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GENERIC UNIQUENESS CONDITIONS FOR THE CANONICAL POLYADIC DECOMPOSITION AND INDSCAL*

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Abstract. We find conditions that guarantee that a decomposition of a generic third-order tensor in a minimal number of rank-1 tensors (canonical polyadic decomposition (CPD)) is unique up to a permutation of rank-1 tensors. Then we consider the case when the tensor and all its rank-1 terms have symmetric frontal slices (INDSCAL). Our results complement the existing bounds for generic uniqueness of the CPD and relax the existing bounds for INDSCAL. The derivation makes use of algebraic geometry. We stress the power of the underlying concepts for proving generic properties in mathematical engineering.

Key words. canonical polyadic decomposition, CANDECOMP/PARAFAC decomposition, INDSCAL, third-order tensor, uniqueness, algebraic geometry

AMS subject classifications. 15A69, 15A23, 14A10, 14A25, 14Q15

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1. Introduction.

1.1. Basic definitions. Throughout the paper $\mathbb{F}$ denotes the field of real or complex numbers. A tensor $\mathcal{T} = (t_{ijk}) \in \mathbb{F}^{I\times J\times K}$ is *rank*-1 if there exist three nonzero vectors $\mathbf{a} \in \mathbb{F}^I$, $\mathbf{b} \in \mathbb{F}^J$, and $\mathbf{c} \in \mathbb{F}^K$ such that $\mathcal{T} = \mathbf{a} \circ \mathbf{b} \circ \mathbf{c}$, in which "$\circ$" denotes the *outer product*. That is, $t_{ijk} = a_i b_j c_k$ for all values of the indices. A *polyadic decomposition* (PD) of a third-order tensor $\mathcal{T}$ expresses $\mathcal{T}$ as a sum of rank-1 terms,

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r, \tag{1.1}$$

where $\mathbf{a}_r \in \mathbb{F}^I$, $\mathbf{b}_r \in \mathbb{F}^J$, $\mathbf{c}_r \in \mathbb{F}^K$ are nonzero vectors. We will write (1.1) as $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$, where $\mathbf{A} = [\mathbf{a}_1 \ \dots \ \mathbf{a}_R] \in \mathbb{F}^{I\times R}$, $\mathbf{B} = [\mathbf{b}_1 \ \dots \ \mathbf{b}_R] \in \mathbb{F}^{J\times R}$, $\mathbf{C} = [\mathbf{c}_1 \ \dots \ \mathbf{c}_R] \in \mathbb{F}^{K\times R}$.

If the number R in (1.1) is minimal, then it is called *the rank of* $\mathcal{T}$ and is denoted by $r_{\mathcal{T}}$. In this case we say that (1.1) is a *canonical polyadic decomposition* (CPD) of $\mathcal{T}$. The CPD was introduced by Hitchcock in [16]. It is also referred to as rank decomposition, canonical decomposition (Candecomp) [2], and the parallel factor model (Parafac) [14, 15].

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It is clear that in (1.1) the rank-1 terms can be arbitrarily permuted and that vectors within the same rank-1 term can be arbitrarily scaled provided the overall rank-1 term remains the same. *The CPD of a tensor is unique* when it is only subject to these trivial indeterminacies.

We call tensors whose frontal slices are symmetric matrices (implying $I = J$) SFS-tensors, where the abbreviation "SFS" stands for "symmetric frontal slices." It is clear that if an SFS tensor $\mathcal{T}$ is rank-1, then $\mathcal{T} = \mathbf{a} \circ \mathbf{a} \circ \mathbf{c}$ for some nonzero vectors $\mathbf{a} \in \mathbb{F}^I$ and $\mathbf{c} \in \mathbb{F}^K$. Similarly to the unstructured case above, one can easily define the SFS-rank, the SFS-CPD, and the uniqueness of the SFS-CPD of an SFS-tensor $\mathcal{T}$ (see [12, section 4] for the exact definitions). Note that the SFS-CPD corresponds to the individual differences in multidimensional scaling (INDSCAL) model, as introduced by Carroll and Chang [2]. To the authors' knowledge, it is still an open question as to whether there exist SFS-tensors with unique SFS-CPD but nonunique CPD.

Blind signal separation (BSS) consists of the splitting of signals into meaningful, interpretable components. The CPD has become a standard tool for BSS: the known mixture of signals corresponds to a given tensor $\mathcal{T}$ and the unknown interpretable components correspond to the rank-1 terms in its CPD. For the interpretation of the components one should be able to assess whether the CPD is unique. The SFS-CPD is a constrained version of the CPD. In the original formulation of the INDSCAL model (or SFS-CPD) the frontal slices of $\mathcal{T}$ were distance matrices. Nowadays, SFS-CPD is widely used in independent component analysis (ICA) where the frontal slices of $\mathcal{T}$ are spatial covariance matrices. The SFS-CPD interpretation of ICA allows one to handle the underdetermined case (more sources than sensors). The (SFS-)CPD based approach has found many applications in signal processing [6, 7], data analysis [19], chemometrics [22], psychometrics [2], etc. We refer the readers to the overview papers [5, 8, 10, 18, 23] and the references therein for background, applications, and algorithms.

The most famous result on uniqueness is due to Kruskal [20]. The k-rank of a matrix $\mathbf{A}$ is defined as the largest integer $k_{\mathbf{A}}$ such that any $k_{\mathbf{A}}$ columns of $\mathbf{A}$ are linearly independent. Kruskal's theorem states that if $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ and

$$R \leq \frac{k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} - 2}{2}, \tag{1.2}$$

then $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T}$ is unique.

Condition (1.2) is an example of a deterministic condition for uniqueness in the sense that the uniqueness of the CPD can be guaranteed for a particular choice of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Checking deterministic conditions can be cumbersome. For instance, in (1.2) the computation of the k-ranks has combinatorial complexity. If the entries of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are drawn from continuous distributions then one can consider uniqueness with probability one or generic uniqueness. Generic conditions are often easy to check; they usually just take the form of a bound on the rank as a function of the tensor dimensions. In this paper we derive new, relaxed conditions for the generic uniqueness of CPD and SFS-CPD. We resort to the following definitions.

DEFINITION 1.1. *Let $\mu_{\mathbb{F}}$ be the Lebesgue measure on $\mathbb{F}^{I\times R} \times \mathbb{F}^{J\times R} \times \mathbb{F}^{K\times R}$. The CPD of an $I \times J \times K$ tensor of rank R is generically unique if*

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{B}, \mathbf{C}) : \text{ the CPD of the tensor } [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R \text{ is not unique } \} = 0.$$

DEFINITION 1.2. *Let $\mu_{\mathbb{F}}$ be the Lebesgue measure on $\mathbb{F}^{I\times R} \times \mathbb{F}^{K\times R}$. The SFS-CPD of an $I \times I \times K$ tensor of SFS-rank R is generically unique if*

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{C}) : \text{ the SFS-CPD of the tensor } [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R \text{ is not unique } \} = 0.$$

1.2. Previous results on generic uniqueness of the CPD. Since the k-rank of a generic matrix coincides with its minimal dimension, the Kruskal theorem implies the following result: if

$$R \leq \frac{\min(I,R)+\min(J,R)+\min(K,R)-2}{2}, \tag{1.3}$$

then the CPD of an $I \times J \times K$ tensor of rank R is generically unique. Without loss of generality we may assume that $2 \leq I \leq J \leq K$. Then (1.3) guarantees generic uniqueness for $R \leq \min(I+J-2, K)$ and $K < R \leq (I+J+K)/2$. Kruskal's condition is not necessary in general. It was shown in [3, Proposition 5.2] that if $3 \leq I \leq J$ and $\mathbb{F} = \mathbb{C}$, then generic uniqueness holds if

$$R \leq (I-1)(J-1) \ \text{ and } \ (I-1)(J-1) \leq K. \tag{1.4}$$

A similar result (involving a different condition in the second part of (1.4)) had been obtained before in [25, Theorem 2.7]. In the following proposition we collect theoretically proven bounds on R that guarantee generic uniqueness of the CPD for the complimentary case where $K \leq R$.

PROPOSITION 1.3. *Let $2 \leq I \leq J \leq K \leq R$. Then each of the following conditions implies that the CPD of an $I \times J \times K$ tensor of rank R is generically unique:*

(i) $R \leq IJK/(I+J+K-2)-K$, $3 \leq I$, $\mathbb{F} = \mathbb{C}$ [1, Corollary 6.2], [25, Corollary 3.7, K is odd];
(ii) $R \leq 2^{\alpha+\beta-2}$, *where α and β are maximal integers such that $2^\alpha \leq I$ and $2^\beta \leq J$* [3, Theorem 1.1];
(iii) $R \leq \frac{I+J+K-2}{2}$ *(follows from Kruskal's bound* (1.3)*).*

The theoretical bounds in Proposition 1.3 can be further relaxed. According to the recent paper [4] the CPD is generically unique (with a few known exceptions) if

$$R \leq \left\lceil \frac{IJK}{I+J+K-2} \right\rceil - 1, \quad IJK \leq 15000, \tag{1.5}$$

where $\lceil x \rceil$ denotes the smallest integer not less than x. The proof of (1.5) involves the computation of the kernel of a certain $IJK \times R(I+J+K)$ matrix for a random example with the given dimensions and number of rank-1 terms. Similarly, Proposition 1.4 below guarantees generic uniqueness of the CPD if at least one of some specially constructed matrices has full column rank. The conditions in Proposition 1.4 are formulated in terms of the Khatri–Rao product of mth compound matrices of $\mathbf{A}$ and $\mathbf{B}$. Recall that *the mth compound matrix* of an $I \times R$ matrix $\mathbf{A}$ (denoted by $\mathcal{C}_m(\mathbf{A})$) is defined for $m \leq \min(I,R)$ and is the $\binom{I}{m} \times \binom{R}{m}$ matrix containing the determinants of all $m \times m$ submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order. We refer the reader to [11] for more details on compound matrices. *The Khatri–Rao product* of the matrices $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_R] \odot [\mathbf{b}_1 \ \ldots \ \mathbf{b}_R] := [\mathbf{a}_1 \otimes \mathbf{b}_1 \ \ldots \ \mathbf{a}_R \otimes \mathbf{b}_R],$$

where "$\otimes$" denotes the *Kronecker product*.

PROPOSITION 1.4 (see [12, Proposition 1.31]). *The CPD of an $I \times J \times K$ tensor of rank R is generically unique if there exist matrices $\mathbf{A}_0 \in \mathbb{F}^{I \times R}$, $\mathbf{B}_0 \in \mathbb{F}^{J \times R}$, and $\mathbf{C}_0 \in \mathbb{F}^{K \times R}$ such that at least one of the following conditions holds:*

(i) $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ *has full column rank, where* $m_{\mathbf{C}} = R - \min(K, R) + 2$*;*
(ii) $\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{B}_0) \odot \mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C}_0)$ *has full column rank, where* $m_{\mathbf{A}} = R - \min(I, R) + 2$*;*
(iii) $\mathcal{C}_{m_{\mathbf{B}}}(\mathbf{C}_0) \odot \mathcal{C}_{m_{\mathbf{B}}}(\mathbf{A}_0)$ *has full column rank, where* $m_{\mathbf{B}} = R - \min(J, R) + 2$.

It was shown in [11, 12] that if (1.3) holds, then (i)–(iii) in Proposition 1.4 hold, i.e., Proposition 1.4 is more relaxed than (1.3). To see if (i)–(iii) hold for given dimensions and rank, it suffices to check a random example (more specifically, in which the entries of $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$ are drawn from continuous probability densities).

1.3. Previous results on generic uniqueness of the SFS-CPD. The generic uniqueness of the SFS-CPD has been less studied. From Kruskal's condition (1.2) it follows that if

$$R \leq \min(I, R) + \frac{\min(K, R)}{2} - 1, \tag{1.6}$$

then the SFS-CPD of an $I \times I \times K$ SFS-tensor of SFS-rank R is generically unique. To the authors' knowledge the following counterpart of Proposition 1.4 is the only known result on the generic uniqueness of the SFS-CPD.

PROPOSITION 1.5 (see [12, Proposition 6.8], [24, $K = R$]). *The SFS-CPD of an* $I \times I \times K$ *SFS-tensor of SFS-rank* R *is generically unique if there exist matrices* $\mathbf{A}_0 \in \mathbb{F}^{I \times R}$ *and* $\mathbf{C}_0 \in \mathbb{F}^{K \times R}$ *such that* $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0)$ *or* $\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C}_0)$ *has full column rank, where* $m_{\mathbf{C}} = R - \min(K, R) + 2$ *and* $m_{\mathbf{A}} = R - \min(I, R) + 2$.

1.4. Contributions of the paper. In this paper we present new generic uniqueness results for the CPD and SFS-CPD. Based on deterministic conditions from [11, 12] (namely, Propositions 3.2, 4.2, and 6.1 further on) we obtain theoretically proven bounds on R.

1.4.1. Results on generic uniqueness of the CPD. The following result complements the conditions for CPD in Proposition 1.3.

PROPOSITION 1.6. *Let*

$$2 \leq I \leq J \leq K \leq R, \tag{1.7}$$

$$R \leq \frac{I + J + 2K - 2 - \sqrt{(I - J)^2 + 4K}}{2} \tag{1.8}$$

or, equivalently,

$$m - 1 \leq I \leq J \leq K \leq R, \tag{1.9}$$

$$R \leq (I + 1 - m)(J + 1 - m) + m - 2, \tag{1.10}$$

where $m = R - K + 2$. *Then the CPD of an* $I \times J \times K$ *tensor of rank* R *is generically unique.*

Since it is often the case in applications that the largest dimension of a tensor exceeds its rank, we explicitly formulate the following special case of Proposition 1.6.

COROLLARY 1.7. *Let* $3 \leq I \leq J \leq R \leq K$ *and* $R \leq (I - 1)(J - 1)$. *Then the CPD of an* $I \times J \times K$ *tensor of rank* R *is generically unique.*

Corollary 1.7 improves the results of [3, Propositions 5.2] and [25, Theorem 2.7] mentioned above (see (1.4)). Namely, the assumption $(I - 1)(J - 1) \leq K$ in (1.4) is relaxed as $R \leq K$ and the statement on generic uniqueness holds both for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$. It is also interesting to note that for $\mathbb{F} = \mathbb{C}$ the decomposition is generically not unique if $R > (I-1)(J-1)$ [3, Proposition 2.2]. In the case where $\mathbb{F} = \mathbb{C}$, Corollary 1.7 can also be obtained by combining [3, Propositions 5.2] and [4, Theorem 4.1].

Let us compare bound (1.8) with Kruskal's bound (1.3), bound (1.4), and bounds from Propositions 1.3–1.4. For $I \geq 3$, bound (1.8) improves (1.3) by $\frac{K-\sqrt{(I-J)^2+4K}}{2}$. For $R = K$, (1.8) coincides with (1.4). Using results of [11, 12] one can show that condition (1.8) is more relaxed than Proposition 1.4. In the following examples we present some cases where (1.8) is more relaxed than any bound from Propositions 1.3–1.4 and compare bound (1.8) with the bound in Proposition 1.3(i).

Example 1.8. By Proposition 1.3(iii) or Proposition 1.4, the CPD of a generic $4 \times 5 \times 6$ tensor is unique for $R \leq 6$ and by Proposition 1.6 and (1.5), generic uniqueness is guaranteed for $R \leq 7$ and $R \leq 9$, respectively.

Example 1.9. One can easily check that if $K = (I-2)(J-2)$, then the right-hand side of (1.8) is equal to $R = (I-2)(J-2)+1$. Thus, by Proposition 1.6, the CPD of an $I \times J \times (I-2)(J-2)$ tensor of rank $(I-2)(J-2)+1$ is generically unique. In particular, the CPD of an $7 \times 8 \times 30$ tensor of rank 31 is generically unique. It can be shown that this result does not follow from Proposition 1.3. By (1.5), generic uniqueness holds for $R \leq 39$. On the other hand, for increasing I and J, bound (1.5) becomes harder and harder to verify. For instance, to guarantee that the CPD of an $I \times J \times (I-2)(J-2)$ tensor of rank $(I-2)(J-2)+1$ is generically unique one should compute the kernel of an $IJ(I-2)(J-2) \times (I+J+(I-2)(J-2))((I-2)(J-2)+1)$ matrix, which quickly becomes infeasible [4].

Example 1.10. We compare the bound in Proposition 1.3(i) with bound (1.8) for $I \times I \times K$ tensors. By formula manipulation it can easily be shown that if $I = J$ and $\min(9, I) \leq K$, then the bound in Proposition 1.3(i) is more relaxed than bound (1.8) if at least $\frac{5}{2} + \sqrt{2K - \sqrt{K} + \frac{21}{4}} \leq I$ and that the bound (1.8) is more relaxed than the bound in Proposition 1.3(i) if at least $I \leq 2 + \sqrt{2K - \sqrt{K} + 3}$.

1.4.2. Results on generic uniqueness of the SFS-CPD. If either $R < I$ or $I \leq R < \min(K, 2I-2)$ or $\max(I, K) \leq R \leq (2I+K-2)/2$, then generic uniqueness of the SFS-CPD follows from (1.6). Other theoretical bounds on R for the cases $2 \leq I < R \leq K$, $2 \leq I \leq K \leq R$, and $2 \leq K \leq I < R$ are stated in Propositions 1.11, 1.12, and 1.13, respectively.

PROPOSITION 1.11. *Let $4 \leq I < R \leq K$ and $R \leq \frac{I^2-I}{2}$. Then the SFS-CPD of an $I \times I \times K$ SFS-tensor of SFS-rank R is generically unique.*

PROPOSITION 1.12. *Let*

$$2 \leq I \leq K \leq R,$$

$$R \leq \frac{2I + 2K + 1 - \sqrt{8K + 8I + 1}}{2} \tag{1.11}$$

or, equivalently,

$$m - 1 \leq I \leq K \leq R,$$

$$R \leq \frac{I^2 + (3-2m)I}{2} + \frac{(m-1)(m-2)}{2}, \tag{1.12}$$

where $m = R - K + 2$. Then the SFS-CPD of an $I \times I \times K$ tensor of rank R is generically unique.

PROPOSITION 1.13. *Let*

$$2 \leq K \leq I \leq R, \tag{1.13}$$

$$R \leq \frac{K + 3I - 1 - \sqrt{(K-I)^2 + 2K + 6I - 3}}{2} \tag{1.14}$$

or, equivalently,

$$\begin{aligned} m-1 \leq K \leq I \leq R, \\ R \leq (I+1-m)(K+1-m), \end{aligned} \tag{1.15}$$

where $m = R - I + 2$. *Then the SFS-CPD of an* $I \times I \times K$ *tensor of rank* R *is generically unique.*

Using results of [11, 12] one can show that for $\min(I, K) \geq 3$, bound (1.14) is more relaxed than the bound in Proposition 1.5, which is known [11, 12] to be more relaxed than Kruskal's condition (1.6). It can also be shown that for $2 \leq I \leq K \leq R$, bound (1.11) is more relaxed than the bound in Proposition 1.5 (and hence (1.6)) in all cases except $(I, K) \in \{(2,2), (3,4), (4,4), (5,6), (6,6), (8,8)\}$.

Example 1.14. Kruskal's condition (1.6) and Proposition 1.5 guarantee that the SFS-CPD of an $8 \times 8 \times 20$ tensor of rank R is generically unique for $R \leq 14$ and $R \leq 20$, respectively [12, Example 6.14]. By Proposition 1.12, uniqueness holds also for $R = 21$. More generally, if $I \geq 5$, then by Proposition 1.12, the SFS-CPD of an $I \times I \times \frac{I^2-3I}{2}$ tensor of rank $\frac{I^2-3I}{2} + 1$ is generically unique.

1.5. Organization of the paper. A number of deterministic conditions for uniqueness of the CPD and SFS-CPD have been obtained in [12]. The main part of the theory in [12] was built around conditions that were denoted as (Km), (Cm), (Um), and (Wm) (each succeeding condition is more relaxed than the preceding one, but harder to use). It was shown that condition (1.3) and the conditions in Propositions 1.4, 1.5 are generic versions of the (Km) and (Cm) based deterministic conditions, respectively. In this paper we obtain generic versions of the (Um) and (Wm) based deterministic conditions from [12]. We proceed as follows. In the first part of section 3 we recall the (Wm) based deterministic condition for uniqueness of the CPD (Proposition 3.2) and in the first parts of sections 4 and 6 we derive two (Um) based conditions for uniqueness of the SFS-CPD (Propositions 4.2 and 6.1, respectively). Then in the second part of sections 3, 4, 6 we interpret the conditions in Propositions 1.6, 1.12, and 1.13 as generic versions of the deterministic Propositions 3.2, 4.2, and 6.1, respectively. Proposition 1.11 is derived from Proposition 1.12 in section 5. Our derivations make use of algebraic geometry. Section 2 contains relevant basic definitions and results. In subsection 2.2.3 we summarize these results in a procedure that may be used in different applications to study generic conditions. Although algebraic geometry based approaches have appeared in, e.g., [1, 3, 4, 25], the power of the algebraic geometry framework is not yet fully acknowledged in mathematical engineering. We hope that our paper has some tutorial value in this respect.

2. Auxiliary results from algebraic geometry. This section is provided to make the paper accessible to readers not familiar with algebraic geometry. We present a well-known algebraic geometry based method to prove that a set $W \subset \mathbb{F}^l$ has measure zero, $\mu_{\mathbb{F}}(W) = 0$. For $\mathbb{F} = \mathbb{C}$ we introduce the notion of dimension of W, $\dim W$, and explain how to compute it. We show that if $\dim W < l$, then $\mu_{\mathbb{C}}(W) = 0$. The method is summarized and illustrated in subsections 2.3 and 2.4, respectively. In subsection 2.3 we also explain that the case $\mathbb{F} = \mathbb{R}$ can be reduced to the case $\mathbb{F} = \mathbb{C}$. In this paper, to prove Proposition 1.6 and Propositions 1.12–1.13, we will use the method for

$$W = \{(\mathbf{A}, \mathbf{B}, \mathbf{C}) : \text{ the CPD of } [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R \text{ is not unique}\} \subset \mathbb{F}^{I \times R} \times \mathbb{F}^{J \times R} \times \mathbb{F}^{K \times R}$$

(W can be considered as a subset of $\mathbb{F}^l$ with $l = (I + J + K)R$) and

$$W = \{(\mathbf{A}, \mathbf{C}) : \text{ the SFS-CPD of } [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R \text{ is not unique}\} \subset \mathbb{F}^{I\times R} \times \mathbb{F}^{K\times R}$$

(W can be considered as a subset of $\mathbb{F}^l$ with $l = (I + K)R$), respectively.

2.1. Zariski topology. A subset $X \subset \mathbb{C}^n$ is *Zariski closed* if there is a set of polynomials $p_1(z_1, \dots, z_n), \dots, p_k(z_1, \dots, z_n)$ such that

$$X = \{(z_1, \dots, z_n) : \ p_1(z_1, \dots, z_n) = 0, \dots, p_k(z_1, \dots, z_n) = 0\}. \tag{2.1}$$

A subset $Y \subset \mathbb{C}^n$ is *Zariski open* if its complement in $\mathbb{C}^n$ is Zariski closed. A subset $Z \subset \mathbb{C}^n$ is *Zariski locally closed* if it equals the intersection of an open and a closed subset. *The Zariski closure* $\overline{W}$ of $W \subset \mathbb{C}^n$ is the smallest closed set such that $W \subset \overline{W}$. For instance, the set

$$Y = \{(z_1, \dots, z_n) : \ q_1(z_1, \dots, z_n) \neq 0, \dots, q_l(z_1, \dots, z_n) \neq 0\}$$

is Zariski open and the set

$$Z = Y \cap X = \{(z_1, \dots, z_n) : p_1(z_1, \dots, z_n) = 0, \dots, p_k(z_1, \dots, z_n) = 0, \\ q_1(z_1, \dots, z_n) \neq 0, \dots, q_l(z_1, \dots, z_n) \neq 0\}.$$

is Zariski locally closed. If $W = (0, 1) \subset \mathbb{C}^1$, then the closure of W in the classical Euclidean topology is $[0, 1]$ and the closure in the Zariski topology is the entire $\mathbb{C}^1$. Indeed, if $\overline{W} = \{z : \ p_1(z) = \cdots = p_k(z) = 0\} \supset (0, 1)$, then $p_i = 0$. Hence, $\overline{W} = \mathbb{C}^1$.

In the sequel we will consider closed and open subsets only in Zariski topology and, for brevity, we drop the term "Zariski." The following lemma follows easily from the above definitions.

LEMMA 2.1.

(i) *The empty set and the whole space* $\mathbb{C}^n$ *are the only subsets of* $\mathbb{C}^n$ *that are both open and closed.*

(ii) *Let* Y *be an open subset of* $\mathbb{C}^n$*. Then* $\overline{Y} = \mathbb{C}^n$.

2.2. Dimension of a subset. With an arbitrary subset $W \subset \mathbb{C}^n$ one can associate a number, called *the dimension of* W, $\dim W \in \{0, 1, \dots, n\}$, such that $\overline{W} \subsetneq \mathbb{C}^n$ if and only if $\dim W < n$. In this subsection we give a definition and discuss how the dimension can be computed.

A closed subset X is *reducible* if it is the union of two smaller closed subsets X_1 and X_2, $X = X_1 \cup X_2$. A closed subset X is *irreducible* if it is not reducible. For instance, the subset $X := \{(z_1, z_2) : \ z_1 z_2 = 0\} \subset \mathbb{C}^2$ is reducible since $X = X_1 \cup X_2$ with $X_i := \{(z_1, z_2) : \ z_i = 0\}$; both X_1 and X_2 are irreducible.

The (topological) dimension of a subset $W \subset \mathbb{C}^n$ is the largest integer d such that there exists a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d \subset \overline{W}$ of distinct irreducible closed subsets of $\overline{W}$. It can be proved that such d always exists and that $d \leq n$. Since the closure of $\overline{W}$ coincides with $\overline{W}$, it follows immediately from the definition of dimension that $\dim W = \dim \overline{W}$. The following properties of the dimension are well known in algebraic geometry.

LEMMA 2.2.

(i) *Let* $W_1 \subset W_2 \subset \mathbb{C}^n$*, then* $\dim W_1 \leq \dim W_2 \leq n$.

(ii) *Let* $W \subset \mathbb{C}^n$*. Then* $\dim W = n$ *if and only if* $\overline{W} = \mathbb{C}^n$.

(iii) *Let* $W \subset W_1 \times W_2$ *and* π_i *be the projection* $W_1 \times W_2 \to W_i$*,* $i = 1, 2$*. Then*

$$\max(\dim \pi_1(W), \ \dim \pi_2(W)) \leq \ \dim W \leq \dim \pi_1(W) + \dim \pi_2(W).$$

(iv) *Let* $W = W_1 \cup \cdots \cup W_k$*. Then* $\dim W = \max(\dim W_1, \dots, \dim W_k)$.

It can be shown that if W is a linear subspace, then $\dim W$ coincides with the well-known definition of dimension in linear algebra, that is, $\dim W$ is equal to the number of vectors in a basis of W. In particular, in the case where W_1 and W_2 are linear subspaces, statements (i)–(iii) in Lemma 2.2 are well known in linear algebra.

In the remaining part of this subsection we explain a method to obtain a bound on the dimension of a set $W \subset \mathbb{C}^l$. The method is summarized in Procedure 2.6, which, together with Lemma 2.7, will serve as the main tool for proving generic properties in this paper.

First, in subsections 2.2.1–2.2.2, we address the following auxiliary problem. Given a set $X = \pi(Z) \subset \mathbb{C}^l$, where the set $Z \subset \mathbb{C}^n$ is of the special form (2.2) and π is the projection $\pi : \mathbb{C}^n \to \mathbb{C}^l$, we want to obtain a bound on $\dim X$.

2.2.1. Construction of set Z and determination of $\dim Z$. Let $p_1, \dots, p_{n-m}$, $q_1, \dots, q_{n-m}$ be polynomials in the variables $z_1, \dots, z_m$. We define the open subset

$$Y = \{(z_1, \dots, z_m) : \ q_1(z_1, \dots, z_m) \neq 0, \dots, q_{n-m}(z_1, \dots, z_m) \neq 0\} \subset \mathbb{C}^m.$$

Then, by Lemma 2.1(ii), $\overline{Y} = \mathbb{C}^m$. Hence, by Lemma 2.2(ii), $\dim Y = m$. By definition, the set $Z \subset \mathbb{C}^n$ is the image of Y under the mapping

$$\phi : \ (z_1, \dots, z_m) \in Y \mapsto \left(z_1, \dots, z_m, \frac{p_1(z_1, \dots, z_m)}{q_1(z_1, \dots, z_m)}, \dots, \frac{p_{n-m}(z_1, \dots, z_m)}{q_{n-m}(z_1, \dots, z_m)} \right) \in Z,$$

that is, $Z = \phi(Y)$. It is clear that the projection of Z onto the first m coordinates of $\mathbb{C}^n$ coincides with Y. Hence, by Lemma 2.2(iii), $\dim Z \geq m$. The following lemma is well known, it states that $\dim Z = m$. In other words, the dimension of the image $\phi(Y)$ cannot exceed the dimension of Y. Note that Lemma 2.3 requires $p_1, \dots, p_{n-m}$, $q_1, \dots, q_{n-m}$ to be polynomials in the variables $z_1, \dots, z_m$.

LEMMA 2.3. *Let $p_1, \dots, p_{n-m}, q_1, \dots, q_{n-m}$ be polynomials in the variables $z_1, \dots, z_m$ and*

$$\begin{aligned} Z = \Big\{ & (z_1, \dots, z_m, z_{m+1}, \dots, z_n) : \\ & z_{m+1} := \frac{p_1(z_1, \dots, z_m)}{q_1(z_1, \dots, z_m)}, \dots, z_n := \frac{p_{n-m}(z_1, \dots, z_m)}{q_{n-m}(z_1, \dots, z_m)}, \\ & q_1(z_1, \dots, z_m) \neq 0, \dots, q_{n-m}(z_1, \dots, z_m) \neq 0, \ z_1, \dots, z_m \in \mathbb{C} \Big\} \subset \mathbb{C}^n. \end{aligned} \tag{2.2}$$

Then $\overline{Z}$ is irreducible and $\dim \overline{Z} = m$.

In this paper we call the variables $z_1, \dots, z_m$ and $z_{m+1}, \dots, z_n$ in (2.2) *"independent" parameters* and *"dependent" parameters*, respectively. Thus, Lemma 2.3 formalizes the fact that $\dim Z$ coincides with the number of its "independent parameters."

2.2.2. Construction of projection π and bound on $\dim \pi(Z)$. Let π be the projection

$$\pi : \ \mathbb{C}^n \to \mathbb{C}^l, \quad \pi(z_1, \dots, z_n) = (z_{k+1}, \dots, z_m, \dots, z_{k+l}), \tag{2.3}$$

for certain k, l such that $k + 1 \leq m \leq k + l \leq n$. We assume additionally that $1 \leq k$ and $l \leq m$. We consider the set $\pi(Z)$. Thus, π drops at least one of the independent

parameters $z_1, \dots, z_m$ but not all of them; π may also drop dependent parameters $z_{m+1}, \dots, z_n$, even all of them. By Lemma 2.2(i), $\dim \pi(Z) \le \dim \mathbb{C}^l = l$. Now we explain that this trivial bound may be further improved so that we obtain $\dim \pi(Z) < l$.

Let f denote the restriction of π to Z,

$$f : Z \to \mathbb{C}^l, \quad f(z_1, \dots, z_n) = (z_{k+1}, \dots, z_{k+l}) \text{ for all } (z_1, \dots, z_n) \in Z, \tag{2.4}$$

yielding that $\pi(Z) = f(Z)$. Denote by $f^{-1}(s_{k+1}, \dots, s_{k+l}) \subset Z$ the preimage of the point $(s_{k+1}, \dots, s_{k+l}) \in f(Z)$:

$$\begin{aligned} f^{-1}(s_{k+1}, \dots, s_{k+l}) &= \{(z_1, \dots, z_n) \in Z : f(z_1, \dots, z_n) = (s_{k+1}, \dots, s_{k+l})\} \\ &= \{(z_1, \dots, z_n) \in Z : \ z_{k+1} = s_{k+1}, \dots, z_{k+l} = s_{k+l}\}. \end{aligned}$$

The following lemma easily follows from the "fiber dimension theorem" [21, Theorem 3.7, p. 78]; it relates the dimension of the preimage with the dimension of the projection.

LEMMA 2.4. *Let Z and f be defined by* (2.2) *and* (2.4)*, respectively. Suppose that*

$$\dim f^{-1}(s_{k+1}, \dots, s_{k+l}) \ge d \ \text{ for all } (s_{k+1}, \dots, s_{k+l}) \in f(Z). \tag{2.5}$$

Then $(\dim \overline{\pi(Z)} =) \dim \overline{f(Z)} \le m - d$.

Thus, to obtain the bound $\dim \pi(Z) < l$, it suffices to show that $d > m - l$.

The results of subsections 2.2.1–2.2.2 are summarized in the following procedure.

PROCEDURE 2.5. *Input: a subset $X = \pi(Z) \subset \mathbb{C}^l$, where $Z \subset \mathbb{C}^n$ $(n \ge l)$ is of the form* (2.2) *and π is of the form* (2.3)*.*
Output: bound on $\dim X$*.*

(i) *Set $m := \dim Z$ (by Lemma* 2.3*).*
(ii) *Find d such that* (2.5) *holds.*
(iii) $\dim X \le m - d$ *(by Lemma* 2.4*).*

2.2.3. A method to obtain a bound on $\dim W \subset \mathbb{C}^l$. In this subsection we consider the following problem: given a set of points $W \subset \mathbb{C}^l$ that satisfy a certain property, we want to show that $\dim W < l$.

First we "parameterize" the problem: we find a larger subset $\widehat{Z} \subset \mathbb{C}^n$ and a projection $\pi : \mathbb{C}^n \to \mathbb{C}^l$ such that $W = \pi(\widehat{Z})$. Our parameterizations are such that the set $\widehat{Z}$ is included into a finite union of subsets $Z_u \subset \mathbb{C}^n$, $\widehat{Z} \subset \bigcup Z_u$, and that all Z_u are of the form (2.2). (For example, if W is the set of 2×2 matrices with zero eigenvalue, then $W = \pi(\widehat{Z})$, where $\widehat{Z} = \{(\mathbf{A}, \mathbf{f}) : \mathbf{Af} = \mathbf{0}, \ \mathbf{f} \ne \mathbf{0}\} \subset \mathbb{C}^{2\times 2} \times \mathbb{C}^2$ and $\pi : \mathbb{C}^{2\times 2} \times \mathbb{C}^2 \to \mathbb{C}^{2\times 2}$. Obviously, $\widehat{Z} := Z_1 \cup Z_2$, where $Z_i = \{(\mathbf{A}, \mathbf{f}) : \mathbf{Af} = \mathbf{0}, \ f_i \ne 0\}$, and it can be verified that, indeed, Z_1 and Z_2 are of the form (2.2).) Since $W = \pi(\widehat{Z})$ and $\widehat{Z} \subset \bigcup Z_u$, from Lemma 2.2(i), (iv) it follows that

$$\dim W = \dim \pi(\widehat{Z}) \le \dim \pi \left(\bigcup Z_u \right) = \dim \bigcup \pi(Z_u) = \max_u \dim \pi(Z_u)$$

or $\dim W \le \max_u \dim X_u$, where $X_u = \pi(Z_u)$. Thus, to obtain a bound on $\dim W$ one should obtain bounds on $\dim X_u$ for all u. This can be done by following the steps in Procedure 2.5 for $X = X_u$.

The results of subsection 2.2 are summarized in the following procedure.

PROCEDURE 2.6. *Input: a set of points $W \subset \mathbb{C}^l$ that satisfy a certain property.*
Output: bound on $\dim W$*.*

Phase I: Parameterization.

(1) *Express* W *as* $\pi(\widehat{Z})$*:* $W = \pi(\widehat{Z})$*, where* $\widehat{Z} \subset \mathbb{C}^n$ *and* π *is of the form* (2.3).

(2) *Express* $\widehat{Z}$ *as part of a finite union* $\widehat{Z} \subset \bigcup Z_u$*, where all* Z_u *are of the form* (2.2).

Phase II: Obtaining a bound on $\dim W$.

(3) *For all values of* u*: apply Procedure* 2.5 *for* $X = X_u = \pi(Z_u)$*, obtain a bound on* $\dim X_u$*,* $\dim X_u \leq l_u$.

(4) $\dim W \leq \max_u l_u$.

In none of the cases in this paper, will the values l_u in step (3) of Procedure 2.6 depend on u. Thus, we will apply Procedure 2.5 only once, e.g., for $u = 1$.

2.3. Zariski closed proper subsets have measure zero. The following lemma is well known. We include a proof since we do not know an explicit reference where such a proof can be found.

LEMMA 2.7. *Let* $W \subset \mathbb{C}^l$*,* $\dim W < l$*, and* $W_{\mathbb{R}} := W \cap \mathbb{R}^l$*. Then* $\mu_{\mathbb{C}}\{W\} = 0$ *and* $\mu_{\mathbb{R}}\{W_{\mathbb{R}}\} = 0$*, where* $\mu_{\mathbb{C}}$ *and* $\mu_{\mathbb{R}}$ *denote the Lebesgue measures on* $\mathbb{C}^l$ *and* $\mathbb{R}^l$*, respectively.*

Proof. We may assume that $\overline{W}$ is defined by (2.1). Then $\overline{W}$ is the zero set of the polynomials $p_1, \ldots, p_k$. The results follow from the well-known fact that the zero set of a nonzero polynomial has measure zero both on $\mathbb{C}^l$ and $\mathbb{R}^l$. ∎

As our overall strategy for showing that a subset $W \subset \mathbb{F}^l$ has measure zero, we will use Procedure 2.6 and Lemma 2.7 as follows. If $\mathbb{F} = \mathbb{C}$, then we follow the steps in Procedure 2.6 to show that $\dim W < l$, and conclude, by Lemma 2.7, that $\mu_{\mathbb{C}}(W) = 0$. If $\mathbb{F} = \mathbb{R}$, then first, we extend $W \subset \mathbb{R}^l$ to a subset $W_{\mathbb{C}} \subset \mathbb{C}^l$ by letting all parameters in W take values in $\mathbb{C}$; second, we follow the steps in Procedure 2.6 to show that $\dim W_{\mathbb{C}} < l$, and conclude, by Lemma 2.7, that $\mu_{\mathbb{R}}(W) = \mu_{\mathbb{R}}(W_{\mathbb{C}} \cap \mathbb{R}^l) = 0$.

2.4. Example. To illustrate our approach we prove the well-known fact that two generic square matrices of the same size do not share eigenvalues.

Example 2.8. Let $W = \{(\mathbf{A}, \mathbf{B}) : \ \mathbf{A} \text{ and } \mathbf{B} \text{ have a common eigenvalue}\}$ be a subset of $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$. We claim that $\mu_{\mathbb{C}}(W) = 0$, where $\mu_{\mathbb{C}}$ is the Lebesgue measure on $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$. By Lemma 2.7 it is sufficient to prove that $\dim W \leq 2n^2 - 1$. To obtain a bound on $\dim W$ we follow the steps in Procedure 2.6.

Phase I: Parameterization.

(1) It is clear that $(\mathbf{A}, \mathbf{B}) \in W$ if and only if there exist $\lambda \in \mathbb{C}$ and nonzero vectors $\mathbf{f}$ and $\mathbf{g}$ such that $\mathbf{Af} = \lambda\mathbf{f}$ and $\mathbf{Bg} = \lambda\mathbf{g}$. Hence, $W = \pi(\widehat{Z})$, where

$$\widehat{Z} = \{(\mathbf{A}, \mathbf{B}, \lambda, \mathbf{f}, \mathbf{g}) : \ \mathbf{Af} = \lambda\mathbf{f},\ \mathbf{Bg} = \lambda\mathbf{g},\ \mathbf{f} \neq \mathbf{0},\ \mathbf{g} \neq \mathbf{0}\}$$

is a subset of $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$ and π is the projection onto the first two factors

$$\pi : \ \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}.$$

(2) We represent $\widehat{Z}$ as a finite union of sets of the form (2.2):

$$\widehat{Z} = \bigcup_{1\leq u,v\leq n} Z_{u,v}, \quad Z_{u,v} := \{(\mathbf{A}, \mathbf{B}, \lambda, \mathbf{f}, \mathbf{g}) : \ \mathbf{Af} = \lambda\mathbf{f},\ \mathbf{Bg} = \lambda\mathbf{g}, \quad f_u \neq 0,\ g_v \neq 0\}.$$

We show that all $Z_{u,v}$ are of the form (2.2). To simplify the presentation we restrict ourselves to the case $u = 1$ and $v = 1$. The general case can be proved in the same

way. Since, by assumption, $f_1 \neq 0$ and $g_1 \neq 0$, we can express $\mathbf{a}_1$ and $\mathbf{b}_1$ via λ, $\mathbf{a}_2, \ldots, \mathbf{a}_n$, $\mathbf{b}_2, \ldots, \mathbf{b}_n$, $\mathbf{f}$, and $\mathbf{g}$. Hence,

$$\begin{aligned} Z_{1,1} &:= \{(\mathbf{A}, \mathbf{B}, \lambda, \mathbf{f}, \mathbf{g}) \in Z,\ f_1 \neq 0,\ g_1 \neq 0\} \\ &= \{(\mathbf{a}_1 = (\lambda \mathbf{f} - \mathbf{a}_2 f_2 - \cdots - \mathbf{a}_n f_n)/f_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \\ &\qquad \mathbf{b}_1 = (\lambda \mathbf{f} - \mathbf{b}_2 g_2 - \cdots - \mathbf{b}_n g_n)/g_1, \mathbf{b}_2, \ldots, \mathbf{b}_n, \lambda, \mathbf{f}, \mathbf{g}) :\ f_1 \neq 0,\ g_1 \neq 0\} \end{aligned}$$

is indeed of the form (2.2), where $z_1, \ldots, z_m$ correspond to the value λ and the entries of $\mathbf{a}_2, \ldots, \mathbf{a}_n$, $\mathbf{b}_2, \ldots, \mathbf{b}_n$, $\mathbf{f}$, $\mathbf{g}$ and where $z_{m+1}, \ldots, z_n$ correspond to the entries of $\mathbf{a}_1$ and $\mathbf{b}_1$.

Phase II: Obtaining a bound on $\dim W$.

(3) To obtain bounds on $\dim \pi(Z_{u,v})$ we follow the steps in Procedure 2.5 for $X = \pi(Z_{u,v})$. W.l.o.g. we again restrict ourselves to the case $u = 1$ and $v = 1$.

(i) By Lemma 2.3, $\dim Z_{1,1} = 1 + n(n-1) + n(n-1) + n + n = 2n^2 + 1$.

(ii) Let $f : Z_{1,1} \to \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ denote the restriction of π to $Z_{1,1}$:

$$f(\mathbf{A}, \mathbf{B}, \lambda, \mathbf{f}, \mathbf{g}) = (\mathbf{A}, \mathbf{B}), \quad (\mathbf{A}, \mathbf{B}, \lambda, \mathbf{f}, \mathbf{g}) \in Z_{1,1}.$$

From the definition of $Z_{1,1}$ it follows that if $(\mathbf{A}, \mathbf{B}, \lambda, \mathbf{f}, \mathbf{g}) \in Z_{1,1}$, then $(\mathbf{A}, \mathbf{B}, \lambda, \alpha\mathbf{f}, \beta\mathbf{g}) \in Z_{1,1}$, where α and β are arbitrary nonzero values. (Indeed, if $\mathbf{A}\mathbf{f} = \lambda\mathbf{f}$ and $\mathbf{B}\mathbf{g} = \lambda\mathbf{g}$, then $\mathbf{A}(\alpha\mathbf{f}) = \lambda(\alpha\mathbf{f})$ and $\mathbf{B}(\beta\mathbf{g}) = \lambda(\beta\mathbf{g})$.) Hence,

$$f^{-1}(\mathbf{A}, \mathbf{B}) \supset \{(\mathbf{A}, \mathbf{B}, \lambda, \alpha\mathbf{f}, \beta\mathbf{g}) :\ \alpha \neq 0,\ \beta \neq 0\}.$$

By Lemma 2.2(iii), $\dim f^{-1}(\mathbf{A}, \mathbf{B}) \geq \dim\{(\alpha\mathbf{f}, \beta\mathbf{g}) :\ \alpha \neq 0,\ \beta \neq 0\} \geq \dim\{(\alpha f_1, \beta g_1) :\ \alpha \neq 0,\ \beta \neq 0\}$. Since $\dim\{(\alpha f_1, \beta g_1) :\ \alpha \neq 0,\ \beta \neq 0\} = \dim \mathbb{C}^2 = 2$, it follows that $\dim f^{-1}(\mathbf{A}, \mathbf{B}) \geq \underline{2 =: d}$.

(iii) By Lemma 2.4, $\dim \overline{\pi(Z_{1,1})} \leq \dim Z_{1,1} - d \leq 2n^2 + 1 - 2 = 2n^2 - 1$. Note that precisely the property that $\mathbf{A}$ and $\mathbf{B}$ share an eigenvalue has allowed us to find a projection that reduces the dimension. What remains is a little effort to show that $d = 2$ implies that having eigenvalues in common is a subgeneric property.

(4) Hence, $\dim W \leq \max_{u,v} \dim \pi(Z_{u,v}) = \dim \pi(Z_{1,1}) = 2n^2 - 1$.

Since $W \subset \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ and $\dim W \leq 2n^2 - 1$, it follows from Lemma 2.7 that $\mu_{\mathbb{C}}(W) = 0$.

3. Uniqueness of the CPD and proof of Proposition 1.6. In what follows, $\omega(\lambda_1, \ldots, \lambda_R)$ denotes the number of nonzero entries of $[\lambda_1\ \ldots\ \lambda_R]^T$. The following condition (Wm) was introduced in [11, 12] in terms of mth compound matrices. In this paper we will use the following (equivalent) definition.

DEFINITION 3.1. *We say that condition* (Wm) *holds for the triplet of matrices* $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathbb{F}^{I \times R} \times \mathbb{F}^{J \times R} \times \mathbb{F}^{K \times R}$ *if* $\omega(\lambda_1, \ldots, \lambda_R) \leq m - 1$ *whenever*

$$r_{\mathbf{A}\mathrm{Diag}(\lambda_1, \ldots, \lambda_R)\mathbf{B}^T} \leq m - 1 \quad \textit{for} \quad [\lambda_1\ \ldots\ \lambda_R]^T \in \mathrm{range}(\mathbf{C}^T).$$

Since the rank of the product $\mathbf{A}\mathrm{Diag}(\lambda_1, \ldots, \lambda_R)\mathbf{B}^T$ does not exceed the rank of any of the factors and since $r_{\mathrm{Diag}(\lambda_1, \ldots, \lambda_R)} = \omega(\lambda_1, \ldots, \lambda_R)$, we have the implication

$$\omega(\lambda_1, \ldots, \lambda_R) \leq m - 1 \quad \Rightarrow \quad r_{\mathbf{A}\mathrm{Diag}(\lambda_1, \ldots, \lambda_R)\mathbf{B}^T} \leq m - 1. \tag{3.1}$$

Condition (Wm) in Definition 3.1 means that the opposite of the implication in (3.1) holds for all $[\lambda_1\ \ldots\ \lambda_R] \in \mathrm{range}(\mathbf{C}^T) \subset \mathbb{C}^R$. We now give a set of deterministic

conditions, among which is a ($\mathrm{W_m}$)-type condition, that guarantee CPD uniqueness. These conditions will be checked for a generic tensor in the proof of Proposition 1.6.

PROPOSITION 3.2 (see [12, Proposition 1.22]). *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *and* $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$. *Assume that*

(i) $\max(\min(k_{\mathbf{A}}, k_{\mathbf{B}} - 1),\ \min(k_{\mathbf{A}} - 1, k_{\mathbf{B}})) + k_{\mathbf{C}} \geq R + 1$;

(ii) *condition* ($\mathrm{W_{m_C}}$) *holds for the triplet* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$;

(iii) $\mathbf{A} \odot \mathbf{B}$ *has full column rank.*

Then $r_{\mathcal{T}} = R$ *and the CPD of tensor* $\mathcal{T}$ *is unique.*

Proof of Proposition 1.6. We show that

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{B}, \mathbf{C}) :\ \text{(i) or (ii) or (iii) of Proposition 3.2 does not hold}\} = 0. \tag{3.2}$$

Since, under condition (1.7),

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{B}, \mathbf{C}) :\ k_{\mathbf{A}} < I \text{ or } k_{\mathbf{B}} < J \text{ or } k_{\mathbf{C}} < K\} = 0,$$

it follows that (3.2) holds if and only if

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{B}, \mathbf{C}) : \text{(i) or (ii) or (iii) of Proposition 3.2 does not hold and } k_{\mathbf{A}} = I,\ k_{\mathbf{B}} = J,\ k_{\mathbf{C}} = K\} = 0.$$

Condition (i): By (1.7)–(1.8),

$$R + 1 \leq I + K + \frac{J - I - \sqrt{(I-J)^2 + 4K}}{2} < I + K \leq J + K,$$

which easily implies that condition (i) of Proposition 3.2 holds for all matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ such that $k_{\mathbf{A}} = I$, $k_{\mathbf{B}} = J$, $k_{\mathbf{C}} = K$.

Condition (iii): By (1.10), (1.9),

$$R \leq IJ - 1 - (m-1)(I + J - m) < IJ - 1.$$

It is well known [17, Theorem 3] that

$$\mu_1\{(\mathbf{A}, \mathbf{B}) :\ \text{(iii) of Proposition 3.2 does not hold}\} = 0$$

if and only if $R \leq IJ$, where μ_1 denotes the Lebesgue measure on $\mathbb{F}^{I \times R} \times \mathbb{F}^{J \times R}$. Fubini's theorem [13, Theorem C, p. 148] allows us to extend this to a statement for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$:

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{B}, \mathbf{C}) :\ \text{(iii) of Proposition 3.2 does not hold}\} = 0.$$

Condition (ii): Let

$$W = \{(\mathbf{A}, \mathbf{B}, \mathbf{C}) : \text{(ii) of Proposition 3.2 does not hold and } k_{\mathbf{A}} = I,\ k_{\mathbf{B}} = J,\ k_{\mathbf{C}} = K\} \subset \mathbb{F}^{I \times R} \times \mathbb{F}^{J \times R} \times \mathbb{F}^{K \times R}. \tag{3.3}$$

To complete the proof of Proposition 1.6 we need to show that $\mu_{\mathbb{F}}\{W\} = 0$. By Lemma 2.7, it is sufficient to prove that, for $\mathbb{F} = \mathbb{C}$, the closure of W is not the entire space $\mathbb{C}^{I \times R} \times \mathbb{C}^{J \times R} \times \mathbb{C}^{K \times R}$, which is equivalent (see discussion in subsection 2.3) to

$$\dim \overline{W} \leq IR + JR + KR - 1. \tag{3.4}$$

To prove bound (3.4) we follow the steps in Procedure 2.6.

Phase I: Parameterization.

(1) We associate W with a certain $\pi(\widehat{Z})$. By Definition 3.1, condition (W$_{\rm m}$) does not hold for the triplet $(\mathbf{A},\mathbf{B},\mathbf{C})$ if and only if there exist values $\lambda_1,\dots,\lambda_R$ and matrices $\widetilde{\mathbf{A}}\in\mathbb{C}^{I\times(m-1)}$, $\widetilde{\mathbf{B}}\in\mathbb{C}^{J\times(m-1)}$ such that

$$\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{B}^T=\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^T,\quad [\lambda_1\ \dots\ \lambda_R]^T\in\mathrm{Range}(\mathbf{C}^T),$$

but $\omega(\lambda_1,\dots,\lambda_R)\geq m$. We claim that, if additionally, $k_{\mathbf{A}}=I$ and $k_{\mathbf{B}}=J$, then $\omega(\lambda_1,\dots,\lambda_R)\geq I$. Indeed, if $m\leq\omega(\lambda_1,\dots,\lambda_R)<I\leq J$, then by the Frobenius inequality,

$$\begin{aligned}m-1&\geq r_{\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^T}=r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{B}^T}\\&\geq r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)}+r_{\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{B}^T}-r_{\mathrm{Diag}(\lambda_1,\dots,\lambda_R)}\\&=\omega(\lambda_1,\dots,\lambda_R)+\omega(\lambda_1,\dots,\lambda_R)-\omega(\lambda_1,\dots,\lambda_R)=\omega(\lambda_1,\dots,\lambda_R)\geq m,\end{aligned}$$

which is a contradiction. Hence, W in (3.3) can be expressed as

$$\begin{aligned}W=\{(\mathbf{A},\mathbf{B},\mathbf{C}):\ &(\mathrm{W}_{\rm m})\text{ does not hold for the triplet }(\mathbf{A},\mathbf{B},\mathbf{C}),\\&k_{\mathbf{A}}=I,\ k_{\mathbf{B}}=J,\ k_{\mathbf{C}}=K\}\\=\{(\mathbf{A},\mathbf{B},\mathbf{C}):\ &\text{there exist }\lambda_1,\dots,\lambda_R\in\mathbb{C},\ \widetilde{\mathbf{A}}\in\mathbb{C}^{I\times(m-1)},\ \text{ and }\ \widetilde{\mathbf{B}}\in\mathbb{C}^{J\times(m-1)}\\&\text{such that }\ \mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{B}^T=\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^T,\qquad(3.5)\\&[\lambda_1\ \dots\ \lambda_R]^T\in\mathrm{Range}(\mathbf{C}^T),\qquad(3.6)\\&k_{\mathbf{A}}=I,\ k_{\mathbf{B}}=J,\ k_{\mathbf{C}}=K,\qquad(3.7)\\&\omega(\lambda_1,\dots,\lambda_R)\geq I\}.\end{aligned}$$

It is now clear that $W=\pi(\widehat{Z})$, where

$$\widehat{Z}=\{(\mathbf{A},\mathbf{B},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{B}}):\ \ (3.5)\text{–}(3.7)\text{ hold}\}$$

is a subset of $\mathbb{C}^{I\times R}\times\mathbb{C}^{J\times R}\times\mathbb{C}^{K\times R}\times\mathbb{C}^{R}\times\mathbb{C}^{I\times(m-1)}\times\mathbb{C}^{J\times(m-1)}$ and π is the projection onto the first three factors

$$\begin{aligned}\pi:\ &\mathbb{C}^{I\times R}\times\mathbb{C}^{J\times R}\times\mathbb{C}^{K\times R}\times\mathbb{C}^{R}\times\mathbb{C}^{I\times(m-1)}\times\mathbb{C}^{J\times(m-1)}\\&\qquad\qquad\to\mathbb{C}^{I\times R}\times\mathbb{C}^{J\times R}\times\mathbb{C}^{K\times R}.\end{aligned}$$

(2) Since

$$\omega(\lambda_1,\dots,\lambda_R)\geq I\ \ \Leftrightarrow\ \ \lambda_{u_1}\cdots\lambda_{u_I}\neq0\ \text{ for some }\ 1\leq u_1<\cdots<u_I\leq R\qquad(3.8)$$

we obtain

$$\widehat{Z}=\bigcup_{1\leq u_1<\cdots<u_I\leq R}\{(\mathbf{A},\mathbf{B},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{B}}):\ (3.5)\text{–}(3.7)\text{ hold, }\lambda_{u_1}\cdots\lambda_{u_I}\neq0\}.$$

Let $\mathbf{A}_{u_1,\dots,u_I}$ denote the submatrix of $\mathbf{A}$ formed by columns $u_1,\dots,u_I$. Since (3.7) is more restrictive than the conditions $\det\mathbf{A}_{u_1,\dots,u_I}\neq0$ and $k_{\mathbf{C}}=K$, it follows that $\widehat{Z}\subset\bigcup_{1\leq u_1<\cdots<u_I\leq R}Z_{u_1,\dots,u_I}$, where

$$\begin{aligned}Z_{u_1,\dots,u_I}:=\{(\mathbf{A},\mathbf{B},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{B}}):\ &(3.5)\text{–}(3.6)\text{ hold, }\lambda_{u_1}\cdots\lambda_{u_I}\neq0,\\&\det\mathbf{A}_{u_1,\dots,u_I}\neq0,\ k_{\mathbf{C}}=K\}.\end{aligned}$$

We show that all $Z_{u_1,\dots,u_I}$ are of the form (2.2). To simplify the presentation we restrict ourselves to the case $(u_1,\dots,u_I)=(1,\dots,I)$. The general case can be proved in the same way. Let the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be partitioned as

$$\mathbf{A}=[\underbrace{\bar{\mathbf{A}}}_{I}\ \underbrace{\bar{\bar{\mathbf{A}}}}_{R-I}],\qquad \mathbf{B}=[\underbrace{\bar{\mathbf{B}}}_{I}\ \underbrace{\bar{\bar{\mathbf{B}}}}_{R-I}],\qquad \mathbf{C}=[\underbrace{\bar{\mathbf{C}}}_{K}\ \underbrace{\bar{\bar{\mathbf{C}}}}_{R-K}],$$

so that $\bar{\mathbf{A}}=\mathbf{A}_{1,\dots,I}$. By (3.5),

$$\bar{\mathbf{B}}=\left[\left(\bar{\mathbf{A}}\mathrm{Diag}(\lambda_1,\dots,\lambda_I)\right)^{-1}\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^T-\bar{\bar{\mathbf{A}}}\mathrm{Diag}(\lambda_{I+1},\dots,\lambda_R)\bar{\bar{\mathbf{B}}}^T\right)\right]^T.$$

By (3.6), there exists $\mathbf{x}\in\mathbb{C}^K$ such that $[\lambda_1\ \dots\ \lambda_R]^T=\mathbf{C}^T\mathbf{x}$ or, equivalently, $[\lambda_1\ \dots\ \lambda_K]^T=\bar{\mathbf{C}}^T\mathbf{x}$ and $[\lambda_{K+1}\ \dots\ \lambda_R]^T=\bar{\bar{\mathbf{C}}}^T\mathbf{x}$. Hence,

$$[\lambda_{K+1}\ \dots\ \lambda_R]^T=\bar{\bar{\mathbf{C}}}^T\bar{\mathbf{C}}^{-T}[\lambda_1\ \dots\ \lambda_K]^T.$$

In other words, by Cramer's rule, each entry of $\bar{\mathbf{B}}$ and each of the values $\lambda_{K+1},\dots,\lambda_R$ can be written as a ratio of two polynomials in the entries of $\mathbf{A}$, $\bar{\bar{\mathbf{B}}}$, $\mathbf{C}$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ and the values $\lambda_1,\dots,\lambda_K$. By the assumptions $\lambda_1\cdots\lambda_I\neq 0$, $\det\bar{\mathbf{A}}\neq 0$, and $k_{\mathbf{C}}=K$ (yielding that $\det\bar{\mathbf{C}}\neq 0$), the denominator polynomial is nonzero.

Hence, $Z_{1,\dots,I}$ is indeed of the form (2.2), where $z_1,\dots,z_m$ correspond to the entries of $\mathbf{A}$, $\bar{\bar{\mathbf{B}}}$, $\mathbf{C}$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ and the values $\lambda_1,\dots,\lambda_K$, and where $z_{m+1},\dots,z_n$ correspond to the entries of $\bar{\mathbf{B}}$ and the values $\lambda_{K+1},\dots,\lambda_R$.

Phase II: Obtaining a bound on $\dim W$.

(3) To obtain bounds on $\dim\pi(Z_{u_1,\dots,u_I})$ we follow the steps in Procedure 2.5 for $X=\pi(Z_{u_1,\dots,u_I})$. W.l.o.g. we again restrict ourselves to the case $(u_1,\dots,u_I)=(1.\dots,I)$.

(i) By Lemma 2.3, $\dim Z_{1,\dots,I}=IR+J(R-I)+KR+K+I(m-1)+J(m-1)$.

(ii) Let $f: Z_{1,\dots,I}\to\mathbb{C}^{I\times R}\times\mathbb{C}^{J\times R}\times\mathbb{C}^{K\times R}$ denote the restriction of π to $Z_{1,\dots,I}$:

$$f(\mathbf{A},\mathbf{B},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{B}})=(\mathbf{A},\mathbf{B},\mathbf{C}),\quad (\mathbf{A},\mathbf{B},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{B}})\in Z_{1,\dots,I}.$$

From the definition of $Z_{1,\dots,I}$ it follows that if $(\mathbf{A},\mathbf{B},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{B}})\in Z_{1,\dots,I}$, then $(\mathbf{A},\mathbf{B},\mathbf{C},\alpha\lambda_1,\dots,\alpha\lambda_R,\widehat{\mathbf{A}}\mathbf{T},\alpha\widehat{\mathbf{B}}\mathbf{T}^{-T})\in Z_{1,\dots,I}$, where α is an arbitrary nonzero value, $\widehat{\mathbf{A}}$ is an arbitrary full column rank matrix such that $\mathrm{range}(\widehat{\mathbf{A}})\supset\mathrm{range}(\widetilde{\mathbf{A}})$, $\mathbf{T}$ is an arbitrary nonsingular $(m-1)\times(m-1)$ matrix, and $\widehat{\mathbf{B}}$ satisfies $\widehat{\mathbf{A}}\widehat{\mathbf{B}}^T=\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^T$. Hence,

$$f^{-1}(\mathbf{A},\mathbf{B},\mathbf{C})\supset\{\mathbf{A},\mathbf{B},\mathbf{C},\alpha\lambda_1,\dots,\alpha\lambda_R,\widehat{\mathbf{A}}\mathbf{T},\alpha\widehat{\mathbf{B}}\mathbf{T}^{-T}:\ \alpha\neq 0,\ \det\mathbf{T}\neq 0\}.$$

By Lemma 2.2(iii), $\dim f^{-1}(\mathbf{A},\mathbf{B},\mathbf{C})\geq\dim\{\alpha\lambda_1,\widehat{\mathbf{A}}\mathbf{T}:\ \alpha\neq 0,\ \det\mathbf{T}\neq 0\}$. Since, by construction, the matrix $\widehat{\mathbf{A}}$ has full column rank and, by assumption, $\lambda_1\neq 0$ and $m-1\leq I$, it follows that $\dim\{\alpha\lambda_1,\widehat{\mathbf{A}}\mathbf{T}:\ \alpha\neq 0,\ \det\mathbf{T}\neq 0\}=1+(m-1)^2=:d$. Thus, $\dim f^{-1}(\mathbf{A},\mathbf{B},\mathbf{C})\geq d$.

(iii) By Lemma 2.4 and (1.10),

$$\begin{aligned}\dim\overline{\pi(Z_{1,\dots,I})}&\leq\dim Z_{1,\dots,I}-d\\&=IR+J(R-I)+KR+K+I(m-1)+J(m-1)-1-(m-1)^2\\&\leq IR+JR+KR-1.\end{aligned}$$

(4) Hence, $\dim W\leq\max_{1\leq u_1<\dots<u_I\leq R}\dim\pi(Z_{u_1,\dots,u_I})\leq IR+JR+KR-1$.

Since $W\subset\mathbb{C}^{I\times R}\times\mathbb{C}^{J\times R}\times\mathbb{C}^{K\times R}$ and $\dim W\leq IR+JR+KR-1$, it follows from Lemma 2.7 that $\mu_{\mathbb{C}}(W)=0$. ∎

4. Uniqueness of the SFS-CPD and proof of Proposition 1.12. The following condition (U$_m$) was introduced in [11, 12] in terms of mth compound matrices. In this paper we will use the following (equivalent) definition.

DEFINITION 4.1. *We say that condition* (U$_m$) *holds for the pair* $(\mathbf{A},\mathbf{B}) \in \mathbb{F}^{I\times R}\times \mathbb{F}^{J\times R}$ *if* $\omega(\lambda_1,\dots,\lambda_R) \le m-1$ *whenever* $r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{B}^T} \le m-1$.

Condition (U$_m$) in Definition 4.1 means that the opposite of the implication in (3.1) holds for all $[\lambda_1 \ \dots \ \lambda_R] \in \mathbb{C}^R$. Thus, if the matrix $\mathbf{C}$ has full column rank (implying range$(\mathbf{C}^T) = \mathbb{C}^R$), then (W$_m$) (see Definition 3.1) is equivalent to (U$_m$). Otherwise, (W$_m$) is more relaxed than (U$_m$):

(4.1) (U$_m$) holds for the pair $(\mathbf{A},\mathbf{B})$ $\quad\Rightarrow\quad$ (W$_m$) holds for the triplet $(\mathbf{A},\mathbf{B},\mathbf{C})$.

The following implication was proved in [11, Lemmas 3.2, 3.7]:

(4.2) (U$_m$) holds for the pair $(\mathbf{A},\mathbf{B})$ $\quad\Rightarrow\quad$ $\mathbf{A}\odot\mathbf{B}$ has full column rank.

The following deterministic condition for uniqueness of the SFS-CPD will be checked for a generic SFS-tensor in the proof of Proposition 1.12.

PROPOSITION 4.2. *Let* $\mathcal{T} = [\mathbf{A},\mathbf{A},\mathbf{C}]_R$ *and* $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$. *Assume that*

(i) $k_{\mathbf{A}} + k_{\mathbf{C}} \ge R+2$*;*

(ii) *condition* (U$_{m_{\mathbf{C}}}$) *holds for the pair* $(\mathbf{A},\mathbf{A})$.

Then $r_{\mathcal{T}} = R$ *and the SFS-CPD of tensor* $\mathcal{T}$ *is unique.*

Proof. We show that (i)–(iii) in Proposition 3.2 hold for $\mathcal{T} = [\mathbf{A},\mathbf{B},\mathbf{C}]_R$ with $\mathbf{B} = \mathbf{A}$: (i) is obvious from $k_{\mathbf{B}} = k_{\mathbf{A}}$ and (ii) and (iii) follow from (4.1) and (4.2), respectively. ∎

Proof of Proposition 1.12. We show that

$$\mu_{\mathbb{F}}\{(\mathbf{A},\mathbf{C}) : \text{ (i) or (ii) of Proposition 4.2 does not hold}\} = 0. \tag{4.3}$$

Since

$$\mu_{\mathbb{F}}\{(\mathbf{A},\mathbf{C}) : \ \mathbf{A} \text{ has a zero minor or } k_{\mathbf{C}} < K\} = 0,$$

it follows that (4.3) holds if and only if

$$\mu_{\mathbb{F}}\{(\mathbf{A},\mathbf{C}) : \text{(i) or (ii) of Proposition 4.2 does not hold, all minors of } \mathbf{A} \text{ are nonzero, and } k_{\mathbf{C}} = K\} = 0.$$

It is clear that if all minors of $\mathbf{A}$ are nonzero, then $k_{\mathbf{A}} = I$.

Condition (i)*:* By (1.11),

$$R+2 \le I + K + \max\left(\frac{-K+2}{2}, \frac{5-\sqrt{8K+8I+1}}{2}\right) \le I+K,$$

which means that condition (i) of Proposition 4.2 holds for all matrices $\mathbf{A}$ and $\mathbf{C}$ such that $k_{\mathbf{A}} = I$ and $k_{\mathbf{C}} = K$.

Condition (ii)*:* Let

$$W = \{(\mathbf{A},\mathbf{C}) : \text{(ii) of Proposition 4.2 does not hold, all minors of } \mathbf{A} \text{ are nonzero, and } k_{\mathbf{C}} = K\} \subset \mathbb{F}^{I\times R}\times\mathbb{F}^{K\times R}. \tag{4.4}$$

To complete the proof of Proposition 1.12 we need to show that $\mu_{\mathbb{F}}\{W\} = 0$. By Lemma 2.7, it is sufficient to prove that for $\mathbb{F} = \mathbb{C}$, the closure of W is not the entire space $\mathbb{C}^{I\times R} \times \mathbb{C}^{K\times R}$, which is equivalent (see discussion in subsection 2.3) to

$$\dim \overline{W} \leq IR + KR - 1. \tag{4.5}$$

To prove bound (4.5) we follow the steps in Procedure 2.6.

Phase I: Parameterization.

(1) We associate W with a certain $\pi(\widehat{Z})$. By Definition 4.1, condition ($\mathrm{U_m}$) does not hold for the pair $(\mathbf{A}, \mathbf{A})$ if and only if there exist values $\lambda_1, \ldots, \lambda_R$, and a matrix $\widetilde{\mathbf{A}} \in \mathbb{C}^{I\times(m-1)}$ such that

$$\mathbf{A}\mathrm{Diag}(\lambda_1, \ldots, \lambda_R)\mathbf{A}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T,$$

but $\omega(\lambda_1, \ldots, \lambda_R) \geq m$. We claim that if additionally $k_{\mathbf{A}} = I$, then $\omega(\lambda_1, \ldots, \lambda_R) \geq I$. Indeed, if $m = m_{\mathbf{C}} \leq \omega(\lambda_1, \ldots, \lambda_R) < I$, then by the Frobenius inequality,

$$\begin{aligned} m - 1 &\geq r_{\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T} = r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\ldots,\lambda_R)\mathbf{A}^T} \\ &\geq r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\ldots,\lambda_R)} + r_{\mathrm{Diag}(\lambda_1,\ldots,\lambda_R)\mathbf{A}^T} - r_{\mathrm{Diag}(\lambda_1,\ldots,\lambda_R)} \\ &= \omega(\lambda_1, \ldots, \lambda_R) + \omega(\lambda_1, \ldots, \lambda_R) - \omega(\lambda_1, \ldots, \lambda_R) = \omega(\lambda_1, \ldots, \lambda_R) \geq m, \end{aligned}$$

which is a contradiction. Hence, W in (4.4) can be expressed as

$$\begin{aligned} W = \{(\mathbf{A}, \mathbf{C}) &: (\mathrm{U_m}) \text{ does not hold for the pair } (\mathbf{A}, \mathbf{A}), \\ &\quad \text{all minors of } \mathbf{A} \text{ are nonzero, and } k_{\mathbf{C}} = K\} \\ = \{(\mathbf{A}, \mathbf{C}) &: \text{there exist } \lambda_1, \ldots, \lambda_R \in \mathbb{C},\ \widetilde{\mathbf{A}} \in \mathbb{C}^{I\times(m-1)} \text{ such that} \\ &\quad \mathbf{A}\mathrm{Diag}(\lambda_1, \ldots, \lambda_R)\mathbf{A}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T, \qquad (4.6) \\ &\quad \text{all minors of } \mathbf{A} \text{ are nonzero}, \qquad (4.7) \\ &\quad k_{\mathbf{C}} = K, \text{ and} \qquad (4.8) \\ &\quad \omega(\lambda_1, \ldots, \lambda_R) \geq I\}. \qquad (4.9) \end{aligned}$$

Obviously, $W = \pi^*(\widehat{Z}^*)$, where $\widehat{Z}^* = \{(\mathbf{A}, \mathbf{C}, \lambda_1, \ldots, \lambda_R, \widetilde{\mathbf{A}}) : \quad (4.6)\text{–}(4.9) \text{ hold}\} \subset \mathbb{C}^{I\times R} \times \mathbb{C}^{K\times R} \times \mathbb{C}^R \times \mathbb{C}^{I\times(m-1)}$ and π^* is the projection onto the first two factors,

$$\pi^* : \mathbb{C}^{I\times R} \times \mathbb{C}^{K\times R} \times \mathbb{C}^R \times \mathbb{C}^{I\times(m-1)} \to \mathbb{C}^{I\times R} \times \mathbb{C}^{K\times R}.$$

In step (2) of Procedure 2.6 we should express $\widehat{Z}^*$ as part of a finite union of sets of the form (2.2). That is, some entries of $\mathbf{A}$, $\mathbf{C}$, $\tilde{\mathbf{A}}$ and some of the values $\lambda_1, \ldots, \lambda_R$ (corresponding to dependent parameters) must be rational functions of the remaining entries of $\mathbf{A}$, $\mathbf{C}$, $\tilde{\mathbf{A}}$ and of the remaining values of $\lambda_1, \ldots, \lambda_R$ (corresponding to independent parameters). Since (4.6) is nonlinear in the entries of $\mathbf{A}$ and $\tilde{\mathbf{A}}$, it is not easy, if at all possible, to find such rational functions. To solve this problem, we further parameterize W, namely, we parameterize an $I \times I$ submatrix of $\mathbf{A}$ by its LDU decomposition, that is, we extend $\widehat{Z}^*$ to a "larger" $\widehat{Z}$ by adding the new variables $\mathbf{L}$, $\mathbf{D}$, and $\mathbf{U}$. In step (2) we will show that the new, LDU-based, parameterization will turn identity (4.6) into a UDU decomposition. The properties of the UDU decomposition will imply that the entries of $\mathbf{U}$ and $\mathbf{D}$ will be rational functions of the remaining parameters involved in (4.6).

By definition, set

$$\begin{aligned}\widehat{Z} := \bigcup_{1\le u_1<\cdots<u_I\le R} \{(\mathbf{A},\mathbf{C},\mathbf{L},\mathbf{D},\mathbf{U},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}}) :\\ \text{(4.6)–(4.8) hold, } \lambda_{u_1}\cdots\lambda_{u_I}\neq 0,\\ \mathbf{A}_{u_1,\dots,u_I}=\mathbf{LDU},\quad \mathbf{D} \text{ is nonsingular and diagonal,}\\ \mathbf{L} \text{ is unit lower triangular, } \mathbf{U} \text{ is unit upper triangular}\}\\ \subset \mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}\times\mathbb{C}^{I\times I}\times\mathbb{C}^{I\times I}\times\mathbb{C}^{I\times I}\times\mathbb{C}^{R}\times\mathbb{C}^{I\times(m-1)},\end{aligned} \tag{4.10}$$

where $\mathbf{A}_{u_1,\dots,u_I}$ denotes the submatrix of $\mathbf{A}$ formed by columns $u_1,\dots,u_I$. Let also π be the projection onto the first two factors

$$\pi:\ \mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}\times\mathbb{C}^{I\times I}\times\mathbb{C}^{I\times I}\times\mathbb{C}^{I\times I}\times\mathbb{C}^{R}\times\mathbb{C}^{I\times(m-1)}\to\mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}.$$

We show that $W=\pi(\widehat{Z})$. Since, by (4.7), all principal minors in the upper left-hand corner of $\mathbf{A}_{u_1,\dots,u_I}$ are nonzero, it follows from properties of the LDU decomposition that there exist $I\times I$ matrices $\mathbf{L}$, $\mathbf{D}$, and $\mathbf{U}$ such that $\mathbf{L}$ is unit lower triangular, $\mathbf{U}$ is unit upper triangular, $\mathbf{D}$ is nonsingular and diagonal, and $\mathbf{A}_{u_1,\dots,u_I}=\mathbf{LDU}$. In other words, the set $\pi(\widehat{Z})$ remains the same if the equality constraint $\mathbf{A}_{u_1,\dots,u_I}=\mathbf{LDU}$ in all subsets in the union in (4.10) is dropped. Hence, by (3.8),

$$\begin{aligned}\pi(\widehat{Z}) = \pi\big(\{(\mathbf{A},\mathbf{C},\mathbf{L},\mathbf{D},\mathbf{U},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}}) :\\ \text{(4.6)–(4.9) hold, } \mathbf{D} \text{ is nonsingular and diagonal,}\\ \mathbf{L} \text{ is unit lower triangular, } \mathbf{U} \text{ is unit upper triangular}\}\big) = W.\end{aligned}$$

(2) Dropping condition (4.7) in all subsets in the union in (4.10), we obtain that $\widehat{Z}\subset\bigcup_{1\le u_1<\cdots<u_I\le R} Z_{u_1,\dots,u_I}$, where

$$\begin{aligned}Z_{u_1,\dots,u_I} := \{(\mathbf{A},\mathbf{C},\mathbf{L},\mathbf{D},\mathbf{U},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}}) :\ \text{(4.6) and (4.8) hold, } \lambda_{u_1}\cdots\lambda_{u_I}\neq 0,\\ \mathbf{A}_{u_1,\dots,u_I}=\mathbf{LDU},\quad \mathbf{D} \text{ is nonsingular and diagonal,}\\ \mathbf{L} \text{ is unit lower triangular, } \mathbf{U} \text{ is unit upper triangular}\}.\end{aligned}$$

We show that all $Z_{u_1,\dots,u_I}$ are of the form (2.2). To simplify the presentation we restrict ourselves to the case $(u_1,\dots,u_I)=(1,\dots,I)$. The general case can be proved in the same way.

For brevity we set $\bar{\mathbf{A}}=\mathbf{A}_{1,\dots,I}$ and $\bar{\bar{\mathbf{A}}}=\mathbf{A}_{I+1,\dots,R}$, so that $\mathbf{A}=[\bar{\mathbf{A}}\ \bar{\bar{\mathbf{A}}}]$ with $\bar{\mathbf{A}}=\mathbf{LDU}$. We show that $Z_{1,\dots,I}$ is of the form (2.2), where $z_1,\dots,z_m$ correspond to the entries of $\bar{\bar{\mathbf{A}}}$, $\mathbf{C}$, $\mathbf{L}$, $\mathbf{D}$, $\widetilde{\mathbf{A}}$ and the values $\lambda_{I+1},\dots,\lambda_R$, where $z_{m+1},\dots,z_n$ correspond to the entries of $\bar{\mathbf{A}}$, $\mathbf{U}$ and the values $\lambda_1,\dots,\lambda_I$, and where the inequality constraint $\lambda_1\cdots\lambda_I\neq 0$ can be replaced by an equivalent inequality constraint which depends only on the entries of $\bar{\bar{\mathbf{A}}}$, $\mathbf{C}$, $\mathbf{L}$, $\mathbf{D}$, $\widetilde{\mathbf{A}}$ and the values $\lambda_{I+1},\dots,\lambda_R$.

Substituting $\mathbf{A}=[\mathbf{LDU}\ \bar{\bar{\mathbf{A}}}]$ into (4.6) we obtain

$$(\mathbf{LDU})\,\mathrm{Diag}(\lambda_1,\dots,\lambda_I)\,(\mathbf{LDU})^T+\bar{\bar{\mathbf{A}}}\mathrm{Diag}(\lambda_{I+1},\dots,\lambda_R)\bar{\bar{\mathbf{A}}}^T=\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T$$

or

$$\mathbf{U}\mathrm{Diag}(\lambda_1,\dots,\lambda_I)\mathbf{U}^T=\mathbf{M},\ \text{where} \tag{4.11}$$

$$\mathbf{M}:=(\mathbf{LD})^{-1}\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T-\bar{\bar{\mathbf{A}}}\mathrm{Diag}(\lambda_{I+1},\dots,\lambda_R)\bar{\bar{\mathbf{A}}}^T\right)(\mathbf{LD})^{-T}. \tag{4.12}$$

It follows from (4.11), the assumption $\lambda_1 \cdots \lambda_I \neq 0$, and properties of the UDU decomposition that each of the values $\lambda_1, \ldots, \lambda_I$ and each entry of $\mathbf{U}$ can be written as the ratio of two polynomials in entries of $\mathbf{M}$, that the denominator polynomial can always be chosen equal to one of the principal minors in the lower right-hand corner of $\mathbf{M}$, and that

$$\begin{aligned} & \lambda_1 \cdots \lambda_I \neq 0 \\ \Leftrightarrow\ & \text{all principal minors in the lower right-hand corner of } \mathbf{M} \text{ are nonzero} \\ \overset{(4.12)}{\Longleftrightarrow}\ & \text{all principal minors in the lower right-hand corner of the matrix} \\ & (\mathbf{LD})^{-1}\left(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T - \bar{\bar{\mathbf{A}}}\mathrm{Diag}(\lambda_{I+1}, \ldots, \lambda_R)\bar{\bar{\mathbf{A}}}^T\right)(\mathbf{LD})^{-T} \text{ are nonzero.} \end{aligned}$$

Hence, $Z_{1,\ldots,I}$ is indeed of the form (2.2).

Phase II: Obtaining a bound on $\mathbf{dim}\, \boldsymbol{W}$.

(3) To obtain bounds on $\dim \pi(Z_{u_1,\ldots,u_I})$ we follow the steps in Procedure 2.5 for $X = \pi(Z_{u_1,\ldots,u_I})$. W.l.o.g. we again restrict ourselves to the case $(u_1, \ldots, u_I) = (1. \ldots, I)$.

(i) By Lemma 2.3, $\dim \overline{Z}_{1,\ldots,I}$ is equal to the sum of the number of entries of $\bar{\bar{\mathbf{A}}}$, $\mathbf{C}$, $\mathbf{L}$, $\mathbf{D}$, $\widetilde{\mathbf{A}}$, and $[\lambda_{I+1}, \ldots, \lambda_R]$,

$$\begin{aligned} \dim \overline{Z}_{1,\ldots,I} &= I(R-I) + KR + (I^2 - I)/2 + I + (m-1)I + (R-I) \\ &= IR + KR + R - \frac{I^2 - I(2m-3)}{2}. \end{aligned}$$

(ii) Let $f : Z_{1,\ldots,I} \to \mathbb{C}^{I\times R} \times \mathbb{C}^{K\times R}$ denote the restriction of π to $Z_{1,\ldots,I}$:

$$f(\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \lambda_1, \ldots, \lambda_R, \widetilde{\mathbf{A}}) = (\mathbf{A}, \mathbf{C}), \quad (\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \lambda_1, \ldots, \lambda_R, \widetilde{\mathbf{A}}) \in Z_{1,\ldots,I}.$$

We need to find d such that $\dim f^{-1}(\mathbf{A}, \mathbf{C}) \geq d$ for all $(\mathbf{A}, \mathbf{C}) \in f(Z_{1,\ldots,I})$.

We make the assumption:

$$\begin{aligned} &\text{if } \ (\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \lambda_1, \ldots, \lambda_R, \widetilde{\mathbf{A}}) \in Z_{1,\ldots,I}, \\ &\text{then } \ (\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_R, \widehat{\mathbf{A}}) \in Z_{1,\ldots,I} \end{aligned} \tag{4.13}$$

for some $\widehat{\lambda}_1, \ldots, \widehat{\lambda}_R$ and full column rank matrix $\widehat{\mathbf{A}}$. That assumption (4.13) indeed holds will be demonstrated later. Since

$$\mathbf{A}\mathrm{Diag}(\alpha^2\widehat{\lambda}_1, \ldots, \alpha^2\widehat{\lambda}_R)\mathbf{A}^T = \alpha^2\mathbf{A}\mathrm{Diag}(\widehat{\lambda}_1, \ldots, \widehat{\lambda}_R)\mathbf{A}^T = \alpha^2\widehat{\mathbf{A}}\widehat{\mathbf{A}}^T = (\alpha\widehat{\mathbf{A}}\mathbf{T})(\alpha\widehat{\mathbf{A}}\mathbf{T})^T$$

for any nonzero α and an $(m-1)\times(m-1)$ matrix $\mathbf{T}$ such that $\mathbf{T}\mathbf{T}^T = \mathbf{I}$, it follows from the definition $Z_{1,\ldots,I}$ that $(\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \alpha^2\widehat{\lambda}_1, \ldots, \alpha^2\widehat{\lambda}_R, \alpha\widehat{\mathbf{A}}\mathbf{T}) \in Z_{1,\ldots,I}$. Thus,

$$\begin{aligned} f^{-1}(\mathbf{A}, \mathbf{C}) \supset \{&\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \alpha^2\widehat{\lambda}_1, \ldots, \alpha^2\widehat{\lambda}_R, \alpha\widehat{\mathbf{A}}\mathbf{T} : \\ &\alpha \in \mathbb{C},\ \mathbf{T} \in \mathbb{C}^{(m-1)\times(m-1)}\ \alpha \neq 0,\ \mathbf{T}\mathbf{T}^T = \mathbf{I}\}. \end{aligned}$$

Hence, by Lemma 2.2(iii) and by Lemma 4.3 below,

$$\begin{aligned} \dim f^{-1}(\mathbf{A}, \mathbf{C}) \geq\ & \dim\{\alpha\widehat{\mathbf{A}}\mathbf{T} :\ \alpha \in \mathbb{C},\ \mathbf{T} \in \mathbb{C}^{(m-1)\times(m-1)},\ \alpha \neq 0,\ \mathbf{T}\mathbf{T}^T = \mathbf{I}\} \\ =\ & \frac{(m-1)(m-2)}{2} + 1 := d. \end{aligned}$$

Let us now prove assumption (4.13). If $r_{\widetilde{\mathbf{A}}} = m-1$, then we just set $\widehat{\mathbf{A}} := \widetilde{\mathbf{A}}$ and $\widehat{\lambda}_i := \lambda_i$. If $r_{\widetilde{\mathbf{A}}} = k \le m-2$, then we construct $\widehat{\lambda}_1, \dots, \widehat{\lambda}_R$ and $\widehat{\mathbf{A}}$ from $\lambda_1, \dots, \lambda_R$, some (generic) values $\mu_1, \dots, \mu_{m-1-k}$, the matrix $\widetilde{\mathbf{A}}$, and certain $m-1-k$ columns of $\mathbf{A}_{1,\dots,I}$ as follows. Let $\mathbf{S}$ be a full column rank $I \times k$ matrix such that $\mathbf{S}\mathbf{S}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T$. Since the columns of $\mathbf{A}_{1,\dots,I}$ form a basis of $\mathbb{C}^I$, the columns of $\mathbf{S}$ can be extended to a basis of $\mathbb{C}^I$ by adding suitable $m-1-k$ columns of $\mathbf{A}_{1,\dots,I}$. In particular, the $I \times (m-1)$ matrix $\widehat{\mathbf{A}} := [\mathbf{S}\ \mathbf{A}_{i_1,\dots,i_{m-1-k}} \mathrm{Diag}(\mu_1, \dots, \mu_{m-1-k})]$ has full column rank for certain indicies $i_1, \dots, i_{m-1-k}$ such that $1 \le i_1 < \cdots < i_{m-1-k} \le I$ and for any nonzero values $\mu_1, \dots, \mu_{m-1-k}$. We construct $\widehat{\lambda}_1, \dots, \widehat{\lambda}_R$ from $\lambda_1, \dots, \lambda_R$ as

$$\begin{aligned} \widehat{\lambda}_{i_j} &= \lambda_{i_j} + \mu_j^2 && \text{for } j \in \{1, \dots, m-1-k\}, \\ \widehat{\lambda}_i &= \lambda_i && \text{for } i \notin \{i_1, \dots, i_{m-1-k}\}. \end{aligned}$$

We choose nonzero $\mu_1, \dots, \mu_{m-1-k}$ such that $\widehat{\lambda}_1 \cdots, \widehat{\lambda}_I \neq 0$. From (4.6) and the identity $\mathbf{S}\mathbf{S}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T$ it follows that

$$\begin{aligned} &\mathbf{A}\mathrm{Diag}(\widehat{\lambda}_1, \dots, \widehat{\lambda}_R)\mathbf{A}^T \\ &\quad = \mathbf{A}\mathrm{Diag}(\lambda_1, \dots, \lambda_R)\mathbf{A}^T + \mathbf{A}_{i_1,\dots,i_{m-1-k}} \mathrm{Diag}(\mu_1^2, \dots, \mu_{m-1-k}^2)\mathbf{A}_{i_1,\dots,i_{m-1-k}}^T \\ &\quad = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T + \mathbf{A}_{i_1,\dots,i_{m-1-k}} \mathrm{Diag}(\mu_1^2, \dots, \mu_{m-1-k}^2)\mathbf{A}_{i_1,\dots,i_{m-1-k}}^T \\ &\quad = [\mathbf{S}\ \mathbf{A}_{i_1,\dots,i_{m-1-k}} \mathrm{Diag}(\mu_1, \dots, \mu_{m-1-k})][\mathbf{S}\ \mathbf{A}_{i_1,\dots,i_{m-1-k}} \mathrm{Diag}(\mu_1, \dots, \mu_{m-1-k})]^T \\ &\quad = \widehat{\mathbf{A}}\widehat{\mathbf{A}}^T. \end{aligned}$$

Thus, $(\mathbf{A}, \mathbf{C}, \mathbf{L}, \mathbf{D}, \mathbf{U}, \widehat{\lambda}_1, \dots, \widehat{\lambda}_R, \widehat{\mathbf{A}}) \in Z_{1,\dots,I}$.

(iii) By Lemma 2.4 and (1.12),

$$\begin{aligned} \dim \overline{\pi(Z_{1,\dots,I})} &\le \dim Z_{1,\dots,I} - d \\ &= IR + KR + R - \frac{I^2 - I(2m-3)}{2} - \frac{(m-1)(m-2)}{2} - 1 \\ &\le IR + KR - 1. \end{aligned}$$

(4) Hence, $\dim W \le \max_{1 \le u_1 < \cdots < u_I \le R} \dim \pi(Z_{u_1,\dots,u_I}) \le IR + KR - 1$.

Since $W \subset \mathbb{C}^{I \times R} \times \mathbb{C}^{K \times R}$ and $\dim W \le IR + KR - 1$, it follows from Lemma 2.7 that $\mu_{\mathbb{C}}(W) = 0$. ∎

The following lemma is well known, it formalizes for a particular case the fact that $\dim Z_{\widehat{\mathbf{A}}}$ coincides with the number of "free parameters" (namely, one parameter for α and $\frac{(m-1)(m-2)}{2}$ parameters for the complex orthogonal matrix $\mathbf{T}$) used to describe $Z_{\widehat{\mathbf{A}}}$.

LEMMA 4.3. *Let the $I \times (m-1)$ matrix $\widehat{\mathbf{A}}$ have full column rank and*

$$Z_{\widehat{\mathbf{A}}} = \{\alpha \widehat{\mathbf{A}}\mathbf{T} :\ \alpha \in \mathbb{C},\ \mathbf{T} \in \mathbb{C}^{(m-1)\times(m-1)},\ \mathbf{T}\mathbf{T}^T = \mathbf{I}\} \subset \mathbb{C}^{I \times (m-1)}.$$

Then $\dim Z = \frac{(m-1)(m-2)}{2} + 1$.

5. Proof of Proposition 1.11. Proposition 1.11 follows immediately from Proposition 1.12. To formalize the derivation, we need the trivial observation expressed in the following lemma.

LEMMA 5.1. *Assume that the SFS-CPD of an $I \times I \times K_1$ SFS-tensor of SFS-rank R_1 is generically unique and let $K_1 \le K_2$, $R_2 \le R_1$. Then the SFS-CPD of an $I \times I \times K_2$ SFS-tensor of SFS-rank R_2 is generically unique.*

Proof of Proposition 1.11. Assume that $4 \leq I \leq K \leq \frac{I^2-I}{2}$. By Proposition 1.12, the SFS-CPD of an $I \times I \times K$ SFS-tensor of SFS-rank $R_1 = K$ is generically unique. The uniqueness for $R = R_2 \leq R_1 = K$ follows from Lemma 5.1.

Assume next that $K \geq K_1 = \frac{I^2-I}{2}$. By Proposition 1.12, the SFS-CPD of an $I \times I \times K_1$ SFS-tensor of SFS-rank $R_1 = \frac{I^2-I}{2}$ is generically unique. Now the result again follows from Lemma 5.1, for $K_2 = K$ and $R_2 = R$. ∎

6. Uniqueness of the SFS-CPD and proof of Proposition 1.13. The following deterministic condition for uniqueness of the SFS-CPD will be checked for a generic SFS-tensor in the proof of Proposition 1.13.

PROPOSITION 6.1. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R$ *and* $m_{\mathbf{A}} := R - r_{\mathbf{A}} + 2$. *Assume that*

(i) $\max(\min(k_{\mathbf{A}}, k_{\mathbf{C}} - 1),\ \min(k_{\mathbf{A}} - 1, k_{\mathbf{C}})) + k_{\mathbf{A}} \geq R + 1$*;*

(ii) *condition* ($\mathrm{U}_{\mathrm{m}_{\mathbf{A}}}$) *holds for the pair* $(\mathbf{A}, \mathbf{C})$.

Then $r_{\mathcal{T}} = R$ *and the SFS-CPD of tensor* $\mathcal{T}$ *is unique.*

Proof. It is clear that if the CPD of the reshaped $I \times K \times I$ tensor $\widetilde{\mathcal{T}} = [\mathbf{A}, \mathbf{C}, \mathbf{A}]_R$ is unique, then the CPD, and in particular, SFS-CPD $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R$ is unique. We show that (i)–(iii) in Proposition 3.2 hold for $\widetilde{\mathcal{T}} = [\mathbf{A}, \mathbf{C}, \mathbf{A}]_R$: (i) is obtained by replacing $k_{\mathbf{B}}$ and $k_{\mathbf{C}}$ by $k_{\mathbf{C}}$ and $k_{\mathbf{A}}$, respectively, and (ii) and (iii) follow from (4.1) and (4.2), respectively. ∎

Proof of Proposition 1.13. We show that

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{C}) :\ \text{(i) or (ii) of Proposition 6.1 does not hold}\} = 0.$$

As in the proofs of Propositions 1.6 and 1.12, it is sufficient to prove that

$$\mu_{\mathbb{F}}\{(\mathbf{A}, \mathbf{C}) : \text{(i) or (ii) of Proposition 6.1 does not hold and } k_{\mathbf{A}} = I,\ k_{\mathbf{C}} = K\} = 0.$$

Condition (i)*:* By (1.13)–(1.14),

$$R + 2 \leq I + K + \frac{I + K - 3 - \sqrt{(I + K - 3)^2 + 8K - 12}}{2} \leq I + K$$

which easily implies that condition (i) of Proposition 6.1 holds for all matrices $\mathbf{A}$ and $\mathbf{C}$ such that $k_{\mathbf{A}} = I$ and $k_{\mathbf{C}} = K$.

Condition (ii)*:* Let

$$W = \{(\mathbf{A}, \mathbf{C}) :\ \text{(ii) of Proposition 6.1 does not hold and } k_{\mathbf{A}} = I,\ k_{\mathbf{C}} = K\} \subset \mathbb{F}^{I \times R} \times \mathbb{F}^{K \times R}. \tag{6.1}$$

As in the proofs of Propositions 1.6 and 1.12, to prove that $\mu_{\mathbb{F}}\{W\} = 0$ we show that for $\mathbb{F} = \mathbb{C}$,

$$\dim \overline{W} \leq IR + KR - 1. \tag{6.2}$$

To prove bound (6.2) we follow the steps in Procedure 2.6.

Phase I: Parameterization.

(1) We associate W with a certain $\pi(\widehat{Z})$. By Definition 4.1, condition (U_{m}) does not hold for the pair $(\mathbf{A}, \mathbf{C})$ if and only if there exist values $\lambda_1, \ldots, \lambda_R$ and matrices

$\widetilde{\mathbf{A}} \in \mathbb{C}^{I\times(m-1)}$, $\widetilde{\mathbf{C}} \in \mathbb{C}^{K\times(m-1)}$ such that

$$\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{C}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{C}}^T,$$

but $\omega(\lambda_1,\dots,\lambda_R) \geq m$. We claim that if additionally $k_{\mathbf{A}} = I$ and $k_{\mathbf{C}} = K$, then $\omega(\lambda_1,\dots,\lambda_R) \geq K$. Indeed, if $m = m_{\mathbf{A}} \leq \omega(\lambda_1,\dots,\lambda_R) < K$, then by the Frobenius inequality,

$$\begin{aligned} m-1 &\geq r_{\widetilde{\mathbf{A}}\widetilde{\mathbf{C}}^T} = r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{C}^T} \\ &\geq r_{\mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)} + r_{\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{C}^T} - r_{\mathrm{Diag}(\lambda_1,\dots,\lambda_R)} \\ &= \omega(\lambda_1,\dots,\lambda_R) + \omega(\lambda_1,\dots,\lambda_R) - \omega(\lambda_1,\dots,\lambda_R) = \omega(\lambda_1,\dots,\lambda_R) \geq m, \end{aligned}$$

which is a contradiction. Hence, W in (6.1) can be expressed as

$$\begin{aligned} W &= \{(\mathbf{A},\mathbf{C}) : (\mathrm{U}_m) \text{ does not hold for the pair } (\mathbf{A},\mathbf{C}),\ k_{\mathbf{A}} = I,\ k_{\mathbf{C}} = K\} \\ &= \{(\mathbf{A},\mathbf{C}) : \text{there exist } \lambda_1,\dots,\lambda_R \in \mathbb{C},\ \widetilde{\mathbf{A}} \in \mathbb{C}^{I\times(m-1)},\ \text{ and } \ \widetilde{\mathbf{C}} \in \mathbb{C}^{K\times(m-1)} \\ &\qquad \text{such that } \ \mathbf{A}\mathrm{Diag}(\lambda_1,\dots,\lambda_R)\mathbf{C}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{C}}^T, \qquad (6.3) \\ &\qquad \omega(\lambda_1,\dots,\lambda_R) \geq K,\ k_{\mathbf{A}} = I,\ k_{\mathbf{C}} = K\}. \qquad (6.4) \end{aligned}$$

It is now clear that $W = \pi(\widehat{Z})$, where

$$\widehat{Z} = \{(\mathbf{A},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{C}}) : \quad (6.3)\text{–}(6.4) \text{ hold}\}$$

is a subset of $\mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}\times\mathbb{C}^R\times\mathbb{C}^{I\times(m-1)}\times\mathbb{C}^{K\times(m-1)}$ and π is the projection onto the first two factors

$$\pi :\ \mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}\times\mathbb{C}^R\times\mathbb{C}^{I\times(m-1)}\times\mathbb{C}^{K\times(m-1)} \to \mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}.$$

(2) Since

$$\omega(\lambda_1,\dots,\lambda_R) \geq K \quad\Leftrightarrow\quad \lambda_{u_1}\cdots\lambda_{u_K} \neq 0 \ \text{ for some } \ 1 \leq u_1 < \cdots < u_K \leq R$$

we obtain

$$\begin{aligned} \widehat{Z} = \bigcup_{1\leq u_1<\cdots<u_K\leq R} \{(\mathbf{A},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{C}}) :\ &(6.3) \text{ holds}, \\ &\lambda_{u_1}\cdots\lambda_{u_K} \neq 0,\ k_{\mathbf{A}} = I,\ k_{\mathbf{C}} = K\}. \end{aligned}$$

Let $\mathbf{C}_{u_1,\dots,u_K}$ denote the submatrix of $\mathbf{C}$ formed by columns $u_1,\dots,u_K$. Dropping the condition $k_{\mathbf{A}} = I$ and relaxing the condition $k_{\mathbf{C}} = K$ as $\det \mathbf{C}_{u_1,\dots,u_K} \neq 0$ we obtain that $\widehat{Z} \subset \bigcup_{1\leq u_1<\cdots<u_K\leq R} Z_{u_1,\dots,u_K}$, where

$$\begin{aligned} Z_{u_1,\dots,u_K} := \{(\mathbf{A},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{C}}) :\ &(6.3) \text{ holds}, \\ &\lambda_{u_1}\cdots\lambda_{u_K} \neq 0,\ \det \mathbf{C}_{u_1,\dots,u_K} \neq 0\}. \end{aligned}$$

We show that all $Z_{u_1,\dots,u_K}$ are of the form (2.2). To simplify the presentation we restrict ourselves to the case $(u_1,\dots,u_K) = (1,\dots,K)$. The general case can be proved in the same way. Let the matrices $\mathbf{A}$ and $\mathbf{C}$ be partitioned as

$$\mathbf{A} = [\underbrace{\bar{\mathbf{A}}}_{K}\ \underbrace{\bar{\bar{\mathbf{A}}}}_{R-K}], \qquad \mathbf{C} = [\underbrace{\bar{\mathbf{C}}}_{K}\ \underbrace{\bar{\bar{\mathbf{C}}}}_{R-K}],$$

so that $\bar{\mathbf{C}} = \mathbf{C}_{1,\dots,K}$. By (6.3),

$$\bar{\mathbf{A}} = \left(\widetilde{\mathbf{A}}\widetilde{\mathbf{C}}^T - \bar{\bar{\mathbf{A}}}\mathrm{Diag}(\lambda_{K+1},\dots,\lambda_R)\bar{\bar{\mathbf{C}}}^T\right)\left(\mathrm{Diag}(\lambda_1,\dots,\lambda_K)\bar{\mathbf{C}}\right)^{-1}.$$

Thus, by Cramer's rule, each entry of $\bar{\mathbf{A}}$ can be written as a ratio of two polynomials in entries of $\bar{\bar{\mathbf{A}}}$, $\mathbf{C}$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{C}}$ and the values $\lambda_1,\dots,\lambda_R$. By the assumptions $\lambda_1\cdots\lambda_K \neq 0$ and $\det\bar{\mathbf{C}} \neq 0$, the denominator polynomial is nonzero.

Hence, $Z_{1,\dots,K}$ is indeed of the form (2.2), where $z_1,\dots,z_m$ correspond to the entries of $\bar{\bar{\mathbf{A}}}$, $\mathbf{C}$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{C}}$ and the values $\lambda_1,\dots,\lambda_R$, and where $z_{m+1},\dots,z_n$ correspond to the entries of $\bar{\mathbf{A}}$.

Phase II: Obtaining a bound on $\mathbf{dim}\,\boldsymbol{W}$.

(3) To obtain bounds on $\dim\pi(Z_{u_1,\dots,u_K})$ we follow the steps in Procedure 2.5 for $X = \pi(Z_{u_1,\dots,u_K})$. W.l.o.g. we again restrict ourselves to the case $(u_1,\dots,u_K) = (1\dots,K)$.

(i) By Lemma 2.3, $\dim Z_{1,\dots,K} = I(R-K)+KR+I(m-1)+K(m-1)+R$.

(ii) From the definition of $Z_{1,\dots,K}$ it follows that if $(\mathbf{A},\mathbf{C},\lambda_1,\dots,\lambda_R,\widetilde{\mathbf{A}},\widetilde{\mathbf{C}}) \in Z_{1,\dots,K}$, then $(\mathbf{A},\mathbf{C},\alpha\lambda_1,\dots,\alpha\lambda_R,\widehat{\mathbf{A}}\mathbf{T},\alpha\widehat{\mathbf{C}}\mathbf{T}^{-T}) \in Z_{1,\dots,K}$, where α is an arbitrary nonzero value, $\widehat{\mathbf{A}}$ is an arbitrary full column rank matrix such that $\mathrm{range}(\widehat{\mathbf{A}}) \supset \mathrm{range}(\widetilde{\mathbf{A}})$, $\mathbf{T}$ is an arbitrary nonsingular $(m-1)\times(m-1)$ matrix, and $\widehat{\mathbf{C}}$ satisfies $\widehat{\mathbf{A}}\widehat{\mathbf{C}}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{C}}^T$. Hence,

$$f^{-1}(\mathbf{A},\mathbf{C}) \supset \{\mathbf{A},\mathbf{C},\alpha\lambda_1,\dots,\alpha\lambda_R,\widehat{\mathbf{A}}\mathbf{T},\alpha\widehat{\mathbf{C}}\mathbf{T}^{-T}:\ \alpha\neq 0,\ \det\mathbf{T}\neq 0\}.$$

By Lemma 2.2(iii), $\dim f^{-1}(\mathbf{A},\mathbf{C}) \geq \dim\{\alpha\lambda_1,\widehat{\mathbf{A}}\mathbf{T}:\ \alpha\neq 0,\ \det\mathbf{T}\neq 0\}$. Since by construction, the matrix $\widehat{\mathbf{A}}$ has full column rank and by assumption $\lambda_1 \neq 0$, it follows that $\dim\{\alpha\lambda_1,\widehat{\mathbf{A}}\mathbf{T}:\ \alpha\neq 0,\ \det\mathbf{T}\neq 0\} = 1+(m-1)^2 =: d$. Thus, $\dim f^{-1}(\mathbf{A},\mathbf{C}) \geq d$.

(iii) By Lemma 2.4 and (1.15),

$$\begin{aligned}\dim\overline{\pi(Z_{1,\dots,K})} &\leq \dim Z_{1,\dots,K} - d\\ &= I(R-K)+KR+I(m-1)+K(m-1)+R-1-(m-1)^2\\ &\leq IR+KR-1.\end{aligned}$$

(4) Hence, $\dim W \leq \max_{1\leq u_1<\dots<u_K\leq R}\dim\pi(Z_{u_1,\dots,u_K}) \leq IR+KR-1$.

Since $W \subset \mathbb{C}^{I\times R}\times\mathbb{C}^{K\times R}$ and $\dim W \leq IR+KR-1$, it follows from Lemma 2.7 that $\mu_{\mathbb{C}}(W) = 0$. ∎

7. Conclusion. We have obtained new conditions guaranteeing generic uniqueness of a CPD and INDSCAL. The overall derivation was based on deterministic conditions for uniqueness previously obtained in [12]. Our bounds improve existing results in the case when one of the tensor's dimensions is significantly larger than the other dimensions. The derivation made use of algebraic geometry. We summarized some results from algebraic geometry in a procedure that may be used in different applications to study generic conditions. It is important that the condition can be represented in terms of a subset of $\mathbb{C}^n$ parameterized by rational functions. The specific realization of the procedure depends on the problem at hand. In this paper we have explained how the procedure works for CPD and INDSCAL. In future work, we will consider the generic uniqueness of block term decompositions [9].

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