TENSOR DECOMPOSITIONS WITH VANDERMONDE FACTOR AND APPLICATIONS IN SIGNAL PROCESSING

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ABSTRACT

Tensor decompositions involving a Vandermonde factor are common in signal processing. For instance, they show up in sensor array processing and in wireless communication. We illustrate that by simultaneously taking the tensor nature and the Vandermonde structure of the problem into account new uniqueness results and numerical methods for computing a tensor decomposition with Vandermonde structure can be obtained.

Index Terms— tensor, canonical polyadic decomposition, block term decomposition, Vandermonde matrix, blind signal separation, blind system identification.

1. INTRODUCTION

Many problems in signal processing can be formulated as tensor decomposition problems involving a Vandermonde factor. Such problems appear in sensor array processing and in wireless communication. However, except for the one-dimensional harmonic retrieval case [17, 18] and the multidimensional harmonic retrieval case [21, 9, 14, 15], the Vandermonde structure has been more or less ignored in the tensor literature. This motivated us to develop identifiability conditions and numerical methods tailored for tensor decompositions with a Vandermonde factor.

This paper will illustrate the use of tensor decompositions with a Vandermonde factor by means of application in blind signal separation of oversampled systems in wireless communication employing a Uniform Linear Array (ULA) [13]. The problem also pops up in many other problems in wireless communication [22, 10, 11] and in array processing [20, 28, 16, 7]. More details on these applications are provided in [26, 27].

The paper is organized as follows. The rest of the introduction will present our notation followed by a quick review of the Canonical Polyadic Decomposition (CPD) and the $(P_r, P_r, 1)$-Block Term Decomposition ($(P_r, P_r, 1)$-BTD). Section 2 discusses the combined CPD and ULA based receiver for oversampled signals in the case of channels with small angle spread and small delay spread. Next, in section 3 we explain how to extend it to the case of large angle spread. Simulations are reported in section 4. Section 5 concludes the paper.

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1.1. Notation

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The rth column vector of $\mathbf{A}$ is denoted by $\mathbf{a}_r$. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao products, defined as

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathbf{A} \odot \mathbf{B} \triangleq \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \mathbf{a}_1 \otimes \mathbf{b}_1 & \dots \end{bmatrix},$$

in which $(\mathbf{A})_{mn} = a_{mn}$. The outer product of N vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ is denoted by $\circ$ such that $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ satisfies

$$\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)}\right)_{i_1, i_2, \dots, i_N} = a^{(1)}_{i_1} a^{(2)}_{i_2} \cdots a^{(N)}_{i_N}.$$

The transpose of the matrix $\mathbf{A}$ is denoted by $\mathbf{A}^T$. The k-rank of a matrix $\mathbf{A}$ is denoted by $k(\mathbf{A})$. It is equal to the largest integer $k(\mathbf{A})$ such that every subset of $k(\mathbf{A})$ columns of $\mathbf{A}$ is linearly independent. More generally, the k'-rank of a partitioned matrix $\mathbf{A}$ is denoted by $k'(\mathbf{A})$. It is equal to to the largest integer $k'(\mathbf{A})$ such that any set of $k'(\mathbf{A})$ submatrices of $\mathbf{A}$ yields a set of linearly independent columns.

1.2. Canonical Polyadic Decomposition

A third order rank-1 tensor $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ is defined as the outer product of non-zero vectors $\mathbf{a} \in \mathbb{C}^I$,$\mathbf{b} \in \mathbb{C}^J$ and $\mathbf{c} \in \mathbb{C}^K$ such that $\mathcal{X}_{ijk} = a_i b_j c_k$. The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that

yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r, \tag{1}$$

where $\mathbf{a}_r \in \mathbb{C}^I$, $\mathbf{b}_r \in \mathbb{C}^J$ and $\mathbf{c}_r \in \mathbb{C}^K$. This decomposition will be referred to as the CPD of $\mathcal{X}$.[1] It can be shown that under mild conditions the CPD of a tensor is unique, i.e., it is unique up to the inherent permutation and scaling ambiguities of the decomposition. We refer to [5, 6] and references therein for a discussion on this subject. As an example, we have the following existing uniqueness result to rely on.

Theorem 1.1 *Let $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ admit the CPD* (1)*. If*

$$(I-1)(J-1) \geq 2R(R-1) \quad \textit{and} \quad K \geq R\,, \tag{2}$$

then the CPD of $\mathcal{X}$ is generically[2] *unique up to the inherent permutation and scaling ambiguities of the decomposition [2].*

For exact CPD, (2) provides the most relaxed bound on R for which an algebraic solution is known [2]. However, in the presence of noise one usually has to resort to optimization-based algorithms possibly initialized by the algebraic solution, see [23] for a discussion on this issue.

1.3. $(P_r, P_r, 1)$-Block Term Decompostion

We say that a tensor $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ admits an R-term $(P_r, P_r, 1)$-BTD if it can be decomposed as [3]:

$$\mathcal{X} = \sum_{r=1}^{R} \left(\mathbf{A}^{(r)}\mathbf{B}^{(r)T}\right) \circ \mathbf{c}^{(r)}, \tag{3}$$

where $\mathbf{A}^{(r)} \in \mathbb{C}^{I\times P_r}$ and $\mathbf{B}^{(r)} \in \mathbb{C}^{J\times P_r}$ have full column rank and $\mathbf{c}^{(r)} \in \mathbb{C}^K$ are non-zero vectors. Note that when $P_r = 1$, $\forall r \in \{1,\ldots,R\}$, then (3) reduces to the CPD (1). We say that the $(P_r, P_r, 1)$-BTD of $\mathcal{X}$ in (3) is unique if all triplets $\left(\{\widehat{\mathbf{A}}^{(r)}\}, \{\widehat{\mathbf{B}}^{(r)}\}, \{\widehat{\mathbf{c}}^{(r)}\}\right)$ satisfying relation (3) are related via

$$\widehat{\mathbf{A}}^{(r)} = \mathbf{A}^{(r)}\alpha_r\mathbf{F}_r\,, \quad \widehat{\mathbf{B}}^{(r)} = \mathbf{B}^{(r)}\beta_r\mathbf{F}_r^{-1}\,, \quad \widehat{\mathbf{c}}^{(r)} = \mathbf{c}^{(r)}\gamma_r$$

up to a permutation of the $(P_r, P_r, 1)$-terms, where $\mathbf{F}_r \in \mathbb{C}^{P_r\times P_r}$ are nonsingular matrices and $\alpha_r, \beta_r, \gamma_r \in \mathbb{C}$ are scalars satisfying $\alpha_r\beta_r\gamma_r = 1$, $\forall r \in \{1,\ldots,R\}$. Uniqueness conditions for the $(P_r, P_r, 1)$-BTD can be found in [3, 4]. For instance, the following result was obtained.

Theorem 1.2 *Let $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$, then the R-term $(P_r, P_r, 1)$-BTD of $\mathcal{X}$ is unique if [3]:*

$$k'(\boldsymbol{A}) = R \quad \textit{and} \quad k'(\boldsymbol{B}) + k(\boldsymbol{C}) \geq R+2\,. \tag{4}$$

[1] The CPD is also known as the PARAFAC decomposition [8] or the CANDECOMP [1].

[2] A tensor decomposition property is called generic if it holds with probability one when the entries of the factor vectors or matrices are drawn from absolutely continuous probability density functions.

2. BLIND SEPARATION OF OVERSAMPLED SIGNALS IN THE CASE OF SMALL ANGLE SPREAD AND SMALL DELAY SPREAD

The connection between tensor decompositions and blind separation of oversampled signals, such as synchronous DS-CDMA, was discovered in [19] and has since then been well-studied. It has been done by formulating the separation as a CPD problem. More precisely, consider a DS-CDMA uplink system equipped with I antennas and R users transmitting synchronously over a flat fading channel. The output of the ith receive antenna at the jth chip period and kth symbol period is given by

$$y_{ijk} = \sum_{r=1}^{R} a_i^{(r)} h_j^{(r)} s_k^{(r)}\,,$$

where $a_i^{(r)}$, $h_j^{(r)}$ and $s_k^{(r)}$ denote the antenna gain response between user r and the ith antenna, the channel gain coefficient associated with user r at the jth chip period and the transmitted symbol from user r at the kth symbol period, respectively. Construct the vectors $\left(\mathbf{a}^{(r)}\right)_i = a_i^{(r)}$, $\left(\mathbf{h}^{(r)}\right)_j = h_j^{(r)}$ and $\left(\mathbf{s}^{(r)}\right)_k = s_k^{(r)}$, then we obtain the CPD

$$\mathbb{C}^{I\times J\times K} \ni \mathcal{Y} = \sum_{r=1}^{R} \mathbf{a}^{(r)} \circ \mathbf{h}^{(r)} \circ \mathbf{s}^{(r)}, \tag{5}$$

A visual representation of the decomposition (5) is given by figure 1. If R is not too large, then due to the CPD uniqueness properties it is possible under rather mild conditions to blindly recover and separate the transmitted symbol sequences $\{\mathbf{s}^{(r)}\}$ based on only the observed data $\mathcal{Y}$.

In many cases the receive array is structured. A commonly used structure is the ULA. Assume that the signals are narrowband and in the far-field. In that case the vectors $\{\mathbf{a}^{(r)}\}$ are Vandermonde structured [12], that is

$$\mathbf{a}^{(r)} = \left[1, z_r, z_r^2, \ldots, z_r^{I-1}\right]^T, \quad z_r \in \mathbb{C}\,,\ \forall r \in \{1,\ldots,R\}\,. \tag{6}$$

It was explained in [13] that by simultaneously taking the CPD structure (5) and the Vandermonde structure (6) into account a more robust separation of the incoming signals $\{\mathbf{s}^{(r)}\}$ is possible. We have shown in [26] that the ULA structure also leads to more relaxed uniqueness conditions. In particular, we demonstrated that the CPD (5) with Vandermonde vectors of the form (6) is generically unique under following condition.

Theorem 2.1 *Let $\mathcal{Y} \in \mathbb{C}^{I\times J\times K}$ admit the CPD* (5) *in which the vectors $\boldsymbol{a}^{(r)}$ are Vandermonde structured. If*

$$\left\lceil \frac{J}{R} \right\rceil + \left\lceil \frac{K}{R} \right\rceil \leq I\,, \tag{7}$$

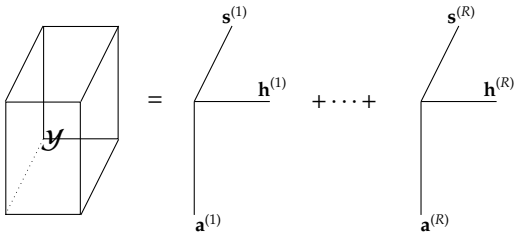


Fig. 1. A visual representation of the CPD of the observation tensor $\mathcal{Y}$ of the form (5) for the case of flat fading channels.

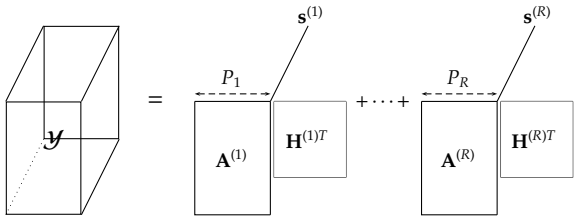


Fig. 2. A visual representation of the $(P_r, P_r, 1)$-BTD of the observation tensor $\mathcal{Y}$ of the form (9) for the case of channels with large angle spread.

then the Vandermonde structured CPD of $\mathcal{Y}$ is in the generic case unique up to intrinsic ambiguities [26].

Moreover, with R bounded as in (7) a CPD with Vandermonde structured $\{\mathbf{a}^{(r)}\}$ can efficiently be computed algebraically, as explained in [26]. Since it takes more of the structure into account, it yields a higher accuracy than the general purpose CPD algorithms. To summarize, by simultaneously exploiting the Vandermonde structure together with CPD structure of (5) better identifiability results and more efficient numerical methods can been obtained which allow us to extract the separate symbol sequences $\{\mathbf{s}^{(r)}\}$ from $\mathcal{Y}$ in (5).

3. BLIND SEPARATION OF OVERSAMPLED SIGNALS IN THE CASE OF LARGE ANGLE SPREAD AND SMALL DELAY SPREAD

In practice the communication channel may suffer from large angle spread and/or large delay spread. Thus, the CPD model (5) is in many practical problems too restrictive. Due to lack of space we limit the discussion to the case of large angle spread and small delay spread. We mention that the authors have also developed identifiability results and numerical methods for the case of large delay spread and possibly also large angle spread in [24, 25].

Assume that the receiver is equipped with ULA and that the channel suffers from large angle spread such that the output of the ith receive antenna at the jth chip and the kth symbol period is

$$y_{ijk} \quad = \quad \sum_{r=1}^{R} \sum_{p=1}^{P_r} a_i^{(p,r)} h_j^{(p,r)} s_k^{(r)}\,, \tag{8}$$

where $a_i^{(p,r)}$ is the ith antenna gain for the pth path of user r, $h_j^{(p,r)}$ is the channel gain coefficient associated with the pth path of user r at the jth chip period and $s_k^{(r)}$ is the transmitted symbol from user r at the kth symbol period, respectively. Construct the matrices $\mathbf{A}^{(r)} \in \mathbb{C}^{I \times P_r}$ and $\mathbf{B}^{(r)} \in \mathbb{C}^{J \times P_r}$ as follows $\left(\mathbf{A}^{(r)}\right)_{ip} = a_i^{(p,r)}$ and $\left(\mathbf{H}^{(r)}\right)_{jp} = h_j^{(p,r)}$ and the vectors $\mathbf{s}^{(r)} \in \mathbb{C}^K$ as follows $\left(\mathbf{s}^{(r)}\right)_k = s_k^{(r)}$, then we obtain the $(P_r, P_r, 1)$-BTD

$$\mathbb{C}^{I \times J \times K} \ni \mathcal{Y} = \sum_{r=1}^{R} \left(\mathbf{A}^{(r)} \mathbf{H}^{(r)T}\right) \circ \mathbf{s}^{(r)}\,. \tag{9}$$

A visual representation of the decomposition (9) can be seen in figure 2. Due to the ULA structure the matrices $\{\mathbf{A}^{(r)}\}$ are Vandermonde structured. Because of the large angle spread ($P_r > 1$) the CPD model (5) and its associated identifiability conditions are not valid anymore. Consequently, the authors have in [27] developed uniqueness conditions and numerical methods tailored to the Vandermonde structured $(P_r, P_r, 1)$-BTD problem (9). For instance the following result has been obtained.

Theorem 3.1 *Let $\mathcal{Y} \in \mathbb{C}^{I \times J \times K}$ admit the $(P_r, P_r, 1)$-BTD* (9) *in which the matrices $\boldsymbol{A}^{(r)}$ are Vandermonde matrices with column vectors of the form* (6)*. Define $P_{\max} = \max_{r \in \{1,\ldots,R\}} P_r$ and $P = \sum_{r=1}^{R} P_r$. If*

$$\left\lceil \frac{P}{J} \right\rceil + \max\left(\left\lceil \frac{P}{K} \right\rceil, P_{\max} \right) \leq I \quad \textit{and} \quad R \leq K\,, \tag{10}$$

then the Vandermonde structured $(P_r, P_r, 1)$-BTD of $\mathcal{Y}$ is in the generic case unique up to intrinsic ambiguities [27].

Again, by simultaneously exploiting the Vandermonde structure of $\mathbf{A}^{(r)}$ together with the $(P_r, P_r, 1)$-BTD structure of (9) better identifiability results than those only relying on the $(P_r, P_r, 1)$-BTD structure such as (4) are obtained. Furthermore, when R is bounded as in (10) a $(P_r, P_r, 1)$-BTD with Vandermonde structured $\{\mathbf{A}^{(r)}\}$ can be computed algebraically. The algorithm allows us to extract the separate symbol sequences $\{\mathbf{s}^{(r)}\}$ from $\mathcal{Y}$ in (9), as explained in [27].

4. SIMULATIONS

We illustrate the computational benefits of taking both the tensor and Vandermonde structure into consideration by means of simulations. We compare the proposed methods in [26, 27] with the Alternating Least Squares (ALS) method. For the ALS method we initialize the factor matrices randomly, or when possible, by means of the Generalized EVD (GEVD). When only one random initialization is used, the ALS method will simply be

referred to as ALS. When the best out of ten random initializations is used, the ALS method will be referred to as ALS-10. When the ALS method is initialized by means of the GEVD, then it will be referred to as ALS-GEVD. The methods proposed in [26, 27] tailored to the Vandermonde structured tensor decomposition problem will be referred to as VDM. When the VDM is followed by a few ALS steps, it is denoted by VDM-ALS.

The distance between $\mathbf{S} = \left[\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(R)}\right] \in \mathbb{C}^{K \times R}$ and its estimate $\widehat{\mathbf{S}}$, is measured as

$$\mathrm{P}(\mathbf{S}) = \min_{\Pi\Lambda} \left\| \mathbf{S} - \widehat{\mathbf{S}}\Pi\Lambda \right\|_F / \|\mathbf{S}\|_F \ ,$$

where $\|\cdot\|_F$ denotes the Frobenius norm, Π denotes a permutation matrix and Λ denotes a diagonal matrix. In order to find Π and Λ we use the greedy least squares column matching algorithm proposed in [19].

Since there may be interchip interference, the entries of the channel vectors $\mathbf{h}^{(r)}$ or $\mathbf{h}^{(p,r)}$ are randomly drawn from a Gaussian distribution with zero mean and unit variance [19]. The entries of the symbol vectors $\mathbf{s}^{(r)}$ are randomly drawn from a QPSK constellation. The generators of the Vandermonde vectors $\mathbf{a}^{(r)}$ or $\mathbf{a}^{(p,r)}$ are randomly chosen on the unit circle. The transmitted data are perturbed by additive Gaussian noise with zero mean and unit variance.

Let us first consider the case in section 2. The model parameters are $I = 3$, $J = 6$, $K = 100$ and $R = 5$. For this configuration it is not necessary to take the Vandermonde structure into account in order to guarantee identifiability. The mean $\mathrm{P}(\mathbf{S})$ values over 100 trials for varying SNR can be seen in figure 3. We observe that below 20 dB SNR the VDM-ALS, ALS-10 and ALS-GEVD methods perform about the same, while the VDM performs slightly worse. However, the ALS-10 and ALS-GEVD methods are computationally much more expensive than the VDM-ALS and VDM methods. The cost of VDM is about the cost of one ALS iteration. In conclusion, the best method for this problem is VDM-ALS.

Consider now the case when $I = 3$, $J = 4$, $K = 100$ and $R = 8$. In this configuration CPD is not unique by itself. Also, initialization by GEVD is not possible. On the other hand, the Vandermonde structured CPD is still unique. The mean $\mathrm{P}(\mathbf{S})$ values over 100 trials for varying SNR can be seen in figure 4. We notice that VDM and VDM-ALS work well while ALS and ALS-10 fail as expected.

Finally, we consider the case in section 3. The model parameters are $I = 4$, $J = 3$, $K = 50$, $R = 3$, $P_1 = 2$, $P_2 = 2$ and $P_3 = 2$. The mean $\mathrm{P}(\mathbf{S})$ values over 200 trials while SNR is varying can be seen in figure 5. We observe that the ALS and ALS-10 do not work. The reason is most likely that the unconstrained $(P_r, P_r, 1)$-BTD is not unique for this problem. [3] On the other hand, the VDM method is still valid for this problem.

5. CONCLUSION

Many problems in wireless communication and in array processing can be formulated as tensor decomposition problems involving a Vandermonde structure. In this short communication we illustrated its use in blind separation of oversampled signals such as DS-CDMA in the case of channels with small delay spread but possibly large angle spread. We argued that by simultaneously taking the tensor and Vandermonde structure of the problem into account it is possible to obtain better identifiability results and computationally efficient numerical methods.

6. REFERENCES

[1] J.D. Carroll and J.-J. Chang, *Analysis of Individual Differences in Multidimensional scaling via an N-way generalization of "Eckart-Young" Decomposition*, Psychometrika, 35(1970), pp. 283-319.

[2] L. De Lathauwer, *A Link between the Canonical Decomposition in Multilinear Algebra and Simultaneous Matrix Diagonalization*, SIAM J. Matrix Anal. Appl., 28(2006), pp. 642–666.

[3] L. De Lathauwer, *Decomposition of a Higher-Order Tensor in Block Terms – Part II: Definitions and Uniqueness*, SIAM J. Matrix Anal. Appl., 30(2008), pp. 1033–1066.

[4] L. De Lathauwer, *Blind Separation of Exponential Polynomials and the Decomposition of a Tensor in Rank-(Lr, Lr, 1) Terms*, SIAM J. Matrix Anal. Appl., 32(2011), pp. 1451–1474.

[5] I. Domanov and L. De Lathauwer, *On the Uniqueness of the Canonical Polyadic Decomposition – Part I: Basic Results and Uniqueness of One Factor Matrix*, Technical Report 12-66, ESAT-SISTA, KU Leuven, Belgium.

[6] I. Domanov and L. De Lathauwer, *On the Uniqueness of the Canonical Polyadic Decomposition – Part II: Overall Uniqueness*, Technical Report 12-72, ESAT-SISTA, KU Leuven, Belgium.

[7] X. Guo and S. Miron and D. Brie and S. Zhu and X. Liao, *A CANDECOMP/PARAFAC Perspective on Uniqueness of DOA Estimation Using a Vector Sensor Array*, IEEE Trans. Signal Process., 59(2011), pp. 3475 – 3481.

[8] R. A. Harshman, *Foundations of the Parafac procedure: Models and conditions for an Explanatory Multimodal Factor Analysis*, UCLA Working Papers in Phonetics, 16(1970), pp. 1–84.

[9] T. Jiang and N. D. Sidiropoulos and J. M. F. ten Berge, *Almost-Sure Identifiability of Multidimensional Harmonic Retrieval*, IEEE Trans. Signal Process., 49(2001), pp. 1849–1859.

[10] T. Jiang and N. D. Sidiropoulos, *A Direct Blind Receiver for SIMO and MIMO OFDM Systems Subject to Unknown Frequency Offset and Multipath*, Proc. SPAWC 2003, June 15-18, Rome, Italy, 2003.

[11] C. A. R. Fernandes and A. L. F. de Almeida and D. B. da Costa, *Unified Tensor Modeling for Blind Receivers in Multiuser Uplink Cooperative Systems*, IEEE Signal Processing Letters, 19(2012), pp. 247–250.

[3] The interested reader can find a necessary and sufficient condition for this unconstrained $(P_r, P_r, 1)$-BTD problem in [4].

[12] H. Krim and M. Viberg, *Two Decades of Array Signal Processing*, IEEE Signal Process. Mag., 14(1996), pp. 67–94.

[13] X. Liu and N. D. Sidiropoulos, *Cramér-Rao Lower Bounds for Low-Rank Decomposition of Multidimensional Arrays*, IEEE Trans. Signal Process., 49(2001), pp. 2074–2086.

[14] X. Liu and N. D. Sidiropoulos, *Almost Sure Identifiability of Multidimensional Constant Modulus Harmonic Retrieval*, IEEE Trans. Signal Process., 50(2002), pp. 2366–2368.

[15] J. Liu and X. Liu, *An Eigenvector-Based Approach for Multidimensional Frequency Estimation with Improved Identifiability*, IEEE Trans. Signal Process., 54(2006), pp. 2074–2086.

[16] D. Nion and N. D. Sidiropoulos, *Tensor Algebra and Multidimensional Harmonic Retrieval in Signal Processing for MIMO Radar*, IEEE Trans. Signal Processing, 58(2010), pp. 5693–5705.

[17] J. M. Papy and L. De Lathauwer and S. Van Huffel, *Exponential Data Fitting using Multilinear Algebra: The Single-Channel and Multi-Channel Case*, Numerical Linear Algebra with Applications, 12(2005), pp. 809–826.

[18] J. M. Papy and L. De Lathauwer and S. Van Huffel, *Exponential Data Fitting using Multilinear Algebra: The Decimative Case*, J. Chemometrics, 23(2009), pp. 341–351.

[19] N. D. Sidiropoulos and G. B. Giannakis and R. Bro, *Blind PARAFAC Receivers for DS-CDMA Systems*, IEEE Trans. Signal Process., 48(2000), pp. 810–823.

[20] N. D. Sidiropoulos and R. Bro and G. B. Giannakis, *Parallel Factor Analysis in Sensor Array Processing*, IEEE Trans. Signal Process., 48(2000), pp. 2377–2388.

[21] N. D. Sidiropoulos, *Generalizing Carathéodory's Uniqueness of Harmonic Parameterization to N Dimensions*, IEEE Trans. Inf. Theory, 47(2001), pp. 1687–1690.

[22] N. D. Sidiropoulos and X. Liu, *Identifiability Results for Blind Beamforming in Incoherent Multipath with Small Delay Spread*, IEEE Trans. Signal Process., 49(2001), pp. 228–236.

[23] L. Sorber and M. Van Barel and L. De Lathauwer, *Optimization-based Algorithms for Tensor Decompositions: Canonical Polyadic Decomposition, Decomposition in rank-$(L_r, L_r, 1)$ Terms and a New Generalization*, Technical Report 12-37, ESAT-SISTA, KU Leuven, Belgium.

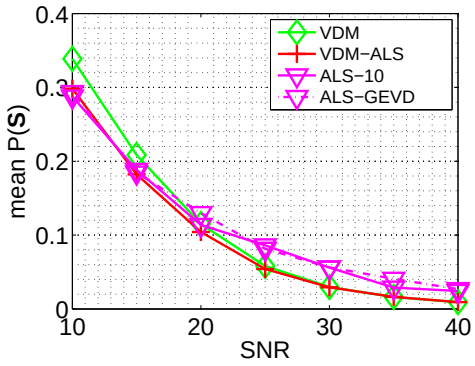
[24] M. Sørensen and L. De Lathauwer, *$(L_r, L_r, 1)$-Block Tensor Decomposition with a Block-Toeplitz Matrix Factor and Applications in Blind Convolutive Signal Separation*, Technical Report 11-216, ESAT-SISTA, KU Leuven, Belgium.

[25] M. Sørensen and L. De Lathauwer, *$(\cdot, L_r, P_r)$-Block Tensor Decomposition with a Block-Toeplitz Matrix Factor and Applications in Blind Convolutive Signal Separation*, Technical Report 11-217, ESAT-SISTA, KU Leuven, Belgium.

[26] M. Sørensen and L. De Lathauwer, *Blind Signal Separation via Tensor Decomposition with Vandermonde Factor – Part I: Canonical Polyadic Decomposition*, Technical Report 12-11, ESAT-SISTA, KU Leuven, Belgium.

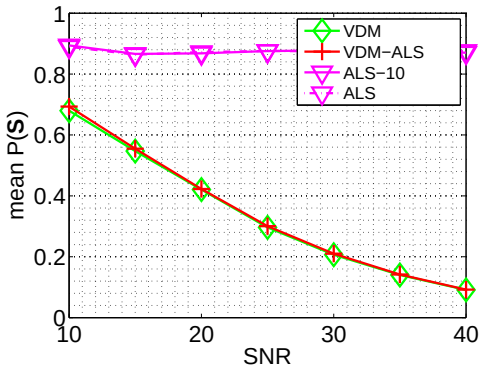
[27] M. Sørensen and L. De Lathauwer, *Blind Signal Separation via Tensor Decomposition with Vandermonde Factor – Part II: Channels with Coherent or Incoherent Multipath and Small Delay Spread*, Technical Report 12-12, ESAT-SISTA, KU Leuven, Belgium.

[28] X. Zhang and D. Xu, *Blind PARAFAC Signal Detection for Polarization Sensitive Array*,EURASIP Journal on Advances in Signal Processing, 2007, Article ID 12025.



(a) Mean P (**S**).

Fig. 3. Mean P (**S**) values over 100 trials while SNR is varying, case $I = 3$, $J = 6$, $K = 100$ and $R = 5$.



(a) Mean P (**S**).

Fig. 4. Mean P (**S**) values over 100 trials while SNR is varying, case $I = 3$, $J = 4$, $K = 100$ and $R = 8$.

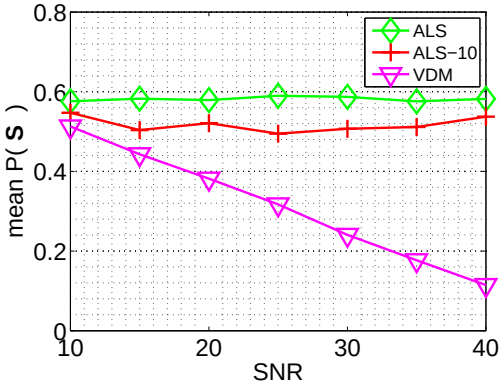


Fig. 5. Mean P (**S**) values over 200 trials while SNR is varying, case $I = 4$, $J = 3$, $K = 50$, $R = 3$, $P_1 = 2$, $P_2 = 2$ and $P_3 = 2$.