin\phi & \rho^2\sin(2\phi) & \dots \end{pmatrix} ,$$

in which ρ and ϕ follow from De Moivre's theorem $\alpha \pm j\beta = \rho(\cos\phi \pm j\sin\phi)$ as $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\phi = \arctan(\beta/\alpha)$.

For instance, let some signal be sampled at a sampling frequency of $f_s = 300Hz$ and let it be corrupted by a 50 Hz power line interference[6], which is to be eliminated from the measurements. We then have $\rho = 1$ and $\phi = \pi/3$ rad, so that the matrix L has the form:

$$L = \begin{pmatrix} 1 & 0.5 & -0.5 & -1 & \dots \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 & 0 & \dots \end{pmatrix} .$$

If there is a pole in 1 (for instance due to the fact that there is an integrator in the system), we have

$$L = (1 \;\; 1 \;\; \dots \;\; 1) \, .$$

Poles on the unit circle: A *necessary* condition for poles to lie on the unit circle, is the fact that the vector y (which contains the coefficients of the characteristic polynomial of the system) is symmetric, i.e. $y_j = y_{q-j+1}, \forall j$. This implies that

$$Ly = 0 \text{ where } L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 & 0 \\ 0 & 0 & 1 & \dots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} .$$

With L being the matrix of linear constraints coefficients, the Riemannian SVD becomes

$$\begin{array}{rcll} \begin{pmatrix} L \\ A \end{pmatrix} v & = & \begin{pmatrix} 0 & 0 \\ 0 & D_v \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tau \, , & u_2^T D_v \, u_2 = 1 \, , \\ & & & v^T D_u \, v = 1 \, , \\ (L^T \;\; A^T) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & = & D_u \;\; v \;\; \tau \, , & v^T v = 1 \, , \end{array}$$

where for the Hankel case, D_u and D_v are as in (9), with an obvious partitioning of u into $(u_1^T \; u_2^T)^T$.

2.5 'Data' Constraints

Another constraint which leads to a Riemannian SVD (we do not work out the details here), is a linear constraint on the elements b_i of the rank deficient approximation.
As an example of such a constraint, consider again the impulse response samples of a linear system, where we now require that

$$\sum_{i=1}^{N} b_i = \sigma_{ss} \, ,$$

[6] ... in Europe. In the USA this would be 60 Hz.

where $\sigma_{ss} \in \mathcal{R}$ is a given real number. Since the b_i are the impulse response samples, such a constraint corresponds to the the requirement that the steady state of the *step response* (we assume here that the approximating system will be stable and that N is sufficiently large) is equal to σ_{ss}. In some applications it is straigtforward to measure the steady state σ_{ss} of the step response. This a priori information can then be exploited in the realization.

2.6 Adding Weights: Relative Errors - Exact Data - Missing Observations

We can extend the objective function (8) by introducing given weights $w_i \geq 0$, $i = 1, \ldots, N$ so that we obtain the 'weighted' STLS problem:

$$\min_{\substack{b_i \in \mathcal{R} \\ y \in \mathcal{R}^q}} \sum_{i=1}^{N} (a_i - b_i)^2 \, w_i \text{ subject to } \begin{cases} By = 0 \, , \\ y^T y = 1 \, , \\ B \text{ Hankel} \, . \end{cases} \tag{16}$$

The weight w_i can be interpreted as *the inverse of the variance of the error on the measured data sample* a_i[7]. So the data need not even be obtained with the same measurement error variance, if only the different variances of these measurement errors are known a priori. If all the variances are equal, we can put all the weights equal to 1.

There are some special cases:

- If the weight $w_i = 0$, then the variance of the measurement error on a_i is infinite, so that we can change it 'as we like'. Said in other words, modifying a_i into b_i doesn't contribute anything to the objective function. This insight can be used to deal with *missing observations*, where a missing observation, say a_i, is replaced by an arbitrary number (e.g. 0), and then weighted in the objective function with a weight $w_i = 0$.
- If the weight $w_i \to \infty$, the measurement error variance goes to 0, meaning that we actually impose $a_i = b_i$, which is a linear constraint on an element of B. It simply means that the data point a_i can not be modified at all. Said in other words, we consider it to be noise-free.
- Finally, if $w_i = 1/a_i^2$, the total least squares criterion in (16), which actually proposes to minimize a sum of absolute errors squared, reduces to a minimization of the sum of squares of (a first order approximation of) the relative errors. This is relevant in some applications, where the scale of the measurements can range over several orders of magnitude.

For all these cases, the solution to (16) is given by the Riemannian SVD (3) with

$$D_u = T_u V T_u'^T \text{ and } D_v = T_v V T_v'^T \text{ where } V = [\text{diag}(w_1, \ldots, w_N)]^{-1} \tag{17}$$

and T_u and T_v are as in (9).
If the noise sequence that perturbs the data is white, zero mean, normal with elementwise variance σ_i^2, we put $V = \text{diag}(\sigma_1^2, \ldots, \sigma_N^2)$.

[7] This interpretation is not only heuristic, but can be made rigorous by referring to maximum likelihood estimation principles.

If one would like to have a relative error criterion, it suffices to put $w_i = 1/a_i^2$.
If we consider a noisy realization problem with 4 'exact' noise-free data points a_1, a_3, a_5, a_6, and the points a_2 and a_4 missing, then, with $p = 4$ and $q = 3$ (so we want an approximation by a $(q-1)$-nd = 2-nd order linear system), we have the Riemannian SVD (just put $w_1 = w_3 = w_5 = w_6 = \infty$ and $w_2 = w_4 = 1$ in (17)):

$$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & a_3 & 0 \\ a_3 & 0 & a_5 \\ 0 & a_5 & a_6 \end{pmatrix} v = \begin{pmatrix} v_2^2 & v_1 v_2 & 0 & 0 \\ v_1 v_2 & v_1^2 + v_3^2 & v_2 v_3 & v_1 v_3 \\ 0 & v_2 v_3 & v_2^2 & v_1 v_2 \\ 0 & v_1 v_3 & v_1 v_2 & v_1^2 \end{pmatrix} u \, \tau \, ,$$

$$\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_3 & 0 & a_5 \\ a_3 & 0 & a_5 & a_6 \end{pmatrix} u = \begin{pmatrix} u_2^2 + u_4^2 & u_1 u_2 + u_3 u_4 & u_2 u_4 \\ u_1 u_2 + u_3 u_4 & u_1^2 + u_3^2 & u_2 u_3 \\ u_2 u_4 & u_2 u_3 & u_2^2 \end{pmatrix} v \, \tau \, ,$$

with $v^T D_u v = u^T D_v u = 1$ and $v^T v = 1$. The missing elements have been replaced by 0 (one could put any other value here as the corresponding term will not be weighted whatsoever in the objective function). It is important to observe that this problem does not necessarily have a solution (said in other words, it is possible that the data a_1, a_3, a_5, a_6, which are supposed to be known exactly, cannot be 'explained' by a 2-nd order linear system's impulse response)[8].

As a concluding note, observe that

- We can interprete the least squares approximation of a given Hankel matrix with missing observations by a rank deficient Hankel matrix as a matrix completion problem[9]. This opens interesting perspectives to investigate whether there are more connections between STLS problems and matrix completion problems.
- In the same way, this particular case of a rank deficient least squares approximation of a given Hankel matrix with missing observations, can also be interpreted as an *interpolation problem*, in which some points are given, and we need to find the best interpolating impulse response of a linear system.

3 Linear System Identification: Errors-in-Variables

Consider the case where, in Figure 1, the system block is replaced by a linear discrete-time time-invariant system. For SISO systems, one of the possible models (besides state space as in (4)) is a so-called difference equation where the output z_k is related to the input t_k by a difference equation of the form

$$\sum_{i=0}^{q} a_i z_{k+i} = \sum_{i=0}^{q} b_i t_{k+i} \, , \; k = 0, \ldots, N \, . \tag{18}$$

[8] We do not treat *existence* problems here. Indeed, not every STLS problem has a solution: As an example, consider the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & ? \end{pmatrix}$. Whatever be the value of the element denoted by ?, the matrix will always be of rank 2, so its rank cannot be reduced.

[9] Completion problems are matrix problems in which certain elements of the matrix are to be determined so that that matrix, when *completed*, satisfies given specified properties.

Using the $z-$transform, we find the linear model

$$z(z) = \frac{b(z)}{a(z)} . t(z) \,, \tag{19}$$

where the rational function $b(z)/a(z)$ is called the transfer function of the linear system, in which the coefficients of the numerator and denominator are b_i resp. a_i.
If all the data are corrupted by noise, i.e. instead of t_k we observe u_k, and instead of z_k, we observe y_k, it makes sense to consider the following STLS problem:

$$\min_{\substack{z_k, w_k, \\ a, b}} \sum_{k=0}^{N+q} [(z_k - y_k)^2 \, \alpha + (t_k - u_k)^2 \, \beta] \text{ subject to } \begin{cases} \text{difference equations (18)} \,, \\ a^T a + b^T b = 1 \,. \end{cases} \tag{20}$$

Here α and β are user-specified weights in the objective function, which allow to trade off the modification 'energy' for the input relatively to that of the output. If $\alpha = \beta = 1$, then inputs and output modifications are 'on the same level'. If $\beta \to \infty$ and $\alpha = 1$, we have what is known in the literature as an *output error model* (all the errors are attributed to output measurements)(see e.g. [12]). If $\beta = 1$ and $\alpha \to \infty$, this might be called an *input error model*[10].
We have been analyzing this problem (without the weights α and β) and some of its intriguing properties in [4], [5]. Suffices to say here that the solution follows from the smallest singular value of the Riemannian SVD:

$$\begin{array}{rcll} (Y \;\; U) \begin{pmatrix} a \\ b \end{pmatrix} & = & D_{ab} \;\; u \;\; \tau \,, & u^T D_{ab} \, u = 1 = (a^T \;\; b^T) D_u \begin{pmatrix} a \\ b \end{pmatrix} \,, \\ \begin{pmatrix} Y^T \\ U^T \end{pmatrix} u & = & D_u \begin{pmatrix} a \\ b \end{pmatrix} \tau \,, & v^T v = 1 \,, \end{array} \tag{21}$$

where the exact expressions for D_a, D_b and D_u can be found in [4], [5]. Y and U are Hankel matrices generated with the measured inputs and outputs while a and b are vectors with the coefficients of the difference equation (18).

Of course, one could extend this problem with all the constraints that we have been considering in the previous section (known poles, linear constraints on a and/or b, including weights, missing observations, etc ...), but we will not consider the details here.

4 ARMA Models

Consider the so-called ARMA identification problem depicted in Figure 3. The input-output model is also given by the difference equation (18), only now the input is not observed but is assumed to be a zero mean white noise sequence c_k with $\mathrm{E}(c_k c_l) = \delta_{kl}$ (Kronecker delta). Also, the outputs are assumed to be observed without error.
Knowing only the outputs y_k, the coefficients a_i and b_j can be determined from the following minimization problem:

$$\min_{a_i, b_j, c_k} \sum_{k=1}^{N} c_k^2 \,, \tag{22}$$

[10] which is a (new ?) class of models, that, up to my knowledge, has not been analyzed so far in the system identification literature.



FIG. 3. *ARMA model identification set-up: unknown white noise is sent through a linear system. Only the outputs are observed (without noise).*

subject to the difference equation (18), with t_k replaced by c_k. Notice that this STLS problem can be derived from (20) by considering the inputs u_k to be identically 0, by letting the weights $\alpha \to \infty$ (we cannot modify the outputs) and $\beta = 1$ (we can change the zero inputs). This reveals quite an interesting relationship between ARMA modeling and 'errors-in-variables' linear system identification, which, up to my knowledge, has not been observed before and will be explored in some future work: we suggest that ARMA model identification can be interpreted as a limiting case of an input-error identification problem.

As a consequence, there is no contribution of a in D_{ab} in (21) and the Riemannian SVD reduces to:

$$\begin{array}{rcll} (Y \;\; 0) \begin{pmatrix} a \\ b \end{pmatrix} & = & D_b \;\; u \;\; \tau \,, & u^T D_b u = 1 = (a^T \;\; b^T) D_u \begin{pmatrix} a \\ b \end{pmatrix} \,, \\ \begin{pmatrix} Y^T \\ 0 \end{pmatrix} u & = & \begin{pmatrix} 0 & 0 \\ 0 & D_u \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tau \,, & (a^T \;\; b^T) \begin{pmatrix} a \\ b \end{pmatrix} = 1 \,. \end{array}$$

in which we leave the derivation of the exact expressions for D_b (only function of b) and D_u to the reader.
These ideas can be generalized to other well-known model classes such as e.g. ARMAX models or Box-Jenkins models, but we refer to some future work for details.

5 Multichannel 'Blind' Finite Impulse Response Identification

To give yet another illustration of the power of modeling problems by rank deficient structured matrices, let's consider the problem of multichannel 'blind' finite impulse response (FIR) identification, depicted in Figure 4. The word 'blind' refers to the fact that the input is not observed at all, it is actually unknown, as are the two finite impulse response sequences, by which the input gets convolved to generate the two output sequences, which are corrupted by additive noise. The problem is now to find the coefficients of the FIR-filters h^1 and h^2 from the observed output. Even in the noiseless case, this seems like an underdetermined problem, because not only the models h^1 and h^2 are unknown, but also the input sequence. However, due to the fact that we have two outputs generated by the same input, and the fact that the filters are finite impulse response filters, one can still find the models. This can be seen as follows. Let t_k be the input sequence, z_k^1 and z_k^2 the output sequences (without noise). Then, using $z-$transforms, we find

$$z^1(z) = h^1(z) . t(z) \;\text{ and }\; z^2(z) = h^2(z) . t(z) \,,$$

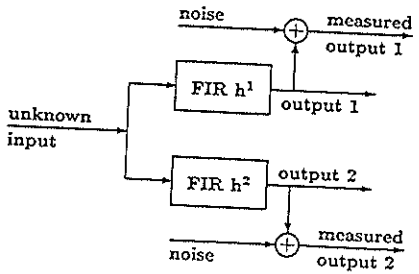

FIG. 4. *Multichannel blind identification problem: The input sequence is unknown, as are the two finite impulse response (FIR) filters h^1 and h^2. The two output sequences are measured but corrupted by additive noise.*

so that we can eliminate $t(z)$ to find

$$z^2(z) = \frac{h^2(z)}{h^1(z)} . z^1(z) .$$

Comparing this to equation (19), we see that z^1 can be considered as an input, and z^2 as the output of a linear dynamic system with transfer function[11] $h^2(z)/h^1(z)$. Since both the input and output are corrupted by noise, the problem now reduces to the 'errors-in-variables' identification problem that we have been considering in Section 3 and its optimal solution follows from the smallest singular value of a Riemannian SVD as in (21).
This key idea can be extended to blind multichannel identification problems with more than 2 FIR channels. In this case, the matrix structure will still be affine, with zero elements (which are to be kept at zero in the structured total least squares problem, by introducing infinite weights as explained in Section 2).
We conclude by saying that recently, there has been quite some interest in blind identification problems in telecommunications (modem egalization using oversampling, i.e. the sampling rate exceeds the baud rate). For a good recent survey, we refer e.g. to [11]. It would also be worth investigating whether this idea is applicable in modeling mechanical vibrational systems with unknown inputs (e.g. transfer functions in a car from real road experiments).

6 Conclusions

The main contribution of this paper lies in its rather informal discussion of the occurrence of structured total least squares problems in many applications of system identification and signal processing. Much remains to be said and explored, including existence questions, algoritmic issues (convergence of methods derived on heuristic grounds such as the 'power method' in [2]) and much experience remains to be gained with these algorithms. Although we have restricted ourselves in this paper to linear (dynamical) systems, the ideas we have put forward also apply to some extent to nonlinear systems, such as e.g. bilinear

[11] Provided that there are no common zeros between the polynomials $h^2(z)$ and $h^1(z)$, which we have to assume.

systems. There too, there is a rank deficiency result for a certain affinely structured matrix (the counterpart of the Hankel matrix for linear systems), built up with the so-called Volterra kernels of the system, which might also be obtained experimentally. Input-output models for bilinear systems allow for products between inputs and outputs, but are linear in the parameters, which implies that (under certain noise assumptions), the identification problem is a structured total least squares problem.
Many of the necessary details have to be worked out in future work, including the connection that we have suggested between STLS problems, matrix completion and interpolation problems, and new interpretations of the relations between several identification methods and approaches.

References

[1] T. J. Abatzoglou, J. M. Mendel, and G. A. Harada, *The constrained total least squares technique and its application to harmonic superresolution*, IEEE Transactions on Signal Processing, SP-39 (1991), no.5, pp. 1070–1087.
[2] B. De Moor, *Structured total least squares and L_2 approximation problems*, Linear Algebra and its Applications, Special issue on Numerical Linear Algebra Methods in Control, Signals and Systems, P. Van Dooren, G. Ammar, N. Nichols, and V. Mehrmann, eds., 188-189 (1993), pp.163–207.
[3] B. De Moor, *Total least squares for affinely structured matrices and the noisy realization problem*, IEEE Trans. on Signal Processing, SP-42 (1994), no. 11, pp. 3104–3113.
[4] B. De Moor, *Dynamic Total Linear Least Squares* , in Proc. of the 10th IFAC Symposium of System Identification, SYSID '94, 4-6 July 1994, Copenhagen, Denmark, Vol.3., pp. 159–164.
[5] B. De Moor and B. Roorda, *L_2-optimal linear system identification: Dynamic total least squares for SISO systems*, in Proc. of 33rd IEEE Conf. on Decision and Control (CDC), Florida, Dec. 1994, pp. 2874–2879.
[6] B. De Moor, *The Riemannian Singular Value Decomposition*, in SVD and Signal Processing III: Algorithms, Architectures, Applications, M. Moonen and B. De Moor, eds., Elsevier Science Publisher, Amsterdam, The Netherlands, 1995, pp. 61–78.
[7] B. De Moor, *Structured total least squares for Hankel matrices*, to be published in 'Communications, Computing, Control and Signal Processing: 2000 ' for the occasion of Thomas Kailath's 60th birthday (also ESAT-SISTA Report 95-54, Department of Electrical Engineering, Katholieke Universiteit Leuven, Belgium, 16 pp.).
[8] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd ed., Johns Hopkins University Press, Baltimore, MD, 1989.
[9] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1980.
[10] P. Lemmerling, B. De Moor, and S. Van Huffel, *On the equivalence of constrained total least squares and structured total least squares*, IEEE Trans. on Signal Processing, SP-44 (1996), no. 11, to appear.
[11] H. Liu, G. Xu, L. Tong, and T. Kailath, *Recent developments in blind channel equalization: From cyclostationarity to subspaces*, Signal Processing, 50 (1996), pp. 83–99.
[12] L. Ljung, *System identification: Theory for the User*, Prentice Hall, Englewood Cliffs, 1987.
[13] S. Van Huffel and J. Vandewalle, *The Total Least Squares Problem: Computational Aspects and Analysis*, SIAM, Philadelphia, PA, 1991.