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Self-fractional Fourier images

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Abstract. Self-fractional Fourier images (SFFIs), which are invariant under the two-dimensional fractional Fourier transform, are considered. Necessary and sufficient conditions for an image to be a SFFI for given angles are formulated. An optical procedure for synthesizing SFFIs from a generator image is developed. The physical meaning of SFFI synthesis is discussed. An image decomposition on the set of orthogonal SFFIs, which simplifies image processing in fractional FT optical systems, is proposed.

1. Introduction

Recently, self-imaging phenomena in optical systems have attracted much interest [1]. Some optical devices such as the Talbot array illuminator [2] and binary logic operators [3] have been developed by using the self-imaging properties of Fresnel diffraction. Applications of optical fractional Fourier systems in phase-space tomography, as filtering and as analysing devices, have also been proposed [4–9]. The purpose of this paper is to discuss specific properties of imaging and self-imaging in two-dimensional fractional Fourier systems that can be useful for optical analysis of complex objects and for the development of new optical devices.

Self-imaging phenomena refer to the fact that an input image is an eigenfunction of the operator describing the given system. In the one-dimensional case, functions which are invariant under the fractional Fourier transform (fractional FT) [10] for some angle are usually called self-fractional Fourier functions (SFFFs) [11–15]. They are eigenfunctions of the corresponding FFT operator. SFFFs cover, as a particular case, self-Fourier functions (SFFs) [16–20], whose Fourier transforms (FTs) are identical with themselves. The problem of generating one-dimensional SFFFs from any factional Fourier transformable function has been considered in [11, 14, 15]. It has also been proven in [15] that any function from L^2 can be represented as a sum of M orthogonal SFFFs for angle $2\pi/M$. This generalized the function decomposition of SFFs (for which $M = 4$, see [17]).

In this paper, we consider two-dimensional SFFFs, that we call self-fractional Fourier images (SFFIs). We find necessary and sufficient conditions for an image to be a SFFI for a given pair of angles. These are different from the one-dimensional case [15], owing to a mutual influence of the coordinates. The Hermite–Gauss mode content of a SFFI for different relationships between the angles for orthogonal coordinates is investigated. Then the procedure for synthe-

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sizing SFFI for different angles from a generator image and its optical realization are considered. We show that an arbitrary image and its fractional FT for angles $(2\pi j/M,\ 2\pi j/M)$, where j and M are integers, can be represented as a linear superposition of M orthogonal SFFIs for the angles $(2\pi/M,\ 2\pi/M)$. It allows one to formulate the image processing in fractional FT systems in terms of the SFFIs.

The paper is organized in the following way. In section 2 the main definitions are given. Section 3 considers the Hermite–Gauss mode content of SFFIs for different angles. The problem of SFFI generation is discussed in section 4. The image decomposition on the SFFIs is carried out in section 5.

2. Some definitions

The two-dimensional fractional FT at angles (α_x, α_y) of a complex function $\Phi(x, y)$ is given as follows [21–23]:

$$[R^{\alpha_x,\alpha_y}\Phi(x,y)](u,v) = \int_{-\infty}^{\infty} \Phi(x,y) K_{\alpha_x,\alpha_y}(x,y,u,v)\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

with the kernel

$$K_{\alpha_x,\alpha_y}(x,y,u,v) = \left(\frac{1-\mathrm{i}\cot(2\pi\alpha_x)}{2\pi}\right)^{1/2} \exp\left(\mathrm{i}\frac{(x^2+u^2)\cos(2\pi\alpha_x) - 2xu}{2\sin(2\pi\alpha_x)}\right)$$
$$\times \left(\frac{1-\mathrm{i}\cot(2\pi\alpha_y)}{2\pi}\right)^{1/2} \exp\left(\mathrm{i}\frac{(y^2+v^2)\cos(2\pi\alpha_y) - 2yv}{2\sin(2\pi\alpha_y)}\right). \tag{2}$$

Observe that, for brevity, we speak about the pair of angles (α_x, α_y), whereas the real angles are the rotational angles $(2\pi\alpha_x,\ 2\pi\alpha_y)$ between the (x, u) and (y, v) axes in the mutually orthogonal planes (x, u) and (y, v) respectively. The kernel of the fractional FT equals the propagator of the non-stationary Schrödinger equation for the harmonic oscillator, except for the factor $\exp[\mathrm{i}\pi(\alpha_x + \alpha_y)]$ [22]. This transform describes in the scalar paraxial approximation the wave propagation through a quadratic refractive index medium including spherical or cylindrical lenses, mirrors, fibres (for example [22, 24–26]). An optical system composed of two orthogonally situated cylindrical thin lenses that realizes the fractional FT of a complex field amplitude for in general non-equal angles (α_x, α_y) has been proposed in [23]. If α_x, α_y or $\alpha_x + \frac{1}{2}$, $\alpha_y + \frac{1}{2}$ are integers the kernel $K_\alpha(x, u)$ reduces to $\delta(x-u)\delta(y-v)$ or $\delta(x+u)\delta(y+v)$ respectively. Thus, the fractional FT for integer α_x, α_y corresponds to the identity operator. For $\alpha_x, \alpha_y = \frac{1}{4}$, equation (1) is the ordinary FT. Moreover, as follows from the additivity property for the fractional FT, namely $R^{\alpha_x,\alpha_y}R^{\beta_x,\beta_y} = R^{\alpha_x+\beta_x,\alpha_y+\beta_y}$ [21], the transforms for the angles (α_x, α_y) and $(\alpha_x + k_x, \alpha_y + k_y)$, where k_x and k_y are integers, are identical. So, we consider the parameters α_x, α_y in the interval $[0, 1[$.

An image $\Phi_{\alpha_x,\alpha_y}(x, y)$ is a SFFI for angles α_x, α_y if it satisfies the following equation:

$$R^{\alpha_x,\alpha_y}[\Phi_{\alpha_x,\alpha_y}(x,y)](u,v) \stackrel{\text{def}}{=} A\Phi_{\alpha_x,\alpha_y}(u,v), \tag{3}$$

where A is a complex constant factor. In other words, $\Phi_{\alpha_x,\alpha_y}(x, y)$ is an eigenfunction of the corresponding fractional FT operator R^{α_x,α_y} with eigenvalue A. Taking into account the Parseval relation for the fractional FT [21], we find that $|A| = 1$,

implying that $A = \exp(-i2\pi\varphi)$. Note that $\varphi \in [0, 1[$ defines all possible eigenvalues. We shall call φ the *index* of a SFFI. With n an integer, it follows from equations (1)–(3) that any arbitrary image is a SFFI for $\alpha_x, \alpha_y = n$, that a symmetric (even or odd) image is SFFI for $\alpha_x, \alpha_y = n + \frac{1}{2}$, and that a self-Fourier image (SFI) is SFFI for $\alpha_x, \alpha_y = n + \frac{1}{4}$.

From the additivity property for the fractional FT, and from equation (3), it immediately follows that, if an image is a SFFI for α_x, α_y with eigenvalue A, it is also a SFFI for angles $k\alpha_x, k\alpha_y (k = 1, 2, \ldots)$ with eigenvalue A^k:

$$\begin{aligned} R^{k\alpha_x, k\alpha_y}[\Phi_{\alpha_x,\alpha_y}(x, y)](u, v) &= AR^{(k-1)\alpha_x,(k-1)\alpha_y}[\Phi_{\alpha_x,\alpha_y}(x, y)](u, v) \\ &= \cdots = A^k \Phi_{\alpha_x,\alpha_y}(u, v). \end{aligned} \tag{4}$$

From the periodicity property for the fractional FT, namely $R^{n,m}[\Phi(x, y)](u, v) = \Phi(u, v)$ for any integers n and m, it follows that, if $k\alpha_x$ and $k\alpha_y$ are integers, then $A^k = 1$.

3. Hermite–Gauss mode content of self-fractional Fourier images

In order to find necessary and sufficient conditions for an image to be a SFFI for given angles, let us represent an arbitrary image $\Phi(x, y)$ as a linear combination of the Hermite–Gauss modes $\Psi_n(u)$:

$$\Phi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{m,n} \Psi_m(x) \Psi_n(y), \tag{5}$$

where

$$f_{m,n} = \int \Phi(x, y) \Psi_m(x) \Psi_n(y)\, dx\, dy, \tag{6}$$

and $\Psi_n(u)$ is expressed through the Hermite polynomial $H_n(u)$ as

$$\Psi_n(u) = (\pi^{1/2} 2^n n!)^{-1/2} \exp\left(-\frac{u^2}{2}\right) H_n(u). \tag{7}$$

This decomposition is very useful for the analysis of the fractional Fourier evolution of an input image because a product of Hermite–Gauss functions $\Psi_m(x), \Psi_n(y)$ is a SFFI for any angle α_x, α_y:

$$R^{\alpha_x,\alpha_y}[\Psi_m(x)\Psi_n(y)](u, v) = \exp[-i2\pi(\alpha_x m + \alpha_y n)]\Psi_m(u)\Psi_n(v), \tag{8}$$

with eigenvalue $\exp(-i2\pi(\alpha_x m + \alpha_y n))$ (for example [27]). Then, taking into account equations (5) and (8), one can write for the fractional FT of an arbitrary image $\Phi(x, y)$ for the angles α_x, α_y

$$R^{\alpha_x,\alpha_y}[\Phi(x, y)](u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \exp[-i2\pi(m\alpha_x + n\alpha_y)] f_{m,n} \Psi_m(u) \Psi_n(v). \tag{9}$$

Let us show that an image which is SFFI for some angles α_x, α_y contains only a specific set of Hermite–Gauss modes which depends on the angles α_x, α_y and on the eigenvalue.

Theorem 1: *An image $\Phi(x,y)$ is a SFFI $\Phi^{\varphi}_{\alpha_x,\alpha_y}(x,y)$ for the angles α_x,α_y with eigenvalue $A=\exp(\mathrm{i}2\pi\varphi)$ if and only if there are only such Hermite–Gauss modes in the decomposition (5) whose indexes m and n are connected by*

$$m\alpha_x+n\alpha_y=k+\varphi(\alpha_x,\alpha_y), \tag{10}$$

where k is an arbitrary integer and $\varphi=(m\alpha_x+n\alpha_y) \bmod 1$ is a constant from the interval $[0,1[$.

Proof: *Sufficient condition.* It is easy to see from equation (9) that, if equation (10) holds for all non-zero coefficients $f_{m,n}$, then

$$\begin{aligned} R^{\alpha_x,\alpha_y}[\Phi(x,y)](u,v) &= \exp[-\mathrm{i}2\pi\varphi(\alpha_x,\alpha_y)]\textstyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\Psi_m(u)\Psi_n(v) \\ &\overset{(5)}{=} \exp[-\mathrm{i}2\pi\varphi(\alpha_x,\alpha_y)]\Phi(u,v). \end{aligned} \tag{11}$$

This means that such $\Phi(x,y)$ is a SFFI $\Phi^{\varphi}_{\alpha_x,\alpha_y}(x,y)$ for angles α_x,α_y with eigenvalue $A=\exp(-\mathrm{i}2\pi\varphi)$.

Necessary condition. Let $\Phi(x,y)$ be a SFFI $\Phi^{\varphi}_{\alpha_x,\alpha_y}(x,y)$ for the angles α_x,α_y with eigenvalue $A=\exp(-\mathrm{i}2\pi\varphi)$. Then, as follows from equations (3), (5) and (9):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{\exp[-\mathrm{i}2\pi(m\alpha_x+n\alpha_y)]-\exp[-\mathrm{i}2\pi\varphi(\alpha_x,\alpha_y)]\}f_{m,n}\Psi_m(u)\Psi_n(v)=0. \tag{12}$$

To satisfy this relation, $f_{m,n}$ or $\exp[-\mathrm{i}2\pi(m\alpha_x+n\alpha_y)]-\exp[-\mathrm{i}2\pi\varphi(\alpha_x,\alpha_y)]$ should be zero. So we come to the equation (10) for non-zero coefficients $F_{m,n}$. □

Corollary 1: *SFFIs for the same angles (α_x,α_y) but different eigenvalues (indices φ) are orthogonal to each other.*

Proof: First of all we should note that two images are non-orthogonal to each other if, as follows from the orthogonality of Hermite–Gauss modes, they have in their decomposition (5) at least one common combination of Hermite–Gauss modes (m,n). Suppose that two SFFIs for the same angles α_x,α_y and different indexes φ_1 and φ_2 are non-orthogonal. Then they have the same combination of Hermite–Gauss modes (m,n) and, as we have learned from theorem 1 above,

$$\begin{aligned} m\alpha_x+n\alpha_y &= k_1+\varphi_1, \\ m\alpha_x+n\alpha_y &= k_2+\varphi_2. \end{aligned} \tag{13}$$

This means that $\varphi_1-\varphi_2=k_2-k_1$, but this is possible only if $\varphi_1=\varphi_2$, because $\varphi_1,\varphi_2\in[0,1[$ and k_2,k_1 are integers. Then our assumption that two SFFIs for the same angles α_x,α_y and different indexes φ_1 and φ_2 are non-orthogonal is wrong. □

Equation (10) determines the set of possible Hermite–Gauss modes of a SFFI for the angles α_x,α_y with index φ. This set depends on φ and the relationship between α_x and α_y. The mutual influence between the Hermite–Gauss modes for coordinates u and v produces a new mode selection rule (10) which differs from the rule for one-dimensional SFFFs [15, 17]. Let us now consider the Hermite–Gauss mode content of a SFFI for various relationships between the angles α_x and α_y.

Case 1: α_y and α_y are *integer-mixable irrationals*, that is irrational numbers for which there exist some integers $a,p\neq 0,q\neq 0$ with p,q relatively prime (possibly also $|p|=|q|=1$), such that

$$p\alpha_x + q\alpha_y = a. \tag{14}$$

Fix an integer pair (m_0, n_0) which determines the index (eigenvalue) of a given SFFI φ as the fractional part of $m_0\alpha_x + n_0\alpha_y : \varphi = (m_0\alpha_x + n_0\alpha_y) \bmod 1$, and define the set of integers $\mathcal{A} = \{z | z \text{ integer such that } m_0 + pz \geqslant 0 \text{ and } n_0 + qz \geq 0\}$. It then follows from lemma 1 in appendix 1 that all solutions for equation (10) are given by

$$(m, n, k) = (m_0 + pl, n_0 + ql, al), l \in \mathcal{A}. \tag{15}$$

So all (α_x, α_y) SFFIs with index φ are then in the form

$$\Phi^{m_0,n_0}_{\alpha_x\alpha_y}(x, y) = \sum_{l\in\mathcal{A}} f_{m_0+pl,n_0+ql}\Psi_{m_0+pl}(x)\Psi_{n_0+ql}(y) \tag{16}$$

with m_0, n_0 arbitrary integers, while the corresponding eigenvalue is $A = \exp[-i2\pi(m_0\alpha_x + n_0\alpha_y)]$.

As we have learned from the one-dimensional case, a SFFF for any given irrational angle is also a SFFF for all possible angles [12, 15]. In the two-dimensional case, an (α_x, α_y) SFFI is also a SFFI for the angles $(j\alpha_x, j\alpha_y)$, $j = 1, 2, \ldots$ (see equation (4)), but not necessarily for all angles like in the one-dimensional case. □

Case 2: $\alpha_x = \alpha_y = \alpha$ is an *irrational* number (this is a special case of case 1). So α_x, α_y are integer-mixable irrationals with $p = -q$, implying that $m + n = m_0 + n_0 = \text{constant} = L \geqslant 0$. Then equation (16) transforms into the (α, α) SFFI

$$\Phi^{L}_{\alpha,\alpha}(x, y) = \sum_{l=0}^{L} f_{l,L-l}\Psi_l(x)\Psi_{L-l}(y), \tag{17}$$

with eigenvalue $A = \exp(-i2\pi\alpha L)$ (index $\varphi = \alpha L \bmod 1$). By applying equation (8), it follows that $\Phi^{L}_{\alpha,\alpha}(x, y)$ is also a SFFI for any angles (β, β) with corresponding eigenvalue $A = \exp(-i2\pi\beta L)$. Also, $\Phi^{0}_{\alpha,\alpha}(x, y) = f_{0,0}\Psi_0(x)\Psi_0(y)$ is a SFFI for all possible angles (β_x, β_y) with eigenvalue $A = 1$. □

Case 3: α_x and α_y are *irrational* numbers which are *not integer mixable*. Lemma 1 in appendix 1 then implies that equation (10) can only hold for one pair (m_0, n_0) with $m_0, n_0 \geqslant 0$. So the SFFIs at angles (α_x, α_y) with index $\varphi = (m_0\alpha_x + n_0\alpha_y) \bmod 1$ are then given by

$$\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x, y) = f_{m_0,n_0}\Psi_{m_0}(x)\Psi_{n_0}(y), \tag{18}$$

Applying equation (8) shows that such images are actually SFFIs for all angles (α_x, α_y) with eigenvalue $A = \exp[-i2\pi(m_0\beta_x + n_0\beta_y)]$. □

Case 4: α_x is irrational and $\alpha_y = N_y/M_y$ is a rational number (N_y, M_y relatively prime non-negative integers such that $N_y < M_y$). According to lemma 2 in appendix 1, all solutions to (10) are of the form $(m, n, k) = (m_0, n_0 + lM_y, lN_y)$ for $l = 0, 1, \ldots$, where we have chosen n_0 as a minimal non-negative integer satisfying $\varphi = (m_0\alpha_x + n_0\alpha_y) \bmod 1$.

The images that are SFFIs for angles (α_x, α_y) are then given by

$$\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y) = \Psi_{m_0}(x)\sum_{l=0}^{\infty} f_{m_0,n_0+lM_y}\Psi_{n_0+lM_y}(y) \tag{19}$$

(for any $m_0, n_0 \geqslant 0$) with eigenvalue $A = \exp[-i2\pi(m_0\alpha_x + n_0\alpha_y)]$.

Thus, a SFFI is a product of Hermite–Gauss mode $\Psi_{m_0}(x)$ and a one-dimensional SFFF $\Phi^{n_0}_{\alpha_y}(y)$ for angle $\alpha_y = N_y/M_y$. From the previous studies [14, 15], it follows that the one-dimensional SFFF for an angle N_y/M_y is also a SFFF for the angle $\beta_y = 1/M_y$, and vice versa. Then, a SFFI $\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y)$ is also a SFFI for the angles $(\beta_x, \beta_y = j/M_y)$, where $j = 1, \ldots, M_y$ with eigenvalue $A = \exp[-i2\pi(\beta_x m_0 + n_0 j/M_y)]$. □

Case 5: $\alpha_x = N_x/M_x$ and $\alpha_y = N_y/M_y$ are rational numbers and N_x, M_x, N_y, M_y are all relatively prime integers. Note that the cascade of M_xM_y fractional FT operators at angles (α_x, α_y) is equivalent to the identity operation. This means that the index φ in equation (10) must satisfy

$$M_xM_y\varphi = L, \tag{20}$$

where L is an integer, because the eigenvalue of the identity operator $A = \exp(-i2\pi M_xM_y\varphi)$ equals unity. So, there are only M_xM_y possible different eigenvalues corresponding to $\varphi = L/M_xM_y, L = 0, 1, \ldots, M_xM_y - 1$.

As follows from corollary 2 in appendix 1, all solutions for non-negative m, n of the equation (10) for such rational angles are given in the form

$$\begin{aligned} m &= m_0 + k_xM_x \\ n &= n_0 + k_yM_y \end{aligned} \tag{21}$$

where k_x, k_y are non-negative integers and m_0, n_0 are minimal non-negative integers satisfying $(m_0\alpha_x + n_0\alpha_y) \bmod 1 = L/M_xM_y \bmod 1$.

So, all $(N_x/M_x, N_y/M_y)$ SFFIs can be written in the form

$$\Phi^{m_0,n_0}_{N_x/M_x,N_y/M_y}(x,y) = \sum_{k_x=0}^{\infty}\sum_{k_y=0}^{\infty} f_{m_0+k_xM_x,n_0+k_yM_y}\Psi_{m_0+k_xM_x}(x)\Psi_{n_0+k_yM_y}(y), \tag{22}$$

while the corresponding eigenvalue is $A = \exp[-i2\pi(m_0N_x/M_x + n_0N_y/M_y)] = \exp(-i2\pi L/M_xM_y)$. Applying the fractional FT operator for angles $(j_x/M_x, j_y/M_y)$ on the image $\Phi^{m_0,n_0}_{N_x/M_x,N_y/M_y}(x,y)$, we see that this image is also a SFFI for angles $(j_x/M_x, j_y/M_y)$, where $j_x, j_y = 0, 1, \ldots$. So, if N_x, M_x, N_y, M_y are relatively prime integers, then a SFFI for angles $(N_x/M_x, N_y/M_y)$ is a SFFI $\Phi^{m_0n_0}_{1/M_x,1/M_y}(x,y)$ for angles $(1/M_x, 1/M_y)$ and vice versa. Moreover, the SFFI for the same angles but different m_0 or n_0 are orthogonal to each other, because they are expanded into disjoint sets of Hermite–Gauss modes (22).

In the case when $M_x = M_y = M$, the procedure (22) yields the particular case of a SFFI for angles $(1/M, 1/M)$

$$\Phi^{m_0,n_0}_{1/M,1/M}(x,y) = \sum_{k_x=0}^{\infty}\sum_{k_y=0}^{\infty} f_{m_0+k_xM,n_0+k_yM}\Psi_{m_0+k_xM}(x)\Psi_{n_0+k_yM}(y). \tag{23}$$

□

Case 6: $\alpha_x = N_x/M_x$ and $\alpha_y = N_y/M_y$ are rational numbers and $N_yM_x/N_xM_y = -p/q$ ($(N_x, M_x), (N_y, M_y), (p, q)$ are relatively prime integers, $pq < 0$). It is easy to see that the cascades of $|q|M_x$ and $|p|M_y$ fractional FT operators at angles

$(N_x/M_x, N_y/M_y)$ are equivalent to the identity operation. Then, the index φ in equation (10) must satisfy the system of equations

$$\begin{aligned} \varphi|q|M_x &= L_x, \\ \varphi|p|M_y &= L_y, \end{aligned} \tag{24}$$

where L_x and L_y are integers from regions $[0, |q|M_x - 1[$ and $[0, |p|M_y - 1[$ correspondingly, because the eigenvalue of the identity operator $A = \exp(-\mathrm{i}2\pi\varphi|q|M_x) = \exp(-\mathrm{i}2\pi\varphi|p|M_y) = 1$. Thus $\varphi = 0$ is always a solution of equation (24). From equation (24) follows that, if $L_x, L_y \neq 0$, then $L_x/L_y = N_x/N_y$ and the number of possible different indices equals the number of elements in the set $\mathcal{N} = \{z \mid z \in \{0, 1, \ldots, |q|M_x - 1\} \cap \{0, N_x/N_y, 2N_x/N_y, \ldots, (|p|M_y - 1)N_x/N_y\}\}$. In particular, if $N_x = N_y$, then $|q|M_x = |p|M_y = J$, and there are only J possible different eigenvalues corresponding to $\varphi = L/J$, $L = 0, 1, \ldots, J - 1$. In another particular case when $M_x = M_y = M$, there are only M different indices $\varphi = L/M, L = 0, 1, \ldots, M - 1$. This follows from the fact that the second subset of $\mathcal{N}$ contains only M integers and all of them belong to the first subset of $\mathcal{N}$.

As follows from lemma 3 in appendix 1, all possible pairs satisfying equation (10) can be written as

$$\begin{aligned} m &= m_0 + k_x M_x + pl, \\ n &= n_0 + k_y M_y + ql, \end{aligned} \tag{25}$$

where $k_x, k_y = 0, 1, \ldots$ and m_0, n_0 are integers satisfying equation (10) with $\varphi = (m_0 N_x/M_x + n_0 N_y/M_y) \bmod 1$ and $l \in \mathcal{D} = \{z \mid z \text{ integer such that } m, n \geqslant 0\}$.

So, we can write the expression for the SFFI for the angles $(N_x/M_x, N_y/M_y)$ with eigenvalue $A = \exp[-\mathrm{i}2\pi(m_0 q + n_0 p)N_x/qM_x]$ as

$$\begin{aligned} \Phi^{m_0,n_0}_{N_x/M_x,N_y/M_y}(x, y) = \sum_{l\in\mathcal{D}} \sum_{k_x=0}^{\infty} \Psi_{m_0+k_xM_x+pl}(x) \\ \times \sum_{k_y=0}^{\infty} f_{m_0+k_xM_x+pl,n_0+k_yM_y+ql} \Psi_{n_0+k_yM_y+ql}(y). \end{aligned} \tag{26}$$

□

Case 7: $\alpha_x = \alpha_y = \alpha = N/M$. As for the one-dimensional SFFF, it is easy to see from the cascade property for the fractional FT that a SFFI for angles $(N/M, N/M)$ is also a SFFI for angles $(1/M, 1/M)$ and $M\varphi = L$. Then one can rewrite equation (10) as

$$n + m = kM + L, \tag{27}$$

where k is an integer and $L = 0, 1, \ldots, M - 1$. Then the positive integers m and n are defined as

$$\begin{aligned} m &= k_x M + m_0, \\ n &= k_y M + L - m_0, \end{aligned}$$

where k_x, k_y are non-negative integers and m_0 is an integer from the set $\mathcal{D}_M = \{z \mid z \text{ integer from the interval } [0, M[\text{ such that } n \geqslant 0\}$.

So for a SFFI for symmetric angles $(1/M, 1/M)$ with eigenvalue $A = \exp(-\mathrm{i}2\pi L/M)$ we can write

$$\Phi^{L}_{1/M,1/M}(x,y) = \sum_{m_0 \in \mathcal{D}_M} \sum_{k_x=0}^{\infty} \Psi_{k_xM+m_0}(x) \sum_{k_y=0}^{\infty} f_{k_xM+m_0,k_yM+L-m_0} \Psi_{k_yM+L-m_0}(y). \quad (28)$$

Note that representation (23) for $M_x = M_y$ is a particular case of (28) for $m_0 =$ constant.

4. Synthesis of self-fractional Fourier images

Here we shall consider the synthesis of a SFFI for given angles (α_x, α_y) from an arbitrary generator image $\Phi(u,v)$. As we have seen in the previous section a SFFI for given angles (α_x, α_y) and eigenvalue $A = \exp(-i2\pi\varphi)$ contains only a specific set of Hermite–Gauss modes. Then the synthesis of a SFFI means the selection of certain Hermite–Gauss modes from the generator image. It has been shown in [11, 14, 15] that a construction of one-dimensional SFFF is based on summing (for rational angles) or integrating (for irrational angles) the fractional FTs of a generator function. Let us generalize this procedure for the two-dimensional case.

First of all we have to note that there are two manners in which the fractional FT operator may act on a generator image. We can apply the fractional FT independently for every coordinate, for example producing $R^{\alpha_x,0}[\Phi(u,v)](x,y)$ first, and then applying R^{0,α_y} to it. This procedure is realized in optics by using cylindrical lenses [4, 23]. In another case the operator acts simultaneously on both coordinates: $R^{\alpha,\alpha}[\Phi(u,v)](x,y)$. The simplest example can be found using spherical lenses. As will be shown below, it is possible that for the synthesis of the SFFI for some types of angle, only one of these manners can be applied. Let us now proceed with this synthesis of SFFIs for various angles (α_x, α_y).

4.1. *Synthesis of a self-fractional Fourier image for irrational angles*

A SFFI for irrational angles (α_x, α_y) with index $\varphi = (m_0\alpha_x + n_0\alpha_y) \bmod 1$ can be constructed from an arbitrary generator image $\Phi(u,v)$ through the following procedure:

$$\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y) = \frac{1}{2\pi} \int_0^1 \exp[i2\pi a(\alpha_x m_0 + \alpha_y n_0)]\, R^{a\alpha_x,a\alpha_y}[\Phi(u,v)](x,y)\, da. \quad (29)$$

Indeed, using equations (5) and (9), this expression becomes

$$\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y) = \frac{1}{2\pi} \int_0^1 \exp\{i2\pi a[\alpha_x(m_0-m) + \alpha_y(n_0-n)]\}\, da \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{m,n} \Psi_m(x) \Psi(y), \quad (30)$$

and, involving the relationship

$$\int_0^1 \exp[i2\pi a(\alpha_x m_0 + \alpha_y n_0 - \alpha_x m - \alpha_y n)]\, da = \begin{cases} 2\pi & \text{if } \alpha_x(m_0-m) + \alpha_y(n_0-n) = -l, \\ 0, & \text{if } \alpha_x(m_0-m) + \alpha_y(n_0-n) \neq -l, \end{cases} \quad (31)$$

where l is an integer, it is easy to see that $\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y)$ contains only Hermite–Gauss modes (m,n) that satisfy the condition

$$\alpha_x m + \alpha_y n = \alpha_x m_0 + \alpha_y n_0 + l. \tag{32}$$

This condition is equivalent to equation (10), where $k = \lfloor m_0\alpha_x + n_0 y \rfloor + l$ (k, l are integers) and $\varphi = (m_0\alpha_x + n_0\alpha_y) \bmod 1$.

In particular, for irrational $\alpha_x = \alpha_y = \alpha$ (case 2), SFFIs with index $\varphi = \alpha L \bmod 1$ can be generated by optical systems producing the fractional FT at identical angles for both coordinates:

$$\Phi^L_{\alpha,\alpha}(x,y) = \frac{1}{2\pi}\int_0^1 \exp(\mathrm{i}\alpha L)\, R^{\alpha,\alpha}[\Phi(u,v)](x,y)\,\mathrm{d}\alpha. \tag{33}$$

We can see this from the following relationship:

$$\begin{aligned}\Phi^L_{\alpha,\alpha}(x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\Psi_m(x)\Psi_n(y)\frac{1}{2\pi}\int_0^1 \exp[\mathrm{i}2\pi\alpha(L-n-m)]\,\mathrm{d}\alpha \\ &= \sum_{l=0}^{L} f_{l,L-l}\Psi_l(x)\Psi_{L-l}(y).\end{aligned} \tag{34}$$

Note that the integration of the fractional FT of generator image on α gives the Gauss beam, which is a SFFI for any angles (α,α) with index $\varphi = 0$:

$$\Phi^0_{\alpha,\alpha}(x,y) = \frac{1}{2\pi}\int_0^1 R^{\alpha,\alpha}[\Phi(u,v)](x,y)\,\mathrm{d}\alpha = f_{0,0}\exp\left(\frac{-(x^2+y^2)}{2}\right). \tag{35}$$

If α_x and α_y are *irrational* numbers which are *not integer mixable* (case 3) a SFFI with index $\varphi = (m_0\alpha_x + n_0\alpha_y) \bmod 1$ constructed through the procedure (29) corresponds to the selection of a unique mode from the generator image for every coordinate, according to the phase shifts m_0, n_0:

$$\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y) = f_{m_0,n_0}\Psi_{m_0}(x)\Psi_{n_0}(y). \tag{36}$$

A linear combination $\sum_{m_0,n_0}\Phi^{m_0,n_0}_{\alpha_x,\alpha_y}(x,y)$ (obtained from equation (29)) allows one to select any set of Hermite–Gauss modes present in the generator function expansion.

In the previous section we have defined the set of Hermite–Gauss modes for a SFFI for the irrational angle α_x and rational angle $\alpha_y = N_y/M_y$ (N_y, M_y relatively prime) (case 4) with index ($\varphi = (\alpha_x m_0 + n_0 N_y/M_y) \bmod 1$ (equation (19)). Let us show that it can be selected from a generator image $\Phi(u,v)$ through the following operation:

$$\begin{aligned}\Phi^{m_0,n_0}_{\alpha_x,N_y/M_y}(x,y) = C\sum_{k_y=1}^{M_y}\int_0^1 \exp\left[\mathrm{i}2\pi\left(\alpha_x m_0 + \frac{n_0 k_y}{M_y}\right)\right] \\ \times R^{\alpha_x,0}[R^{0,(k_y-1)/M_y}[\Phi(u,v)]](x,y)\,\mathrm{d}\alpha_x.\end{aligned} \tag{37}$$

Representing the generator image in the form (5) and using equation (9) we have

$$\begin{aligned}\Phi^{m_0,n_0}_{\alpha_x,N_y/M_y}(x,y) = C\int_0^1 \exp[\mathrm{i}2\pi\alpha_x(m_0-m)]\,\mathrm{d}\alpha_x \sum_{k_y=1}^{M_y}\exp\left[\mathrm{i}2\pi\left(\frac{n_0 k_y - m(k_y-1)}{M_y}\right)\right] \\ \times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\Psi_m(x)\Psi_n(y).\end{aligned} \tag{38}$$

Then, applying equation (31) and the well known formula

$$\sum_{k=1}^{M} \exp\left(\frac{i2\pi(k-1)(L-m)}{M}\right) = \begin{cases} M, & L = m + M, \\ & j = 0, 1, \ldots, \\ 0, & L \neq m + Mj, \end{cases} \tag{39}$$

one obtains equation (19) by choosing $C = 2\pi \exp(i2\pi n_0/M_y)$.

4.2. *Synthesis of a self-fractional Fourier image for rational angles*

If both angles $\alpha_x = N_x/M_x$ and $\alpha_y = N_y/M_y$ are rational, then, a SFFI for the angles $(N_x/M_x, N_y/M_y)$ can be constructed from a generator image $\Phi(u, v)$ through the procedure expressed by the following formula:

$$\Phi^{m_0,n_0}_{N_x/M_x,N_y/M_y}(x, y) = C \sum_{k=1}^{M_xM_y} \exp\left[i2\pi(k-1)\left(\frac{m_0N_x}{Mx} + \frac{n_0N_y}{M_y}\right)\right] \times R^{(k-1)N_x/M_x,(k-1)N_y/M_y}[\Phi(u, v)](x, y), \tag{40}$$

where C is an arbitrary complex constant and $m_0 \in [0, M_x - 1]$ and $n_0 \in [0, M_y - 1]$. It is easy to see from equations (5), (8) and (39) that $\Phi^{m_0,n_0}_{N_x/M_x,N_y/M_y}(x, y)$ contains only the Hermite–Gauss modes (m, n) which satisfy the condition

$$\frac{mN_x}{M_x} + \frac{nN_y}{M_y} = \frac{m_0N_x}{M_x} + \frac{n_0N_y}{M_y} + l,$$

and as follows from theorem 1 above, it is an $(N_x/M_x, N_y/M_y)$ SFFI with eigenvalue

$$A = \exp\left[-i2\pi\left(\frac{N_xm_0}{M_x} + \frac{N_yn_0}{M_y}\right)\right].$$

If N_x, M_x, N_y, M_y are all relatively prime (case 5), then one can produce from a generator image M_xM_y orthogonal SFFIs for angles $(1/M_x, 1/M_y)$ with different eigenvalues $A = \exp(-i2\pi L/M_xM_y)$ $(L = 0, 1, \ldots, M_xM_y - 1)$. Here we consider the SFFIs differing in C as identical. As we have seen in the previous section, a SFFI for angles $(N_x/M_x, N_y/M_y)$ is also a SFFI for angles $(1/M_x, 1/M_y)$ and vice versa.

If $\alpha_x = \alpha_y = \alpha = N/M$ (case 7), equation (40) reduces to

$$\Phi^L_{1/M,1/M}(x, y) = C \sum_{k=1}^{M} \exp\left(\frac{i2\pi Lk}{M}\right) R^{(k-1)/M,(k-1)/M}[\Phi(u, v)](x, y). \tag{41}$$

Here we use the note from the previous section that a SFFI for angles $(N/M, N/M)$ is also a SFFI for angles $(1/M, 1/M)$ and vice versa. Substituting in equation (41) the generator image $\Phi(u, v)$ in the form (5), and using equation (8), we get

$$\begin{aligned} \Phi^L_{1/M,1/M}(x, y) &= C \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\Psi_m(x)\Psi_n(y) \sum_{k=1}^{M} \exp\left(\frac{i2\pi}{M}[Lk - (k-1)(n+m)]\right) \\ &= C \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\Psi_m(x)\Psi_n(y) \exp\left(\frac{i2\pi L}{M}\right) \\ &\quad \times \sum_{k=1}^{M} \exp\left(\frac{i2\pi}{M}(k-1)(L-n-m)\right). \end{aligned} \tag{42}$$

The summation over k gives 0 for any (n, m) pair except when $n+m-L=jM$, where j is an integer; in this case it gives M. Then choosing for normalization $C=M^{-1}\exp(-\mathrm{i}2\pi L/M)$, we obtain equation (28).

So, by choosing different $L=0,\ldots,M-1$, we can construct M mutually orthogonal SFFIs for angles $(1/M, 1/M)$ as given by equation (28), where $f_{m,n}$ are corresponding coefficients in the decomposition (6) of generator image $\Phi(u,v)$.

Note that the list of procedures for SFFI synthesis from the generator image, as given above, is not complete. One can select a partial set of the Hermite–Gauss modes satisfying the self-imaging condition (10) for given angles. For example, the procedure

$$\Phi^{m_0,n_0}_{1/M,1/M}(x,y)=C\sum_{k_x=1}^{M}\sum_{k_y=1}^{M}\exp\left(\frac{\mathrm{i}2\pi(m_0k_x+n_0k_y)}{M}\right)\times R^{(k_x-1)/M,0}[R^{0,(k_y-1)/M}[\Phi(u,v)]](x,y) \tag{43}$$

yields the particular case of a SFFI for angles $(1/M, 1/M)$ with eigenvalue $A=\exp[-\mathrm{i}2\pi(m_0+n_0)/M]$:

$$\Phi^{m_0,n_0}_{1/M,1M}(x,y)=C\sum_{k_x=0}^{\infty}\sum_{k_y=0}^{\infty}f_{k_xM+m_0,k_yM+n_o}\Psi_{k_xM+m_0}(x)\Psi_{k_yM+n_0}(y).$$

Moreover, it is easy to see that a SFFI for angles $(1/M, 1/M)$ with index $\varphi=L/M$ generated by equation (41) is a sum of M SFFIs for angle $(1/M, 1/M)$ generated by equation (43) ($M_x=M_y=M$ and $L=m_0+n_0$):

$$\Phi^{L}_{1/M,1/M}(x,y)=\sum_{L_x=0}^{M-1}\Phi^{L_x,L-L_x}_{1/M,1/M}(x,y). \tag{44}$$

Thus, we can conclude that the synthesis of a SFFI from a generator image means a selection of some its Hermite–Gauss modes. One can choose a certain set of modes which depends on the angles and the eigenvalue of a SFFI and the procedure of its generation. Then the fractional FT systems can be used as devices for Hermite–Gauss mode filtering. An optical configuration for the synthesis of SFIs ($M=4$) with eigenvalue 1, from a generator image, has been proposed in [20]. It can be generalized for the case of SFFIs for different rational angles and different indices. For the generation of SFFIs for non-equal angles, cylindric lenses could be used.

5. Image decomposition into self-fractional Fourier images

It has been shown in [17], for the one-dimensional case, that any Fourier transformable function is the linear combination of four SFFs. Similarly, it has been proved in [15] that any one-dimensional function from L^2 can be represented as a linear combination of M orthogonal SFFFs for the angle $1/M$. Let us now investigate how an arbitrary image can be represented as a linear combination of orthogonal SFFIs for given angles (α_x, α_y).

First of all we note that the expansion of $\Phi(x,y)$ into the set of Hermite–Gauss functions is the representation of $\Phi(x,y)$ as an infinite sum of SFFIs for irrational angles, which are not integer mixable.

Let us consider the image decomposition into SFFIs for rational angles. Representing an arbitrary image $\Phi(x,y)$ in the form (5), and rewriting it, we obtain

$$\Phi(x,y)=\sum_{m_0=0}^{M_x-1}\sum_{n_0=0}^{M_y-1}\sum_{k_x=0}^{\infty}\sum_{k_y=0}^{\infty} f_{m_0+k_xM_x,n_0+k_yM_y}\Psi_{m_0+k_xM_x}(x)\Psi_{n_0+k_yM_y}(y)$$
$$\overset{(22)}{=}\sum_{m_0=0}^{M_x-1}\sum_{n_0=0}^{M_y-1}\Phi^{m_0,n_0}_{1/M_x,1/M_y}(x,y), \tag{45}$$

where $\Phi^{m_0,n_0}_{1/M_x,1/M_y}(x,y)$ is a SFFI for angles $(1/M_x,1/M_y)$ with eigenvalue $A=\exp[-\mathrm{i}2\pi(m_0/M_x+n_0/M_y)]$, $m_0\in[0,M_x[$ and $n_0\in[0,M_y[$. Because the SFFIs for the same angles but for different m_0 or n_0 are expanded into disjoint sets of Hermite–Gauss modes, they are orthogonal to each other. This means that any image is a linear superposition of M_xM_y orthogonal SFFIs for the angles $(1/M_x,1/M_y)$.

If $M_x=M_y=M$, the SFFIs $\Phi^{m_0,n_0}_{1/M_x,1/M_y}(x,y)$ with $m_0+n_0=L$, where L is a constant, have the same eigenvalue $A=\exp(-\mathrm{i}2\pi L/M)$. Then their sum $\Phi^{L}_{1/M,1/M}(x,y)$ given by

$$\Phi^{L}_{1/M,1/M}(x,y)=\sum_{n_0\in D_M}\Phi^{L-n_0,n_0}_{1/M,1/M}(x,y) \tag{46}$$

(see equation (28)) is also a SFFI for angles $(1/M,1/M)$ with eigenvalue A ($D_M=\{z\,|\,z$ integer from the interval $[0,M[$, such that $L-n_0\geqslant 0\}$).This means that an arbitrary image can be represented as a linear superposition of M orthogonal SFFIs for angles $(1/M,1/M)$

$$\Phi(x,y)=\sum_{L=0}^{M-1}\Phi^{L}_{1/M,1/M}(x,y). \tag{47}$$

In particular, for $M=2$ the sum (47) is the well known decomposition into the even $(L=0)$ and odd $(L=1)$ images; for $M=4$ it is the linear combination of four SFIs.

The fractional FT of $\Phi(x,y)$ for angles $(j/M,j/M)$ is also represented as a linear combination of these M orthogonal SFFIs for angles $(1/M,1/M)$:

$$R^{j/M,j/M}[\Phi(x,y)](u,v)=\sum_{L=0}^{M-1}\exp\left(\frac{-\mathrm{i}2\pi jL}{M}\right)\Phi^{L}_{1/M,1/M}(u,v). \tag{48}$$

The image decomposition into orthogonal SFFIs for some angles $(1/M,1/M)$ can be applied in fractional FT image analysis. The map of the fractional FTs or its moduli (Radon–Wigner transforms) for the angles in the region $[0,\pi]$ is used for phase-space tomography, characterization of scale-symmetric images, and filtering of chirp-modulated signals [4–9]. The fractional FTs of an image are usually calculated or measured at some discrete set of rotation angles $\alpha=j/M$. Then equation (48) can simplify the analysis of the fractional FT map.

6. Conclusions

Based on the periodic and additive properties of the fractional FT we have shown that an expansion of a SFFI for some angles α_x,α_y into Hermite–Gauss

modes contains only specific modes which depend on the angle combination and on the eigenvalue. A procedure for synthesis of a SFFI for different angles and eigenvalues, from an arbitrary generator image, has been developed. It selects certain Hermite–Gauss modes of the generator function dependent on the angles and the eigenvalue. Then the synthesis procedures can be used as a mode-filtering device. Their optical realization could be similar to the optical synthesis of SFI discussed in [20]. The image decomposition into orthogonal SFFIs proposed here simplifies the analysis of its fractional Fourier evolution and can be useful in the applications of the fractional Fourier systems in phase-space tomography, as filtering device and in the analysis of complex images.

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Appendix 1

Lemma 1: *Let α_x, α_y be irrational constants, and m_0, n_0 be constant non-negative integers. Define the constant*

$$\varphi = m_0\alpha_x + n_0\alpha_y. \tag{49}$$

Then, a necessary and sufficient condition for

$$m\alpha_x + n\alpha_y - k = \varphi \tag{50}$$

to have more than one solution in the non-negative integers m, n and in the integer k, is that

$$p\alpha_x + q\alpha_y = a, \tag{51}$$

where p, q are relatively prime non-zero integer constants, and a is an integer, and the set $\mathcal{A} = \{z \mid z \text{ integer such that } m_0 + pz \geqslant 0 \text{ and } n_0 + az \geqslant 0\}$ contains more than one element (cardinality $|\mathcal{A}| > 1$)

The solutions to equation (50) *are then given by*

$$\begin{aligned} m &= m_0 + pl, \\ n &= n_0 + ql, \quad l \in \mathcal{A}, \\ k &= al. \end{aligned} \tag{52}$$

Observe that $\mathcal{A}$ is a set of consecutive integers which contains 0 *(because $(m_0, n_0, 0)$ is a solution of equation* (50)*), and so it contains necessarily $+1$ or -1 if more than one solution exists. Note also that $\mathcal{A}$ is finite when $pq < 0$, and infinite when $pq > 0$.*

Proof: *Necessary condition.* Observe that $(m_0, n_0, 0)$ is a solution of equation (50). Let (m, n, k) be another solution. Then

$$\begin{aligned} m_x + n\alpha_y &= k + \varphi, \\ m_0\alpha_x + n_0\alpha_y &= \varphi \end{aligned} \tag{53}$$

Subtracting these two equations gives

$$(m - m_0)\alpha_x + (n - n_0)\alpha_y = k, \tag{54}$$

where α_x and α_y are irrational constants, k is an integer and φ is a constant. Observe that $n = n_0$ implies that $(m - m_0)\alpha_x = k$ and, as α_x is being irrational, this implies that $m = m_0$, $k = 0$. This means that no solution of equation (50) can be found of the form (m, n_0, k) which differs from $(m_0, n_0, 0)$. In the same way, no solution of the form (m_0, n, k) can be found which differs from $(m_0, n_0, 0)$. So, having more than one solution of equation (50) implies that, for this solution, $m \neq m_0$ and $n \neq n_0$.

Then, equation (54) can be written as

$$\alpha_y = \frac{k}{n - n_0} - \frac{m - m_0}{n - n_0}\alpha_x = \frac{j}{r} - \frac{p}{q}\alpha_x, \tag{55}$$

where j, r and p, q are unique pairs of relatively prime non-zero integers ($|p| = |q| = 1$ is also possible). Equation (55) implies that only solution (m, n, k) of equation (50) should be of the form

$$(m, n, k) = \left(m_0 + pl, n_0 + ql, (n - n_0)\frac{j}{r}\right), \tag{56}$$

with $l \in \mathcal{A}$. However, an additional condition for equation (56) to be a solution is that $j/r = a/q$, for some integer a. Indeed, substitute equations (55) and (56) in equation (50) to find that

$$\begin{aligned} m\alpha_x + n\alpha_y &= \left(\frac{m_0 - n_0 p}{q}\right)\alpha_x + \frac{n_0 j}{r} + \frac{jql}{r} \\ &= m_0\alpha_x + n_0\alpha_y + \frac{jql}{r} \\ &= \varphi + \frac{jql}{r} \overset{(2)}{=} \varphi + k. \end{aligned} \tag{57}$$

So $k = jql/r$ has to be an integer for all $l \in \mathcal{A}$, and thus for $l = 1$ or $l = -1$. As j and r are relatively prime, this means that $j/r = a/q$, for some integer a. This implies that $k = al$ ($a = 0$ is a possibility that corresponds to $k = 0$ and solutions $(m_0 + pl, n_0 + ql, 0)$), and equation (55) becomes $\alpha_y = (a - p\alpha_y)/q$, which is equivalent to equation (51).

Sufficient condition. Let $p\alpha_x + q\alpha_y = a$, with (p, q) relatively prime integer constants and a an integer constant. Let m_0, n_0 be non-negative integer constants such that $\mathcal{A}$ contains more than one element. Then it is easily checked that the triples (53) are solutions of equation (50). From the necessary condition part of the proof, it follows that these are all the possible solutions. Since $|\mathcal{A}| > 1$, there is more than one solution. □

Corollary 2: *When $\alpha_y \neq (a - p\alpha_x)/q$ or $|\mathcal{A}| = 1$, then equation (50) has the unique solution $(m_0, n_0, 0)$.*

Example: $\alpha_y = (1 + 2\alpha_x)/3$ (i.e. $p = -2, q = 3$). With $(m_0, n_0) = (1, 2)$, this implies that $\mathcal{A} = \{0\}$, and so $(m, n, k) = (1, 2, 0)$ is the only solution of equation (50). □

Lemma 2: *Let α_x be an irrational constant, and $\alpha_y = N_y/M_y$ a rational constant (N_y, M_y relatively prime; $|M_y| = |N_y| = 1$ is also possible). Let m_0, n_0 be non-nega-*

tive integers. Define the irrational constant $\varphi^* = M_y(m_0\alpha_x + n_0\alpha_y) = M_y\varphi$. *Then all solutions of the equation*

$$m\alpha_x + n\alpha_y - k = \varphi \tag{58}$$

in the non-negative integers m, n *and in the integer* k *are given by*

$$(m, n, k) = (m_0, n_0 + lM_y, lN_y), l \in \mathcal{B}, \tag{59}$$

where $\mathcal{B} = \{z \mid z \text{ integer}, n_0 + zM_y \geqslant 0\}$.

Proof: Rewrite equation (58) as

$$mM_y\alpha_x + nN_y - kM_y = \varphi^* \tag{60}$$

and observe that $(m_0, n_0, 0)$ is indeed a solution:

$$m_0M_y\alpha_x + n_0N_y = \varphi^*. \tag{61}$$

Subtracting the last two equations gives

$$(m - m_0)M_y\alpha_x + (n - n_0)N_y = kM_y. \tag{62}$$

As α_x is irrational, and all other numbers are integers, equation (62) implies that $m = m_0$ and $(n - n_0)N_y = kM_y$. However, that fact that N_y and M_y are relatively prime then implies that $n - n_0 = lM_y$ and $k = lM_y$. As $n = n_0 + lM_y \geqslant 0$, this proves equation (59).

Under the condition of lemma 2, the only necessary and sufficient condition for the existence of more than one solution is that $|\mathcal{B}| > 1$. In case $\alpha_y > 0$, this is always satisfied. □

Lemma 3: *Let* $\alpha_x = N_x/M_x, \alpha_y = N_y/M_y$ *be rational constants* (N_x, M_x *and* N_y, M_y *are relatively prime*), *and* m_0, n_0 *constant non-negative integers. Define the constant*

$$\varphi = m_0\alpha_x + n_0\alpha_y. \tag{63}$$

Then, any solution for

$$m\alpha_x + n\alpha_y - k = \varphi \tag{64}$$

in the non-negative integers m, n *and in the integer* k, *is given by*

$$\begin{aligned} m &= m_0 + k_xM_x + pl, \\ n &= n_0 + k_yM_y + ql, \\ k &= k_xN_x + k_yN_y, \end{aligned} \tag{65}$$

where p, q *are relatively prime non-zero integer constants such that*

$$\frac{p}{q} = -\frac{N_yM_x}{N_xM_y}, \tag{66}$$

and $k_x, k_y, l \in Z$ *such that* $m, n \geqslant 0$.

Proof: Direct substitution shows that the triples (m, n, k) as given in equation (65) are solutions of equation (64). We shall prove that all solutions of equation (64) can be put in the form given by equation (65). Let (m, n, k) and (m', n', k') be two different solutions of equation (64). Using equation (64) for these solutions, and subtracting the resulting equations gives

$$(m' - m)\alpha_x + (n' - n)\alpha_y = k' - k. \tag{67}$$

Formulated in an alternative way, if (m, n, k) is a solution, then any other solution can be written as $(m + z_n, n + z_y, k + z_k)$, where (z_x, z_y, z_k) is a solution of

$$z_x\alpha_x + z_y\alpha_y = z_k. \tag{68}$$

Observe that equation (68) is equivalent to

$$\frac{N_x}{M_x}\left(z_x - z_y\frac{p}{q}\right) = k, \tag{69}$$

for $q \neq 0$, As the right-hand side of equation (69) is an integer and M_x, N_x are relatively prime, M_x must divide $(z_x - z_y p/q)$, that is

$$z_x - z_y\frac{p}{q} = k_x M_x, \tag{70}$$

where k_x is an integer. Moreover q must divide z_y because p, q are relatively prime and $z_x, k_x M_x$ are integers. This yields

$$z_y = ql,$$
$$z_x = pl + k_x M_x,$$

where l is an integer, and we conclude that, if (m, n, k) is a solution of equation (64), then any other solution is of the form

$$(m + k_x M_x + pl, n + ql, k + N_x k_x). \tag{71}$$

Using symmetrical arguments, we can also show that any solution can be obtained in the form

$$(m + pl', n + k_y M_y + ql', k + N_y k_y). \tag{72}$$

Both these forms can be transformed into each other. Applying equation (72) to the solution given by equation (71) produces the equivalent symmetrical form in x and y: $(m + k_x M_x + p(l + l'), n + k_y M_y + q(l + l'), k + N_x k_x + N_y k_y)$. Let $(m, n, k) = (m_0, n_0, 0)$, and this produces the form (65).

Corollary: *If N_x, M_x, N_y, M_y are all relatively prime, then $p = N_y M_x$ and $q = -N_x M_y$, and the set of l solutions $(m_0 + pl, n_0 + ql, 0) = (m_0 + j_x M_x l, n_0 + j_y M_y l, 0)$, where $j_x/j_y = N_y/N_x$, is included in the set of k_x, k_y solutions $(m_0 + k_x M_x\ n_0 + k_y M_y, N_x k_x + N_y k_y)$ (take $k_x = N_y l, k_y = -N_x l$). In other cases, the set of l solutions contains the elements differing from the set of k_x, k_y solutions. For example, if $M_x = M_y$, then $p = N_y$, and $q = -N_x$.*

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