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Model-free subspace-based LQG-design

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ABSTRACT

When only input/output data of a system are available the classical way to design a linear quadratic Gaussian controller consists of mainly three separate parts. First a system identification step is performed to find the system parameters. With these parameters a Kalman filter is designed to find an estimate of the state of the system. Finally, this state is then used in an LQ-controller. In literature these three steps are hardly ever considered as one joint problem. Based on techniques from the field of subspace system identification the present paper gives a new, much more direct method to implement finite horizon LQG-controller. The three steps of the LQG-controller design, i.e. the system identification, the Kalman filter and the LQ-controller are replaced by one single QR-decomposition. The equivalence between the new subspace based approach and the classical approach is proven.

Keywords: *subspace identification, system identification, LQG control, LQ-control, Kalman filter.*

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1 Introduction

A lot of the modern control methods, such as LQG control, use state space models to design a control system. When such a model is not available, the LQG design thus requires a first step of system identification from input/output data. In addition, once a state-space model is available, the state of the system must be estimated using a step of Kalman filter design. Finally the third step then consists of designing an LQ-controller. In this paper we show that one can bypass these three steps by designing an LQG controller directly from input/output data using subspace techniques.

We will be interested in the LQG-controller design for linear time-invariant systems that can be described by the following state-space equations:

$$x_{k+1} = Ax_k + Bu_k + Ke_k, \quad (1)$$

$$y_k = Cx_k + Du_k + e_k \quad (2)$$

where the input $u_k \in \mathbb{R}^m$, the output $y_k \in \mathbb{R}^l$, the state $x_k \in \mathbb{R}^n$, K the steady-state Kalman filter gain and the stationnary, ergodic, white noise $e_k \in \mathbb{R}^l$ has the following covariance matrix:

$$\mathbf{E}[e_p e_q^T] = S\delta_{pq}.$$

The available input-output data set is supposed to be measured in open-loop. The main problem we are interested in is the following:

Given open-loop measurements of the inputs u_k and the outputs y_k of the unknown system (1)-(2) find an LQG-controller for this system.

In the literature the solution to the above problem is hardly ever considered as one identification/control problem. Most authors concentrate on either the identification of the unknown system given a set of input and output measurements or on the LQG-design given the system parameters A, B, C, D, K and S.

In the present paper a new method is derived to design LQG controllers. The method is much more direct than the existing techniques in the sense that the three typical steps in calculating an LQG-controller, i.e. system identification, Kalman filter design and LQ-regulator design, are replaced by one single operation. More specifically, in the classical LQG-framework Riccati equations have to be solved for the Kalman filter and the LQ-regulator, whereas now it suffices to calculate a QR decomposition of matrices constructed out of input and output data to find the controller parameters. The model parameters A, B, C, D, K and R are not estimated ! This idea is represented graphically in Figure 1.

The results in this paper are heavily based on results from the field of subspace system identification [1]. Subspace identification is a system identification method that appeared in the late eighties and early nineties. Due to the

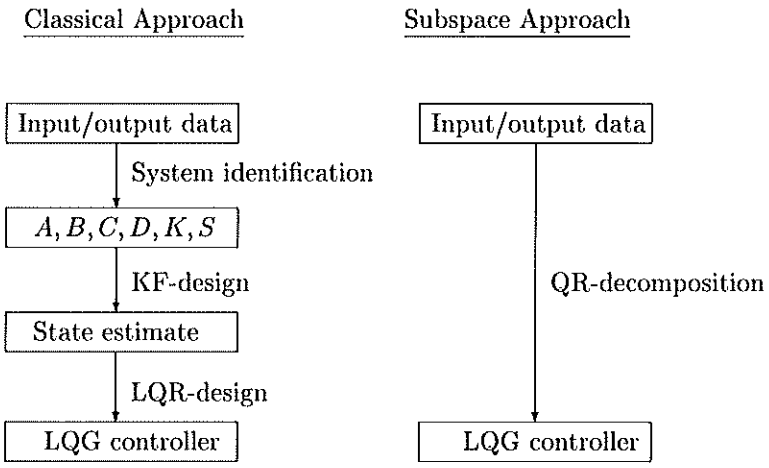


Figure 1: **Main result of the paper: based on subspace system identification, a new algorithm is proposed for efficient implementation of LQG-controllers. In the classical LQG-framework, first a system identification step is performed to find the state space matrices A, B, C, D, K and S. In a second step, these matrices are used to design a Kalman filter which gives an estimate of the state. Finally, this state estimate is used in a linear quadratic controller. The main result of the paper is that these three steps can be short-circuited and replaced by one single QR-decomposition of matrices constructed out of open-loop input and output data of the system.**

interesting features when compared to other system identification methods and due to the multiple successful applications, many papers have been published in this area. Among others we cite the pioneering papers [2], [3], [4], the overview paper [5] and the book [1]. Some real-life applications of subspace identification are described in [6], [7], [8], [9]. One of the largely underestimated features that make subspace identification so different from other system identification methods is the fact that *first* a Kalman filter estimate of the states can be found even *before* the system parameters are determined [1]. It is clear that this interesting feature could be heavily exploited for control purposes. Up to now no research has been done in this area. However, some researchers tried to combine control and subspace identification. For instance in [10] a link between subspace identification and model predictive control is presented. In [11] a method is proposed for the joint design and control by the way of subspace identification. None of them make the link with the Kalman filter states obtained from the subspace identification algorithm.

The idea of computing an optimal LQG-controller directly from data, without any use of a model, has been developed by Hjalmarsson and collaborators (see [12], [13]) using a technique called Iterative Feedback Tuning (IFT) that is entirely different from the subspace-based technique developed here. In [12], [13] a parametrized controller structure is used and implemented on the actual plant. Its parameters are optimized iteratively by collecting data on the closed loop system, with the present controller active, and by computing an unbiased estimate of the gradient of the LQG-criterion from these closed-loop data. This iterative scheme is shown to converge to at least a local minumum of the LQG-criterion under weak conditions. IFT has been successfully applied to a range of mechanical and industrial control problems, and has been shown to work also on classes of nonlinear systems. In some ways, the fundamental difference between IFT-controller design and the subspace-based controller design developed here parallels the fundamental difference between parametric identification based on criterion minimization, and subspace-based identification based on projections. The advantages and drawbacks of the two identification methods in terms of optimality, parametric versus non-parametric, local minima versus uniqueness of the solution, etc are transported from the realm of identification to the realm of control design.

Since the results presented here are inspired by subspace identification we start the paper by giving a short overview of some basic results of subspace identification in Section 2. The idea of subspace identification is explained, the link with Kalman filtering is emphasized and the importance of the QR-decomposition is illustrated. Section 3 contains the main results of the paper. First the LQG-problem is presented, the classical framework of solving it is recalled and the new subspace based approach is derived. In Section 4 it is shown that both approaches are equivalent. We end the paper in Section 6 with some conclusions and open problems.

2 Brief overview of linear subspace system identification

In this section an overview of linear subspace identification is given. Only the results that are relevant for this paper are given. We start by giving some definitions of the block-Hankel matrices constructed from the input and output measurements of the system. Further the specific projection of the row space of these matrices and the link with Kalman filtering is recalled. Finally we illustrate the importance of the QR-decomposition for a numerically efficient implementation of this projection.

2.1 Problem statement

The problem treated in linear subspace identification is the following:

Given measurements of the inputs u_k and the outputs y_k of the unknown system (1)-(2) find an estimate of the system matrices A, B, C, D and the noise related matrices S and K.

2.2 Matrix input-output equations

The starting point of all subspace identification algorithms is the following set of matrix input-output equations:

$$Y_f = \Gamma_i X_f + H_i U_f + H_i^s E_f, \tag{3}$$
$$Y_p = \Gamma_i X_p + H_i U_p + H_i^s E_p. \tag{4}$$

They represent the effect of the state x_k, the deterministic input u_k and the unknown stochastic input e_k on the outputs y_k and can be directly derived from the state-space equations (1)-(2). In what follows the different terms in these equations are defined.

First of all there are the data block-Hankel matrices:

$$U_p \stackrel{\text{def}}{=} \begin{pmatrix} u_0 & u_1 & \dots & u_{j-1} \\ u_1 & u_2 & \dots & u_j \\ \dots & \dots & \dots & \dots \\ u_{i-1} & u_i & \dots & u_{i+j-2} \end{pmatrix}, \tag{5}$$

$$U_f \stackrel{\text{def}}{=} \begin{pmatrix} u_i & u_{i+1} & \dots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & \dots & u_{i+j} \\ \dots & \dots & \dots & \dots \\ u_{2i-1} & u_{2i} & \dots & u_{2i+j-2} \end{pmatrix} \tag{6}$$

where the indices p and f stand for past and future. In a similar way the block-Hankel matrices Y_p, Y_f and E_p, E_f can be defined for the outputs y_k and the output noise e_k. It should be noted that, for statistical reasons, it is assumed that the number of columns j goes to infinity. We will also use the following short-hand notation:

$$W_p = \begin{pmatrix} Y_p \\ U_p \end{pmatrix}.$$

The past and future state sequences are defined as:

$$X_p \stackrel{\text{def}}{=} \begin{pmatrix} x_0 & x_1 & \dots & x_{j-1} \end{pmatrix}, \tag{7}$$

$$X_f \stackrel{\text{def}}{=} \begin{pmatrix} x_i & x_{i+1} & \dots & x_{i+j-1} \end{pmatrix}. \tag{8}$$

Further we also have the following system related matrices:

$$\Gamma_i \stackrel{\text{def}}{=} \begin{pmatrix} C \\ CA \\ \dots \\ CA^{i-1} \end{pmatrix},$$

$$H_i \stackrel{\text{def}}{=} \begin{pmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \dots & \dots & \dots & \dots \\ CA^{i-2}B & CA^{i-3}B & \dots & D \end{pmatrix},$$

$$H_i^s \stackrel{\text{def}}{=} \begin{pmatrix} I_l & 0 & \dots & 0 \\ CK & I_l & \dots & 0 \\ \dots & \dots & \dots & \dots \\ CA^{i-2}K & CA^{i-3}K & \dots & I_l \end{pmatrix}$$

where Γ_i is the extended observability matrix, H_i and H_i^s the matrices containing the impulse response of the system due to the deterministic input u_k and the unknown stochastic input e_k respectively.

2.3 Orthogonal and oblique projections

A very commonly used tool in subspace identification are orthogonal and oblique projections of the row space of matrices. Suppose we are interested in the projection of the row space of $\mathcal{A} \in \mathbb{R}^{p\times j}, \mathcal{B} \in \mathbb{R}^{q\times j}$ and $\mathcal{C} \in \mathbb{R}^{r\times j}$. The rows of $\mathcal{A}$ can be decomposed as a linear combination of the rows of $\mathcal{B}$, $\mathcal{C}$ and a third matrix orthogonal to $\mathcal{B}$ and $\mathcal{C}$. This can be written as:

$$\mathcal{A} \;=\; L_{\mathcal{B}}\mathcal{B} + L_{\mathcal{C}}\mathcal{C} + L_{\mathcal{B}^\perp,\mathcal{C}^\perp} \begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix}^\perp.$$

where $L_{\mathcal{B}}, L_{\mathcal{C}}$ and $L_{\mathcal{B}^\perp,\mathcal{C}^\perp}$ are the corresponding weights in the linear combination.

- The *orthogonal projection* of the row space of $\mathcal{A}$ into the row space of $\mathcal{B}$ is denoted by $\mathcal{A}/\mathcal{B}$ and defined as:

$$\mathcal{A}/\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A}\mathcal{B}^{\dagger}\mathcal{B} = L_{\mathcal{B}}\mathcal{B}.$$

- The *oblique projection* of the row space of $\mathcal{A}$ into the row space of $\mathcal{B}$ along the row space of $\mathcal{C}$ is denoted by $\mathcal{A}/_{\mathcal{C}}\ \mathcal{B}$. This oblique projection can be interpreted through the following recipe: Project the row space of $\mathcal{A}$ orthogonally on the joint row space of $\mathcal{B}$ and $\mathcal{C}$ and decompose the result along the row space of $\mathcal{B}$. Mathematically, the orthogonal projection of the row space of $\mathcal{A}$ on the joint row space of $\mathcal{B}$ and $\mathcal{C}$ can be stated as:

$$\mathcal{A}/\begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix} = \mathcal{A}\begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix}^{\dagger}\begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix}.$$

This leads to the following definition for the oblique projection:

$$\mathcal{A}/_{\mathcal{C}}\ \mathcal{B} \stackrel{\text{def}}{=} \mathcal{A}\left[\begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix}^{\dagger}\right]_{(:,1:q)}\mathcal{B}.$$

We also have that:

$$\mathcal{A}/\begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix} = \underbrace{\mathcal{A}/_{\mathcal{C}}\ \mathcal{B}}_{L_{\mathcal{B}}\mathcal{B}} + \underbrace{\mathcal{A}/_{\mathcal{B}}\ \mathcal{C}}_{L_{\mathcal{C}}\mathcal{C}}.$$

2.4 Basics of subspace identification

The subspace identification problem can also be interpreted as follows: given the past inputs and outputs W_p and the future inputs U_f, find an optimal prediction of the future outputs Y_f. If we use a linear predictor:

$$\widehat{Y}_f = L_w W_p + L_u U_f \tag{9}$$

the least squares prediction $\widehat{Y}_f$ of Y_f can be found from the following least squares problem:

$$\min_{L_w, L_u} \|Y_f - \begin{pmatrix} L_w & L_u \end{pmatrix}\begin{pmatrix} W_p \\ U_f \end{pmatrix}\|_F^2.$$

The solution to this problem is the orthogonal projection of the row space of Y_f into the row space spanned by W_p and U_f:

$$\begin{aligned} \widehat{Y}_f &= Y_f/\begin{pmatrix} W_p \\ U_f \end{pmatrix} \\ &= \underbrace{Y_f/_{U_f}\ W_p}_{L_w W_p} + \underbrace{Y_f/_{W_p}\ U_f}_{L_u U_f}. \end{aligned} \tag{10}$$

The link of the above projection with Kalman filtering is extremely important. Each of the two projections in the right hand side can be given the following interpretation:

- The first projection is directly linked to the Kalman filter states and the observability matrix of the system:

$$Y_f /_{U_f} W_p = \Gamma_i \widehat{X}_f \tag{11}$$

where Γ_i is the extended observability matrix and $\widehat{X}_f$ a Kalman filter estimate of the state sequence X_f. In [1] it was shown that the estimated state sequence $\widehat{X}_f$ is the solution to a bank of Kalman filters acting column-wise on the matrix W_p. Any arbitrary element $\widehat{x}_{q+i}$ of $\widehat{X}_f$ is a Kalman filter estimate of the corresponding element x_{q+i} in X_f:

$$\begin{aligned}
\widehat{x}_{q+i} &= A\widehat{x}_{q+i-1} + Bu_{q+i-1} \\
&\quad + K_{q+i-1}(y_{q+i-1} - C\widehat{x}_{q+i-1} - Du_{q+i-1}), && (12) \\
K_{q+i-1} &= (KS + A\Sigma_{q+i-1}C^T)(S + C\Sigma_{q+i-1}C^T)^{-1}, && (13) \\
\Sigma_r &= A\Sigma_{r-1}A^T + KSK^T - (KS + A\Sigma_{r-1}C^T) \\
&\quad (S + C\Sigma_{r-1}C^T)^{-1}(KS + A\Sigma_{r-1}C^T)^T && (14)
\end{aligned}$$

with the initial conditions:

$$\widehat{x}_q = \left[\widehat{X}_p\right]_{(:,q)} \quad , \quad \Sigma_q = \frac{1}{j}\widehat{X}_p\widehat{X}_p^T. \tag{15}$$

and

$$\widehat{X}_p = X_p /_{U_f} U_p \tag{16}$$

and the inputs u_k and outputs y_k the elements of the corresponding column in the data Hankel matrices U_p and Y_p.

This quite abstract but important concept is represented in a more intuitive way in Figure 2. Any column q of $Y_f /_{U_f} W_p$ can therefore be considered as the free response of the system if it is released from the state $\widehat{x}_{q+i}$.

- The second projection of the right hand side in (10) can be given a similar interpretation. Any column q of $Y_f /_{W_p} U_f$ is the response of the system if it were driven by the input sequence $\left(\begin{array}{ccc} u_{q+i} & \cdots & u_{q+2i} \end{array} \right)$ (i.e. the q-th column of the future inputs U_f) and starting from the initial state $\widehat{x}_{q+i}$. This initial state should not be confused with the state obtained from the

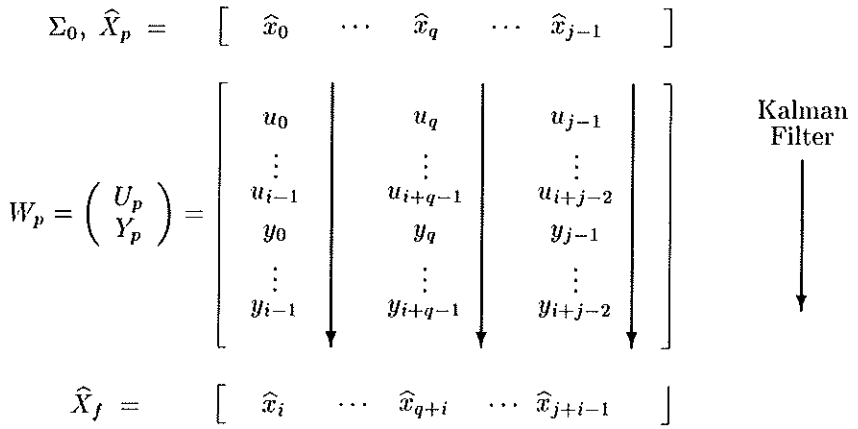


Figure 2: **Interpretation of the estimated state sequence $\widehat{X}_f$, obtained from the subspace identification, as the solution to a bank of Kalman filters. If the system matrices A, B, C, D, K and S were known, the state $\widehat{x}_i$ could be determined from the Kalman filter (12)-(14) as follows: Start the filter at time 0, with an initial state estimate $\widehat{x}_0$ and initial state covariance matrix Σ_0. Now iterate the Kalman filter over i time steps (the vertical arrow down). The Kalman filter will then return a state estimate $\widehat{x}_i$. When repeating this procedure for each of the j columns we obtain the estimated state sequence $\widehat{X}_f$. We thus speak about a *bank* of Kalman filters. The major observation in subspace algorithms is that the system matrices A, B, C, D, K and S do not have to be known to determine the state sequence $\widehat{X}_f$. It can be determined directly from the input-output data as in the projection (11). For further details we refer to [1].**

Kalman filter equations (12)-(??). Although it can also be considered as the solution of the Kalman filter equations (12)-(14), the initial conditions Σ_q and $\widehat{x}_q$, the inputs u_k and the outputs y_k that are used are different from the ones used in (15)-(16). Actually $\widehat{x}_{q+i}$ is the solution to the Kalman filter equations (12)-(16) if the initial state $\widehat{x}_q$ were the q-th element of $\widehat{X}_p = X_p /_{U_p} U_f$, the initial state covariance matrix $\Sigma_q = \frac{1}{j}\widehat{X}_p\widehat{X}_p^T$ and the inputs u_k and outputs y_k were zero.

2.5 Numerical implementation

The numerical implementation of the projection (10) can be done in a very efficient way by making the following QR-decomposition:

$$\begin{pmatrix} W_p \\ U_f \\ Y_f \end{pmatrix} = \begin{pmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{pmatrix}. \tag{17}$$

By posing:

$$L = \begin{pmatrix} R_{31} & R_{32} \end{pmatrix} \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}^{\dagger}$$

it is easy to show that (10) can be written as:

$$Y_f /_{\begin{pmatrix} W_p \\ U_f \end{pmatrix}} = L \begin{pmatrix} W_p \\ U_f \end{pmatrix}.$$

Using Matlab notations, we then have:

$$L_w = L(:, 1 : i(m+l)), \tag{18}$$

$$L_u = L(:, i(m+l)+1 : end). \tag{19}$$

These two matrices will play a very important role in the next section where the link between subspace identification and LQG is made. It will be shown that they are of direct relevance in the LQG-design. Without going into the details we already mention that L_w determines the Kalman filter whereas L_u fixes the LQR parameters.

3 LQG: classical vs. subspace approach

In this section the main results of the paper are presented. We start by recalling the LQG-problem and split it up into three separate subproblems namely the system identification, the Kalman filter and the LQ-controller design. Further we recall the classical solution to this LQG-problem on the one hand and present a new subspace based way of solving it. Finally the equivalence of both methods is analyzed and formally proven for two particular cases: the case where the horizon is infinite and the case where the inputs are zero-mean white.

3.1 LQG-control problem

Throughout the whole paper we will assume that the last measured output and implemented input are y_0 and u_0, respectively. The first control input to be found is thus u_1. The LQG-control problem as treated in the present paper can be stated as:

LQG-Control Problem

Given a set of open-loop measurements of the inputs u_k and the outputs y_k, $k \leq 0$, of the unknown system (1)-(2), find the input sequence $u_f = (u_1, \ldots u_N)$ such that the following quadratic cost function J is minimized over the horizon N:

$$J = \sum_{k=1}^{N} \{(\widehat{y}_k - r_k)^T Q_k (\widehat{y}_k - r_k) + u_k^T R_k u_k\} \tag{20}$$

where $\widehat{y}_k$ is the k-step-ahead optimal predicted output given past inputs and outputs and future inputs up to time k. The reference output trajectory is described by r_k and the matrices $Q_k \in \mathbb{R}^{l \times l}$ and $R_k \in \mathbb{R}^{m \times m}$ are user-defined non-negative definite weightings of the outputs and the inputs.

Figure 3 illustrates the principle of finite horizon LQG-control. The first term in this expression penalizes the fact that the predicted outputs $\widehat{y}_k$ are different from the reference outputs r_k while the second term keeps the energy in the inputs u_k at a reasonable level. We will call the *forward horizon* the number of time steps over which the system output is predicted and the control input is computed. The *backward horizon* is the set of past input/output data points that are used to predict the outputs on the forward horizon. Although a generalization to the case of horizons with different lengths is possible, we will assume for simplicity that backward and forward horizon have the same length.

The above formulation of the LQG-problem differs from the one that is usually found in literature in mainly two ways:

- A receding finite horizon will be used instead of an infinite horizon, with the length of this horizon being equal to the number of block rows i in the future data Hankel matrices (6) in the previously explained subspace identification algorithm.
- A second difference is that we consider an output tracking based performance index J instead of a state based criterion. It can be shown that such output based criterion can be reformulated as a more classical state-based criterion: see e.g. [14]. We will come back to this point in the following subsection.

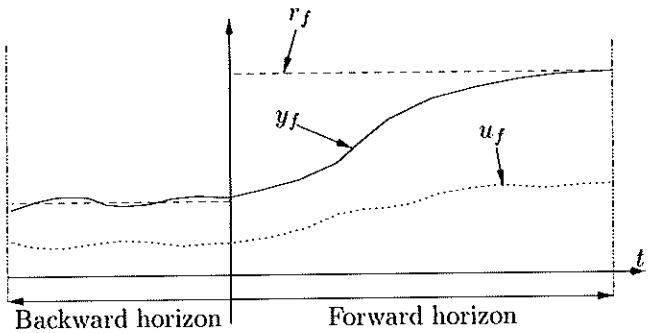


Figure 3: **This figure illustrates the principle of finite horizon linear quadratic gaussian control. The aim is to find the future control sequence u_f that minimizes the performance criterion J. r_f is the future reference output trajectory, y_f the optimal prediction of the future outputs given the future input sequence u_f.**

3.2 LQG - The classical approach

The classical way of solving the LQG-problem as presented in Section 3.1 is to split it up into three separate subproblems: system identification, Kalman filter design and LQ-controller design [15], [14]. The system identification step can be stated as:

System Identification

Given open-loop measurements of the inputs u_k and the outputs y_k of the unknown system (1)-(2) find an estimate of the system matrices A, B, C, D and the noise related matrices S and K.

Different methods are available in the literature to solve this problem.

Since LQG is a state space based control method the knowledge of the state is mandatory. The second step of the LQG-controller then consists of finding a one-step-ahead prediction $\widehat{x}_1$ of the state, given the system parameters A, B, C, D, K, S and measurements of the past inputs and outputs. This state estimate will then be used in an optimal state feedback control law. The optimal state prediction is obtained with a Kalman filter which is described by the following equations:

Kalman filter

Given

- *the system related matrices A, B, C, D, K and S*

- *measurements of the inputs* u_k *and outputs* y_k *for* $k \in \{-N+1, \ldots, 0\}$
- *an initial state estimate* $\widehat{x}_{-N+1}$ *and initial state error covariance matrix* Σ_{-N+1}

the optimal one-step-ahead state prediction $\widehat{x}_1$ *is the solution to the following recursive Kalman filter equations:*

$$\begin{aligned} \widehat{x}_{q+1} &= A\widehat{x}_q + Bu_q + K_q(y_q - C\widehat{x}_q - Du_q), && (21)\\ K_q &= (KS + A\Sigma_q C^T)(S + C\Sigma_q C^T)^{-1}, && (22)\\ \Sigma_{q+1} &= A\Sigma_q A^T + KSK^T - (KS + A\Sigma_q C^T) \\ &\quad (S + C\Sigma_q C^T)^{-1}(KS + A\Sigma_q C^T)^T. && (23) \end{aligned}$$

Now that the system parameters A, B, C, D, K, S and the one-step-ahead predicted state estimate $\widehat{x}_1$ are known they can be used to design a linear quadratic controller. The classical solution to the LQ-controller is given in what follows. For simplicity, we restrict ourselves to the case where the future reference output trajectory r_f is zero. The extension to the more general case of arbitrary reference output is treated in [14]. In the next subsection the arbitrary reference outputs case will be presented for the subspace based LQG-controller.

LQ-controller

Given

- *the system related matrices* A, B, C *and* D *and the weighting matrices* Q_k *and* R_k
- *the optimal one-step-ahead state prediction* $\widehat{x}_1$
- *an initial condition* P_0 *for LQR Riccati equation*

the input u_1 *that minimizes the performance criterion (20) is the solution to the following recursive LQR equations:*

$$\begin{aligned} u_1 &= L_1\widehat{x}_1, && (24)\\ L_1^T &= -(C^TQ_1D + A^TP_NB)(R_1 + D^TQ_1D + B^TP_NB)^{-1}, && (25)\\ P_{q+1} &= A^TP_qA + C^TQ_{N-q}C - (C^TQ_{N-q}D + A^TP_qB) \\ &\quad (R_{N-q} + D^TQ_{N-q}D + B^TP_qB)^{-1}(C^TQ_{N-q}D + A^TP_qB)^T. && (26) \end{aligned}$$

A more graphical interpretation of the Kalman filter and the LQ controller of the above described LQG-controller is given in Figure 4. It is assumed that the system parameters A, B, C, D and K are known.

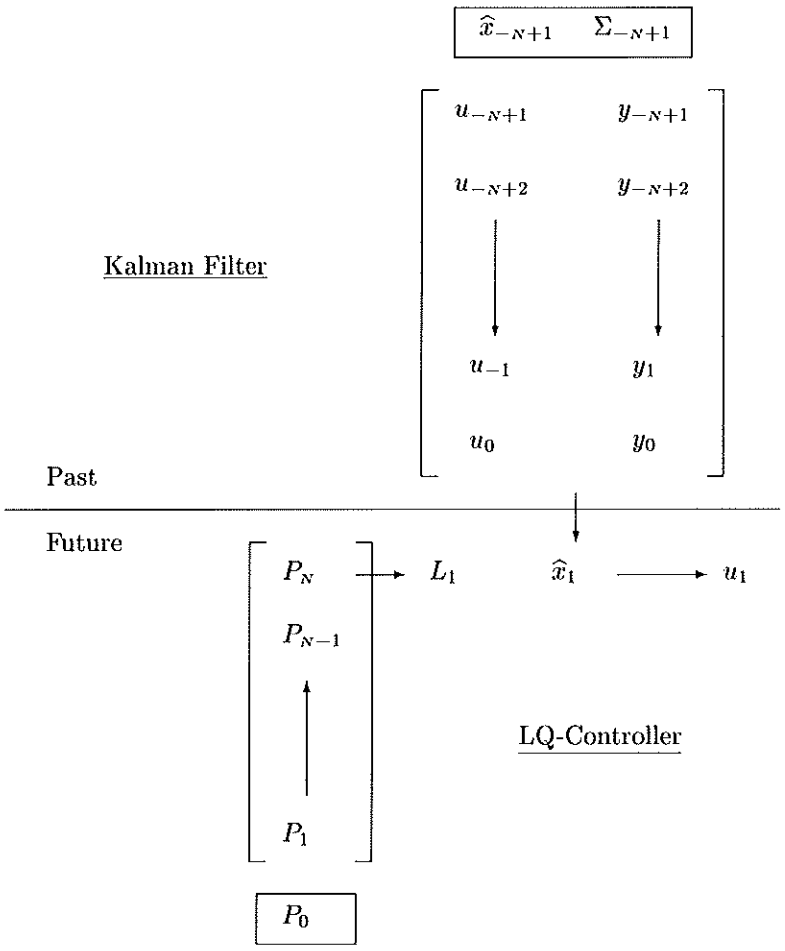


Figure 4: **Graphical interpretation of how an LQG-controller works. The last implemented control input and measured output are u_0 and y_0. The aim of the LQG-controller is to find the controller output u_1 such that the performance criterion (20) is minimized. The horizontal line represents the separation between the past and the future i.e. Kalman filter and the LQ-controller. If the system matrices A, B, C, D, K and S are known the Kalman filter uses the N last known values of the inputs u_k and the outputs y_k to determine the one-step-ahead predicted Kalman filter state $\widehat{x}_1$. In order to initialize the Kalman filter equations (21)-(23) an initial state estimate $\widehat{x}_{-N+1}$ and an initial state error covariance matrix Σ_{-N+1} are required. The solution of the LQ-controller (24)-(26) is the control input u_1 at time step 1. It requires the optimal one-step-ahead predicted state estimate of x_1, which is provided by the Kalman filter, and P_0 which is the initial condition for the LQ-controller Riccati equation. The final LQG-controller output u_1 is then the LQ-controller gain L_1, derived from the LQ-controller Riccati equation, multiplied by the one-step-ahead predicted Kalman filter state $\widehat{x}_1$.**

3.3 LQG - A subspace approach

What inspired us to make the link between subspace identification and LQG-design is the following idea. In state-space based controllers the state is used as an intermediate variable between the past and the future to find the control input. Indeed, first the state estimate that minimizes the prediction criterion (Kalman filter), given past inputs and outputs, is calculated. In a second step the inputs that minimize the control criterion (LQ-controller), given the state and the reference output, are calculated. This remarkable duality between the Kalman filter (21)-(23) and the LQ-controller (24)-(26) was already pointed out in [15]. Although not as explicit as in classical LQG-design, the above idea is also clearly present in subspace system identification where the link between Kalman filtering and the projection of the future outputs Y_f into the row space of the past inputs U_p, outputs Y_p and future inputs U_f is well-known (see also Section 2). The interesting thing is that in subspace identification the Kalman filter states can be found without the knowledge of the system parameters and thus without having to solve the Kalman filter equations. This property will be heavily exploited in the rest of the paper.

In this subsection we study the LQG-problem from another point of view. Instead of solving the performance criterion J in the classical way i.e. by solving a two-point boundary value problem, we start from the input-output equation (3). If we suppose that the output reference trajectory r_f is zero and that the future input sequence u_f and the corresponding estimated future output sequence $\widehat{y}_f$ are defined as:

$$\widehat{y}_f \stackrel{\text{def}}{=} \begin{pmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \vdots \\ \widehat{y}_N \end{pmatrix} \quad , \quad u_f \stackrel{\text{def}}{=} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

the optimal control criterion (20) can also be written as:

$$J = \widehat{y}_f^T Q \widehat{y}_f + u_f^T R u_f$$

where the matrices $Q \in \mathbb{R}^{Nl \times Nl}$ and $R \in \mathbb{R}^{Nm \times Nm}$ are defined as:

$$Q = \begin{pmatrix} Q_1 & 0 & \dots & 0 \\ 0 & Q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_N \end{pmatrix} \quad , \quad R = \begin{pmatrix} R_1 & 0 & \dots & 0 \\ 0 & R_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_N \end{pmatrix}.$$

From (9) we then know that the optimal predicted output sequence $\widehat{y}_f$ can be expressed as:

$$\widehat{y}_f = L_w w_p + L_u u_f$$

where

$$w_p = \begin{pmatrix} y_p \\ u_p \end{pmatrix}$$

with $u_p \in \mathbb{R}^{im}$ and $y_p \in \mathbb{R}^{il}$ the i last known values of the inputs and the outputs:

$$y_p = \begin{pmatrix} y_{-N+1} \\ \vdots \\ y_{-1} \\ y_0 \end{pmatrix} \quad , \quad u_p = \begin{pmatrix} u_{-N+1} \\ \vdots \\ u_{-1} \\ u_0 \end{pmatrix}.$$

The performance criterion J can now be written as:

$$J = (L_w w_p + L_u u_f)^T Q (L_w w_p + L_u u_f) + u_f^T R u_f.$$

which is a problem that can be solved by putting the trace of the derivative of J with respect to the input sequence u_f to zero:

$$\frac{\delta J}{\delta u_f} = 0.$$

It is easy to show that the subspace based LQG control law then becomes:

$$u_f = -(R + L_u^T Q L_u)^{-1} L_u^T Q L_w w_p. \tag{27}$$

Observing this equation we can make some important remarks:

1. It is no coincidence that the term "subspace identification" is not present in the title of the paper. Although the results presented here are clearly based on techniques from the field of subspace identification, the system identification step is inherent in the way of solving the control design problem in the sense that the system parameters A, B, C, D, K and S are never explicitely calculated. This is the reason why we refered to it as model-free LQG-design.

2. Contrarily to the classical LQG-framework where only the first control input is calculated, the subspace-based LQG-controller finds the control inputs over the whole forward horizon. Although only the first input is implemented, the knowledge of the inputs over this horizon can be very interesting for instance when one has to meet input constraints. The readers familiar with Model Predictive Control will certainly have noticed the link between the optimal MPC law and the subspace-based LQG control law (27). The relation between LQG and MPC has already been pointed out in [14].

3. Except for the user-defined parameters Q and R, the different steps in the design of the LQG-controller, i.e. the system identification step, the Kalman filter and the LQ-controller, are replaced by one single QR-decomposition (see also Figure 1). Moreover, the system matrices A, B, C, D and the noise covariance matrices K and S do not have to be calculated explicitely ! The only parameters that are of importance are L_u and L_w.

4. In Section 2 we have seen that there is a link between the subspace projection (10) and the Kalman filter state of the system:

$$L_w w_p = \Gamma_N \widehat{x}_1$$

where Γ_N is the extended observability matrix and $\widehat{x}_1$ is a Kalman filter estimate. Moreover, if the number of block rows in the data block-Hankel matrices is infinite or if the inputs are zero-mean white, it can be seen from subspace identification theory [1] that:

$$L_u = H_N \tag{28}$$

where H_N is the Toeplitz matrix containing the Markov parameters of the system. In that particular case the subspace based LQG control law becomes:

$$u_f = -(R + H_N^T Q H_N)^{-1} H_N^T Q \Gamma \widehat{x}_1.$$

5. Contrary to the classical LQG-framework the extension to output tracking is straightforward. Suppose that the desired output trajectory r_f is given, the performance criterion simply becomes:

$$J = (\widehat{y}_f - r_f)^T Q (\widehat{y}_f - r_f) + u_f^T R u_f$$

and the control law (27):

$$u_f = (R + L_u^T Q L_u)^{-1} L_u^T Q (r_f - L_w w_p).$$

4 Equivalence between the two approaches

If we look at the subspace based LQG control law (27) on the one hand and the classical LQG control laws (23)-(24), one might wonder what is the link between the two. Are they equivalent and under what circumstances ? A basic difference between the two is that the classical LQG-problem requires that the user does a system identification, chooses initial conditions for the Kalman filter and for the LQ-controller. For the subspace based LQG-controller no choice of initial conditions has to be made and the system identification step is inherent to the way of solving the problem. The only thing one has to do is a QR-decomposition of data block-Hankel matrices. Of course the following fundamental question then arises:

Basic Problem:

Do there exist initial conditions $\widehat{x}_{-N+1}$ and Σ_{-N+1} of the Kalman filter (21)-(23) and an initial condition P_0 of the linear quadratic regulator (24)-(26) that give the same control u_1 as the subspace based LQG-controller (27) (see also Figure 4) ?

The above problem is important in that it makes the direct link between LQG and subspace identification. In the general case we do not have the answer to the above question yet. However under some particular conditions we were able to derive a solution. Two cases are considered namely the infinite horizon case ($i \to \infty$) and the case where the open-loop inputs are white zero-mean.

4.1 Infinite horizon

In the case the backward horizon is infinite it is possible to prove that the classical and the subspace based LQG-approach are equivalent as defined in the basic problem above. Indeed, if the horizon is infinite the initial conditions of the Kalman filter (21)-(23) and the linear quadratic controller (24)-(26) do not influence the controller output u_1. However, we still have to prove that the equations (27) and (21)-(26) deliver the same expression for u_1.

Theorem 1
If the horizon is infinite ($N \to \infty$), the subspace based LQG-controller (27) and classical LQG-controller equations (21)-(26) produce the same value for u_1. The choice of the initial conditions Σ_{-N+1}, $\widehat{x}_{-N+1}$ and P_0 does not play a role and can thus be made arbitrarily.

Proof:

As mentioned in Section 2 the $L_w w_p$-term in the LQG-controller equation (27) equals $\Gamma_N \widehat{x}_1$ where Γ_N is the extended observability matrix and $\widehat{x}_1$ the steady state Kalman filter estimate corresponding to the Kalman filter equations (21)-(23). In the particular case where the horizon is infinite, the Kalman filter is steady-state. The Kalman filter is steady-state, not only in the sense that the Kalman gain is the steady-state Kalman gain, but also in the sense that the influence of the error on the initial state estimate $\widehat{x}_{-N+1}$ has disappeared. This can be seen from Figure 2 for an infinite i-index, i.e. the number of block rows in the data block-Hankel matrices. As mentioned in remark 4 on the LQG-controller equation (27) we have that $L_u = H_N$ in the case the horizon is infinite. We refer to [1] for further details.

The second part of the proof consists in showing that u_1, the first element of

the future control sequence u_f

$$u_f = -(R + H_N^T Q H_N)^{-1} H_N^T Q \Gamma_N \widehat{x}_1,$$

equals the solution of the LQR-equations (24)-(26). We refer to Appendix A for the mathematical details. ■

4.2 White inputs

Let us now consider the case where the inputs of the open-loop identification data are white and zero-mean. This is a quite common situation since white signals are often used as inputs for system identification purposes. In this particular case the equivalence between the classical and the subspace based LQG-controller can also be made.

Theorem 2
If the input data for open-loop identification are white and zero-mean, the subspace based LQG-controller (27) and the classical LQG-controller equations (24)-(23) with initial conditions $\Sigma_{-N+1} = 0$, $\widehat{x}_{-N+1} = 0$ *and* $P_0 = 0$ *are equivalent.*

Proof: The proof is very similar to that of Theorem 1. The $L_w w_p$-term in the LQG-controller equation (27) also equals $\Gamma_N \widehat{x}_1$ where Γ_N is the extended observability matrix and $\widehat{x}_1$ the Kalman filter estimate corresponding to the Kalman filter equations (21)-(23). The initial conditions of the Kalman filter are determined by the projection (16):

$$\widehat{X}_p = X_p /_{U_f} U_p.$$

If the inputs u_k are white zero-mean this projection is 0:

$$\widehat{X}_p = 0,$$

which implies that the initial Kalman filter estimate $\widehat{x}_{-N+1}$ is zero and the initial error covariance matrix (15):

$$\begin{aligned} \Sigma_0 &= \frac{1}{j} \widehat{X}_p \widehat{X}_p^T \\ &= 0. \end{aligned}$$

The proof of (24)-(26) and the initial condition $P_0 = 0$ is the same as in the proof of Theorem 1 (see Appendix A). ■

5 Algorithm

The final subspace identification based LQG-algorithm can now be implemented as in Figure 5.

Subspace based LQG-controller

1. Construct the data block-Hankel matrices Y_f, U_f and $W_p = \binom{Y_p}{U_p}$ from the open-loop data.

2. Calculate the R-component of the following QR-decomposition:

$$\begin{pmatrix} W_p \\ U_f \\ Y_f \end{pmatrix} = \begin{pmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{pmatrix}.$$

3. Calculate

$$L = \begin{pmatrix} R_{31} & R_{32} \end{pmatrix} \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}^{\dagger}$$

and derive the LQG-controller parameters:

$$\begin{aligned} L_w &= L(:, 1 : N(m+l)), \\ L_u &= L(:, N(m+l)+1 : end). \end{aligned}$$

4. Construct the controller inputs:

$$w_p = \begin{pmatrix} y_{k-N+1}^T & \cdots & y_{k-1}^T & y_k^T \mid u_{k-N+1}^T & \cdots & u_{k-1}^T & u_k^T \end{pmatrix}^T.$$

5. Implement the first input u_{k+1} of the LQG-control sequence u_f:

$$u_f = -(R + L_u^T Q L_u)^{-1} L_u^T Q L_w w_p.$$

6. To implement the following control step construct the new control input vector w_p with the newly measured system output y_{k+1} and repeat from step 4.

Figure 5: **Algorithm for the subspace-based LQG-controller. The first 3 steps, from which the controller parameters L_w and L_u are derived, only have to be performed once. The next 3 steps represent the implementation of the controller.**

6 Conclusions and open problems

In the present paper we have derived a new algorithm for the calculation of linear quadratic gaussian controllers based on input/output data only, i.e. without any plant model. Even though the derivation is based on expressions from subspace identification theory, the algorithm bypasses the identification step altogether. The main result is that given (open-loop) input/output data of the system, one can directly derive the controller parameters from one single QR-decomposition. It was proven that in the case the horizon is infinite or in the case the inputs are white zero-mean, the model-free subspace-based LQG-controller is equivalent to the classical LQG-controller equations.

Although the present paper makes some important links between subspace identification and LQG-design, there are still a certain number of open problems.

- The model-free subspace-based and the classical model-based LQG-controller are equivalent only if the estimate L_u of the Toeplitz matrix H_N containing the Markov parameters of the system is unbiased (28):

$$L_u = H_N.$$

  If the horizon is infinite ($N \to \infty$) or the inputs are zero-mean white it is possible to show that this condition holds. It would be interesting to know if the method also works under more general conditions.

- Another open problem is the fact that the subspace identification algorithm, as presented here, only works with open-loop data. It would be of great interest to extend the current results to the case where only closed-loop data is available. In that case the LQG-parameters L_u and L_w of the subspace based LQG-controller could be updated on-line. The main operation of the method is indeed only based on a QR-decomposition which is a linear algebra tool that is particularly well suited for updating. It would then be very possible to extend the present subspace based LQG-controller to an adaptive framework as in [14].

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Extension towards specific applications, FWO-Onderzoeksgemeenschappen: *Identification and Control of Complex Systems, Advanced Numerical Methods for Mathematical Modeling*); **Belgian Federal Government**(Interuniversity Attraction Pole IUAP IV/02: *Modeling, Identification, Simulation and Control of Complex Systems*, Interuniversity Attraction Pole IUAP IV/24: IMechS: Intelligent Mechatronic Systems); **European Commission:** (Human Capital and Mobility: *SIMONET: System Identification and Modeling Network, SCIENCE-ERNSI: European Research Network for System Identification.*)

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A Proof of Theorem 1

Feedback gain equation (25)

We show that the first element u_1 of

$$u_f = -(R + H_N^T Q H_N)^{-1} H_N^T Q \Gamma_N \widehat{x}_1$$

corresponds to the solution of the LQR equations (24)-(26).

Let us focus on the term $R + H_N^T Q H_N$. Since R is a block diagonal matrix it can be split up into the first block element R_1 and the remaining part $\bar{R}$. The same holds for Q. Exploiting the structure of H_N we can write that:

$$\begin{aligned} R + H_N^T Q H_N &= \begin{pmatrix} R_1 & 0 \\ 0 & \bar{R} \end{pmatrix} + \begin{pmatrix} D & 0 \\ \Gamma_{N-1} B & H_{N-1} \end{pmatrix}^T \begin{pmatrix} Q_1 & 0 \\ 0 & \bar{Q} \end{pmatrix} \cdot \\ &\qquad \begin{pmatrix} D & 0 \\ \Gamma_{N-1} B & H_{N-1} \end{pmatrix} \\ &= \begin{pmatrix} R_1 + D^T Q_1 D + B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} B & B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} \\ H_{N-1}^T \bar{Q} \Gamma_{N-1} B & \bar{R} + H_{N-1}^T \bar{Q} H_{N-1} \end{pmatrix}. \end{aligned}$$

Now we use the matrix inversion theorem to calculate the inverse of $R + H_N^T Q H_N^T$. With the Schur complement Δ

$$\Delta \stackrel{\text{def}}{=} R_1 + D^T Q_1 D + B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} B - B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} \cdot (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} H_{N-1}^T \bar{Q} \Gamma_{N-1} B$$

we have that

$$(R + H_N^T Q H_N^T)^{-1} = \begin{pmatrix} \Delta^{-1} & -\Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} \cdot (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \\ -(\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \cdot H_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} & (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \left(I + H_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} \cdot B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \right) \end{pmatrix} \quad (29)$$

We now see that the first element u_1 of u_f can be written as:

$$\begin{aligned} u_1 &= -\begin{pmatrix} \Delta^{-1} & -\Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} D & 0 \\ \Gamma_{N-1} B & H_{N-1} \end{pmatrix}^T \begin{pmatrix} Q_1 & 0 \\ 0 & \bar{Q} \end{pmatrix} \begin{pmatrix} C \\ \Gamma_{N-1} B \end{pmatrix} \widehat{x}_1 \\ &= -\Delta^{-1} \left(D^T Q_1 C + B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} B - \right. \\ &\quad \left. B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} H_{N-1} \bar{Q} \Gamma_{N-1} B \right) \widehat{x}_1. \end{aligned}$$

For the moment we assume that P_N can be written as the following expression which will be proven in a couple of minutes:

$$P_N = \Gamma_N^T \left(Q - Q H_N (R + H_N^T Q H_N)^{-1} H_N^T Q \right) \Gamma_N \quad (30)$$

We then have that Δ becomes

$$\Delta = R_1 + D^T Q_1 D + B^T P_{N-1} B.$$

Finally for the control input u_1 we have

$$\begin{aligned} u_1 &= -\left(R_1 + D^T Q_1 D + B^T P_{N-1} B \right)^{-1} \left(D^T Q_1 C + B^T P_{N-1} A \right) \widehat{x}_1 \\ &= L_1 \widehat{x}_1. \end{aligned}$$

This proves the feedback gain equation of the LQ-controller.

Riccati equation (26)

What remains to be proven now is that (30) equals the LQR Riccati equation (26) :

$$\begin{aligned} P_N &= A^T P_{N-1} A + C^T Q_1 C - (C^T Q_1 D + A^T P_{N-1} B) \cdot \\ &\quad (R_1 + D^T Q_1 D + B^T P_{N-1} B)^{-1} (C^T Q_1 D + A^T P_{N-1} B)^T \end{aligned}$$

Since (26) is a recursive equation, the proof is one by induction: first we prove that (26) equals (30) for $N = 1$. In as second step we assume that they are equal for $N - 1$ and show that this is also true for N, which then completes the proof.

$\underline{N=1}$:

For $N=1$ (30) is:

$$\begin{aligned} P_1 &= \Gamma_1^T \left(Q_N - Q_N H_1 (R_N + H_1^T Q_N H_1)^{-1} H_1^T Q_N\right) \Gamma_1 \\ &= C^T Q_N C - C^T Q_N D (R_N + D^T Q_N D)^{-1} D^T Q_N C \end{aligned}$$

which corresponds to (26) with $N=1$ and $P_0 = 0$.

$\underline{N-1 \rightarrow N}$:

Now we assume that (26) equals (30) for $N-1$

$$P_{N-1} = \Gamma_{N-1}^T \left(\bar{Q} - \bar{Q} H_{N-1} (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} H_{N-1}^T \bar{Q}\right) \Gamma_{N-1}, \quad (31)$$

and we prove that this also holds for N. We know that:

$$\begin{aligned} P_N &= \Gamma_N^T \left(Q - Q H_N (R + H_N^T Q H_N)^{-1} H_N^T Q\right) \Gamma_N \\ &= \begin{pmatrix} C \\ \Gamma_{N-1} A \end{pmatrix}^T \left[\begin{pmatrix} Q_1 & 0 \\ 0 & \bar{Q} \end{pmatrix} - \begin{pmatrix} Q_1 & 0 \\ 0 & \bar{Q} \end{pmatrix} \begin{pmatrix} D & 0 \\ \Gamma_{N-1} B & H_{N-1} \end{pmatrix} \cdot \right. \\ &\quad \left(\begin{array}{cc} \Delta^{-1} & -\Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} \cdot \\ & (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \\ -(\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} \cdot & (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1} (I + \\ H_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} & H_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} \\ & (\bar{R} + H_{N-1}^T \bar{Q} H_{N-1})^{-1}) \end{array} \right) \cdot \\ &\quad \left. \begin{pmatrix} D & 0 \\ \Gamma_{N-1} B & H_{N-1} \end{pmatrix}^T \begin{pmatrix} Q_1 & 0 \\ 0 & \bar{Q} \end{pmatrix} \right] \begin{pmatrix} C \\ \Gamma_{N-1} A \end{pmatrix} \\ &= -C^T Q_1 D \Delta^{-1} D^T Q_1 C^T + C^T Q_1 C \\ &\quad - A^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} D^T Q_1 C \\ &\quad + A^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1} \bar{Q} H_{N-1})^{-1} H_{N-1} \bar{Q} \Gamma_{N-1} B \Delta^{-1} D^T Q_1 C \\ &\quad + C^T Q_1 D \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1} \bar{Q} H_{N-1})^{-1} B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} A \\ &\quad - C^T Q_1 D \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} A \\ &\quad - A^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} A \\ &\quad + A^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1} \bar{Q} H_{N-1})^{-1} H_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} A \\ &\quad + A^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1} \bar{Q} H_{N-1})^{-1} H_{N-1}^T \bar{Q} \Gamma_{N-1} A \\ &\quad - A^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1} \bar{Q} H_{N-1})^{-1} \cdot \\ &\quad \left[I + H_{N-1}^T \bar{Q} \Gamma_{N-1} B \Delta^{-1} B^T \Gamma_{N-1}^T \bar{Q} H_{N-1} (\bar{R} + H_{N-1} \bar{Q} H_{N-1})^{-1} \right] H_{N-1}^T \bar{Q} \Gamma_{N-1} A \\ &\quad + A^T \Gamma_{N-1}^T \bar{Q} \Gamma_{N-1} A. \end{aligned}$$

Since we assumed that (31) holds and with:

$$\Delta = R_1 + D^T Q_1 D + B^T P_{N-1} B,$$

we find the final expression for P_N:

$$\begin{aligned} P_N &= A^T P_{N-1} A + C^T Q_1 C - (C^T Q_1 D + A^T P_{N-1} B) \cdot \\ &\quad (R_1 + D^T Q_1 D + B^T P_{N-1} B)^{-1} (C^T Q_1 D + A^T P_{N-1} B)^T. \end{aligned}$$