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Self-imaging in fractional Fourier transform systems

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Abstract

The self-imaging phenomenon in a fractional Fourier transform optical system is described in the framework of self-fractional Fourier functions. The main properties of these functions are investigated. The possible application of fractional Fourier transform optical systems in optical signal processing is discussed. © 1998 Elsevier Science B.V. All rights reserved.

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In spite of the fact that self-imaging phenomena of periodic wavefronts during Fresnel diffraction have attracted much interest in the optical community, the analogical effects in other first-order optical systems are still not well investigated. The purpose of this paper is to give a summarizing overview of the main properties of the self-imaging phenomenon in fractional Fourier transform (fractional FT) optical systems [1–4]. This self-imaging phenomenon is closely related to self-fractional Fourier functions (SFFFs) [5–9], which reproduce themselves under the fractional FT.

The fractional FT at an angle α of a function $f(x)$ is given as follows [4,10,15]:

$$R^{\alpha}[f(x)](u) = \int_{-\infty}^{\infty} f(x) K_{\alpha}(x,u)\,\mathrm{d}x, \tag{1}$$

with the kernel

$$K_{\alpha}(x,u) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\,\exp\left(i\frac{(x^2+u^2)\cos\alpha - 2xu}{2\sin\alpha}\right). \tag{2}$$

This transform describes, in paraxial approximation of the diffraction theory, except for a phase shift $\alpha/2$, the evolution of the complex field amplitude during propagation through a quadratic refractive index medium [1–4]. Thus the complex field amplitude in a plane which is transversal to the propagation axis at a distance described by the parameter α, is a fractional FT of the complex field amplitude at the input plane $f(x)$ (for which $\alpha = 0$). We call such an optical system a fractional FT system and we suppose that α changes form 0 to 2π. If α or $\alpha + \pi$ is a multiple of 2π the kernel $K_{\alpha}(x,u)$ reduces to $\delta(x-u)$ or $\delta(x+u)$ respectively. Thus, the fractional FT at an angle $2\pi n$ (n is an integer) corresponds to the identity operator. For $\alpha = \pi/2$, relationship (1) is the ordinary Fourier transform (FT).

A function $f_{\alpha}(x)$ is a self-fractional Fourier function for angle α (α-SFFF) if it satisfies the following equation:

$$R^{\alpha}[f_{\alpha}(x)](u) \underset{\text{def}}{=} A f_{\alpha}(u), \tag{3}$$

where A is a complex constant factor. In other words, $f_{\alpha}(x)$ is an eigenfunction of the corresponding fractional FT operator R^{α} with eigenvalue A. It immediately follows from the additive property for fractional FT: $R^{\alpha}R^{\beta} = R^{\alpha+\beta}$ [15] and from (3) that, if a function is a SFFF for α with eigenvalue A, it is also a SFFF one for αk ($k = 1,2,\ldots$) with eigenvalue A^k. It means that if the input complex field amplitude is an α-SFFF, then it produces a self-image at the planes $k\alpha$ of the fractional FT optical system. So,

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self-imaging in the fractional FT optical system is related to the SFFFs for different values of α. Let us consider the main SFFF-properties which can be useful for the analysis of self-imaging in a fractional FT optical system.

Properties of SFFFs for different values of α. From (1), (2) and (3) it follows that any arbitrary function is a SFFF for $\alpha = 2\pi n$, where n is an integer; a symmetric (even or odd) function is SFFF for $\alpha = \pi n$. SFFF for $\alpha = \pi n/2$ cover, as a particular case, self-Fourier functions (SFFs) [11–14]. A SFF is a function whose Fourier transform (FT) is identical to the function itself, except for a (complex) constant factor.

It is well known from quantum mechanics (see, for example, Ref. [16]) that the Hermite-Gauss functions $\Psi_n(u)$

$$\Psi_n(u) = \left(\sqrt{\pi}\,2^n n!\right)^{-1/2} \exp(-u^2/2)\, H_n(u), \tag{4}$$

where $H_n(u)$ are the Hermite polynomials, are SFFFs for any angle α:

$$R^{\alpha}[\Psi_n(u)](x) = \exp(-i\alpha n)\Psi_n(x), \tag{5}$$

with eigenvalue $\exp(-i\alpha n)$. Note that the kernel of the fractional FT equals the propagator of the non-stationary Schrödinger equation for the harmonic oscillator, except for the factor $\exp(i\alpha/2)$.

It has been proven [7] for the one-dimensional case that, if $\alpha/2\pi$ is an irrational number, the Hermite-Gauss functions $\Psi_n(u)$ are the only solutions of (3) with eigenvalues $A = \exp(-i\alpha n)$.

If $\alpha/2\pi$ is rational $\alpha = 2\pi N/M$, where N and M are relatively prime integers and $N < M$, then a SFFF for any angle $\alpha = 2\pi N/M$ is also one for the angle $2\pi/M$ and vice versa [6]. It was found in Ref. [6] that there are only M different eigenvalues for the SFFFs for angle $\alpha = 2\pi N/M$: $A = \exp(-i2\pi L/M)$, where $L = 1, \ldots, M$. A SFFF for angle $2\pi/M$ with eigenvalue $A = \exp(-i2\pi L/M)$, which we denote by $f^L_{2\pi/M}(u)$, can be represented through a sum of Hermite-Gauss modes in the following way [9,7]:

$$f^L_{2\pi/M}(u) = \sum_{m=0}^{\infty} g_{L+mM}\Psi_{L+mM}(u). \tag{6}$$

All SFFFs for angle $2\pi/M$ with eigenvalue A are also SFFFs for angle $2\pi k/M$ with eigenvalue A^k. Then $(2\pi/M)$-SFFFs for $M = 4p$, where $p \in N$, are SFFs; and all $(2\pi/M)$-SFFFs for even M must be even or odd.

Orthogonality. SFFFs for the same angle with different eigenvalues are orthogonal.

Note that Hermite-Gauss functions are orthonormal to each other. If $\alpha/2\pi$ is an irrational number the eigenvalues of the different Hermite-Gauss functions are always different. SFFFs for the same angle $2\pi/M$ but with different eigenvalues (dependent on L) are expanded into disjoint sets of Hermite-Gauss functions (6), so they are orthogonal to each other.

Non-bandlimitedness. It has been shown in Ref. [17] that if a signal is bandlimited for the fractional FT for some angle α, then it is not bandlimited for the fractional FT for any $\beta \neq \alpha \pm \pi n$ (n being any integer). It means that only an SFFF for angle π can be bandlimited. SFFFs for other angles are not bandlimited, because they reproduce themselves (except for a constant factor A) under the fractional FT for any angle $\beta \neq \alpha \pm \pi n$.

Addition. It easy to see from definition (3) and from the linearity of the fractional FT, that a sum of SFFFs for the same angle α with equal eigenvalues A, is also a SFFF for this angle with eigenvalue A. From this follows that, if $f_\alpha(x)$ is a α-SFFF with eigenvalue A, then its complex conjugate $f^*_\alpha(x)$ is also a α-SFFF with eigenvalue A.

Fractional FT of an SFFF. The fractional FT for any angle β of an α-SFFF with eigenvalue A, is a SFFF for angle α with eigenvalue A. Indeed,

$$R^{\alpha}\left[R^{\beta}[f_\alpha(x)]\right](u) = R^{\beta}\left[R^{\alpha}[f_\alpha(x)]\right](u) = AR^{\beta}[f_\alpha(x)](u). \tag{7}$$

The fractional FT for angle $-\alpha$ of an α-SFFF with eigenvalue A, is given by

$$R^{-\alpha}[f_\alpha(x)](u) = A^* f_\alpha(x). \tag{8}$$

If $\alpha/2\pi$ is an irrational number, this follows from (5). For $\alpha = 2\pi/M$ it follows by applying $-\alpha = 2\pi(M-1)/M = (M-1)\alpha$, that an eigenvalue for angle $-\alpha$ is given by $A_{-2\pi/M} = (A_{2\pi/M})^{M-1} = \exp(i2\pi L(M-1)/M) = \exp(-i2\pi L/M) = A^*$.

Using the additive property for the fractional FT we derive that

$$R^{\beta}[f_\alpha(x)](u) = R^{\alpha-(\alpha-\beta)}[f_\alpha(x)](u) = AR^{\beta-\alpha}[f_\alpha(x)](u). \tag{9}$$

In particular, for $\beta = \alpha/2$ we have that the fractional FT of $f_\alpha(x)$ at angles $\alpha/2$ and $-\alpha/2$ are identical, except for a constant phase factor which depends on the eigenvalue:

$$R^{\alpha/2}[f_\alpha(x)](u) = AR^{-\alpha/2}[f_\alpha(x)](u). \tag{10}$$

Then, using the R^{α}-definitions (1), (2), one gets for $\alpha \neq 2\pi n$:

$$\int_{-\infty}^{\infty} f_\alpha(x)\sin\left(\frac{x^2+u^2}{2}\cot\alpha/2 - \frac{xu}{\sin\alpha/2} + \frac{\alpha-\pi}{4} - \frac{\varphi}{2}\right)\mathrm{d}x = 0, \tag{11}$$

where φ is the phase of the eigenvalue $A = \exp(i\varphi)$ of $f_\alpha(x)$. This relationship is a generalization of well known

formulas for even $f_\pi^0(x)$ and odd $f_\pi^1(x)$ functions (which are SFFFs for $\alpha=\pi$)

$$\int_{-\infty}^{\infty} f_\pi^0(x)\sin(xu)\,dx=0,$$

$$\int_{-\infty}^{\infty} f_\pi^1(x)\cos(xu)\,dx=0, \tag{12}$$

in the case of SFFFs for arbitrary angle α. Because relationship (11) holds for all values u we can put $u=0$, producing the particular case

$$\int f_\alpha(x)\sin\left(\left(x^2\cot\alpha/2-\varphi\right)/2+(\alpha-\pi)/4\right)dx=0.$$

Rotational symmetry of Wigner distribution function. It is well known that the fractional FT induces a rotation of the Wigner distribution (see for example Ref. [15]). As follows from the definition of SFFFs (3) the Wigner distributions of $f_\alpha(x)$ and of $R^\alpha[f_\alpha(u)](x)$ are identical:

$$W_{R^\alpha[f_\alpha(u)]}(x,\omega)=|A|^2\int f_\alpha(x-x_0/2)f_\alpha^*(x-x_0/2)\times\exp(-i\omega x_0)\,dx_0=W_{f_\alpha}(x,\omega). \tag{13}$$

So, the Wigner distribution of an α-SFFF is invariant under rotation by angle α.

Radon-Wigner transform for SFFF. The Radon-Wigner transform of a function $f(x)$ is the squared modulus of its fractional FT $|R^\alpha[f(x)](u)|^2$ [18]. The Radon-Wigner Transform in the region of angles $\alpha\in[0,\pi]$ completely determines the amplitude and phase of $f(x)$ and is used for the analysis of complex signals [19] as well as in phase-space tomography [20]. It is related to the intensity distribution in the corresponding α-planes of the fractional FT system.

It follows from (3) that the Radon-Wigner transform of an α-SFFF is periodic with respect to the angle, with period α, because for any integer n

$$\left|R^{n\alpha+\beta}\left[f_\alpha(x)\right](u)\right|^2=\left|R^\beta\left[f_\alpha(x)\right](u)\right|^2. \tag{14}$$

Thus, any α-SFFF can be completely defined from its Radon-Wigner transform in the region of angles $[0,\alpha]$.

Shifting SFFFs. A shifted SFFF is not a SFFF. Nevertheless, the fractional Fourier transform for angle α of the shifted α-SFFF:

$$R^\alpha\left[f_\alpha(x+y)\right](u)=\exp(i\varphi)\exp\left(i\frac{y^2}{4}\sin2\alpha+iuy\sin\alpha\right)f_\alpha(u+y\cos\alpha) \tag{15}$$

produces the shifted replica of the α-SFFF with an additional quadratic phase factor. It means that we get self-imaging of the intensity distribution with some translation in the observation plane.

In particular, if the function is a SFFF for angle $\alpha=\pi/2$, then

$$R^{\pi/2}\left[f_{\pi/2}(x+y)\right](u)=\exp(i\varphi)\exp(iuy)f_{\pi/2}(u). \tag{16}$$

Correspondingly, for a SFFF for angle $\alpha=\pi$, one has

$$R^{\pi}\left[f_{\pi}(x+y)\right](u)=\exp(i\varphi)f_{\pi}(u-y). \tag{17}$$

The Radon-Wigner transform of a shifted α-SFFF for any β is given by

$$\left|R^\beta\left[f_\alpha(x+y)\right](u)\right|^2=\left|R^\beta\left[f_\alpha(x)\right](u+y\cos\beta)\right|^2. \tag{18}$$

So, for any integer n, the Radon-Wigner transforms of a shifted α-SFFF at angles β and $n\alpha+\beta$, are the same except for a coordinate shift:

$$\begin{aligned}&\left|R^{n\alpha+\beta}\left[f_\alpha(x+y)\right](u)\right|^2\\&=\left|R^{\beta+n\alpha}\left[f_\alpha(x)\right](u+y\cos(n\alpha+\beta)\right|^2\\&=\left|R^\beta\left[f_\alpha(x)\right](u+y\cos(n\alpha+\beta))\right|^2.\end{aligned} \tag{19}$$

It follows that a shifted α-SFFF can also be defined from its Radon-Wigner transform in the region of angles $[0,\alpha]$.

As follows from the shift property for the fractional FT [15], for any function $f(x)$ we have

$$\begin{aligned}&\int R^\alpha\left[f(x+y)\right](u)\,du\\&=\exp\left(i\frac{y^2}{4}\sin2\alpha\right)\int\exp(iuy\sin\alpha)F_\alpha(u+y\cos\alpha)\,du\\&=\exp\left(-i\frac{y^2}{4}\sin2\alpha\right)F_{\alpha-\pi/2}(y\sin\alpha).\end{aligned} \tag{20}$$

Applying this when $f(x)$ is a SFF $f_{\pi/2}(x)$, we observe that

$$\int R^{\pi/2}\left[f_{\pi/2}(x+y)\right](u)\,du=f_{\pi/2}(y), \tag{21}$$

and for $\alpha=0$ we get

$$\int f_{\pi/2}(u)\,du=\exp(i3\varphi)f_{\pi/2}(0). \tag{22}$$

Scaling of SFFFs. The scaling property for the fractional FT can be written as

$$\begin{aligned}&R^\beta\left[f(\lambda x)\right](u)\\&=\exp(i(\beta-\alpha)/2)\sqrt{\gamma\lambda^{-1}}\\&\quad\times\exp\left(\frac{i}{2}u^2\cot\beta\left(1-\gamma^2\lambda^{-2}\right)\right)R^\alpha\left[f(x)\right](\gamma u),\end{aligned} \tag{23}$$

where $\alpha,\ \beta \neq \pi n/2$ and

$$\gamma = \frac{\sin\alpha}{\lambda \sin\beta}, \quad \lambda^2 = \frac{\tan\alpha}{\tan\beta}. \tag{24}$$

Define $\beta = \beta_n$ such that $\tan\beta_n = \lambda^{-2}\tan n\alpha$, where n is integer and $n\alpha \neq \pi k/2$. Then the fractional FT of a scaled replica of an α-SFFF at angle β_n can be written as

$$R^{\beta_n}[f_\alpha(\lambda x)](u) = \exp[i(\varphi n + (\beta - \alpha)/2)]\sqrt{\gamma\lambda^{-1}} \times \exp\left(\frac{i}{2}u^2 \cot\beta_n(1 - \gamma^2\lambda^{-2})\right) f_\alpha(\gamma x). \tag{25}$$

Although the scaling of an α-SFFF destroys its self-reproducibility under the fractional FT for angles αn, another scaled replica $f_\alpha(\gamma x)$ is reproduced under the fractional FT for angles β_n, but with an additional quadratic phase which depends on α and on the scaling factor λ. The intensity distributions at the planes corresponding to the angles β_n are similar to the intensity distribution of the input:

$$\left|R^{\beta_n}[f_\alpha(\lambda x)](u)\right|^2 = \gamma\lambda^{-1}\left|f_\alpha(\gamma x)\right|^2. \tag{26}$$

A scaled SFF is not exactly a SFF, but applying the FT on it reproduces its scaled image

$$R^{\pi/2}[f_{\pi/2}(\lambda x)](u) = \frac{\exp(i\varphi)}{\lambda} f_{\pi/2}(\lambda^{-1}u). \tag{27}$$

Note also that if $f_{\pi/2}(x)$ is a SFF with eigenvalue $\exp(i\varphi)$ then

$$f^{(\lambda)}_{\pi/2}(x) \underset{\text{def}}{=} \sqrt{\lambda} f_{\pi/2}(\lambda x) + \sqrt{\lambda^{-1}} f_{\pi/2}(\lambda^{-1}u) \tag{28}$$

is also a SFF with eigenvalue $\exp(i\varphi)$.

Synthesis of SFFFs. It has been shown in Refs. [5,6,9] that a SFFF of order M can be constructed from any generator function $g(u) \in L^2$ through the following procedure,

$$f^L_{2\pi/M}(u) = C\sum_{k=1}^{M} \exp\left(\frac{i2\pi Lk}{M}\right) R^{2\pi(k-1)/M}[g(x)](u), \tag{29}$$

where C is an arbitrary complex constant. The synthesis of the SFFF from a generator $g(u)$ has the physical meaning of a mode selection. Thus the SFFF of order M and index L contains only the modes $L + mM$ of the generator function, expanded in Hermite-Gauss functions, where $m = 0,1,\ldots$. For infinite M, this reduces to a single mode which is the L-Hermite-Gauss function. In particular, for $L = 0$, we have that the integration of the fractional FT of any function $g(u)$ from L^2 on α gives the Gauss function

$$\frac{1}{2\pi}\int_0^{2\pi} R^\alpha[g(u)](x)\,d\alpha = g_0 \exp(-x^2/2), \tag{30}$$

where

$$g_0 = \pi^{-1/4}\int_{-\infty}^{\infty} g(u)\exp(-u^2/2)\,du.$$

An optical configuration for synthesis of the SFFs from an arbitrary image by using a Sagnac-Type interferometer coupled with a modified Mach-Zehnder interferometer has been proposed in Ref. [21]. A similar optical configuration can be used for the optical synthesis of SFFF for different rational angles.

Function decomposition on the SFFFs. It has been proved in Ref. [9] that any function $g(u)$ from L^2 can be decomposed into a sum of M self-fractional Fourier functions $f^L_{2\pi/M}(u)$ which are orthogonal to each other

$$g(u) = \sum_{L=0}^{M-1}\left(\sum_{m=0}^{\infty} g_{L+mM}\Psi_{L+mM}(u)\right) = \sum_{L=0}^{M-1} f^L_{2\pi/M}(u). \tag{31}$$

In particular, for $M = 2$, the sum (31) is the well-known decomposition into the even $(L = 0)$ and odd $(L = 1)$ functions, for $M = 4$ it is the linear combination of four self-Fourier functions [12]. In the limit case $M \to \infty$ we have the Hermite-Gauss expansion. Note that, because of (31) and the property that a SFFF for $\alpha = 2\pi/M$ is also a SFFF for $k\alpha$, the fractional FT of $g(u)$ at angle $2\pi k/M$ is a linear combination of the SFFFs in (6) with some phase shifts

$$R^{2\pi k/M}[g(u)](x) = \sum_{L=0}^{M-1} f^L_{2\pi/M}(u)\exp\left(-\frac{i2\pi kL}{M}\right), \tag{32}$$

where $k = 0,1,\ldots$.

The decomposition of the optical image on the set of SFFFs can be useful for the analysis of complex images, their processing through the fractional FT optical system, and in the development of filtering devices. In particular, the fractional FT optical systems can be used for filtering certain Hermite-Gauss modes and therefore for the generation of optical signals with desirable properties. For example, the following procedure (see relationships (10) and (31))

$$\left(R^{\pi/M} - \exp\left(\frac{i2\pi L_0}{M}\right)R^{-\pi/M}\right)[g(u)](x) = g(x) - f^{L_0}_{2\pi/M}(x)$$

results in removing $f^{L_0}_{2\pi/M}(x)$ from an initial optical signal $g(x)$.

We finally note that the properties of the SFFFs discussed in this paper stimulate the search of new applications of fractional FT systems in optical engineering.

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