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Cellular automata, quasigroups and symmetries

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Summary. We investigate cellular automata (CA) with a local rule $\phi : G^2 \to G$, where the local rule defines a quasigroup structure (Latin square) on the finite set G. If the quasigroup is semisymmetric or totally symmetric, some top-down equilateral triangular subsets of the CA-orbits, the so-called ∇-configurations, exhibit certain symmetries. The most interesting symmetries are the rotational and the total (dihedral) symmetries, which may be considered in conjunction with certain automorphisms.

We first explore the conditions for quasigroups to be symmetric (or for local CA-rules to allow symmetric ∇-configurations), and how to construct symmetric quasigroups by prolongation, i.e., by steadily increasing the order of the quasigroup, thereby conserving the symmetry. Then we study rotationally or totally symmetric ∇-configurations. We begin with the existence of symmetric ∇-configurations of arbitrary size N and $N \equiv 0, 1 \bmod 3$, and show that symmetric ∇-configurations of size $N \equiv 2 \bmod 3$ exist under mild conditions on ϕ. We present explicit formulas for the number of distinct symmetric ∇-configurations. By studying the combined group action of the dihedral (or rotational) group and the automorphism group of the quasigroup G on the ∇-configurations of size N, we are able to classify and count the number of different equivalence classes of ∇-configurations.

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Introduction

Let G be a finite set, then $\underline{c} = (c(k))_{k\in\mathbb{Z}} \in G^{\mathbb{Z}}$ represents a one-dimensional bidirectional sequence with values in G. A one-dimensional cellular automaton (CA) is a discrete dynamical system defined by the mapping

$$A_\phi : G^{\mathbb{Z}} \to G^{\mathbb{Z}} : \ (A_\phi \underline{c})(k) = \phi(c(k-M), c(k-M+1), \ldots, c(k+N)), \qquad (1)$$

with $M, N \in \mathbb{Z}$, $M < N$, and where ϕ is a mapping from $G^{N-M+1} \to G$, known as the *local rule* of the automaton, e.g., [21]. The mapping (1) is applied iteratively, starting with the *initial state* $\underline{c}_0$. The state of the automaton after m generations is $\underline{c}_m = A^m \underline{c}_0$. Its *orbit* is the two-dimensional sequence $O_A(\underline{c}_0) = (\underline{c}_m)_{m\in\mathbb{N}}$. Such an orbit has a geometrical-graphical representation associated with it: the *orbit pattern.*

When the state-evolution of the CA is given by $\phi(c(k), c(k+1))$, as will be the

case in this paper, this pattern consists of touching circles ("*cells*") placed at the vertices of an equidistant triangular grid, and with coordinates $(m, k) \in \mathbb{N} \times \mathbb{Z}$, as shown in Fig. 1(a). Each circle is filled with a color (or other symbol) in one-one

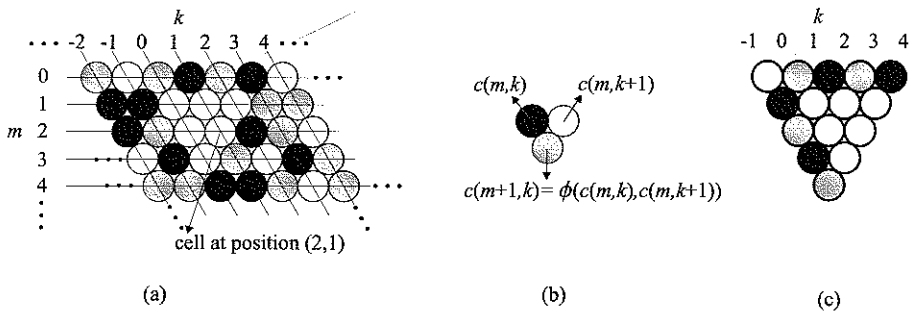


Figure 1. (a): A partial orbit-pattern of a cellular automaton; (b): an elementary triangular configuration; (c): the ∇_ϕ-configuration generated by the sequence $\underline{c}_0(-1, 5) = [c_0(-1), c_0(0), c_0(1), c_0(2), c_0(3), c_0(4)]$ in the orbit pattern.

correspondence with its state $c \in G$. Any elementary triangular configuration in the orbit pattern (a top-down triangular arrangement of three adjacent cells as shown in Fig. 1(b)) reflects a realization of the local rule (called also *local configuration* or *local matching pattern* in the context of orbit patterns).

When we consider a connected finite subset of $\underline{c}_0$, i.e., $\underline{c}_0(j, N) = [c_0(j), c_0(j+1), \ldots, c_0(j+N)]$, it is clear that only a finite segment of the orbit pattern can be generated from it by using the local rule: it contains all cells in an equilateral top-down triangular domain with as top-row the cells corresponding to $\underline{c}_0(j, N)$ (see Fig. 1(c)). Let us call this the ∇_ϕ-configuration generated by $\underline{c}_0(j, N)$. The top row $\underline{c}_0(j, N)$ itself is called *generator sequence.*

When $G = \mathbb{Z}_2$, the set of integers modulo 2, and

$$\phi(c(k), c(k+1)) = (c(k) + c(k+1)) \bmod 2, \tag{2}$$

the local configuration pattern is invariant under the group of geometric operations that leave an equilateral triangle invariant. It implies that – in this case – a ∇_ϕ-configuration remains a ∇_ϕ- configuration after application of such an operation (but with possibly different generator sequences). This fact allows, and even suggests, the question about the existence, the nature and the number of ∇_ϕ-configurations that exhibit certain symmetries (e.g., rotational symmetry, and total symmetry (referring to the dihedral symmetry of an equilateral triangle)). This question was investigated in [2] for this particular case. Figure 2 gives an idea about the abundance of symmetric ∇_ϕ-configurations in the orbit of the CA satisfying rule (2), with random initial configuration.

In this paper, we generalize the problem and look for the complete class of local CA-rules of the form $\phi : G^2 \to G$ that may produce symmetric ∇_ϕ-configurations.

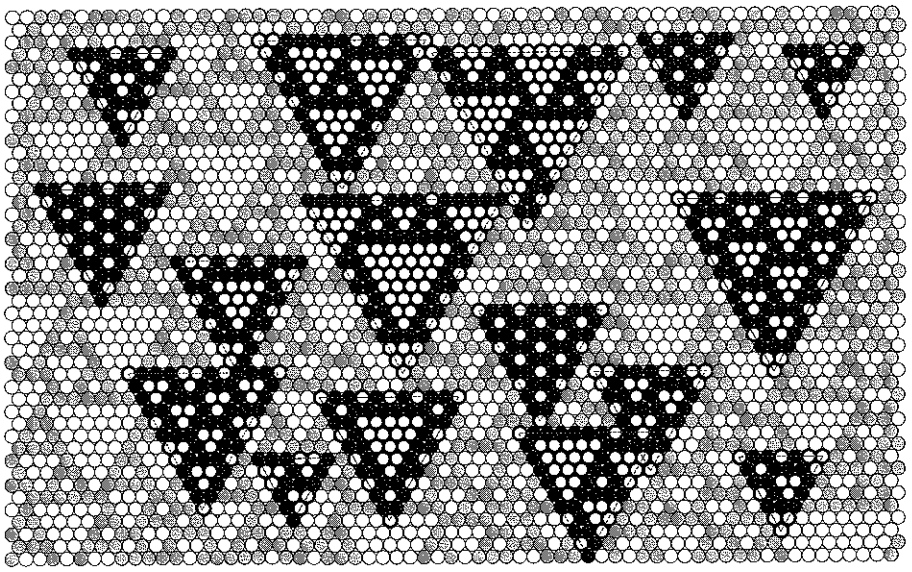


Figure 2. A partial orbit-pattern of the two-valued cellular automaton with rule $\phi(c(k), c(k+1)) = (c(k) + c(k+1)) \bmod 2$ and random initial configuration (top row). Here both gray and black cells are in state 1, white cells are in state 0. The highlighted triangular configurations exhibit either reflective, rotational or total symmetry (not all symmetric ∇_ϕ-configurations are shown).

The central issues are the existence and the nature of such rules, and the nature and number of symmetric patterns that may be obtained from them. The paper is organized in the following way.

In the first part, we introduce the notion of Latin squares and quasigroups, see [4, 5], which provide the proper framework for a study of CA-rules $\phi : G^2 \to G$ which may produce symmetric ∇_ϕ-configurations. It turns out that this is possible when the CA-rules are defined by semisymmetric or totally symmetric quasigroups.

In the second section, we study semisymmetric quasigroups in more detail based on the work of A. Sade [13, 14, 15, 16, 17, 18].

In the third section, we count the number of rotationally invariant (totally invariant) ∇_ϕ-configurations generated by a CA. Moreover, we study the group action of several relevant groups on symmetric configurations and provide tools to count the number of different equivalence classes w.r.t. the group action.

In the fourth section we investigate the possible symmetries of a ∇_ϕ-configuration closer. We classify ∇_ϕ-configurations according to their stabilizer and construct a matrix which enables us to count the different number of configurations with a prescribed stabilizer.

1. Basic facts

In this section we present the notion and latin squares and quasigroups, and some facts concerning symmetries in cellular automata with rules of the form $\phi: G^2 \to G$.

We start with the basic facts on latin squares and quasigroups that are necessary for the further developments in this paper (see [4] for a more extensive treatment).

From now on G denotes a finite set of cardinality N, which we usually think of as $G = \{0, 1, \ldots, N-1\}$. A matrix $L \in G^{N \times N}$ is called a *latin square of order* $|G|$ if each row and each column contains each element of G only once. Closely related to Latin squares are binary operations, i.e., a map $\star : G \times G \to G$, $(x, y) \mapsto x \star y$ with a certain property.

Definition 1.1. A finite set G with a binary operation $\star$ is called a *quasigroup* if for all $a, b \in G$ there exist unique $x, y \in G$ such that $a \star x = b$ and $y \star a = b$.

If the map $\star$ defines a quasigroup structure on G, then the Cayley table, i.e., the $|G| \times |G|$ matrix with entries $l_{x,y} = x \star y$, is a Latin square.

The symmetric group of G is denoted by $\mathrm{Sym}(G)$ and it contains all bijective maps $\alpha : G \to G$.

Definition 1.2. Let $\star_1$ and $\star_2$ be quasigroup structures on G. The quasigroup structures are *isotopic* if there exist $\alpha, \beta, \gamma \in \mathrm{Sym}(G)$ such that $\alpha(x) \star_1 \beta(y) = \gamma(x \star_2 y)$ holds for all $x, y \in G$.

More important for our purposes is the notion of isomorphic quasigroups. Two quasigroups $(G, \star_1)$ and $(G, \star_2)$ are isomorphic if they are isotopic such that $\alpha = \beta = \gamma$.

Of particular interest for our later considerations is the automorphism group of a quasigroup.

Definition 1.3. Let $(G, \star)$ be a quasigroup. A bijection $\sigma : G \to G$ is called an *automorphism* (of $(G, \star)$) if $\sigma(x) \star \sigma(y) = \sigma(x \star y)$ holds for all $x, y \in G$.

The group of all automorphisms is denoted by $\mathrm{Aut}(G, \star)$, If $(G, \star)$ is a quasigroup, then we can define new quasigroup structures on G in several ways. We are particularly interested in the following two constructions.

If $(G, \star)$ is a quasigroup, then the binary operation $\star_r$ defined by $a \star_r b = c$, where c is the unique solution of $b \star x = a$ is called the *right rotated quasigroup.* The binary operation $\star_l$ defined by $a \star_l b = c$, where c is the unique solution of $x \star a = b$ is called the *left rotated quasigroup.* In [4] the right rotated quasigroup is called the *first translate* of $(G, \star)$ and the left rotated quasigroup is called *the second translate* of $(G, \star)$. Some authors, e.g., [8, 14], also use the term *transpose*

of $(G, \star)$ for the right rotated quasigroup.

The basic idea behind the construction of the rotated quasigroup structures is the following. If x and y are given, then we denote the relation $z = x \star y$ symbolically as an *elementary* $\star$-triangle

$$\begin{array}{ccc} x & & y \\ & z = x \star y & \end{array}$$

The right rotation of the above pattern, i.e., $\begin{smallmatrix} z & x \\ & y \end{smallmatrix}$ is an elementary $\star_r$-triangle. From the definitions above it follows that $x(\star_r)_r y = x \star_l y$. Therefore the following definition, e.g., [4], makes sense.

Definition 1.4. The quasigroup $(G, \star)$ is called *semisymmetric* if $x \star y = x \star_r y$ holds for all $x, y \in G$.

Equivalently we can say that a quasigroup is semisymmetric if and only if $(x \star y) \star x = y$ holds for all $x,\ y \in G$.

The semisymmetry of a quasigroup has therefore a well defined geometric interpretation, namely any rotation of a $\star$-triangle yields a $\star$-triangle.

If $(G, \star_1)$ and $(G, \star_2)$ are isomorphic quasigroups, then the property of being semisymmetric is invariant under isomorphisms. This is no longer true for isotopic quasigroups, [18].

The quasigroup $(G, \star)$ is commutative if $x \star y = y \star x$ holds for all $x, y \in G$. From our geometric point of view the commutativity corresponds to reflections of elementary $\star$-configurations about the vertical axis.

Definition 1.5. The quasigroup $(G, \star)$ is called *totally symmetric* if it is commutative and semisymmetric.

We are particularly interested in cellular automata which are induced by local rules $\star : G^2 \to G$, $(x, y) \mapsto x \star y$ that allow symmetric $\nabla_\star$-configurations.

Like for quasigroups, the relation $x \star y = z$ is visualized in an elementary $\star$-triangle.

A rotation to the right of a $\star$-triangle does not necessarily lead to a triangle that is defined by a binary operation on $G \times G$.

In order to ensure that the left and the right rotations of all elementary $\star$-triangles define new local rules, $\star_l$ and $\star_r$, respectively, it is necessary and sufficient to assume that $\star$ defines a quasigroupstructure on G. Thus, any semisymmetric (totally symmetric) quasigroup $(G, \star)$ defines a CA with a rotationally symmetric (totally symmetric) local rule, and vice-versa. This implies that, in order to study rotational and total symmetries in $\nabla_\star$-configuration, it is sufficient to study local rules which are given by a semisymmetric or totally quasigroup $(G, \star)$ (and which then generate $\nabla_\star$-configurations).

2. Symmetric quasigroups

In this section, we study symmetric quasigroups or, equivalently, special cellular automata induced by these quasigroups, in a more detailed manner. Semisymmetric quasigroups were extensively studied by A. Sade, [13, 14, 15, 16, 17, 18].

Of special interest in the theory of cellular automata are so called additive or linear cellular automata, see e.g., [1, 7, 9, 10, 19, 20] and the literature cited there. Therefore we begin with the introduction of affine semisymmetric quasigroups.

Finally we show how to construct semisymmetric quasigroups (not necessarily affine) of rank $N+1$ from semisymmetric quasigroups of rank N.

Affine quasigroups

From now on let G be a finite commutative ring with 0 and 1. A binary operation which is given by $x \star y = r_1 x + r_2 y + r_3$ is called affine. If the binary operation induces a quasigroup structure on G, then we say $(G, \star)$ is an affine quasigroup.

The following lemma, see [16], provides us with a criterion for an affine quasigroup to be semisymmetric.

Lemma 2.1. *Let $(G, \star)$ be an affine quasigroup, i.e., $x \star y = r_1 x + r_2 y + r_3$, then the quasigroup is semisymmetric if and only if*

$$r_1^3 = -1, \ r_2 = r_1^{-1}, \ \textit{and} \ r_3(1 + r_1) = 0.$$

In particular, the affine operation $x \star y = -x - y + g$, where $g \in G$, defines for every ring G an affine quasigroup which is semisymmetric. This trivial observation establishes the existence of semisymmetric quasigroups of arbitrary rank.

In fact the above observation can be slightly generalized. If G is a finite group, then $x \star y = x^{-1} y^{-1}$ defines a semisymmetric quasigroup structure on G.

We briefly turn our attention to isomorphic affine quasigroups. Among all possible bijective maps $\sigma : G \to G$ there are the affine maps $\sigma(g) = ag + b$, where $a \in G^*$ and $b \in G$. If $\star_1$ and $\star_2$ are isomorphic via an affine map, we say that $\star_1$ and $\star_2$ are affine isomorphic.

Sade, in [14], provides us with a complete list of semisymmetric (non isomorphic) quasigroups up to order 7. We will only mention two of them, which will serve as examples troughout the paper.

Examples. 1. Let $G = \mathbb{Z}_3$, then $x \star y = 2x + 2y$ defines a semisymmetric quasigroup, in fact a totally symmetric quasigroup. Its Cayley table is given by

$x \star_0 y$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

The automorphism group is

$$\mathrm{Aut}(\mathbb{Z}_3, \star_0) = \{\sigma(x) = ax + b \mid a \in \mathbb{Z}_3^*, b \in \mathbb{Z}_3\}.$$

2. If $G = \mathbb{F}_4 = \{0, 1, \zeta, 1+\zeta\}$, where $\zeta^2 = 1 + \zeta$, the field with four elements, then $x \star_a y = \zeta x + (1+\zeta)y$ defines a semisymmetric quasigroup structure on $\mathbb{F}_4$. Its Cayley table is given by

$x \star_a y$	0	1	2	3
0	0	3	1	2
1	2	1	3	0
2	3	0	2	1
3	1	2	0	3

where we identify ζ with 2 and $1+\zeta$ with 3. Its automorphism group is described as

$$\mathrm{Aut}(\mathbb{F}_4, \star_a) = \{\sigma(x) = ax + b \mid a \in \mathbb{F}_4^*, b \in \mathbb{F}_4\},$$

which is isomorphic to A_4, the group of even permutations of 4 symbols.

Construction of semisymmetric quasigroups

As already mentioned, the construction of all latin squares or equivalently all quasigroups of arbitrary order is difficult and for large values of the order almost impossible. In [4], the authors discuss several ways to obtain new quasigroups from old. We will briefly review the so-called method of *prolongation* ([4] pp. 28) which produces a quasigroup of order $|G|+1$ from a quasigroup $(G, \star)$, after which we extend the method for semisymmetric quasigroups.

Definition 2.2. Let $(G, \star)$ be a quasigroup and let $\theta : G \to G$ be in $\mathrm{Sym}(G)$. The permutation θ is called a complete mapping (of $(G, \star)$) if the map $\eta : G \to G$ defined as $\eta(g) = g \star \theta(g)$ is also an element of $\mathrm{Sym}(G)$.

If θ is a complete mapping of $(G, \star)$, then the set $\{(g, \theta(g)) \mid g \in G\}$ is called a *transversal* of the latin square $(x \star y)_{x,y \in G}$. Generally, a transversal T of a latin square of order N is a set of N cells, one in each row and one in each column such that the cells contain pairwise different symbols.

Complete mappings provide a tool to build a quasigroup of order $|G|+1$ out of a quasigroup $(G, \star)$. Before we give the formal definition we look at an example. Let $G = \mathbb{Z}_3$ and define $x \star y = x + y$. The Cayley table is given by

$x \star_\theta y$	0	1	2
0	0	1	$\underline{2}$
1	$\underline{1}$	2	0
2	2	$\underline{2}$	1

For $G' = G \cup \{3\}$ we define the operation $\star_\theta : G' \times G' \to G'$ by the following table

$x \star_\theta y$	0	1	2	3
0	0	1	$\underline{3}$	2
1	$\underline{3}$	2	0	1
2	2	$\underline{3}$	1	0
3	1	0	2	3

which clearly shows that $(G', \star_\theta)$ is a quasigroup. The construction is as follows: Take the Cayley table for $(G, \star)$ and add a column called 3 to the right and add a row called 3 to the bottom. Replace each underlined symbol by 3 and move the deleted symbol to the right column and to the bottom row, respectively. Eventually, define $3 \star_\theta 3$ as 3. That yields the Cayley table of $(G', \star_\theta)$.

Algebraically we define the prolongation as follows. Let $(G, \star)$ be a quasigroup and let $\theta \in \mathrm{Sym}(G)$ be a complete mapping. For $G' = G \cup \{\Xi\}$, where $\Xi \notin G$, we define

$$x \star_\theta y = \begin{cases} x \star y & \text{if } x, y \in G,\ y \neq \theta(x) \\ \Xi & \text{if } \theta(x) = y \\ \eta(x) & \text{if } x \in G,\ y = \Xi \\ \eta(\theta^{-1}(y)) & \text{if } x = \Xi,\ y \in G \\ \Xi & \text{if } x = y = \Xi. \end{cases} \tag{3}$$

The quasigroup $(G', \star_\theta)$ is called the prolongation of $(G, \star)$ w.r.t. θ.

For our purposes we need criteria which ensure the prolongation to be semisymmetric if the underlying quasigroup is semisymmetric.

Theorem 2.3. *Let $(G, \star)$ be a semisymmetric quasigroup. If $\theta \in \mathrm{Sym}(G)$ satisfies $x = \theta(x) \star \theta^2(x)$ for all $x \in G$, then the prolongation $(G \cup \{\Xi\}, \star_\theta)$ is a semisymmetric quasigroup.*

Moreover, if $(G \cup \{\Xi\}, \star_\rho)$ is a semisymmetric prolongation of $(G, \star)$, then $x = \rho(x) \star \rho^2(x)$ holds for all $x \in G$.

Proof. If $\theta \in \mathrm{Sym}(G)$ satisfies $x = \theta(x) \star \theta^2(x)$, then θ is a complete mapping. Indeed, the semisymmetry of $(G, \star)$ implies that

$$x \star \theta(x) = (\theta(x) \star \theta^2(x)) \star \theta(x) = \theta^2(x) = \eta(x) \tag{4}$$

is a permutation of G. Moreover, η is the inverse of θ [apply (4) with x replaced by $\theta(x)$ to obtain $\theta(x) \star \theta^2(x) = \eta(\theta(x))$, or $\theta(x) \star \eta(x) = \eta(\theta(x))$, while through semisymmetry (4) also reads as $\theta(x) \star \eta(x) = x$; this yields $\eta(\theta(x)) = x$].

It remains to show that $(G \cup \{\Xi\}, \star_\theta)$ is semisymmetric. If x, y, $z \in G$ such that $x \star_\theta y = z$, the semisymmetry follows from the invariance of $(G, \star)$. Suppose now that $x \star_\theta y = \Xi$, i.e., either $x = \Xi$ and $y = \Xi$ or $x \in G$ and $y = \theta(x)$. In the first case the semisymmetry is clear. In the second case, we have $x \star_\theta \theta(x) = \Xi$ and we compute $\Xi \star_\theta x \overset{(3)}{=} \eta(\theta^{-1}(x)) = \eta^2(x) = \theta(x)$ and $(\Xi \star_\theta x) \star_\theta \Xi = \theta(x) \star_\theta \Xi \overset{(3)}{=} \eta(\theta(x)) = x$. Therefore $(G \cup \{\Xi\}, \star_\theta)$ is a semisymmetric quasigroup.

The second assertion follows immediately from the above proof. If a prolongation is semisymmetric, then the semisymmetry implies $\eta(\rho(x)) = x$, and therefore $x = \rho(x) \star \rho^2(x)$. □

Under certain conditions it is even possible to extend automorphisms of $(G, \star)$ to the prolongation.

Corollary 2.4. *Let $(G, \star)$ be a semisymmetric quasigroup and $(G \cup \{\Xi\}, \star_\theta)$ be a semisymmetric prolongation of $(G, \star)$. An automorphism $\alpha \in \mathrm{Aut}(G, \star)$ prolongs to an automorphism α_θ which fixes Ξ if and only if α commutes with θ, i.e., $\alpha \circ \theta = \theta \circ \alpha$.*

All examples studied so far suggest the conjecture that any automorphism of a prolongation is indeed a prolongation of an automorphism. We have no proof for this conjecture.

3. Group operations on $\nabla_\star$-configurations

In this section, which is the main body of the article, we study symmetries of group operations on $\nabla_\star$-configurations occurring in cellular automata defined by a semisymmetric or totally symmetric quasigroup $(G, \star)$.

Rotationally symmetric $\nabla_\star$-configurations

In this section $(G, \star)$ denotes a semisymmetric quasigroup. We study $\nabla_\star$-configurations, which are generalizations of elementary $\star$-triangles, and their symmetries. A $\nabla_\star$-configuration of size N, is an equilateral triangular array of elements in G, i.e., of the form

$$\begin{array}{ccccccccc} g_1^1 & & g_2^1 & & \cdots & & g_{N-1}^1 & & g_N^1 \\ & g_1^2 & & g_2^2 & & \cdots & & g_{N-1}^2 & \\ & & g_1^3 & & \cdots & & g_{N-2}^3 & & \\ & & & \ddots & & \iddots & & & \\ & & & & g_1^N & & & & \end{array}$$

such that $g_j^{i+1} = g_j^i \star g_{j+1}^i$ holds, whenever defined. For $N = 0$ we speak of the empty $\nabla_\star$-configuration. For $N = 1$, any $g \in G$ is considered as a $\nabla_\star$-configuration and for $N = 2$, a $\nabla_\star$-configuration is just an elementary $\star$-triangle.

When the size N needs explicit mentioning, we use $\nabla_\star(N)$. The set of $\nabla_\star(N)$-configurations is denoted by $C(N, \star)$.

It is quite obvious that a $\nabla_\star(N)$-configuration is already defined by its first row. Therefore there exist $|G|^N$ different $\nabla_\star(N)$-configurations.

Since $(G,\star)$ is a quasigroup, a $\nabla_\star$-configuration is also determined by $(g_1^j)_{j=1}^N$ as well as by $(g_{N-j+1}^j)_{j=1}^N$. In other words, a $\nabla_\star$-configuration is determined by its entries on the sides of the triangle. Thus there exists three bijective maps $p_t : G^N \to C(N,\star)$, where $p_t(g_1,\dots,g_N) = g_1\ g_2\ \dots\ g_N$, and $p_l : G^N \to C(N,\star)$, where

$$p_r(g_1,\dots,g_N) = \begin{array}{ccc} \cdots & & g_1 \\ & \iddots & \\ g_N & & \end{array}$$

and $p_l : G^N \to C(N,\star)$, where

$$p_l(g_1,\dots,g_N) = \begin{array}{ccc} g_N & & \cdots \\ & \ddots & \\ & & g_1 \end{array},$$

which represent the parametrization of a $\nabla_\star(N)$-configuration by its top row, its right side, and its left side, respectively.

If we want to specify the entries of, e.g., the top row, we write

$$\nabla_\star = p_t(g_1,\dots,g_N)$$

or any other of the two remaining sides of the configuration.

If a $\nabla_\star(N)$-configuration has size $N \geq 3$, then we define the *reduced* configuration $\nabla_\star^{red}(N-3)$, by removing the three sides from $\nabla_\star(N)$-configuration.

Since a $\nabla_\star$-configuration is an equilateral triangle decorated with elements of G we can apply the symmetries of the triangle to the configuration. The symmetry group of the triangle is denoted as $\mathcal{D}$ and consists of the reflections and rotations, where the rotations are denoted by r (rotation clockwise) and l (counterclockwise).

The subgroup of rotations is denoted by $\mathcal{R}$.

The symmetry group $\mathcal{D}$ acts in an obvious way on the triangular array $\nabla_\star$. For $s \in \mathcal{D}$ we denote by $s\nabla_\star$ the transform of $\nabla_\star$ under the operation of s.

Of crucial importance for all the following is the next lemma which provides us with a recursive procedure to extend a $\nabla_\star$-configuration of size N to a $\nabla_\star$-configuration of size $N+1$.

Lemma 3.1. *Let $(G,\star)$ be a quasigroup and let $\nabla_\star = p_t(g_1,\dots,g_N)$ be a $\nabla_\star(N)$-configuration. For every $x \in G$ and $j = 1,\dots,N+1$ there exist a unique configuration $\nabla_\star^{x,j} = p_t(h_1,h_2,\dots,h_{N+1})$ such that $h_j = x$ and the second row of $\nabla_\star^{x,j}$ is equal to the first row of ∇.*

Proof. Let $x \in G$ be fixed and let us assume that $h_j = x$. We have to find the values of h_{j-1} and h_{j+1}, i.e., locally we have

$$\begin{array}{ccccc} h_{j-1} & & x & & h_{j+1} \\ & g_{j-1} & & g_j & \end{array}$$

which gives that h_{j-1} is the unique solution y of the equation $y * x = g_{j-1}$ and h_{j+1} is the unique solution y of the equation $x * y = g_j$. Therefore $h_{j-1} = x \star_l g_{j-1}$ and $h_{j+1} = g_j \star_r x$. Continuing this procedure to the left and to the right we obtain the desired result. □

Note that for semisymmetric quasigroups we have that $h_{j-1} = x \star g_{j-1}$ and $h_{j+1} = g_j \star x$.

If the quasigroup $(G, \star)$ is semisymmetric, i.e., $\star$ is rotationally symmetric, then $\mathcal{R}$ operates on the set $C(N, \star)$ since $\star = \star_l$. We call a $\nabla_\star$-configuration of a semisymmetric quasigroup *rotationally symmetric* if $\nabla_\star = r\nabla_\star$.

Note that the condition $r\nabla_\star = \nabla_\star$ yields $l\nabla_\star = \nabla_\star$, and vice versa.

Due to the semisymmetry of the quasigroup, we can state that a $\nabla_\star$-configuration is rotationally invariant if and only if two sides of the triangle are equal.

As a next step, we present a method to construct in a recursive manner rotationally symmetric $\nabla_\star$-configurations of size $N+3$ from rotationally symmetric $\nabla_\star$-configurations of size N. In fact, any rotationally symmetric $\nabla_\star(N)$-configuration is an extension of a rotationally symmetric configuration of size $N-3$.

Theorem 3.2. *Let $(G, \star)$ be a semisymmetric quasigroup. If $\nabla_\star = p_t(g_1, g_2, \dots g_N)$ is a rotationally symmetric configuration of size N, then for any $x \in G$ the $\nabla_\star$-configuration defined by*

$$\nabla_\star^x(N+3) = p_t(\eta, x, \xi_1, \xi_2, \dots, \xi_N, \eta),$$

where $\xi_0 = x$ and $\xi_{j+1} = g_{j+1} \star \xi_j$, $j = 0, \dots, N-1$, and $\eta = x \star \xi_N$, is rotationally symmetric and of size $N+3$.

Moreover, any rotationally symmetric $\nabla_\star$-configuration of size $N+3$ is of the above form.

Proof. For $x \in G$ fixed and for ξ_j be defined as above, we have by Lemma 3.1 that $(x, \xi_1, \dots, \xi_N)$ is the unique extension of the configuration $(g_1, \dots, g_N)$.

We now extend the row $(x, \xi_1, \dots, \xi_N)$ to the left and to the right by η, i.e., we consider the new top row given by $(\eta, x, \xi_1, \dots, \xi_N, \eta)$. The thus defined $\nabla_\star^x(N+3)$-configuration starts with

$$\begin{array}{ccccccc} \eta & & x & & \dots & \xi_N & \eta \\ & \xi_N & & g_1 & \dots & g_N & \\ & & & & & & x \end{array}$$

where the entries on the left and right of the second row are a consequence of the rotational symmetry of $\star$.

Due to the rotational symmetry of $\nabla_\star$ we have $p_t^{-1}(\nabla_\star) = p_r^{-1}(\nabla_\star)$ and due to the semisymmetry of $(G, \star)$, we have that the last two rows of $\nabla_\star(N+3)$ are given by

$$\begin{array}{ccc} x & & \xi_N \\ & \eta & \end{array}.$$

Therefore $p_t^{-1}(\nabla_\star(N+3)) = p_r^{-1}(\nabla_\star(N+3))$, which proves the assertion.

It remains to show that any rotationally symmetric configuration is obtained in this way. If $\nabla_\star = p_t(g_1, g_2, \ldots, g_{N+3})$ is a rotationally symmetric configuration of size $N+3$, then $\nabla_\star^{red}$ is a rotationally symmetric configuration of size N. Our above considerations show that the value of g_2 and $\nabla_\star^{red}$ completely determines the configuration $\nabla_\star$. Therefore any rotationally symmetric configuration is obtained by extending a rotationally symmetric configuration of size N. □

The previous theorem shows that all rotationally symmetric $\nabla_\star$-configurations of any size can be constructed from rotationally symmetric $\nabla_\star$-configurations of size 0,1,2 by increasing the size in steps of 3. The following lemma determines the rotationally symmetric $\nabla_\star$-configurations of size 0, 1, 2.

Lemma 3.3. *Let $(G, \star)$ be a semisymmetric quasigroup. Then the number of rotationally symmetric $\nabla_\star$-configurations of size $0, 1, 2$ are given by $|\nabla_\star(0)| = 1$, $|\nabla_\star(1)| = |G|$, $|\nabla_\star(2)| = |\mathrm{Fix}(\star)|$, where $\mathrm{Fix}(\star) = \{x \mid x \star x = x,\ x \in G\}$.*

Proof. The results follow from the following observations:
- the only $\nabla_\star$-configuration of size 0 is the "empty" configuration
- any $g \in G$ defines a $\nabla_\star(1)$-configuration which is rotationally symmetric
- if $x \in \mathrm{Fix}(\star)$, then $\nabla_\star = p_t(x, x)$ defines a rotationally symmetric $\nabla_\star$-configuration; and any rotationally symmetric configuration of size 2 is of this form. □

The next theorem tells us how many rotationally symmetric $\nabla_\star(N)$-configurations there are.

Theorem 3.4. *Let $(G, \star)$ be a semisymmetric quasigroup. The number $\mathrm{conf}_{\mathcal{R}}(N)$ of rotationally symmetric $\nabla_\star$-configurations of size N is given by*

$$\mathrm{conf}_{\mathcal{R}}(N) = \begin{cases} |G|^K & \text{if } N = 3K \\ |G|^{K+1} & \text{if } N = 3K+1 \\ |\mathrm{Fix}(\star)||G|^K & \text{if } N = 3K+2. \end{cases}$$

Proof. By Theorem 3.2, we have $\mathrm{conf}_{\mathcal{R}}(N+3) = \mathrm{conf}_{\mathcal{R}}(N)|G|$. Using the initial values $\mathrm{conf}_{\mathcal{R}}(0) = 1$, $\mathrm{conf}_{\mathcal{R}}(1) = |G|$, and $\mathrm{conf}_{\mathcal{R}}(2) = |\mathrm{Fix}(\star)|$ from Lemma 3.3, we obtain the above formulas. □

This generalizes the result stated in [2]. Note that the existence of rotationally symmetric $\nabla_\star$-configurations of size $N = 3K$, $N = 3K+1$, respectively, is guaranteed by the above theorem without any further assumptions on the semisymmetric quasigroup. However, the existence of rotationally symmetric $\nabla_\star$-configurations of size $3K+2$ requires the existence of elements $x \in G$ such that $x \star x = x$.

Since the group of rotations $\mathcal{R} = \{\mathrm{id}, r, l\}$ operates on the $\nabla_\star$-configurations, we can apply the Lemma of Cauchy–Frobenius, see, e.g., [12] p. 53 (sometimes

referred to as Burnside's Theorem, see, e.g., [6]) to count the number of different orbits of $\nabla_\star$-configurations under these $\mathcal{R}$-operations.

Lemma 3.5. *Let $\mathcal{R}$ operate on the set of $\nabla_\star$-configurations of size N and let $(G,\star)$ be semisymmetric. The number of different $\mathcal{R}$-equivalence classes is given by*

$$\frac{1}{3}\left(|G|^N + 2\,\mathrm{conf}_{\mathcal{R}}(N)\right). \tag{5}$$

Proof. According to the Cauchy–Frobenius Lemma, if Γ is a group acting on a finite set X, then the number of Γ-equivalence classes of X, i.e., the number of orbits of X under the action of Γ, is given by $\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}|\mathrm{Fix}(\gamma)|$, where $\mathrm{Fix}(\gamma)$ is the set of elements in X which are fixed under γ. With $\Gamma = \mathcal{R} = \{\mathrm{id}, r, l\}$, and $X = C(N,\star)$, this becomes $\frac{1}{|\mathcal{R}|}(|\mathrm{Fix}(\mathrm{id})| + |\mathrm{Fix}(r)| + |\mathrm{Fix}(l)|)$.

As all elements of $C(N,\star)$ remain fixed under id, and their number is $|G|^N$, and the elements which are fixed by r as well as by l are the rotationally symmetric $\nabla_\star$-configurations whose number is $\mathrm{conf}_{\mathcal{R}}(N)$ as given by Theorem 3.4, this last expression yields (5). □

As a consequence of Lemma 3.5 we have a partial solution to the existence of rotationally symmetric $\nabla_\star$-configurations of size $3K+2$.

Theorem 3.6. *Let $(G,\star)$ be a semisymmetric quasigroup. If $|G| \not\equiv 0 \bmod 3$, then there exist rotationally symmetric $\nabla_\star$-configurations of size $3K+2$.*

Proof. By Theorem 3.4, it suffices to consider the case $N = 2$. By Lemma 3.5, the number of $\mathcal{R}$-equivalence classes of $\nabla_\star$-configurations of size 2 is given by $\frac{1}{3}\left(|G|^2 + 2\mathrm{conf}_{\mathcal{R}}(2)\right)$. Since $|G|$ is not divisible by 3 it follows that $\mathrm{conf}_{\mathcal{R}}(2)$ is different from 0 and therefore $\mathrm{Fix}(\star) \neq \emptyset$. □

Lemma 3.5 provides also a criterion for a a quasigroup to be non-semisymmetric.

Corollary 3.7. *Let $(G,\star)$ be a quasigroup.*

1. *If $|G| \equiv 0 \bmod 3$ and $\mathrm{Fix}(\star) \not\equiv 0 \bmod 3$, then $(G,\star)$ is not semisymmetric.*
2. *If $|G| \not\equiv 0 \bmod 3$ and $\mathrm{Fix}(\star) \equiv 0 \bmod 3$, then $(G,\star)$ is not semisymmetric.*

Proof. We prove the first assertion. If $(G,\star)$ is a semisymmetric quasigroup, then $\mathcal{R}$ operates on the $\nabla_\star$-configurations of size 2 and therefore $\frac{1}{3}(|G|^2 + 2\mathrm{conf}_{\mathcal{R}}(2))$ is a natural number. This is a contradiction if $(G,\star)$ satisfies the given requirements. □

In particular, if $(G,\star)$ is a semisymmetric quasigroup such that $|G| \equiv 0 \bmod 3$, then $\mathrm{Fix}(\star)$ has to be a multiple of 3. That shows that the semisymmetric quasigroup $(\mathbb{Z}_{3m}, x\star y = -x-y)$, where $m \geq 1$, has at least 3 elements g such

that $g \star g = g$ ($0 \in \mathrm{Fix}(\star)$ and therefore $|\mathrm{Fix}(\star)| \geq 3$).

Lemma 3.5 partitions the set of $\nabla_\star$-configurations into classes of configurations that are equivalent modulo operations in $\mathcal{R}$. However, one could additionally consider $\nabla_\star(N)$-configurations which are also equivalent under certain symbol permutations. This will further reduce the number of equivalence classes. We will deal with this in the next sections.

The effect of automorphisms

As in the previous section, let $(G, \star)$ denote a semisymmetric quasigroup. The group $\mathrm{Aut}(G, \star)$ operates in an obvious way on the set of $\nabla_\star$-configurations. Let $\sigma \in \mathrm{Aut}(g, \star)$ and let $\nabla_\star = p_t(g_1, \ldots, g_N)$ be a configuration with local rule $\star$, then $\sigma\nabla_\star = p_t(\sigma(g_1), \ldots, \sigma(g_N))$ is also a configuration with local rule $\star$, since σ is an automorphism of $(G, \star)$. I.e., $\sigma\nabla_\star$ is also a $\nabla_\star$-configuration.

Two $\nabla_\star$-configurations $\nabla_{\star,1}, \nabla_{\star,2}$ are $\mathrm{Aut}(G, \star)$-equivalent if there exists a $\sigma \in \mathrm{Aut}(G, \star)$ such that $\sigma\nabla_{\star,1} = \nabla_{\star,2}$.

Lemma 3.8. *Let $(G, \star)$ be a quasigroup. Then the number $\mathrm{conf}_{\mathrm{Aut}(G,\star)}(N)$ of $\mathrm{Aut}(G, \star)$-equivalence classes of $\nabla_\star$-configurations of size N is given by*

$$\mathrm{conf}_{\mathrm{Aut}(G,\star)}(N) = \frac{1}{|\mathrm{Aut}(G, \star)|} \sum_{\sigma \in \mathrm{Aut}(G,\star)} |\mathrm{Fix}(\sigma)|^N,$$

where $\mathrm{Fix}(\sigma) = \{g \in G \mid \sigma(g) = g\}$.

Proof. Since any $\nabla_\star$-configuration is determined by its first row, the action of $\mathrm{Aut}(G, \star)$ on the $\nabla_\star$-configurations is modeled by the action of $\mathrm{Aut}(G, \star)$ on G^N. The assertion then follows from the Lemma of Cauchy–Frobenius, [12], p. 53. □

Examples. 1. For $(\mathbb{Z}_3, \star)$, where $x \star y = 2(x + y)$, the automorphism group is S_3, the symmetric group over 3 elements. We have seen before that $\mathrm{Aut}(\mathbb{Z}_3, \star) = \{ax + b \mid a \in \mathbb{Z}_3^*, b \in \mathbb{Z}_3\}$.Therefore, the number of different $\mathrm{Aut}(\mathbb{Z}_3, \star)$-equivalence classes is $\frac{1}{6}(3^N + 3)$.

2. For $(\mathbb{F}_4, \star_0)$, where $x \star_0 y = x + y$, the automorphism group is given by$\mathrm{Aut}(\mathbb{F}_4, \star_0) = \{\sigma \mid \sigma(0) = 0\}$. Therefore the number of different $\nabla_\star$-configurations is given by $\frac{1}{6}(4^N + 3 \cdot 2^N + 2)$.

The proof of Theorem 3.2 in combination with Theorem 3.4 allows us to compute the number of $\mathrm{Aut}(G, \star)$-equivalence classes restricted to the rotationally symmetric $\nabla_\star$-configurations.

Corollary 3.9. *Let $(G,\star)$ be a semisymmetric quasigroup. The number of $\mathrm{Aut}(G,\star)$-equivalence classes of rotationally symmetric $\nabla_\star$-configurations is given by*

1. *For $N=3K$:* $\frac{1}{|\mathrm{Aut}(G,\star)|}\sum_\sigma |\mathrm{Fix}(\sigma)|^K$.
2. *For $N=3K+1$:* $\frac{1}{|\mathrm{Aut}(G,\star)|}\sum_\sigma |\mathrm{Fix}(\sigma)|^{K+1}$.
3. *For $N=3K+2$:* $\frac{1}{|\mathrm{Aut}(G,\star)|}\sum_\sigma |\mathrm{Fix}(\sigma,\star)||\mathrm{Fix}(\sigma)|^K$,

where $\mathrm{Fix}(\sigma,\star)=\{x\in\mathrm{Fix}(\sigma)\mid x\star x=x\}$.

As a next step we consider $\nabla_\star$-configurations which are fixed under the combined action of $\mathcal{R}$ and $\mathrm{Aut}(G,\star)$. We therefore consider the group

$$\mathcal{S}=\mathrm{Aut}(G,\star)\times\mathcal{R}$$

which operates on the $\nabla_\star$-configurations in the following way

$$(\sigma,s)\nabla_\star=\sigma(s(\nabla_\star)).$$

Note that $\sigma\circ s=s\circ\sigma$ holds for all $\nabla_\star$-configurations and all $\sigma\in\mathrm{Aut}(G,\star)$, $s\in\mathcal{R}$. Therefore the operation of $\mathcal{S}$ is a group operation on the set of $\nabla_\star$-configurations. A $\nabla_\star$-configuration is called (σ,s)-invariant if $(\sigma,s)\nabla_\star=\nabla_\star$.

In order to count the $\mathcal{S}$-equivalence classes we have to determine the (σ,s)-invariant $\nabla_\star$-configurations where neither σ nor s is the identity.

If $\nabla_\star$ is a (σ,s)-invariant configuration, then the reduced configuration $\nabla_\star^{red}$ is also (σ,s)-invariant. Like for the rotationally symmetric configurations, we can construct (σ,s)-invariant configurations in a recursive way. To do so, we start with a list of small (σ,l)-invariant $\nabla_\star$-configurations. The list provides the initialization for the recursive counting of all (σ,l)-invariant $\nabla_\star$-configurations.

1. All (σ,l)-invariant $\nabla_\star$-configurations of size 2 are of the form

$$\begin{array}{cc} a & \sigma^2(a) \\ & \sigma(a) \end{array}$$

where $a\in D_l(\sigma,\star)=\{g\in\mathrm{Fix}(\sigma^3)\mid g\star\sigma^2(g)=\sigma(g)\}$.

2. Let $a\in\mathrm{Fix}(\sigma^3)$. Then every (σ,l)-invariant $\nabla_\star$-configuration of size 3 is of the form

$$\begin{array}{ccc} a\star\sigma(a) & a & \sigma^2(a)\star a \\ & \sigma(a) \quad \sigma^2(a) & \\ & \sigma(a)\star\sigma^2(a) & \end{array}$$

3. Let $a\in\mathrm{Fix}(\sigma)$ and let $b\in\mathrm{Fix}(\sigma^3)$. Then every (σ,l)-invariant $\nabla_\star$-configuration of size 4 has a first row of the form

$$b\star\sigma(a\star b)\quad b\quad a\star b\quad \sigma^2(b)\star(a\star b)$$

If in the above list the role of σ and σ^2 is interchanged, i.e., σ is replaced by σ^2 and vice versa, we obtain the (σ,r)-invariant $\nabla_\star$-configurations. Note that the

condition for the case $N = 2$ then changes to $a \in D_r(\sigma, \star) = \{g \in \mathrm{Fix}(\sigma^3) \mid g \star \sigma(g) = \sigma^2(g)\}$. Note further that all entries of $\nabla_\star$ except possibly one (the center of the triangle) are elements of $\mathrm{Fix}(\sigma^3)$.

Like for the rotationally symmetric $\nabla_\star$-configurations, we can give a simple criterion for (σ, l)-invariance. A $\nabla_\star$-configuration is (σ, l)-invariant if and only if $p_t^{-1}(\nabla_\star) = \sigma(p_r^{-1}(\nabla_\star))$, i.e, the top row is the σ-image of the right side. A similar criterion holds for (σ, r)-invariant $\nabla_\star$-configurations.

Theorem 3.10. *Let $\nabla_\star = p_t(g_1, \ldots, g_N)$ be a (σ, l)-invariant $\nabla_\star$-configuration. If $x \in \mathrm{Fix}(\sigma^3)$, then the $\nabla_\star$-configuration $\nabla_\star^x(N+3)$ given by*

$$\nabla_\star^x(N+3) = p_t(\sigma(\eta), x, \xi_1, \ldots, \xi_N, \eta),$$

where $\xi_0 = x$, $\xi_{j+1} = g_{j+1} \star \xi_j$, $j = 0, \ldots, N-1$ and $\eta = \sigma^2(x) \star \xi_N$, is (σ, l)-invariant.

Proof. By our construction, we have that $(\sigma(\eta), x, \xi_1, \ldots, \xi_N, \eta)$ defines a $\nabla_\star$-configuration which has $(\sigma(\xi_N), g_1, g_2, \ldots, g_N, \sigma^2(x))$ as a second row. Taking into account that $g_N = \sigma^2(g_1)$ and the right side of the configuration $\nabla_\star$ is the σ^2 image of the top row of $\nabla_\star$, we obtain the assertion. □

Of course, a similar result holds for (σ, r)-invariant configurations. The recursive construction of (σ, l)-invariant $\nabla_\star$-configurations implies the next theorem.

Theorem 3.11. *If $(\sigma, l) \in \mathcal{S}$, then the number $\mathrm{conf}_{(\sigma,l)}(N)$ of (σ, l)-invariant $\nabla_\star$-configurations of size N is given by*

$$\mathrm{conf}_{(\sigma,l)}(N) = \begin{cases} |\mathrm{Fix}(\sigma^3)|^K & \text{if } N = 3K \\ |\mathrm{Fix}(\sigma)||\mathrm{Fix}(\sigma^3)|^K & \text{if } N = 3K+1 \\ |D_l(\sigma, \star)||\mathrm{Fix}(\sigma^3)|^K & \text{if } N = 3K+2 \end{cases}$$

where $D_l(\sigma, \star) = \{g \in \mathrm{Fix}(\sigma^3) \mid g \star \sigma^2(g) = \sigma(g)\}$.

If we replace l by r in the above theorem, we obtain the same assertion for (σ, r)-invariant configurations. Note that the cardinality of $D_l(\sigma, \star)$ and $D_r(\sigma, \star) = \{g \in \mathrm{Fix}(\sigma^3) \mid g * \sigma(g) = \sigma^2(g)\}$ are not necessarily the same, see Example 2 below.

Corollary 3.12. *If $(G, \star)$ is a totally symmetric quasigroup, then $D_l(\sigma, \star) = D_r(\sigma, \star)$ for all $\sigma \in \mathrm{Aut}(G, \star)$.*

The above theorem also applies for the case $\sigma = \mathrm{id}$ and we have Theorem 3.4 as a special case.

An other application is the following corollary.

Corollary 3.13. *The number of different $\mathcal{S}$-equivalence classes of $\nabla_\star$-configurations of size N is given by*

1. *For $N = 3K$:*

$$\frac{1}{3|\mathrm{Aut}(G,\star)|}\left[\sum_\sigma |\mathrm{Fix}(\sigma)|^N + 2\sum_\sigma |\mathrm{Fix}(\sigma^3)|^K\right].$$

2. *For $N = 3K+1$:*

$$\frac{1}{3|\mathrm{Aut}(G,\star)|}\left[\sum_\sigma |\mathrm{Fix}(\sigma)|^N + 2\sum_\sigma |\mathrm{Fix}(\sigma)||\mathrm{Fix}(\sigma^3)|^K\right].$$

3. *For $N = 3K+2$:*

$$\frac{1}{3|\mathrm{Aut}(G,\star)|}\left[\sum_\sigma |\mathrm{Fix}(\sigma)|^N + \sum_\sigma (|D_l(\sigma,\star)| + |D_r(\sigma,\star)|)|\mathrm{Fix}(\sigma^3)|^K\right].$$

The proof is a consequence of the above theorem and the Cauchy–Frobenius Lemma.

Examples. 1. Let the quasigroup be given as $(\mathbb{Z}_3,\star)$, where $x\star y = 2(x+y)$. The automorphism group is $\mathrm{Aut}(\mathbb{Z}_3,\star) = \{\sigma(x) = ax+b \mid a\in\mathbb{Z}_3^*, b\in\mathbb{Z}_3\}$. Moreover, $(\mathbb{Z}_3,\star)$ is totally symmetric and thus we have $D(\sigma,\star) = D_l(\sigma,\star) = D_r(\sigma,\star)$. Using the above formulae and the following table

σ	$\mathrm{Fix}(\sigma)$	$\mathrm{Fix}(\sigma^3)$	$D(\sigma,\star)$
$\sigma(x) = x$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
$\sigma(x) = x+1$	$\emptyset$	$\{0,1,2\}$	$\{0,1,2\}$
$\sigma(x) = x+2$	$\emptyset$	$\{0,1,2\}$	$\{0,1,2\}$
$\sigma(x) = 2x$	$\{0\}$	$\{0\}$	$\{0\}$
$\sigma(x) = 2x+1$	$\{2\}$	$\{2\}$	$\{2\}$
$\sigma(x) = 2x+2$	$\{1\}$	$\{1\}$	$\{1\}$

we obtain as number of $\mathcal{S}$-equivalence classes $\frac{1}{18}(3^N + 2\cdot 3^{K+1} + 9)$ for $N = 3K$. For $N = 3K+1$ one counts $\frac{1}{18}(3^N + 2\cdot 3^{K+1} + 9)$ $\mathcal{S}$-equivalence classes. Finally, for $N = 3K+2$ we have $\frac{1}{18}(3^N + 2\cdot 3^{K+2} + 9)$ $\mathcal{S}$-equivalence classes.

2. The quasigroup $(\mathbb{F}_4,\star)$, where $x\star_a y = \zeta x + (1+\zeta)y$, is semisymmetric but not totally symmetric. As we have already seen, the automorphism group is given by $\mathrm{Aut}(\mathbb{F}_4,\star_a) = \{\sigma(x) = ax+b \mid a\in\mathbb{F}_4^*, b\in\mathbb{F}_4\}$. That enables us to compute

the following table

σ	$\mathrm{Fix}(\sigma)$	$\mathrm{Fix}(\sigma^3)$	$D_l(\sigma,\star)$	$D_r(\sigma,\star)$
$\sigma(x)=x$	$\mathbb{F}_4$	$\mathbb{F}_4$	$\mathbb{F}_4$	$\mathbb{F}_4$
$\sigma(x)=x+1$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\sigma(x)=x+\zeta$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\sigma(x)=x+1+\zeta$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\sigma(x)=\zeta x$	$\{0\}$	$\mathbb{F}_4$	$\{0\}$	$\mathbb{F}_4$
$\sigma(x)=\zeta x+1$	$\{\zeta\}$	$\mathbb{F}_4$	$\{\zeta\}$	$\mathbb{F}_4$
$\sigma(x)=\zeta x+\zeta$	$\{1+\zeta\}$	$\mathbb{F}_4$	$\{1+\zeta\}$	$\mathbb{F}_4$
$\sigma(x)=\zeta x+\zeta+1$	$\{1\}$	$\mathbb{F}_4$	$\{1\}$	$\mathbb{F}_4$
$\sigma(x)=(1+\zeta)x$	$\{0\}$	$\mathbb{F}_4$	$\mathbb{F}_4$	$\{0\}$
$\sigma(x)=(1+\zeta)x+1$	$\{1+\zeta\}$	$\mathbb{F}_4$	$\mathbb{F}_4$	$\{1+\zeta\}$
$\sigma(x)=(1+\zeta)x+\zeta$	$\{1\}$	$\mathbb{F}_4$	$\mathbb{F}_4$	$\{1\}$
$\sigma(x)=(1+\zeta)x+1+\zeta$	$\{\zeta\}$	$\mathbb{F}_4$	$\mathbb{F}_4$	$\{\zeta\}$

Note that $D_l(\sigma,\star)\neq D_r(\sigma,\star)$ for certain σ. Using the above data we obtain $\frac{1}{36}(4^N+18\cdot 4^K+8)$ for the number of $\mathcal{S}$-equivalence classes of size $N=3K$. For $N=3K+1$, we count $\frac{1}{36}(4^N+6\cdot 4^{K+1}+8)$ different equivalence classes, and finally for $N=3K+2$ we have $\frac{1}{36}(4^N+3\cdot 4^{K+2}+8)$ different equivalence classes.

We end this section with a list of representatives of $\mathcal{S}$-equivalence classes of $\nabla_{\star_a}$-configurations in $(\mathbb{F}_4,\star_a)$, for $N=1,2,3,4,5$ (see Fig. 3).

Totally symmetric configurations

If a $\nabla_\star$-configuration is obtained from a totally symmetric quasigroup, it is possible, as mentioned before, that the configuration itself is totally symmetric; i.e., is invariant under all operations in $\mathcal{D}$, the symmetry group of an equilateral triangle.

The group $\mathcal{D}$ operates in an obvious manner on the set $C(N,\star)$. To ensure that $s\nabla_\star$ is also a $\nabla_\star$-configuration for all $s\in\mathcal{D}$, we have to assume that $(G,\star)$ is a totally symmetric quasigroup (see Definition 1.5).

We begin the study of totally symmetric $\nabla_\star$-configurations with a characterization of $\nabla_\star$-configurations which are invariant under reflections about the vertical axis. It is plain that a $\nabla_\star$-configuration is invariant under the reflection about the vertical axis if and only if the top row is a palindrome. Moreover, if any of the other sides is a palindrome, then the $\nabla_\star$-configuration is invariant under reflection corresponding to that side. That gives

$$\frac{1}{6}\left(|G|^N+3|G|^{\lceil\frac{N}{2}\rceil}+2\,\mathrm{conf}_{\mathcal{R}}(N)\right).$$

as the number of $\mathcal{D}$-equivalence classes.

Like for the rotationally symmetric $\nabla_\star$-configurations, we want to count the number of totally symmetric configurations of size N. Again we try to obtain

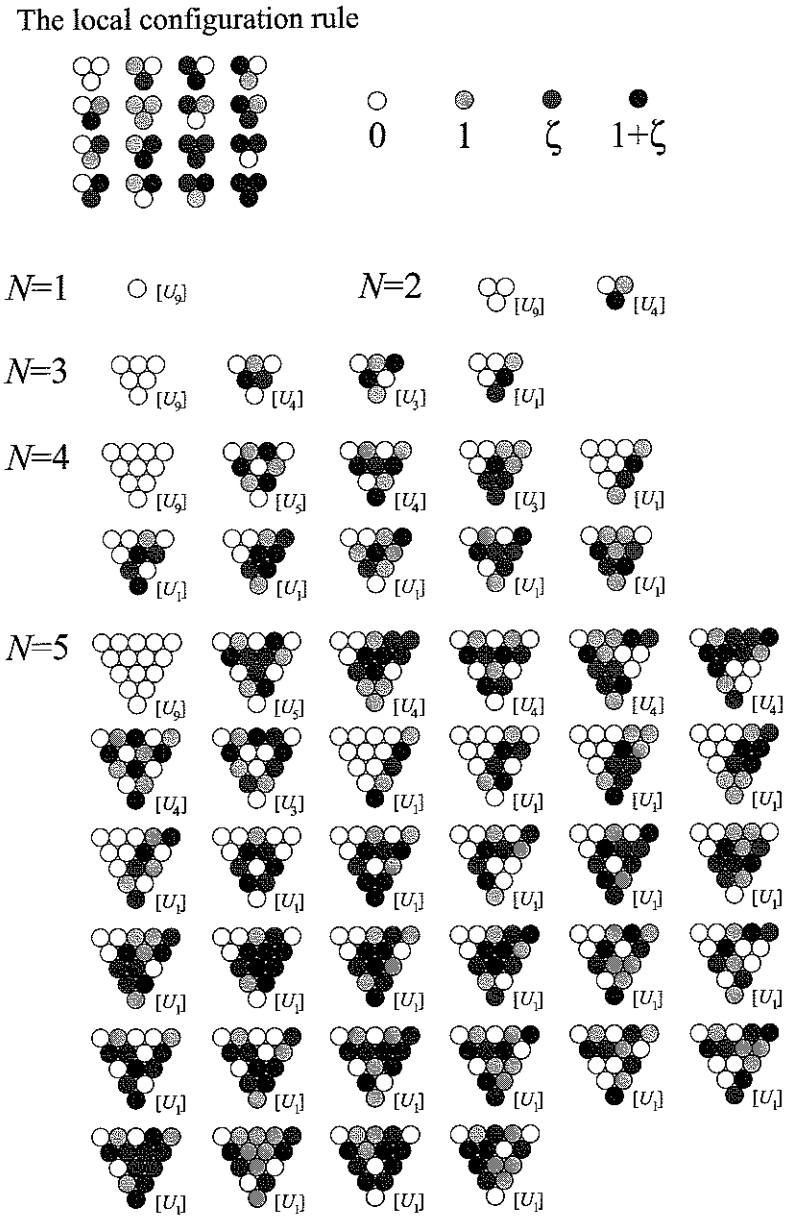


Figure 3. The different $\nabla_\star$-configurations (modulo $\mathcal{S} = \mathrm{Aut}(\mathbb{F}_4, \star) \times \mathcal{R}$) of size $N = 1, 2, 3, 4, 5$ satisfying the local rule $x \star y = \zeta x + (1 + \zeta)y$ in $\mathbb{F}_4$. On top: the elementary $\star$-triangles (local rule configurations) and the greycode for the cell-states $0, 1, \zeta, 1 + \zeta \in \mathbb{F}_4$. The $[U_j]$ indicate the symmetry class to which the related configuration belongs (see final section, Example 1).

results by extending a totally symmetric configuration of size N to a totally symmetric configuration of size $N+3$. However, as the next lemma shows, things become a bit more complicated.

Lemma 3.14. *Let $(G,\star)$ be a totally symmetric quasigroup and let $\mathrm{conf}_{\mathcal{D}}(N)$ denote the number of totally symmetric $\nabla_\star$-configurations of size $N \geq 4$.*

If N is an even number, i.e., $N = 2K$, then

$$\mathrm{conf}_{\mathcal{D}}(N+3) = \mathrm{conf}_{\mathcal{D}}(N)|G|. \tag{6}$$

If N is an odd number, i.e., $N = 2K+1$, then

$$\mathrm{conf}_{\mathcal{D}}(N+3) = \mathrm{conf}_{\mathcal{D}}(N-3)|G|. \tag{7}$$

Proof. Let $N = 2K$ and let $\nabla_\star = p_t(g_1,\ldots,g_k,g_k,\ldots,g_1)$ be a totally symmetric $\nabla_\star$-configuration. As in the proof of Theorem 3.2, we extend the configuration $\nabla_\star$ to a totally symmetric configuration of size $N+3$. To this end we consider

$$\begin{array}{ccccccccccc} \eta & & h_1 & \ldots & h_K & & h_{K+1} & & h_K & \ldots & h_1 & & \eta \\ & h_1 & & g_1 & \ldots & g_K & & g_K & \ldots & g_1 & & h_1 & \end{array}$$

We observe that due to the total symmetry of $(G,\star)$ the value of h_{K+1} completely determines the top row except for the value of η. If we set $\eta = h_1 \star h_1$, the top row determines a totally symmetric $\nabla_\star$-configuration. Thus, any totally symmetric $\nabla_\star$-configuration of size $N = 2K$ and a choice of h_{K+1} yield a totally symmetric configuration of size $N+3$. On the other hand, if $\nabla_\star$ is a totally symmetric configuration of size $N+3$, then the above considerations imply that it is determined by the middle value h_{K+1} of the top row and by $\nabla_\star^{red}$. Therefore Equation (6) follows, taking into account that h_{K+1} can take $|G|$ different values.

Let $N = 2K+1$, then the top row of $\nabla_\star$ is of the form

$$g_1 \cdots g_K \; g_{K+1} \; g_K \cdots g_1$$

and our above considerations show that $\nabla_\star$ is determined by $\nabla_\star^{red}$ and g_{K+1}. The extension of $\nabla_\star$ yields a top row of the form

$$\eta \; h_1 \; \ldots \; h_{K+1} \; h_{K+1} \; \ldots \; h_1 \; \eta,$$

where $\eta = h_1 \star h_1$ and h_{K+1} is chosen such that $h_{K+1} \star h_{K+1} = g_{K+1}$. In general, it is not g_{K+1} that can be freely chosen in the expansion, as it may occur that there is no h_{K+1} such that $h_{K+1} \star h_{K+1} = g_{K+1}$. It is actually h_{K+1} that can be freely chosen, which in its turn determines g_{K+1}. But in this way, we grow the $\nabla_\star^{red}$-configuration of size $N-3$ to a $\nabla_\star$-configuration of size $N+3$. Also, any $\nabla_\star(N+3)$ where N=2K+1 is related to a $\nabla_\star(N-3)$ via the selection of one extra element h_{K+1}. Therefore, we obtain (7). □

The proof of the previous lemma actually shows how to construct larger $\nabla_\star$-configurations from smaller ones, as stated in

Corollary 3.15. *Totally symmetric $\nabla_\star(N)$-configurations can be expanded in size according to the following growth-sequences, where transitions along the right-oriented arrows use* (7) *and the downward-oriented arrows use* (6)*:*

$$\begin{array}{ccccccccc}
\nabla_\star(0) & \to & \nabla_\star(6) & \to & \nabla_\star(12) & \to & \nabla_\star(18) & \to & \nabla_\star(24) \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\nabla_\star(3) & & \nabla_\star(9) & & \nabla_\star(15) & & \nabla_\star(21) & & \nabla_\star(27) \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\nabla_\star(2) & \to & \nabla_\star(8) & \to & \nabla_\star(14) & \to & \nabla_\star(20) & \to & \nabla_\star(26) \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\nabla_\star(5) & & \nabla_\star(11) & & \nabla_\star(17) & & \nabla_\star(23) & & \nabla_\star(29) \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\nabla_\star(4) & \to & \nabla_\star(10) & \to & \nabla_\star(16) & \to & \nabla_\star(22) & \to & \nabla_\star(28) \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\nabla_\star(7) & & \nabla_\star(13) & & \nabla_\star(19) & & \nabla_\star(25) & & \nabla_\star(31) \cdots
\end{array}$$

$\nabla_\star(1)$ stands isolated. $\nabla_\star$-configurations of odd size can not generally be extended in an arbitrary way.

Thus the number of totally symmetric $\nabla_\star$-configurations of sizes 0,1,2, and 4 is given by $|\nabla_\star(0)| = 1$, $|\nabla_\star(1)| = |G|$, $|\nabla_\star(2)| = |\mathrm{Fix}(\star)|$, $|\nabla_\star(4)| = |G|$. The first three values are immediate consequences of Lemma 3.3, as for these sizes, there is no difference between rotationally symmetric and totally symmetric configurations. To prove the fourth assertion we consider a totally symmetric triangular array of size 4, i.e.,

$$\begin{array}{ccccccc}
y & & x & & x & & y \\
& x & & z & & x & \\
& & x & & x & & \\
& & & y & & &
\end{array}$$

Taking into consideration that $z = x \star x$ and $y = x \star x$ by the total symmetry of the quasigroup, we obtain that the complete $\nabla_\star(4)$-configuration is determined by x, and so $|\nabla_\star(4)| = |G|$.

We can now state the main result of this section

Theorem 3.16. *Let $(G, \star)$ be a totally symmetric quasigroup. The number $\mathrm{conf}_{\mathcal{D}}(N)$ of totally symmetric $\nabla_\star$-configurations is given by*

$$\mathrm{conf}_{\mathcal{D}}(N) = \begin{cases} |G|^K & \text{if } N = 6K \\ |G|^{K+1} & \text{if } N = 6K+1 \\ |\mathrm{Fix}(\star)||G|^K & \text{if } N = 6K+2 \\ |G|^{K+1} & \text{if } N = 6K+3 \\ |G|^{K+1} & \text{if } N = 6K+4 \\ |\mathrm{Fix}(\star)||G|^{K+1} & \text{if } N = 6K+5. \end{cases}$$

Proof. The assertion follows from the growth-scheme in Corollary 3.15, starting with the initial values for the small sizes: $\mathrm{conf}_{\mathcal{D}}(0) = 1$, $\mathrm{conf}_{\mathcal{D}}(1) = |G|$, $\mathrm{conf}_{\mathcal{D}}(2) = |\mathrm{Fix}(\star)|$, $\mathrm{conf}_{\mathcal{D}}(4) = |G|$. □

Examples. The quasigroup $(\mathbb{Z}_4, \star)$, where $x \star y = 3x + 3y$, is totally symmetric and we have $\mathrm{Fix}(\star) = \{0\}$. Therefore Theorem 3.16 reads as

$$\mathrm{conf}_{\mathcal{D}}(N) = \begin{cases} 4^{\lfloor \frac{N}{6} \rfloor} & \text{if } N \text{ is even} \\ 4^{\lfloor \frac{N}{6} \rfloor + 1} & \text{if } N \text{ is odd.} \end{cases}$$

The same formula holds for the totally symmetric quasigroup $(\mathbb{F}_4, \star_0)$, where $x \star_0 y = x + y$, since $\mathrm{Fix}(\star_0) = \{0\}$. Thus it is possible that two non-isomorphic quasigroups have the same number of different totally symmetric configurations of any size N.

Like for the rotationally symmetric configurations, we will now study the $\nabla_\star$-configurations which are invariant under elements of the group

$$\mathcal{S}_c = \mathrm{Aut}(G, \star) \times \mathcal{D},$$

where $\mathrm{Aut}(G, \star)$ denotes the automorphism group of $(G, \star)$.

The main result is then

Theorem 3.17. *Let $(G, \star)$ be a totally symmetric quasigroup and let (σ, r) and $(\sigma, \alpha) \in \mathcal{S}_c$. The number of $\nabla_\star$-configurations $\mathrm{conf}_{\sigma \times \mathcal{D}}$ which are (σ, α)- and (σ, r)-invariant (and thus $\sigma \times \mathcal{D}$-invariant) is given by*

$$\mathrm{conf}_{\sigma \times \mathcal{D}}(N) = \begin{cases} |\mathrm{Fix}(\sigma)|^K & \text{if } N = 6K \\ |\mathrm{Fix}(\sigma)|^{K+1} & \text{if } N = 6K+1 \\ |\mathrm{Fix}(\star) \cap \mathrm{Fix}(\sigma)||\mathrm{Fix}(\sigma)|^K & \text{if } N = 6K+2 \\ |\mathrm{Fix}(\sigma)|^{K+1} & \text{if } N = 6K+3 \\ |\mathrm{Fix}(\sigma)|^{K+1} & \text{if } N = 6K+4 \\ |\mathrm{Fix}(\star) \cap \mathrm{Fix}(\sigma)||\mathrm{Fix}(\sigma)|^{K+1} & \text{if } N = 6K+5. \end{cases}$$

Proof. Like in the proof of Theorem 3.16, we apply Corollary 3.15, taking into account modified values for the cardinalities of the initial $\nabla_\star$-configurations. Indeed, any assignment of elements of G to a ∇-triangle in such a way that the assignment is (σ, α)-, where α is the reflection about the vertical axis, and (σ, r)-invariant requires that the entries g satisfy $g = \sigma^2(g)$ for the (σ, α)-invariance and $g = \sigma^3(g)$ for the (σ, r)-invariance. Therefore all entries have to be fixed points of σ. This is the reason why the element that determines an extension of $\nabla_\star(N)$ to $\nabla_\star(N+3)$ or $\nabla_\star(N+6)$, can only take $|\mathrm{Fix}(\sigma)|$ values (instead of $|G|$ in Theorem 3.16). Moreover, if $N \equiv 2 \bmod 3$ and $\nabla_\star(N)$ is (σ, α)- and (σ, r)-invariant, then we have the additional requirement that $g \star g = g$. □

Finally, again as a consequence of the Cauchy-Frobenius Lemma and of our previous considerations we obtain

Corollary 3.18. *Let $(G,\star)$ be a totally symmetric quasigroup. Then the number of $\mathcal{S}_c$-equivalence classes of $\nabla_\star(N)$-configurations is given by*

$$\frac{1}{6|\mathrm{Aut}(G,\star)|}\left(|G|^N + 2\sum_{\sigma}\mathrm{conf}_{(\sigma,l)}(N) + 3\sum_{\sigma}\mathrm{conf}_{(\sigma,\alpha)}(N)\right),$$

where α is the reflection about the vertical axis.

Note that due to the total symmetry of $(G,\star)$ we have that $\mathrm{conf}_{(\sigma,l)}(N) = \mathrm{conf}_{(\sigma,r)}(N)$, that is generally not true if $(G,\star)$ is only semisymmetric [(see Theorem 3.11 for the value of $\mathrm{conf}_{(\sigma,l)}(N)$]. Furthermore, we have $\mathrm{conf}_{(\sigma,\alpha)}(N) = \mathrm{conf}_{(\sigma,\beta)}(N) = \mathrm{conf}_{(\sigma,\gamma)}(N) = |\mathrm{Fix}(\sigma^2)|^{\lceil \frac{N}{2} \rceil}$.

4. Symmetry classes of $\nabla_\star$-orbits

In the previous sections, our attention went mainly to equivalence classes of $\nabla_\star$-configurations modulo $\mathcal{R}$ [or $\mathcal{D}$] and modulo $\mathcal{S} = \mathrm{Aut}(G,\star) \times \mathcal{R}$ [or $\mathcal{S}_c = \mathrm{Aut}(G,\star) \times \mathcal{D}$] for semisymmetric [totally symmetric] quasigroups $(G,\star)$. More in particular, we concentrated on rotationally symmetric and totally symmetric $\nabla_\star$-configurations.

In this section, we give a more detailed description of all symmetries in $\mathcal{S}$- (or $\mathcal{S}_c$-) equivalent $\nabla_\star$-configurations, i.e., in the orbit of the $\nabla_\star$-configuration under the group operation defined by either $\mathcal{S}$ or $\mathcal{S}_c$.

A way to study the richness of the symmetries of a $\nabla_\star$-configuration is to consider the stabilizer of the configuration. Indeed, the stabilizer consists of all elements of the group $\mathcal{S}$ [or $\mathcal{S}_c$] which fix the given configuration, and therefore define an invariance (or symmetry) of the configuration. The cardinality of the stabilizer is a measure for the number of symmetries of a given configuration, see, e.g., [3, 12]. Also, the stabilizer of a configuration is a subgroup of $\mathcal{S}$ [or $\mathcal{S}_c$], and the structure of the subgroup provides us with additional information on the configuration. In particular, it is possible to have two configurations such that their respective stabilizers have equal cardinality, but different structures as subgroups.

Moreover, the previous sections have shown that by carefully extending a configuration of size N to a configuration of size $N+3$ (or possibly of size $N+6$ when $\mathcal{S}_c$-operations are considered), the extended configuration inherits certain symmetries from the smaller configuration. In this section, we study the change of the symmetry properties of a configuration of size N if extended to a configuration of size $N+3$ (or $N+6$, respectively). We construct matrices with integer coefficients which describe precisely the possible changes in the type of symmetry.

In the first part, we study the problem from a fairly general point of view, after which we apply the results to ∇_*-configurations and their extensions.

The general case

Let Γ be a finite group which operates from the left on the finite sets X, Y, respectively. We denote the operation by γx and γy, respectively. Of course, that also defines an operation on $X \times Y$ which is denoted by $\gamma(x, y) = (\gamma x, \gamma y)$. In our context we will refer to the pair $(x, y) \in X \times Y$ as the extension of x by y.

If $x \in X$, then the set

$$\{\gamma \in \Gamma \mid \gamma x = x\}$$

is called the stabilizer of x (w.r.t. Γ) and is denoted by $\mathrm{Stab}_\Gamma(x)$. The stabilizer is a subgroup of Γ. We have the important property

$$\mathrm{Stab}_\Gamma(\gamma x) = \gamma \mathrm{Stab}_\Gamma(x)\gamma^{-1},$$

or, two Γ-equivalent elements $x, x' \in X$ (i.e., $x' = \gamma x$ for some $\gamma \in \Gamma$) have conjugate stabilizers.

The set of Γ-equivalence classes of X is denoted by X/Γ and an element of X/Γ is written as Γx, $x \in X$.

We denote the set of subgroups of Γ by $\mathcal{SUB}(\Gamma)$. $[\mathcal{SUB}](\Gamma)$ represents the set of conjugacy classes of subgroups; its elements are denoted by

$$[W] = \{\gamma W \gamma^{-1} \mid \gamma \in \Gamma\},$$

where W is a subgroup of Γ.

If we consider the operation of Γ on $X \times Y$, we have the obvious relation

$$\mathrm{Stab}_\Gamma((x, y)) = \mathrm{Stab}_\Gamma(x) \cap \mathrm{Stab}_\Gamma(y).$$

The main idea is the following. If $x \in X$ is fixed, then we study the map

$$\begin{aligned} \kappa_x : Y &\to [\mathcal{SUB}](\Gamma) \\ y &\mapsto [\mathrm{Stab}_\Gamma((x, y))] \end{aligned}$$

which, in a certain sense, indicates the change of the stabilizer of x when x is *extended* by y.

Lemma 4.1. *If $x \in X$ and $[W] \in [\mathcal{SUB}](\Gamma)$, then $\kappa_{\gamma x}^{-1}([W]) = \gamma \kappa_x^{-1}([W])$ holds for all $\gamma \in \Gamma$.*

Proof. We have $\kappa_{\gamma x}^{-1}([W]) = \{y \mid \mathrm{Stab}(\gamma x, y) \in [W]\} = \{y \mid \gamma \mathrm{Stab}(x, \gamma^{-1} y)\gamma^{-1} \in [W]\} = \{\gamma y' \mid \mathrm{Stab}(x, y') \in [W]\} = \gamma \kappa_x^{-1}([W])$. □

Thus, the number of elements in $\kappa_x^{-1}([W])$ depends only on the equivalence class Γx.

The proof of the following lemma is straightforward.

Lemma 4.2. *Two extensions (x, y_1), (x, y_2) are Γ-equivalent if and only if there exists an $h \in \mathrm{Stab}_\Gamma(x)$ such that $h(x, y_1) = (x, y_2)$.*

If $\kappa_x^{-1}([W]) \neq \emptyset$, the set $\{(x, y) \mid y \in \kappa_x^{-1}([W])\}$ contains all extensions of x such that the stabilizer of the extension belongs to $[W]$. Taking into account Lemma 4.2, we can then state

Corollary 4.3. *The number of distinct (w.r.t. Γ) extensions of x with stabilizer in $[W]$ is given by $\left|\kappa_x^{-1}([W])/\mathrm{Stab}_\Gamma(x)\right|$.*

We prove that this number depends only on Γx.

Corollary 4.4. *For all $x \in X$ and all $\gamma \in \Gamma$, it holds that*

$$\left|\kappa_x^{-1}([W])/\mathrm{Stab}_\Gamma(x)\right| = \left|\kappa_{\gamma x}^{-1}([W])/\mathrm{Stab}_\Gamma(\gamma x)\right|.$$

Proof. If $y_1', y_2' \in \kappa_{\gamma x}^{-1}([W])$, then, due to Lemma 4.1, there exists $\gamma \in \Gamma$ and y_1, $y_2 \in \kappa_x^{-1}([W])$ such that $y_1' = \gamma y_1$ and $y_2' = \gamma y_2$. Furthermore, if $(\gamma x, y_1')$ and $(\gamma x, y_2')$ are Γ-equivalent, then there is a $\rho \in \Gamma$ such that $\rho(\gamma x, y_1') = (\gamma x, y_2')$. Then it follows that there exists an $h \in \mathrm{Stab}_\Gamma(x)$ such that $\rho = \gamma h \gamma^{-1}$ and $hy_1 = y_2$. Therefore $\gamma h \gamma^{-1}(y_1') = y_2'$, which proves the assertion. □

We are now prepared to define the announced matrix. Since Γ is a finite group, the set $\mathcal{H}(\Gamma, X, Y) = \{[\mathrm{Stab}_\Gamma(x)] \mid x \in X\} \cup \{[\mathrm{Stab}_\Gamma(x, y)] \mid x \in X, y \in Y\}$ is finite.

As a direct consequence of Corollary 4.4 we obtain:

Lemma 4.5. *Let $[W]$, $[V] \in \mathcal{H}(\Gamma, X, Y)$, then*

$$b([W], [V]) = \begin{cases} \left|\kappa_x^{-1}([W])/\mathrm{Stab}_\Gamma(x)\right| & \text{if there is } x \in X \text{ with } \mathrm{Stab}_\Gamma(x) \in [V] \\ 0 & \text{otherwise,} \end{cases}$$

is well defined.

Observe that, according to the definition of κ_x and of $b([W], [V])$, we can also write, for any $V \in [V]$ such that $V = \mathrm{Stab}_\Gamma(x)$ for some $x \in X$:

$$\begin{aligned} b([W], [V]) &= |\{y \in Y \mid \mathrm{Stab}_\Gamma(x, y) \in [W]\}/\mathrm{Stab}_\Gamma(x)| \\ &= |\{y \in Y \mid \mathrm{Stab}_\Gamma(y) \cap V \in [W]\}/V|. \end{aligned} \tag{8}$$

For a practical computation of the values of $b([W], [V])$ using (8), we observe that $\mathrm{Stab}_\Gamma(y) \cap V = \mathrm{Stab}_V(y)$, i.e., it suffices to study the operation of V on the set Y. By (8), $\mathrm{Stab}_V(y)$ has to be of the same order as W, and $\mathrm{Stab}_V(y)$ is a subgroup of V, therefore we have

Corollary 4.6. *If $|W|$ does not divide $|V|$, then $b([W],[V]) = 0$.*

From now on let $[H_1], \ldots, [H_K]$ be an enumeration of $\mathcal{H}$ such that $|H_1| \leq |H_2| \leq \ldots \leq |H_K|$.

Definition 4.7. The matrix $E = E(\Gamma, X \times Y) = (b_{ij})_{i,j=1,\ldots,K}$, where

$$b_{ij} = b([H_i],[H_j])$$

is called the *extension matrix* (w.r.t. the action of Γ on $X \times Y$).

As a direct consequence of this definition, of Lemma 4.5 and of Corollary 4.3, we can state

Lemma 4.8. *If $x \in X$ has $\mathrm{Stab}_\Gamma(x) \in [H_j]$, then the element b_{ij} at the j-th column and i-th row of the extension matrix counts the number of non Γ-equivalent extensions (x,y) of x such that $\mathrm{Stab}_\Gamma((x,y)) \in [H_i]$.*

Theorem 4.9. *Let $[H_1], \ldots, [H_K]$ be an enumeration of $\mathcal{H}(\Gamma, X, Y)$ and let $\underline{c} = (c_1, \ldots, c_K)^T$, where $c_j = |\{x \in X \mid \mathrm{Stab}_\Gamma(x) \in [H_j]\}/\Gamma|$. If $E = E(\Gamma, X \times Y)$ denotes the extension matrix, then*

$$\underline{c}' = (c_1', \ldots, c_K')^T = E\underline{c},$$

satisfies

$$c_i' = |\{(x,y) \mid \mathrm{Stab}_\Gamma((x,y)) \in [H_i]\}/\Gamma|$$

for all $i = 1, \ldots, K$.

Proof. Since $c_j = |\{x \in X \mid \mathrm{Stab}_\Gamma(x) \in [H_j]\}/\Gamma|$ is the number of different orbits Γx with conjugate stabilizer $\mathrm{Stab}_\Gamma(x) \in [H_j]$, and $b_{ij} = b([H_i],[H_j])$ is the number of Γ-equivalent extension classes with stabilizer $\in [H_i]$ obtained from any element in such an orbit, the sum $c_i' = \sum_{j=1}^K b([H_i],[H_j])c_j$ gives the number of distinct classes of Γ-equivalent extensions in $(X \times Y)/\Gamma$ such that the stabilizer of the extensions in these classes are in $[H_i]$. □

By its construction in Definition 4.7, the extension matrix $E(\Gamma, X \times Y)$ depends on the sets X and Y; but a closer look reveals that we can define a generalized extension matrix by 'forgetting' that the subgroup V in the definition of $b([W],[V])$ is a stabilizer of some $x \in X$. If we do so, we have to consider all conjugacy classes of subgroups of Γ (which might be more than $\mathcal{H}$) and compute new $b'([W],[V])$. However, although we might have to deal with more equivalence classes of subgroups we obtain a matrix $E' = E'(\Gamma, Y)$ which allows us to compute the possible extensions for a (left) group operation of Γ on any finite set X. We now formulate a property which is the generalization of Lemma 4.5 interpreted through the version given in (8).

Lemma 4.10. *If* $[W]$, $[V]$ *are two elements of* $[\mathcal{S}UB](\Gamma)$, *then*

$$b'([W],[V]) = |\{y \in Y \mid \mathrm{Stab}_\Gamma(y) \cap V \in [W]\}/V| \tag{9}$$

does not depend on the choice of $V \in [V]$.

Proof. Let $V' = \gamma V \gamma^{-1}$ and let $y \in Y$ be such that $\mathrm{Stab}_\Gamma(y) \cap V \in [W]$. That implies $\gamma \mathrm{Stab}_\Gamma(y)\gamma^{-1} \cap \gamma V \gamma^{-1} \in [W]$, or equivalently, $\mathrm{Stab}_\Gamma(\gamma y) \cap V' \in [W]$. If $y_1 = \rho y$, $\rho \in V$, then $\gamma y_1 = \gamma \rho y = \mu(\gamma y)$ with $\mu = \gamma \rho \gamma^{-1} \in V'$, i.e., if y_1 is in the V-equivalence class Vy, then γy_1 is in the V'-equivalence class $V'(\gamma y)$, from which the assertion follows. □

From now on $[U_1], \ldots, [U_K]$ is an enumeration of $[\mathcal{S}UB](\Gamma)$ such that $|U_1| \leq |U_2| \leq \ldots \leq |U_K|$; in particular, $U_1 = \{\mathrm{id}\}$ and $U_K = \Gamma$.

Definition 4.11. The matrix $E' = E'(\Gamma, Y) = (b_{ij})_{i,j=1,\ldots,K}$, where

$$b'_{ij} = b'([U_i],[U_j]) \tag{10}$$

is called the *generalized extension matrix* of Y (w.r.t. the action of Γ on Y).

The partial order $W \subset V$ for subgroups of Γ induces a partial order for the equivalence classes of subgroups.

The equivalence class $[W]$ *is less* than $[V]$ (denoted as $[W] \prec [V]$) if there exists $W \in [W]$ and $V \in [V]$ such that $W \subset V$ and $W \neq V$.

Corollary 4.12. *If* $[W]$, $[V] \in [\mathcal{S}UB(\Gamma)]$ *and* $[V] \prec [W]$, *then* $b'([W],[V]) = 0$.

Furthermore, note that the divisibility properties as already mentioned in Corollary 4.6, still hold: If $|W|$ does not divide $|V|$, then $b'([W],[V]) = 0$.

Note that, for more practical purposes, the following expression which is equivalent to (9), through (10), is useful:

$$b'_{ij} = \left|\left\{U_j y \in Y/U_j \mid \mathrm{Stab}_{U_j}(y) \in [U_i]\right\}\right|, \tag{11}$$

where U_j is a representative of $[U_j]$ and $U_j y$ denotes the orbit of $y \in Y$ (under the action of U_j on Y).

Following the notion in [12], p. 131, we define

Definition 4.13. Let Γ act on X (from the right). The vector $\underline{s} = \underline{s}(\Gamma, X) = (s_1, \ldots, s_K)$, where

$$s_i = |\{x \in X \mid \mathrm{Stab}_\Gamma(x) \in [U_i]\}/\Gamma|,$$

$i = 1, \ldots, K$ is called the *strata counter* of X (w.r.t. the action of Γ).

The importance of the strata counter is demonstrated by the next theorem.

Theorem 4.14. *Let Γ act on X and Y, respectively, and let Γ act component-wise on $X \times Y$. If $E'(\Gamma, Y)$ is the generalized extension matrix of Y and $\underline{s}(X)$ is the strata counter of X, then the strata counter $\underline{s}(\Gamma, X \times Y)$ satisfies*

$$\underline{s}(\Gamma, X \times Y) = E'(\Gamma, Y)\underline{s}(\Gamma, X).$$

The proof is straightforward and follows the same arguments the proof of Theorem 4.9.

The following corollary will be the cornerstone for what comes next.

Corollary 4.15. *Let Γ act on the sets X, Y_i, $i = 1, \ldots, L$. If Γ acts on the product $X \times Y_1 \times \ldots \times Y_L$ component-wise, then the strata counter of the product is given by*

$$\underline{s}(\Gamma, X \times Y_1 \times \ldots \times Y_L) = E'(\Gamma, Y_L) \ldots E'(\Gamma, Y_1)\underline{s}(\Gamma, X).$$

Finally, we want to mention that the matrix $E'(\Gamma, Y)$ is related to the Burnside matrices of subgroups of Γ, see [12], p. 136. It can be shown that the j-th column of $E'(\Gamma, Y)$ is (after a proper extension with 0's) the strata counting vector of the operation of U_j on Y. It is this strata counting vector which can be computed using the Burnside matrix of U_j.

Counting orbits and symmetries in $\nabla_\star$-configurations

We can now apply the above results to $\nabla_\star$-configurations of size N. In the previous sections, we demonstrated and used the fact that a $\nabla_\star$-configuration is uniquely determined by its top row, or by its left edge, or by its right edge. .

In order to count the distinct stabilizer classes of $\nabla_\star$-configurations (w.r.t. the action of $\mathcal{S} = \mathrm{Aut}(G, \star) \times \mathcal{R}$ or $\mathcal{S}_c = \mathrm{Aut}(G, \star) \times \mathcal{D}$), we need a different way of parametrizing the $\star$-configurations. Unfortunately, the parameterization depends on the group under consideration.

We first consider the group $\mathcal{S} = \mathrm{Aut}(G, \star) \times \mathcal{R}$ operating on the $\nabla_\star$-configurations, where $(G, \star)$ is a semisymmetric quasigroup. We show that it is possible to parametrize the set of configurations in a "product-like" way and then apply the above methods.

In a second step, when $(G, \star)$ is a totally symmetric quasigroup, we consider the group $\mathcal{S}_c = \mathrm{Aut}(G, \star) \times \mathcal{D}$ and its action on the $\nabla_\star$-configurations. Again it is possible to obtain a "product-like" parametrization, although more complicated than for the $\mathcal{S}$-case, which allows us to compute the different orbit types.

We conclude the section with a discussion of two examples. These examples also show how to reduce the amount of work for computing the coefficients b_{ij}.

We begin with the proper parametrization of $C(N, \star)$, the complete set of $\nabla_\star$-configurations, for the case $\mathcal{S}$. It will turn out that $C(N, \star)$ is given by products

of small configurations, i.e., $C(J,\star)$, $J=0,1,2$, and powers of the set

$$T(3)=\left\{\begin{array}{cc} & a \\ c & b \end{array}\;\middle|\; a,b,c\in G\right\}.$$

This set is acted on by the group $\mathcal{S}$ in an obvious way.

Lemma 4.16. *For all $N=3N'+J\in\mathbb{N}$, where $J=0,1,2$ there exists a bijective map*

$$p_N : C(J,\star)\times T(3)^{N'}\to C(3N'+J,\star)$$

such that

$$(\sigma,s)p_N(\underline{\pi})=p_N((\sigma,s)\underline{\pi})$$

holds for all $(\sigma,s)\in\mathcal{S}$ and all $\underline{\pi}\in C(J,\star)\times T(3)^{N'}$, where $\mathcal{S}$ acts componentwise on $C(J,\star)\times T(3)^{N'}$.

Proof. If $N'=0$ and if $T(3)^0$ is supposed to be the empty set, we set $p_J=\mathrm{id}$. Obviously p_J, $J=0,1,2$, commutes with the group action.

Now assume that the assertion is true for N', i.e., there exists a $p_{N'}: C(J,\star)\times T(3)^{N'}\to C(3N'+J,\star)$ which is bijective and commutes with the group action. For $\underline{\pi}\in C(J,\star)\times T(3)^{N'}$ we denote

$$p_N(\underline{\pi})=\begin{array}{ccccccccc} g_1^1 & & g_2^1 & & \cdots & & g_{N-1}^1 & & g_N^1 \\ & g_1^2 & & g_2^2 & & \cdots & & g_{N-1}^2 & \\ & & g_1^3 & & \cdots & & g_{N-2}^3 & & \\ & & & \ddots & & \iddots & & & \\ & & & & g_1^N & & & & \end{array}$$

and for $(\underline{\pi},\tau)$, where $\tau=\begin{pmatrix} & a \\ c & b \end{pmatrix}$, we define the "incomplete" configuration as

$$\begin{array}{ccccccccccc} a & & & & \cdots & & & & & & \\ & g_1^1 & & g_2^1 & & \cdots & & g_{N-1}^1 & & g_N^1 & b \\ & & g_1^2 & & g_2^2 & & \cdots & & g_{N-1}^2 & & \\ & & & g_1^3 & & \cdots & & g_{N-2}^3 & & & \\ & \ddots & & \ddots & & \iddots & & \iddots & & & \\ & & & & g_1^N & & & & & & \\ & & & & c & & & & & & \end{array}.$$

By Lemma 3.1, the incomplete configuration has a unique extension to a $\nabla_\star$-configuration of size $N+3$, see also the proof of Theorem 3.2. The $\nabla_\star$-configuration obtained in the above way is $p_{N+3}((\underline{\pi},\tau))$. Due to the uniqueness of the extension, p_{N+3} is bijective, and due to the construction, p_{N+3} commutes with the group action. □

Thus, a study of the group action on $C(J,\star)\times T(3)^K$ is equivalent to the study of the group action of $\mathcal{S}$ on $C(J+3K,\star)$. The advantage of the group action on $C(J,\star)\times T(3)^K$ lies in the product structure of the latter.

Theorem 4.17. *If $N=3N'+J\in\mathbb{N}$, where $J=0,1,2$, then the strata counter of $C(N,\star)$ w.r.t. the action of $\mathcal{S}$ is given by*

$$\underline{s}(\mathcal{S},C(N,\star))=E'(\mathcal{S},T(3))^{N'}\underline{s}(\mathcal{S},C(J,\star)).$$

Proof. By Lemma 4.16, the action of $\mathcal{S}$ on $C(N,\star)$ is equivalent to the action of $\mathcal{S}$ on $C(J,\star)\times T(3)^{N'}$. Since $\mathcal{S}$ acts componentwise on $C(J,\star)\times T(3)^{N'}$, an application of Corollary 4.15 yields the assertion. □

For an application of this theorem, see Example 1 below.

In a similar way as for $\mathcal{S}$, we study the action of $\mathcal{S}_c=\mathrm{Aut}(G,\star)\times\mathcal{D}$ on $C(N,\star)$. Again we need a proper representation of $C(N,\star)$. In other words, we want to represent $C(N,\star)$ as a product of simpler sets such that the group operation splits into componentwise operations.

As we have seen already in the proof of Lemma 3.14, we have to distinguish between configurations of odd and even size. The product representation that we are looking for will not only use $T(3)$, but also the set of "hexagonally arranged" symbols

$$T(6)=\left\{\begin{array}{ccc} & a & b \\ f & & c \\ & e & d \end{array}\;\middle|\; a,b,c,d,e,f\in G\right\}.$$

The group $\mathcal{S}_c$ acts on $T(6)$ in an obvious way.

Lemma 4.18. *For any $N=6N'+J\in\mathbb{N}$, where $N\geq 2$ and $J=0,1,\ldots,5$, there exists a bijective map $p_N: X_N\to C(N,\star)$, where X_N is given by*

1. *If $N=6N'\geq 6$, then $X_N=T(6)^{N'}$; 2. if $N=6N'+1\geq 7$, then $X_N=C(4,\star)\times T(6)^{N'-1}\times T(3)$; 3. if $N=6N'+2\geq 2$, then $X_N=C(2,\star)\times T(6)^{N'}$; 4. if $N=6N'+3\geq 3$, then $X_N=T(6)^{N'}\times T(3)$; 5. if $N=6N'+4\geq 4$, then $X_N=C(4,\star)\times T(6)^{N'}$; 6. if $N=6N'+5\geq 5$, then $X_N=C(2,\star)\times T(6)^{N'}\times T(3)$.*

Moreover, if $\mathcal{S}_c$ acts componentwise on X_N, then

$$(\sigma,s)p_N(\underline{\pi})=p_N((\sigma,s)\underline{\pi})$$

holds for all $(\sigma,s)\in\mathcal{S}_c$ and all $\underline{\pi}\in X_N$.

Proof. The proof is by induction on N'. For $N=2$ and $N=4$ the assertion is true if we set $p_2=\mathrm{id}$, $p_4=\mathrm{id}$. For $N=3$ and an element

$$\begin{array}{ccc} a & b & c \\ & d & e \\ & & f \end{array}$$

in $C(3,\star)$, the element

$$\begin{array}{ccc} & b & \\ d & & e \end{array}$$

in $T(3)$ determines the $\nabla_\star$-configuration uniquely. This defines $p_3 : T(3) \to C(3,\star)$ such that p_3 commutes with the group action.
The case $N = 5$ is dealt with in the final induction argument presented below.
For $N = 6$ and an element of $C(6,\star)$ one easily sees that the $\nabla_\star$-configuration

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ & x_7 & x_8 & x_9 & x_{10} & x_{11} \\ & & x_{12} & x_{13} & x_{14} & x_{15} \\ & & & x_{16} & x_{17} & x_{18} \\ & & & & x_{19} & x_{20} \\ & & & & & x_{21} \end{array}$$

is uniquely determined by

$$\begin{array}{ccc} & x_3 \; x_4 & \\ x_{12} & & x_{15} \\ x_{16} & & x_{18} \end{array}$$

which can be considered as an element of $T(6)$. This defines $p_6 : T(6) \to C(6,\star)$ such that p_6 commutes with the group action.
The case $N = 7$ is a consequence of the final induction argument.

The final argument is of course an induction argument and uses the already observed phenomena in Corollary 3.15.

Let us assume that the assertion holds for N' and let $N = 6(N'+1)+J$, where $J = 0,1,2,3,4,5$. Then we distinguish two cases.

The first case is where N is an even number, i.e., $J = 0,2,4$. Let $\nabla \in C(N,\star)$ and assume that

$$\nabla_\star = p_\xi(g_1^\xi, \ldots, g_N^\xi), \tag{12}$$

where p_ξ for $\xi = t,r,l$ denotes the parametrization of $\nabla_\star$ by its top row, right side and left side, respectively. As a consequence of Lemma 3.1, we have that the values of

$$\theta = \begin{array}{ccc} & g^t_{\frac{N}{2}} \; g^t_{\frac{N}{2}+1} & \\ g^l_{\frac{N}{2}+1} & & g^r_{\frac{N}{2}} \\ g^l_{\frac{N}{2}} & & g^r_{\frac{N}{2}+1} \end{array}$$

together with $(\nabla_\star^{red})^{red}$ completely determine $\nabla_\star$. Since the above values as part of the triangular array $\nabla_\star$ form an element of $T(6)$, we have

$$X_N = X_{N-6} \times T(6)$$

and $p_N(\underline{\pi},\theta)$ is the unique extension of $p_{N-6}(\underline{\pi})$ by θ. By our construction, we have that p_N commutes with the group operation.

Finally we consider the case $N = 6(N'+1)+J$ is odd, i.e., $J = 1,3,5$. With the same parametrization of $\nabla_\star$ as in (12) above, the middle entries of the sides of $\nabla_\star$ as given by

$$\tau = \begin{matrix} & g^t_{\frac{N+1}{2}} & \\ g^l_{\frac{N+1}{2}} & & g^r_{\frac{N+1}{2}} \end{matrix}$$

and the configuration $\nabla^{red}_\star \in C(N-3,\star)$ completely determine $\nabla_\star$. Since these middle entries form an element of $T(3)$, and $N-3$ is an even number, we have

$$X_N = X_{N-3} \times T(3)$$

and $p_N(\underline{\pi},\tau)$ is the unique extension of $p_{N-3}(\underline{\pi})$ by $\tau \in T(3)$. By construction, p_N commutes with the group operation. □

Due to the product structure of the sets X_N as given in the previous lemma, we are able to apply Corollary 4.15 for the determination of the strata counter of $C(N,\star)$. This results in the final

Theorem 4.19. *Let $N = 6N'+J$, $N \neq 1$ and $J = 0,1,\ldots,5$, and denote by $A = E'(\mathcal{S}_c, T(6))$, $B = E'(\mathcal{S}_c, T(3))$ the generalized extension matrices. Then we have*

1. *If $N = 6N' \geq 6$, then $\underline{s}(\mathcal{S}_c, C(N,\star)) = A^{N'-1}\underline{s}(\mathcal{S}_c, C(6,\star))$;* 2. *if $N = 6N'+1 \geq 7$, then $\underline{s}(\mathcal{S}_c, C(N,\star)) = BA^{N'-1}\underline{s}(\mathcal{S}_c, C(4,\star))$;* 3. *if $N = 6N'+2 \geq 2$, then $\underline{s}(\mathcal{S}_c, C(N,\star)) = A^{N'}\underline{s}(\mathcal{S}_c, C(2,\star))$;* 4. *if $N = 6N'+3 \geq 9$, then $\underline{s}(\mathcal{S}_c, C(N,\star)) = BA^{N'-1}\underline{s}(\mathcal{S}_c, C(6,\star))$ and $\underline{s}(\mathcal{S}_c, C(3,\star)) = \underline{s}(\mathcal{S}_c, T(3))$;* 5. *if $N = 6N'+4 \geq 4$, then $\underline{s}(\mathcal{S}_c, C(N,\star)) = A^{N'}\underline{s}(\mathcal{S}_c, C(4,\star))$;* 6. *if $N = 6N'+5 \geq 5$, then $\underline{s}(\mathcal{S}_c, C(N,\star)) = BA^{N'}\underline{s}(\mathcal{S}_c, C(2,\star))$.*

The case $N = 1$ is covered by $\underline{s}(\mathcal{S}_c, C(1,\star)) = \underline{s}(\mathcal{S}_c, G) = \underline{s}(\mathrm{Aut}(G,\star), G)$.

This theorem is illustrated in the second of the two elaborate concluding examples which follow next.

Examples

Example 1. Consider the semisymmetric quasigroup $(G,\star) = (\mathbb{F}_4,\star)$, with $x\star y = \zeta x + (1+\zeta)y$. In what follows, it will be convenient to replace the elements $0,1,\zeta,1+\zeta$ by the symbols $0,1,2,3$, respectively. As already mentioned before, $\mathrm{Aut}(\mathbb{F}_4,\star) = A_4$, the group of even permutations of 4 symbols. Using the cycle notation for permutations, $A_4 = \{(0),(01)(23),(02)(13),(03)(12),(012),(132),(031),(023),(021),(032),(123),(013)\}$ (see, e.g., [6]). Then $\mathcal{S} = \mathrm{Aut}(\mathbb{F}_4,\star)\times\mathcal{R}$ has order 36, and has 28 subgroups contained in 12 equivalence classes $[U_j], j = 1,\ldots,12$ of $\mathcal{S}$-conjugate subgroups. These classes are presented in the following list, satisfying $|U_i| \leq |U_j|$ when $i < j$. A representative of $[U_j]$ is denoted by U_j. The symbols

$(0), \mathrm{id}_{\mathcal{R}}, \mathrm{id}_{\mathcal{S}}$ denote the identities in A_4, $\mathcal{R}$ and $\mathcal{S}$ respectively.
Class $[U_1]$ contains 1 group of order 1: $U_1 = \{(0)\mathrm{id}_{\mathcal{R}}\} = \{\mathrm{id}_{\mathcal{S}}\}$;
class $[U_2]$ contains 3 groups of order 2: $U_2 = \{(0), (01)(23)\} \times \{\mathrm{id}_{\mathcal{R}}\}$;
class $[U_3]$ contains 4 groups of order 3: $U_3 = \{\mathrm{id}_{\mathcal{S}}, (012)r, (021)l\}$;
class $[U_4]$ contains 4 groups of order 3: $U_4 = \{\mathrm{id}_{\mathcal{S}}, (021)r, (012)l\}$;
class $[U_5]$ contains 1 group of order 3: $U_5 = \{(0)\} \times \mathcal{R}$;
class $[U_6]$ contains 4 groups of order 3: $U_6 = \{(0), (012), (021)\} \times \{\mathrm{id}_{\mathcal{R}}\}$;
class $[U_7]$ contains 1 group of order 4: $U_7 = \{(0), (01)(23), (02)(13), (03)(12)\} \times \{\mathrm{id}_{\mathcal{R}}\}$;
class $[U_8]$ contains 3 groups of order 6: $U_8 = \{(0), (01)(23)\} \times \mathcal{R}$;
class $[U_9]$ contains 4 groups of order 9: $U_9 = \{(0), (012), (021)\} \times \mathcal{R}$;
class $[U_{10}]$ contains 1 group of order 12: $U_{10} = \{(0), (01)(23), (02)(13), (03)(12)\} \times \mathcal{R}$;
class $[U_{11}]$ contains 1 group of order 12: $U_{11} = A_4 \times \{\mathrm{id}_{\mathcal{R}}\}$;
class$[U_{12}]$ contains 1 group of order 36: $U_{12} = \mathcal{S}$ $(= A_4 \times \mathcal{R})$.

The generalized extension matrix $E'(\mathcal{S}, T(3))$, where $T(3) = \left\{ \begin{smallmatrix} a & \\ c & b \end{smallmatrix} \mid a, b, c \in \mathbb{F}_4 \right\}$, can be obtained from applying the practical form of the b'_{ij}-definition (see (11)). This gives

$$E'(\mathcal{S}, T(3)) = \begin{bmatrix}
\boxed{64} & \boxed{32} & \boxed{20} & \boxed{20} & \boxed{20} & \boxed{21} & \boxed{16} & \boxed{10} & \boxed{6} & \boxed{5} & \boxed{5} & \boxed{1} \\
0 & \boxed{0} & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{0} & 0 & \boxed{0} & \boxed{0} & \boxed{0} \\
0 & 0 & \boxed{4} & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & \boxed{4} & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & \boxed{4} & 0 & 0 & \boxed{2} & \boxed{1} & \boxed{1} & 0 & \boxed{0} \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{0} & 0 & \boxed{1} & \boxed{0} \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{0} & \boxed{0} & \boxed{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0}
\end{bmatrix}$$

The entries which are not placed in a frame are those which are already known to be zero from the $\prec$- ordering of the $[U_j]$ (see Corollary 4.12). As will be shown next, there is no need to consider the whole matrix.

In order to derive the strata counters for all N-sized $C(N, *)$-configurations, we need – according to Theorem 4.17 – the strata counters for the configurations of size $0, 1, 2$. Using Definition 4.13 in its more practical form $|\{\mathcal{S}x \mid x \in C(J, *), \mathrm{Stab}_{\mathcal{S}}(x) \in [U_i]\}|$, we obtain respectively

$\underline{s}^{(0)} = \underline{s}(\mathcal{S}, C(0,*)) = [000000000001]^T$ (observe that $C(0,*)$ is the empty configuration);

$\underline{s}^{(1)} = \underline{s}(\mathcal{S}, C(1,*)) = [000000001000]^T$ (the $C(1,*)$-configuration being the single-cell configurations 1, 2, 3, 4 which are all $\mathcal{S}$-equivalent);

$\underline{s}^{(2)} = \underline{s}(\mathcal{S}, C(2,*)) = [000100001000]^T$ (the $C(2,*)$-representatives were shown in Fig. 3: One is invariant under the operations of $\{\mathrm{id}_{\mathcal{S}}, (013)r, (031)l\} \in [U_4]$, the other is invariant under the group $\{(0),(123),(132)\} \times \mathcal{R}\} \in [U_9]$).

Straightforward application of Theorem 3.47 shows that for $j = 11, 10, 8, 7, 6, 2$, it holds that $s_j^{(N)} = 0$ for all $N \geq 0$, meaning that no ∇_*-configuration has symmetry-class $[U_j]$ for $j = 11, 10, 8, 7, 6, 2$. Therefore only the classes $[U_j]$, $j = 1, 3, 4, 5, 9, 12$ remain to be considered, and the generalized extension matrix E' of size 12×12 reduces to the following 6×6-matrix E'_{red}, whose rows and columns correspond to these $[U_j]$-classes in the given order $j = 1, 3, 4, 5, 9, 12$:

$$E'_{\mathrm{red}} = \begin{bmatrix} 64 & 20 & 20 & 20 & 6 & 1 \\ 0 & 4 & 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding reduced strata counters are

$$\underline{s}^{(0)}_{\mathrm{red}} = [000001]^T; \quad \underline{s}^{(1)}_{\mathrm{red}} = [000010]^T; \quad \underline{s}^{(2)}_{\mathrm{red}} = [001010]^T.$$

Using Theorem 4.17 in a recursive way: $\underline{s}^{(N+3)}_{\mathrm{red}} = E'_{\mathrm{red}} \cdot \underline{s}^{(N)}_{\mathrm{red}}$, and starting with $\underline{s}^{(0)}_{\mathrm{red}}$, $\underline{s}^{(1)}_{\mathrm{red}}$, $\underline{s}^{(2)}_{\mathrm{red}}$ produces difference equations in the components of the strata counter vectors $\underline{s}^{(N)}_{\mathrm{red}}$ which generate the following closed form solutions for the number of distinct (w.r.t. $\mathcal{S}$) configurations which are symmetric in the classes $[U_j]$, $j = 1, 3, 4, 5, 9$ (notice that $[U_{12}]$ has been left aside as it happens that the corresponding strata counter component is 0 for $N \geq 1$):

for $k \geq 1$:

$$\underline{s}^{(3k)}_{\mathrm{red}} = [\, s_1^{(3k)} \ \tfrac{4^k-1}{3} \ \tfrac{4^k-1}{3} \ \tfrac{4^{k-1}-1}{3} \ 1\,]^T,$$

$$\text{with } s_1^{(3k)} = 2 \cdot 64^{k-1} - 4^{k-1} - \tfrac{14}{63}[64^{k-1} - 1];$$

for $k \geq 0$:

$$\underline{s}^{(3k+1)}_{\mathrm{red}} = [\, s_1^{(3k+1)} \ \tfrac{4^k-1}{3} \ \tfrac{4^k-1}{3} \ \tfrac{4^k-1}{3} \ 1\,]^T,$$

$$\text{with } s_1^{(3k+1)} = \tfrac{6}{63}[64^k - 1] + \tfrac{4}{3}[64^{k-1} - 4^{k-1}] + \tfrac{20}{63}[64^{k-1} - 1];$$

$$\underline{s}^{(3k+2)}_{\mathrm{red}} = [\, s_1^{(3k+2)} \ \tfrac{4^k-1}{3} \ \tfrac{4^{k+1}-1}{3} \ \tfrac{4^k-1}{3} \ 1\,]^T,$$

$$\text{with } s_1^{(3k+2)} = 26 \cdot 64^{k-1} + \tfrac{24}{9}[64^{k-1} - 4^{k-1}] - \tfrac{14}{63}[64^{k-1} - 1].$$

The $s_1^{(N)}$-component in these strata counter vectors corresponds to the number of distinct (w.r.t. $\mathcal{S}$) $\nabla_*(N)$-configurations that show no particular symmetry (i.e.,

are only invariant under the $\mathrm{id}_{\mathcal{S}}$-operation). The second and third components give the number of symmetries in the related classes $[U_3]$ and $[U_4]$ respectively (these classes contain order-3 groups whose operations leave a configuration invariant under a rotation combined with a symbol permutation, the classes are however not conjugate). Note the difference in the number of $\nabla_\star(N)$-configurations for both classes when $N = 3k + 2$. This can be seen in Fig. 3 for N=5, where there are five configurations which belong to $[U_4]$, and only one configuration belonging to $[U_3]$. The fourth component gives the number of configurations which are invariant under rotation. There is always only one configuration (modulo $\mathcal{S}$) of any size that has $[U_9]$-symmetry, e.g., the "all-white" configurations shown in Fig. 3 with symmetry group $\{(1), (234), (243)\} \times \mathcal{R}$. The sum of all components in the strata counter gives the total number of distinct (w.r.t. $\mathcal{S}$) configurations. It is easy to verify, that in the limit for $N \to \infty$, the ratio of the number of configurations showing some interesting symmetry over the number of nonsymmetric ones vanishes, although the number of symmetric configurations itself grows exponentially.

Note that, when one is interested in the configurations which are symmetric in a particular class $[U_i]$, one should actually also consider all configurations in the classes $[U_j]$ for which $[U_j] \succeq [U_i]$ (because groups in $[U_i]$ are subgroups of groups in $[U_j]$). This also implies that the number of different $\mathcal{S}$-equivalent configurations which are $[U_i]$-symmetric, equals the sum of the strata counter components that correspond to all $[U_j]$ configurations that are $\succeq [U_i]$. Thus, in the example, the $[U_9]$-symmetric configuration is also rotationally symmetric ($[U_5] \preceq [U_9]$).

Example 2. Consider the totally symmetric quasigroup $(G, *) = (\mathbb{Z}_4, *)$ with $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $x * y = 3(x + y) \bmod 4$. Now $\mathrm{Aut}(G, *) = S_2 = \{(1), (13)\}$ – the symmetric group of permutations of the two symbols 1 and 3 –, and $\mathcal{S}_c = \mathrm{Aut}(G, *) \times \mathcal{D}$. The group $\mathcal{S}_c$ has 16 subgroups in 10 equivalence classes $[U_j]$, $j = 1, 2, \ldots, 10$, as displayed in the following list ($\mathrm{id}_{\mathcal{D}}$ is the identity in $\mathcal{D}$; α, β, γ denote the reflections in $\mathcal{D}$, r and l the rotations; $\mathrm{id}_{\mathcal{S}_c}$ is the identity in $\mathcal{S}_c$):

Class $[U_1]$ contains 1 group of order 1: $U_1 = \{(0)\mathrm{id}_{\mathcal{D}}\} = \{\mathrm{id}_{\mathcal{S}_c}\}$;
class $[U_2]$ contains 3 groups of order 2: $U_2 = \{(0)\mathrm{id}_{\mathcal{D}}, (13)\alpha\}$;
class $[U_3]$ contains 3 groups of order 2: $U_5 = \{(0)\} \times \{\mathrm{id}_{\mathcal{D}}, \alpha\}$;
class $[U_4]$ contains 1 group of order 2: $U_8 = \{(0), (13)\} \times \{\mathrm{id}_{\mathcal{D}}\}$;
class $[U_5]$ contains 1 group of order 3: $U_9 = \{(0)\} \times \mathcal{R}$;
class $[U_6]$ contains 3 groups of order 6: $U_6 = \{(0), (13)\} \times \{0, \alpha\}$;
class $[U_7]$ contains 1 group of order 6: $U_7 = \{(0)\} \times \mathcal{D}$;
class $[U_8]$ contains 1 group of order 6: $U_8 = \{(0), (13)\} \times \mathcal{R}$;
class $[U_9]$ contains 1 group of order 6: $U_9 = \{\mathrm{id}_{\mathcal{S}_c}, (0)r, (0)l, (13)\alpha, (13)\beta, (13)\gamma\}$;
class $[U_{10}]$ contains 1 group of order 12: $U_{10} = \{(0), (13)\} \times \mathcal{D}$.

The generalized extension matrices $A = E'(\mathcal{S}_c, T(6))$ and $B = E'(\mathcal{S}_c, T(3))$ to be used in Theorem 4.19 are calculated based on the "practical" equivalent of their definition as given in (11) (which is also applicable for $T(6)$ instead of $T(3)$). This

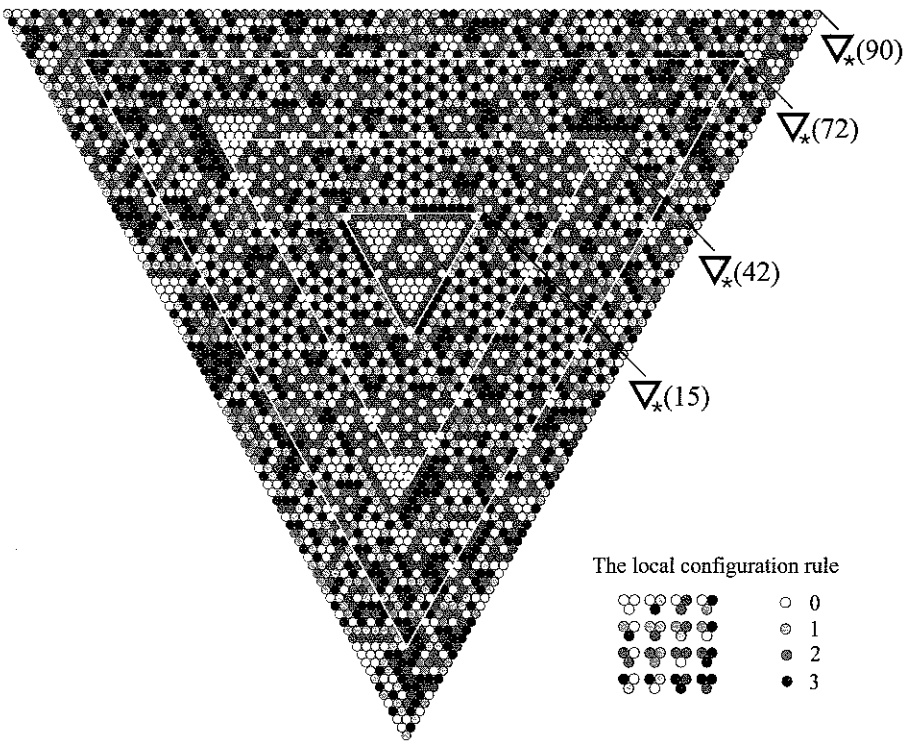


Figure 4. A $\nabla_\star(90)$-configuration in $(\mathbb{Z}_4, \star)$ with $x \star y = 3(x+y) \bmod 4$. The inner reduced $\nabla_\star$-configurations of size $N = 3, 6, 9, 12, 15$ are totally symmetric, and are also invariant under permutation of 1 and 3 (symmetry class $[U_{10}]$, see discussion of Example 2). The inner $\nabla_\star$-configurations up to size $N = 42$ are not only rotationally symmetric, but are also invariant under median reflection and simultaneous permutations of 1 and 3 (symmetry class $[U_9]$). From size $N = 45, 48 \ldots$ up to $N = 72$, the configurations are only rotationally symmetric (symmetry class $[U_5]$). From $N = 75$ on, the configuration has no symmetry at all (symmetry class $[U_1]$).

gives:

$$A = \begin{bmatrix}
\boxed{4096} & \boxed{2016} & \boxed{2016} & \boxed{2016} & \boxed{1360} & \boxed{980} & \boxed{650} & \boxed{670} & \boxed{650} & \boxed{308} \\
0 & \boxed{64} & 0 & 0 & 0 & \boxed{28} & 0 & 0 & \boxed{60} & \boxed{27} \\
0 & 0 & \boxed{64} & 0 & 0 & \boxed{28} & \boxed{60} & 0 & 0 & \boxed{27} \\
0 & 0 & 0 & \boxed{64} & 0 & \boxed{38} & 0 & \boxed{20} & 0 & \boxed{7} \\
0 & 0 & 0 & 0 & \boxed{16} & 0 & \boxed{6} & \boxed{6} & \boxed{6} & \boxed{2} \\
0 & 0 & 0 & 0 & 0 & \boxed{8} & 0 & 0 & 0 & \boxed{6} \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{4} & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{4} & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{4} & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2}
\end{bmatrix}$$

and

$$B = \begin{bmatrix} \boxed{64} & \boxed{28} & \boxed{24} & \boxed{28} & \boxed{20} & \boxed{10} & \boxed{4} & \boxed{9} & \boxed{7} & \boxed{1} \\ 0 & \boxed{8} & 0 & 0 & 0 & \boxed{2} & 0 & 0 & \boxed{6} & \boxed{2} \\ 0 & 0 & \boxed{16} & 0 & 0 & \boxed{6} & \boxed{12} & 0 & 0 & \boxed{5} \\ 0 & 0 & 0 & \boxed{8} & 0 & \boxed{2} & 0 & \boxed{2} & 0 & \boxed{0} \\ 0 & 0 & 0 & 0 & \boxed{4} & 0 & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{0} \\ 0 & 0 & 0 & 0 & 0 & \boxed{4} & 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{4} & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} & 0 & \boxed{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} & \boxed{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} \end{bmatrix}$$

The initial strata counter vectors are

$$\underline{s}(\mathcal{S}_c, T(6)) = [308\ 27\ 27\ 7\ 2\ 6\ 1\ 1\ 1\ 2]^T,$$
$$\underline{s}(\mathcal{S}_c, C(4,*)) = [14\ 5\ 5\ 1\ 2\ 2\ 1\ 1\ 1\ 2]^T,$$
$$\underline{s}(\mathcal{S}_c, C(2,*)) = [0\ 1\ 2\ 0\ 0\ 1\ 0\ 0\ 0\ 1]^T.$$

Unlike in Example 1, these matrices and strata counters can not be reduced: Given any of the $[U_j]$-classes, there are always ∇_*-configurations that exhibit the associated symmetry.

Figure 4 depicts a $\nabla_*(90)$-configuration for $(\mathbb{Z}_4, \star)$ which has no symmetry at all, while the "inner" reduced configurations may exhibit symmetries of higher order. It also illustrates that the symmetry either remains constant or decreases while a ∇_*-configuration "grows".

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