Blind Signal Separation via Tensor Decomposition with Vandermonde Factor Part I: Canonical Polyadic Decomposition

Mikael Sørensen and Lieven De Lathauwer, *Senior Member, IEEE*

***Abstract*—Several problems in signal processing have been formulated in terms of the Canonical Polyadic Decomposition of a higher-order tensor with one or more Vandermonde structured factor matrices. We first propose new, relaxed uniqueness conditions. We explain that, under these conditions, the number of components may simply be estimated as the rank of a matrix. We propose an efficient algorithm for the computation of the factors that only resorts to basic linear algebra. We demonstrate the use of the results for various applications in wireless communication and array processing.**

***Index Terms*—tensor, polyadic decomposition, parallel factor (PARAFAC), canonical decomposition (CANDECOMP), Vandermonde matrix, blind signal separation, polarization sensitive array, DS-CDMA, OFDM.**

I. Introduction

Canonical Polyadic Decomposition (CPD) of a higher-order tensor is its decomposition in a minimal number of rank-1 terms. Compared to decomposition of a matrix in rank-1 terms, CPD is unique under mild conditions [8]. This makes CPD a natural tool for signal separation [5]. In particular, orthogonality constraints like in Singular Value Decomposition (SVD) are not per se required for uniqueness. Nevertheless, imposing such constraints may increase the number of rank-1 terms for which uniqueness can be established, increase the accuracy of estimates, reduce the computational cost, facilitate interpretability, and so on [38]. In this paper we consider CPD with a different type of constraint, namely Vandermonde structure. We mean that the columns of at least one factor matrix take the form of exponentials. This is relevant for a broad class of signal processing problems, as is clear from the extensive literature on one-dimensional [26], [27] and multidimensional harmonic retrieval [31], [17], [24], [22]. We also mention that the Vandermonde structure may result from the use of a sinusoidal carrier in telecommunication applications, or from the use of a Uniform Linear Array (ULA) in array processing [40], [33], [42] and in wireless communication [23], [9]. The Vandermonde structure can also be caused by Doppler effects, carrier frequency offset or small delay spread in array processing and in wireless communication [25], [35], [15].

The paper is organized as follows. The rest of the introduction presents our notation, followed by a brief review of the CPD. In section II some existing properties of Vandermonde structured CPD and some techniques for handling exponential signals are briefly reviewed. Section III consists of five parts. In Subsection III-A we present new uniqueness results for CPD with a Vandermonde factor matrix. The derivation is constructive in the sense that it also provides us with an algorithm for the actual computation of the factors. The method can be understood as a generalization of well-known ESPRIT [28] from the matrix to the tensor case. The derivation also shows that in the noise-free case (or for a sufficiently high Signal-to-Noise Ratio (SNR)), the number of rank-1 terms can be estimated as the rank of a matrix. This is to be contrasted with the general CPD setting, in which the estimation of the number of components is NP-hard [13], [14]. The rank result is presented in Subsection III-B and the algorithm in Subsection III-C. In Subsection III-D we discuss situations in which more than one factor matrix is Vandermonde structured. Subsection III-E pays special attention to the case of Vandermonde vectors with multiplicity larger than one. In Sections IV–VII we discuss several applications in wireless communication and array processing. Section IV concerns MIMO radar. Section V concerns the blind separation of signals impinging on a polarization sensitive array. In section VI we explain the use in wireless communication systems employing a ULA at the receiver. Section VII explains that the results lead to improved identifiability results in MIMO OFDM with carrier frequency offset. Simulations are reported in section VIII. We conclude in section IX.

Part of this work was presented in the conference paper [37].

A. Notation

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The rth column vector of $\mathbf{A}$ is denoted by $\mathbf{a}_r$. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao products, defined as

M. Sørensen and L. De Lathauwer are with KU Leuven - E.E. Dept. (ESAT) - SCD-SISTA, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium, and the Group Science, Engineering and Technology, KU Leuven Kulak, E. Sabbelaan 53, 8500 Kortrijk, Belgium, {Mikael.Sorensen, Lieven.DeLathauwer}@kuleuven-kulak.be.

$$\mathbf{A}\otimes\mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathbf{A}\odot\mathbf{B} \triangleq \begin{bmatrix} \mathbf{a}_1\otimes\mathbf{b}_1 & \mathbf{a}_2\otimes\mathbf{b}_2 & \dots \end{bmatrix},$$

in which $(\mathbf{A})_{mn} = a_{mn}$. Sometimes we denote the Khatri-Rao product of N matrices $\{\mathbf{A}^{(n)}\}_{n=1}^N$ by

$$\bigodot_{n=1}^{N} \mathbf{A}^{(n)} = \mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(N)}.$$

The Hadamard product is given by

$$\mathbf{A}*\mathbf{B} \triangleq \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots \\ a_{21}b_{21} & a_{22}b_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $\circ$ denote the outer product of N vectors $\mathbf{a}^{(n)}\in\mathbb{C}^{I_n}$ such that $\mathbf{a}^{(1)}\circ\mathbf{a}^{(2)}\circ\cdots\circ\mathbf{a}^{(N)}\in\mathbb{C}^{I_1\times I_2\times\cdots\times I_N}$ satisfies

$$\left(\mathbf{a}^{(1)}\circ\mathbf{a}^{(2)}\circ\cdots\circ\mathbf{a}^{(N)}\right)_{i_1,i_2,\dots,i_N} = a_{i_1}^{(1)}a_{i_2}^{(2)}\cdots a_{i_N}^{(N)}.$$

We sometimes denote the Cartesian product of N vector spaces $\{\mathbb{C}^{I_n}\}_{n=1}^N$ by $\mathbb{C}^{\bigtimes_{n=1}^{N} I_n} = \mathbb{C}^{I_1\times\cdots\times I_N}$.

Matlab index notation will be used to extract submatrices from a given matrix. For example $\mathbf{A}(1:k,:)$ denotes the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k of $\mathbf{A}$. Let $\mathbf{T}\in\mathbb{C}^{I\times R}$, then $\overline{\mathbf{T}} = \mathbf{T}(2:I,:)\in\mathbb{C}^{(I-1)\times R}$ and $\underline{\mathbf{T}} = \mathbf{T}(1:I-1,:)\in\mathbb{C}^{(I-1)\times R}$, i.e., $\overline{\mathbf{T}}$ and $\underline{\mathbf{T}}$ are obtained by deleting the top and bottom row of $\mathbf{T}$, respectively.

The identity matrix, all-zeros vector and all-ones vector are denoted by $\mathbf{I}_R\in\mathbb{C}^{R\times R}$, $\mathbf{0}_R\in\mathbb{C}^R$ and $\mathbf{1}_R = [1,\dots,1]^T\in\mathbb{C}^R$, respectively. Let $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^\dagger$, $\mathrm{Re}\{\cdot\}$, $\mathrm{Im}\{\cdot\}$ and $\mathrm{Col}(\cdot)$ denote the transpose, conjugate, conjugate-transpose, Moore-Penrose pseudo-inverse, real part, imaginary part and column space of a matrix, respectively.

Given $\mathbf{A}\in\mathbb{C}^{I\times J}$, then $\mathrm{Vec}(\mathbf{A})\in\mathbb{C}^{IJ}$ will denote the column vector defined by $(\mathrm{Vec}(\mathbf{A}))_{i+(j-1)I} = (\mathbf{A})_{ij}$. Given $\mathbf{a}\in\mathbb{C}^{IJ}$, the reverse operation is $\mathrm{Unvec}(\mathbf{a}) = \mathbf{A}\in\mathbb{C}^{I\times J}$ such that $(\mathbf{a})_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $D_k(\mathbf{A})\in\mathbb{C}^{J\times J}$ denote the diagonal matrix holding row k of $\mathbf{A}\in\mathbb{C}^{I\times J}$ on its diagonal.

Let $r(\mathbf{A})$ denote the rank of $\mathbf{A}$. The k-rank of a matrix $\mathbf{A}$ is denoted by $k(\mathbf{A})$. It is equal to the largest integer $k(\mathbf{A})$ such that every subset of $k(\mathbf{A})$ columns of $\mathbf{A}$ is linearly independent.

The k-th compound matrix of $\mathbf{A}\in\mathbb{C}^{I\times R}$ is denoted by $C_k(\mathbf{A})\in\mathbb{C}^{C_I^k\times C_R^k}$, where $C_m^k = \frac{m!}{k!(m-k)!}$. It is the matrix containing the determinants of all $k\times k$ submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order, see [7] for a detailed explanation.

B. Canonical Polyadic Decomposition (CPD)

Given is an Nth-order tensor $\mathcal{X}\in\mathbb{C}^{I_1\times\cdots\times I_N}$. We say that $\mathcal{X}$ is a rank-1 tensor if it is equal to the outer product of some non-zero vectors $\mathbf{a}^{(n)}\in\mathbb{C}^{I_n}$, $n\in\{1,\dots,N\}$, such that $\mathcal{X}_{i_1\dots i_N} = \prod_{n=1}^N a_{i_n}^{(n)}$. More generally, the rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)}\circ\cdots\circ\mathbf{a}_r^{(N)}. \tag{1}$$

The decomposition (1) will in this paper be referred to as the CPD of $\mathcal{X}$. Let us stack the vectors $\{\mathbf{a}_r^{(n)}\}$ into the matrices

$$\mathbf{A}^{(n)} = \begin{bmatrix} \mathbf{a}_1^{(n)}, & \cdots & , \mathbf{a}_R^{(n)} \end{bmatrix}\in\mathbb{C}^{I_n\times R},\ n\in\{1,\dots N\}. \tag{2}$$

The matrices $\mathbf{A}^{(n)}$ will be referred to as factor matrices.

The following matrix representation of a CPD of $\mathcal{X}\in\mathbb{C}^{I_1\times I_2\times I_3}$ will be used throughout the paper. Let $\mathbf{X}^{(i_1\cdot\cdot)}\in\mathbb{C}^{I_2\times I_3}$ denote the matrix such that $\mathbf{X}^{(i_1\cdot\cdot)}_{i_2i_3} = \mathcal{X}_{i_1i_2i_3}$, then

$$\mathbb{C}^{I_1I_2\times I_3}\ni\mathbf{X}_{(1)} \triangleq \begin{bmatrix} \mathbf{X}^{(1\cdot\cdot)} \\ \vdots \\ \mathbf{X}^{(I_1\cdot\cdot)} \end{bmatrix} = \left(\mathbf{A}^{(1)}\odot\mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T}.$$

Similarly, let the matrix $\mathbf{X}^{(\cdot i_2\cdot)}\in\mathbb{C}^{I_3\times I_1}$ be constructed such that $\mathbf{X}^{(\cdot i_2\cdot)}_{i_3i_1} = \mathcal{X}_{i_1i_2i_3}$, then

$$\mathbb{C}^{I_2I_3\times I_1}\ni\mathbf{X}_{(2)} \triangleq \begin{bmatrix} \mathbf{X}^{(\cdot 1\cdot)} \\ \vdots \\ \mathbf{X}^{(\cdot I_2\cdot)} \end{bmatrix} = \left(\mathbf{A}^{(2)}\odot\mathbf{A}^{(3)}\right)\mathbf{A}^{(1)T}.$$

For Nth-order tensors, the following matrix representation will also be used throughout the paper. Consider $\mathcal{T}\in\mathbb{C}^{I_1\times\cdots\times I_N}$ with rank R, then

$$\begin{aligned}\mathbf{X}^{[P]} &\triangleq \begin{bmatrix} \mathcal{X}_{1,\dots,1,1,\dots,1} & \mathcal{X}_{1,\dots,1,1,\dots,2} & \cdots & \mathcal{X}_{1,\dots,1,I_{P+1},\dots,I_N} \\ \mathcal{X}_{1,\dots,2,1,\dots,1} & \mathcal{X}_{1,\dots,2,1,\dots,2} & \cdots & \mathcal{X}_{1,\dots,2,I_{P+1},\dots,I_N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_{I_1,\dots,I_P,1,\dots,1} & \mathcal{X}_{I_1,\dots,I_P,1,\dots,2} & \cdots & \mathcal{X}_{I_1,\dots,I_P,I_{P+1},\dots,I_N} \end{bmatrix} \\ &= \left(\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(P)}\right)\left(\mathbf{A}^{(P+1)}\odot\cdots\odot\mathbf{A}^{(N)}\right)^T. \end{aligned} \tag{3}$$

A CPD of $\mathcal{X}\in\mathbb{C}^{I_1\times\cdots\times I_N}$ is said to be unique if the rank-1 terms are unique up to permutation of the terms and scaling/counterscaling of vectors within the same term. Formally, the CPD is unique if all N-tuplets $\left(\widehat{\mathbf{A}}^{(1)},\dots,\widehat{\mathbf{A}}^{(N)}\right)$ satisfying (1) are related via

$$\widehat{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)}\mathbf{P}\Delta_{\mathbf{A}^{(n)}},\quad \forall n\in\{1,\dots N\},$$

where $\mathbf{P}$ is a permutation matrix and $\{\Delta_{\mathbf{A}^{(n)}}\}$ are diagonal matrices satisfying $\prod_{n=1}^N\Delta_{\mathbf{A}^{(n)}} = \mathbf{I}_R$. Unlike the decomposition of a matrix in rank-1 terms, CPD is unique under mild conditions, which makes it an important tool for blind signal separation. A well-known uniqueness result is the following.

Theorem I.1. *Let $\mathcal{X}\in\mathbb{C}^{I_1\times\cdots\times I_N}$ be a tensor with rank R. If*

$$\sum_{n=1}^{N} k\left(\mathbf{A}^{(n)}\right)\geq 2R+(N-1). \tag{4}$$

then the CPD of $\mathcal{X}$ is unique [19], [39], [32].

In the generic case[1], condition (4) becomes

[1]A CPD property is called generic if it holds with probability one when the entries of the factor matrices are drawn from absolutely continuous probability density functions.

$$\sum_{n=1}^{N} \min(I_n, R) \geq 2R + (N-1). \quad (5)$$

Condition (4) is sufficient but not necessary. For instance, for the case where one of the factor matrices of the CPD has full column rank, say $\mathbf{A}^{(3)}$, the following more relaxed uniqueness condition has been obtained [6], [16], [8].

Theorem I.2. *Consider the tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R and CPD representation $\boldsymbol{X}_{(1)} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right)\boldsymbol{A}^{(3)T}$. If*

$$\begin{cases} \boldsymbol{A}^{(3)} \text{ has full column rank,} \\ C_2\left(\boldsymbol{A}^{(1)}\right) \odot C_2\left(\boldsymbol{A}^{(2)}\right) \text{ has full column rank,} \end{cases}$$

then the CPD of $\mathcal{X}$ is unique. Furthermore, if

$$2R(R-1) \leq I_1(I_1-1)I_2(I_2-1) \quad \text{and} \quad R \leq I_3, \quad (6)$$

then the CPD of $\mathcal{X}$ is unique in the generic case [6].

Under the conditions in Theorem I.2 the CPD can be computed by means of linear algebra [6]. In general, one resorts to optimization-based methods [36].

One has recently obtained a relaxation of Theorem I.2 to the case where none of the involved factor matrices needs to have full column rank [7], [8].

Theorem I.3. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ be a tensor with rank R and CPD representation*

$$\boldsymbol{X}_{(1)} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right)\boldsymbol{A}^{(3)T},$$

where $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n \times R}$. If

$$\begin{cases} \min\left(k\left(\boldsymbol{A}^{(1)}\right) + k\left(\boldsymbol{A}^{(3)}\right), k\left(\boldsymbol{A}^{(2)}\right) + k\left(\boldsymbol{A}^{(3)}\right)\right) \geq R+1, \\ \max\left(k\left(\boldsymbol{A}^{(1)}\right) + k\left(\boldsymbol{A}^{(3)}\right), k\left(\boldsymbol{A}^{(2)}\right) + k\left(\boldsymbol{A}^{(3)}\right)\right) \geq R+2, \\ C_{R-r(\boldsymbol{A}^{(3)})+2}\left(\boldsymbol{A}^{(1)}\right) \odot C_{R-r(\boldsymbol{A}^{(3)})+2}\left(\boldsymbol{A}^{(2)}\right) \text{ has full column rank,} \end{cases}$$

then the CPD of $\mathcal{X}$ is unique.

II. CPD with Vandermonde Factor Matrix

The factor matrix $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$ is said to be Vandermonde structured if

$$\mathbf{A}^{(n)} = \left[\mathbf{a}_1^{(n)}, \ldots, \mathbf{a}_R^{(n)}\right], \quad \mathbf{a}_r^{(n)} = \left[1, z_{n,r}, \ldots, z_{n,r}^{I_n-1}\right]^T, \quad (7)$$

where the scalars $\{z_{n,r}\}$ are called the generators of $\mathbf{A}^{(n)}$. We say that a CPD is Vandermonde structured if one or more of the factor matrices $\{\mathbf{A}^{(n)}\}$ are Vandermonde matrices.

In a Vandermonde structured CPD the scaling/counterscaling indeterminacies do not involve the Vandermonde factors. Consider $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R and constructed from the factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1, \ldots, N\}$. Assume that the factors $\mathbf{A}^{(p)}$, $1 \leq p \leq P \leq N$ are Vandermonde matrices. Let the N-tuplet $\left(\widehat{\mathbf{A}}^{(1)}, \ldots, \widehat{\mathbf{A}}^{(N)}\right)$ yield an alternative decomposition of $\mathcal{X}$. Formally, the Vandermonde structured CPD of $\mathcal{X}$ is said to be unique if

$$\begin{aligned} \widehat{\mathbf{A}}^{(p)} &= \mathbf{A}^{(p)}\mathbf{P}, \quad \forall p \in \{1, \ldots, P\} \\ \widehat{\mathbf{A}}^{(q)} &= \mathbf{A}^{(q)}\mathbf{P}\Delta_q, \quad \forall q \in \{P+1, \ldots, N\}, \end{aligned}$$

where $\mathbf{P} \in \mathbb{C}^{R \times R}$ is a permutation matrix and $\Delta_q \in \mathbb{C}^{R \times R}$ are diagonal matrices with property $\prod_{q=P+1}^{N} \Delta_q = \mathbf{I}_R$.

Subsection II-A reviews an existing uniqueness condition for a CPD with a Vandermonde factor matrix. Next, subsection II-B presents a direct variant of ESPRIT [28]. In subsection II-C we discuss the common technique of spatial smoothing.

A. Existing Uniqueness Result

The following uniqueness result for a CPD with a Vandermonde factor matrix was obtained in [35]:

Theorem II.1. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R be constructed from the factor matrices $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_1 \times R}$, $n \in \{1,2,3\}$. Assume that $\boldsymbol{A}^{(1)}$ is a Vandermonde matrix of the form (7) with distinct generators, then the Vandermonde structured CPD of $\mathcal{X}$ is unique if*

$$\min\left(2R, I_1 + k\left(\boldsymbol{A}^{(3)}\right)\right) + k\left(\boldsymbol{A}^{(2)}\right) \geq 2(R+1). \quad (8)$$

In the generic case, condition (8) becomes

$$\min\left(2R, I_1 + \min(I_3, R)\right) + \min(I_2, R) \geq 2(R+1). \quad (9)$$

B. ESPRIT

Consider $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with CPD represented by $\mathbf{X}_{(2)} = \left(\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right)\mathbf{A}^{(1)T}$. If $\mathbf{A}^{(1)}$ is a tall ($I_1 > R$) Vandermonde matrix and $\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}$ has full column rank, then we might just ignore the Khatri-Rao structure and compute $\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}$ and $\mathbf{A}^{(1)}$ by means of classical ESPRIT [28]. We state this for arbitrary order in the following theorem.

Theorem II.2. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R be constructed from the factor matrices $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1, \ldots, N\}$. Let $\boldsymbol{A}^{(1)}$ be a tall ($I_1 > R$) Vandermonde matrix of the form (7) with distinct generators $\{z_r\}_{r=1}^{R}$. If the matrix $\bigodot_{n=2}^{N} \boldsymbol{A}^{(n)}$ has full column rank, then the Vandermonde structured CPD of $\mathcal{X}$ is unique. Generically, this is true if*

$$\prod_{n=2}^{N} I_n \geq R \quad \text{and} \quad I_1 > R. \quad (10)$$

C. Spatial Smoothing

Spatial smoothing is a technique commonly applied in sensor array processing to overcome the problems that can occur when some of the involved matrices are rank deficient [30]. It has been extended to the tensorial case in for instance [40], [31], [35] and will be reviewed next.

Given is a tensor

$$\mathbb{C}^{\bigtimes_{n=1}^{N} I_n} \ni \mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(N)}, \quad (11)$$

where we assume that the factor matrices $\mathbf{A}^{(p)} \in \mathbb{C}^{I_p \times R}$, $1 \leq p \leq P < N$, are Vandermonde matrices. From (3) we get the following matrix representation of (11):

$$\mathbb{C}^{\prod_{p=1}^{P} I_p \times \prod_{q=P+1}^{N} I_q} \ni \mathbf{X}^{[P]} = \left(\bigodot_{p=1}^{P} \mathbf{A}^{(p)}\right)\left(\bigodot_{q=P+1}^{N} \mathbf{A}^{(q)}\right)^T. \quad (12)$$

If the matrix $\bigodot_{q=P+1}^{N} \mathbf{A}^{(q)}$ does not have full column rank, i.e., its rank is less than R, then the matrix $\mathbf{X}^{[P]}$ also has

rank less than R. The goal of spatial smoothing is to obtain a matrix that has rank R.

Construct a tensor $\mathcal{Y} \in \mathbb{C}^{\overset{P}{\underset{p=1}{\times}} L_p \times \overset{P}{\underset{p=1}{\times}} K_p \times \overset{N}{\underset{q=P+1}{\times}} I_q}$ by writing the first P Vandermonde matrices in (11) as products of two Vandermonde matrices. Let

$$\begin{aligned}\mathcal{Y}_{k_1,\ldots,k_P,i_{P+1},\ldots,i_Q,l_1,\ldots,l_P,i_{Q+1},\ldots,i_N} &= \mathcal{X}_{l_1+k_1-1,\ldots,l_P+k_P-1,i_{P+1},\ldots,i_N} \\ &= \sum_{r=1}^{R}\prod_{p=1}^{P} z_{p,r}^{l_p+k_p-2}\prod_{q=P+1}^{N} a_{i_q,r}^{(q)} \\ &= \sum_{r=1}^{R}\prod_{s=1}^{P} z_{s,r}^{l_s-1}\prod_{t=1}^{P} z_{t,r}^{k_t-1}\prod_{q=P+1}^{N} a_{i_q,r}^{(q)}\end{aligned}$$

where $l_p \in \{1,\ldots,L_p\}$, $k_p \in \{1,\ldots,K_p\}$ and $L_p + K_p = I_p + 1$, $p \in \{1,\ldots,P\}$, then from matrix representation (3) we get

$$\mathbf{Y}^{[P]} = \left(\bigodot_{p=1}^{P}\mathbf{A}^{(K_p,p)} \odot \bigodot_{q=P+1}^{Q}\mathbf{A}^{(q)}\right)\left(\bigodot_{p=1}^{P}\mathbf{A}^{(L_p,p)} \odot \bigodot_{q=Q+1}^{N}\mathbf{A}^{(q)}\right)^T \quad (13)$$

with $\mathbf{A}^{(K_p,p)} \in \mathbb{C}^{K_p\times R}$ and $\mathbf{A}^{(L_p,p)} \in \mathbb{C}^{L_p\times R}$ given by

$$\begin{aligned}\mathbf{A}^{(K_p,p)} &= \left[\mathbf{a}_1^{(K_p,p)},\ldots,\mathbf{a}_R^{(K_p,p)}\right], \quad \mathbf{a}_r^{(K_p,p)} = \left[1, z_{p,r},\ldots,z_{p,r}^{K_p-1}\right]^T, \\ \mathbf{A}^{(L_p,p)} &= \left[\mathbf{a}_1^{(L_p,p)},\ldots,\mathbf{a}_R^{(L_p,p)}\right], \quad \mathbf{a}_r^{(L_p,p)} = \left[1, z_{p,r},\ldots,z_{p,r}^{L_p-1}\right]^T.\end{aligned}$$

Hence, if a matrix representation (12) of the tensor $\mathcal{X}$ in (11) is such that it involves a rank deficient matrix $\bigodot_{q=P+1}^{N}\mathbf{A}^{(q)}$, then due to (13) it may be possible to replace it by a matrix representation that only involves full column rank matrices $\bigodot_{p=1}^{P}\mathbf{A}^{(K_p,p)} \odot \bigodot_{q=P+1}^{Q}\mathbf{A}^{(q)}$ and $\bigodot_{p=1}^{P}\mathbf{A}^{(L_p,p)} \odot \bigodot_{q=Q+1}^{N}\mathbf{A}^{(q)}$.

III. New Results for Vandermonde Structured CPD

Subsection III-A presents new uniqueness conditions for a CPD with a Vandermonde factor matrix. Subsection III-B briefly discusses the estimation of the rank. Subsection III-C provides an algorithm for the actual computation. Subsection III-D discusses extensions to cases where more than factor matrix is Vandermonde structured. In subsection III-E we discuss the special case where some of the generators of the Vandermonde matrix are repeated.

A. Uniqueness

If $\mathbf{A}^{(1)}$ is a Vandermonde matrix and the matrix $\mathbf{A}^{(3)}$ has full column rank, then we can obtain a uniqueness result that is more relaxed than Theorems II.1 and II.2. This result is given as Theorem III.2 below. The proof of the deterministic part is a Khatri-Rao variant of the derivation of ESPRIT [28]. The proof of the generic version is based on the following lemma.

Lemma III.1. *Let $A^{(1)} \in \mathbb{C}^{I_1\times R}$ be a Vandermonde matrix and let $A^{(2)} \in \mathbb{C}^{I_2\times R}$, then the matrix $A^{(1)} \odot A^{(2)}$ generically has rank* $\min(I_1 I_2, R)$.

Proof: The result follows from Theorem 3 and Corollary 1 in [17]. ∎

Theorem III.2. *Let $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ with rank R be constructed from the factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n\times R}$, $n \in \{1,2,3\}$. Let $\mathbf{A}^{(1)}$ be a Vandermonde matrix with distinct generators $\{z_r\}_{r=1}^R$. If the matrices $\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ have full column rank, then the Vandermonde structured CPD of $\mathcal{X}$ is unique. Generically, this is true if*

$$\min\left((I_1-1)I_2, I_3\right) \geq R. \quad (14)$$

Proof: Consider the matrix representation $\mathbf{X}_{(1)} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T}$ of $\mathcal{X}$ and let $\mathbf{X}_{(1)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact Singular Value Decomposition (SVD) of $\mathbf{X}_{(1)}$, in which $\mathbf{U} \in \mathbb{C}^{I_1I_2\times R}$, $\Sigma \in \mathbb{C}^{R\times R}$ and $\mathbf{V} \in \mathbb{C}^{I_3\times R}$. Since we assume that the matrices $\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ have full column rank, we know that there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ such that

$$\mathbf{U}\mathbf{M} = \mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}. \quad (15)$$

Due to the Vandermonde structure of $\mathbf{A}^{(1)}$ we have

$$\left(\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}\right)\mathbf{Z} = \overline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}, \quad (16)$$

where $\mathbf{Z} = D_1([z_1,\ldots z_R])$. The Vandermonde structure of $\mathbf{A}^{(1)}$ and relation (15) imply that

$$\mathbf{U}_1\mathbf{M} = \underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)} \quad (17)$$

$$\mathbf{U}_2\mathbf{M} = \overline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}, \quad (18)$$

where

$$\begin{aligned}\mathbf{U}_1 &= \mathbf{U}(1:(I_1-1)I_2,:) \in \mathbb{C}^{(I_1-1)I_2\times R} \\ \mathbf{U}_2 &= \mathbf{U}(I_2+1:I_1I_2,:) \in \mathbb{C}^{(I_1-1)I_2\times R}.\end{aligned}$$

By combining (16), (17) and (18) we get the following chain of equalities:

$$\mathbf{U}_2\mathbf{M} = \overline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)} = \left(\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}\right)\mathbf{Z} = \mathbf{U}_1\mathbf{M}\mathbf{Z}. \quad (19)$$

Thus, from (19) we get $\mathbf{U}_2 = \mathbf{U}_1\widehat{\mathbf{Z}}$, where $\widehat{\mathbf{Z}} = \mathbf{M}\mathbf{Z}\mathbf{M}^{-1}$. Assuming that $\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}$ has full column rank, $\mathbf{U}_1$ and $\mathbf{U}_2$ have full column rank and $\widehat{\mathbf{Z}} = \mathbf{U}_1^\dagger\mathbf{U}_2$. From the EigenValue Decomposition (EVD) $\widehat{\mathbf{Z}} = \mathbf{M}\mathbf{Z}\mathbf{M}^{-1}$ the generators $\{z_r\}_{r=1}^R$, $\mathbf{A}^{(1)}$ and $\mathbf{M}$ are obtained, up to permutation ambiguities.

Given $\mathbf{A}^{(1)}$ and $\mathbf{M}$, the next step is to find $\mathbf{A}^{(2)}$. Due to the scaling ambiguity of the problem, we can without loss of generality assume that the column vectors of $\mathbf{A}^{(2)}$ are scaled such that from (15) we get

$$\mathbf{a}_r^{(2)} = \left(\mathbf{a}_r^{(1)H} \otimes \mathbf{I}_{I_2}\right)\mathbf{U}\mathbf{m}_r, \quad r \in \{1,\ldots,R\}. \quad (20)$$

Given $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ follows directly from [18]:

$$\mathbf{A}^{(3)T} = \left(\left(\mathbf{A}^{(1)H}\mathbf{A}^{(1)}\right) * \left(\mathbf{A}^{(2)H}\mathbf{A}^{(2)}\right)\right)^{-1}\left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)^H\mathbf{X}_{(1)}.$$

The generic version follows from the above reasoning together with Lemma III.1. ∎

To appreciate the new Theorem III.2, let us compare it with Theorems I.1, I.2, II.2 and II.1 in a few cases. Assume that $\mathbf{A}^{(1)}$ is Vandermonde structured, $\mathbf{A}^{(3)}$ has full column rank and $I_1 = I_2$. The maximal value R for which the conditions guarantee uniqueness in the generic case are given in Table I for various I_1 values. From Table I it is clear that Theorem I.1 is not appropriate when $\mathbf{A}^{(3)}$ has full column rank and $\mathbf{A}^{(1)}$ is Vandermonde structured. We also notice that in the generic

$I_1 = I_2$	2	3	4	5	6	7	8	9	10
Theorem I.1	**3**	5	7	9	11	13	15	17	19
Theorem I.2	**3**	5	10	15	22	31	41	52	65
Theorem II.1	**3**	5	7	9	11	13	15	17	19
Theorem II.2	1	2	3	4	5	6	7	8	9
Theorem III.2	**3**	**7**	**13**	**21**	**31**	**43**	**57**	**73**	**91**

TABLE I

MAXIMUM VALUE R FOR WHICH THEOREMS I.1, I.2, II.1, II.2 AND III.2 GUARANTEE UNIQUENESS IN THE GENERIC CASE OF A CPD WITH $I_1 = I_2$, $I_3 \geq R$ AND $\mathbf{A}^{(1)}$ VANDERMONDE STRUCTURED.

case Theorem II.1 yields the same result as Theorem I.1. Theorem I.2 yields a more relaxed uniqueness condition than Theorems I.1 and II.1. The reason is that Theorem I.2 takes into account that $\mathbf{A}^{(3)}$ has full column rank. Finally, we notice that the new Theorem III.2 yields an even more relaxed uniqueness condition than Theorem I.2. The reason is that Theorem III.2 simultaneously takes into account that $\mathbf{A}^{(1)}$ is Vandermonde structured and $\mathbf{A}^{(3)}$ has full column rank. In Theorem II.2 R is bounded by the dimension of the Vandermonde factor.

We can also combine Theorem III.2 with spatial smoothing. In this way we can obtain CPD uniqueness in cases where $\mathbf{A}^{(3)}$ does not have full column rank. We have the following uniqueness result.

Corollary III.3. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R be constructed from $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1,2,3\}$, where $\boldsymbol{A}^{(1)}$ is a Vandermonde matrix with distinct generators $\{z_r\}_{r=1}^{R}$. Consider the matrix representation $\boldsymbol{X} = \left(\boldsymbol{A}^{(K_1,1)} \odot \boldsymbol{A}^{(2)}\right)\left(\boldsymbol{A}^{(L_1,1)} \odot \boldsymbol{A}^{(3)}\right)^T$ with $K_1 + L_1 = I_1 + 1$. If*

$$\begin{cases} r\left(\underline{\boldsymbol{A}}^{(K_1,1)} \odot \boldsymbol{A}^{(2)}\right) = R \\ r\left(\boldsymbol{A}^{(L_1,1)} \odot \boldsymbol{A}^{(3)}\right) = R \end{cases} \tag{21}$$

for some $K_1 + L_1 = I_1 + 1$, then the Vandermonde structured CPD of $\mathcal{X}$ is unique. Generically, this is true if and only if

$$\left\lceil \frac{R}{I_2} \right\rceil + \left\lceil \frac{R}{I_3} \right\rceil \leq I_1 \,. \tag{22}$$

Proof: Let us first prove condition (21). Let $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{X}$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R \times R}$ such that

$$\mathbf{UM} = \mathbf{A}^{(K_1,1)} \odot \mathbf{A}^{(2)} \tag{23}$$

$$\mathbf{V}^*\Sigma\mathbf{N} = \mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(3)}, \quad \mathbf{N} = \mathbf{M}^{-T}. \tag{24}$$

Due to Theorem III.2 we know that $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{M}$ follow from (23). From (24) we obtain $\mathbf{A}^{(3)}$ as follows

$$\mathbf{a}_r^{(3)} = \left(\frac{\mathbf{a}_r^{(L_1,1)}}{\mathbf{a}_r^{(L_1,1)H}\mathbf{a}_r^{(L_1,1)}} \otimes \mathbf{I}_{I_3} \right) \mathbf{V}^*\Sigma\mathbf{n}_r \,, \quad r \in \{1,\ldots,R\} \,.$$

Let us now prove that (22) is generically equivalent to (21). Due to Lemma III.1 we know that condition (21) is generically true if

$$\min\left((K_1 - 1)\, I_2, L_1 I_3\right) \geq R \,, \tag{25}$$

i.e., $K_1 - 1 \geq \left\lceil \frac{R}{I_2} \right\rceil$ and $L_1 \geq \left\lceil \frac{R}{I_3} \right\rceil$ in which K_1 and L_1 can be chosen such that $K_1 - 1 + L_1 = I_1$. This is equivalent to (21). ∎

Corollary III.3 allows us to deal with Vandermonde structured CPD problems where none of the factor matrices have full column rank. Let us compare it with Theorems I.1, I.3 and II.1 in a few cases. Assume that $\mathbf{A}^{(1)}$ is Vandermonde structured and $I_1 = I_2 = I_3$. The maximal value R for which the conditions guarantee uniqueness in the generic case are given in Table II for various I_1 values. It is clear that Corollary III.3 yields a significantly more relaxed bound on R than Theorems I.1, I.3 and II.1.

$I_1 = I_2 = I_3$	2	3	4	5	6	7	8	9	10
Theorem I.1	**2**	**3**	5	6	8	9	11	12	14
Theorem I.3	**2**	**3**	5	6	8	9	11	13	15
Theorem II.1	**2**	**3**	5	6	8	9	11	12	14
Corollary III.3	**2**	**3**	**8**	**10**	**18**	**21**	**32**	**36**	**50**

TABLE II

MAXIMUM VALUE R FOR WHICH THEOREMS I.1, II.1 AND III.2 AND COROLLARY III.3 GUARANTEE UNIQUENESS IN THE GENERIC CASE OF A CPD WITH $I_1 = I_2 = I_3$ AND $\mathbf{A}^{(1)}$ VANDERMONDE STRUCTURED.

Corollary III.3 also allows us to deal with Vandermonde structured CPD problems where $\min_{m \neq n} I_m I_n < R$. This is in contrast with the unconstrained CPD where $\min_{m \neq n} I_m I_n \geq R$ is a necessary uniqueness condition [23].

B. Rank

In general the determination of tensor rank is NP-hard [13], [14]. Vandermonde structure of a factor matrix simplifies the determination of tensor rank considerably. We assume that $\mathbf{A}^{(1)}$ is Vandermonde structured.

Let us first assume that $\mathbf{A}^{(3)}$ has full column rank. If $\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}$ also has full column rank, then the (tensor) rank of $\mathcal{X}$ is equal to the (matrix) rank of its matrix representation $\mathbf{X}_{(1)} = (\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)})\mathbf{A}^{(3)T}$. Lemma III.1 implies that this is generically true for R bounded as $\min(I_3, I_1 I_2) \geq R$.

If $\mathbf{A}^{(3)}$ does not have full column rank, then we first apply spatial smoothing as mentioned in Corollary III.3 and work with $\mathbf{X} = \left(\mathbf{A}^{(K_1,1)} \odot \mathbf{A}^{(2)}\right)\left(\mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(3)}\right)^T$ with $K_1 + L_1 = I_1 + 1$. Lemma III.1 implies that generically the (tensor) rank of $\mathcal{X}$ is equal to the (matrix) rank of the latter matrix representation if $\min(L_1 I_3, K_1 I_2) \geq R$.

C. Algorithm

It is also important to take the Vandermonde structure of the problem into account in the actual computation of the tensor decomposition. Unconstrained optimization methods such as the popular Alternating Least Squares (ALS) algorithm [2], [12] fail when the rank is high, even in the exact case. In this respect, an important observation is that the proof of Theorem III.2 is constructive, i.e., it provides us with an algorithm to compute the decomposition when the uniqueness conditions are satisfied. This algorithm only resorts to standard linear algebra and it is guaranteed to return the exact solution in the noise-free case. If $\mathbf{A}^{(3)}$ does not have full column rank, then we first apply spatial smoothing as mentioned in Corollary III.3 and work with $\mathbf{X} = \left(\mathbf{A}^{(K_1,1)} \odot \mathbf{A}^{(2)}\right)\left(\mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(3)}\right)^T$ with $K_1 + L_1 = I_1 + 1$.

An outline of the proposed method for computing a CPD with a Vandermonde factor matrix is given as Algorithm 1. Note that the algorithm is computationally very cheap. It basically has the same cost as one ALS iteration.

Algorithm 1 Outline of the procedure for computing a CPD with a Vandermonde factor matrix.

Input: $\mathbf{X}_{(1)} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T}$

Choose pair (K_1, L_1) subject to $K_1 + L_1 = I_1 + 1$:

$$\mathbf{X} = \left(\mathbf{A}^{(K_1,1)} \odot \mathbf{A}^{(2)}\right)\left(\mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(3)}\right)^T$$

Compute SVD $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^H$.

Determine R from Σ.

Build $\mathbf{U}_1 = \mathbf{U}(1 : (K_1-1)I_2, :)$ and $\mathbf{U}_2 = \mathbf{U}(I_2+1 : K_1 I_2, :)$.

Determine $\{z_r\}_{r=1}^R$ and $\mathbf{M}$ from EVD $\mathbf{U}_1^\dagger\mathbf{U}_2 = \mathbf{M}\mathbf{Z}\mathbf{M}^{-1}$.

Build $\mathbf{a}_r^{(1)} = \left[1, z_r, \ldots, z_r^{I_1-1}\right]^T, \quad r \in \{1, \ldots, R\}$.

Compute $\mathbf{a}_r^{(2)} = \left(\mathbf{a}_r^{(K_1,1)H} \otimes \mathbf{I}_{I_2}\right)\mathbf{U}\mathbf{m}_r, \quad r \in \{1, \ldots, R\}$.

if $\left(\left(\mathbf{A}^{(1)H}\mathbf{A}^{(1)}\right) * \left(\mathbf{A}^{(2)H}\mathbf{A}^{(2)}\right)\right)^{-1}$ exists **then**

Compute $\mathbf{F} = \left(\left(\mathbf{A}^{(1)H}\mathbf{A}^{(1)}\right) * \left(\mathbf{A}^{(2)H}\mathbf{A}^{(2)}\right)\right)^{-1}$.

Compute $\mathbf{A}^{(3)T} = \mathbf{F}\left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)^H \mathbf{X}_{(1)}$.

else

Compute $\mathbf{N} = \mathbf{M}^{-T}$.

Compute $\mathbf{a}_r^{(3)} = \left(\frac{\mathbf{a}_r^{(L_1,1)}}{\mathbf{a}_r^{(L_1,1)H}\mathbf{a}_r^{(L_1,1)}} \otimes \mathbf{I}_{I_3}\right)\mathbf{V}^*\Sigma\mathbf{n}_r$, $r \in \{1, \ldots, R\}$.

end if

Output: $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$

D. More than One Vandermonde Factor

Algorithm 1 can also be applied if two of the factor matrices of the CPD, say $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$, are Vandermonde structured, as explained in the following corollary.

Corollary III.4. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R be constructed from $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1, 2, 3\}$, where $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are Vandermonde matrices with generators $\{z_{1,r}\}_{r=1}^R$ and $\{z_{2,r}\}_{r=1}^R$, respectively. Assume that the generators of $\boldsymbol{A}^{(1)}$ are distinct. Consider the matrix representation $\boldsymbol{X} = \left(\boldsymbol{A}^{(K_1,1)} \odot \boldsymbol{A}^{(K_2,2)}\right)\left(\boldsymbol{A}^{(L_1,1)} \odot \boldsymbol{A}^{(L_2,2)} \odot \boldsymbol{A}^{(3)}\right)^T$ with $K_1+L_1 = I_1+1$ and $K_2 + L_2 = I_2 + 1$. If the matrices $\underline{\boldsymbol{A}}^{(K_1,1)} \odot \boldsymbol{A}^{(K_2,2)}$ and $\boldsymbol{A}^{(L_1,1)} \odot \boldsymbol{A}^{(L_2,2)} \odot \boldsymbol{A}^{(3)}$ have full column rank, then the Vandermonde structured CPD of $\mathcal{X}$ is unique. Generically, this is true if*

$$\min\left((K_1 - 1)\, K_2, L_1 L_2 I_3\right) \geq R \quad \textit{and} \quad K_n + L_n = I_n + 1\,. \tag{26}$$

Proof: Consider

$$\mathbf{X} = \left(\mathbf{A}^{(K_1,1)} \odot \mathbf{A}^{(K_2,2)}\right)\left(\mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(L_2,2)} \odot \mathbf{A}^{(3)}\right)^T.$$

Ignoring the Vandermonde structure of $\mathbf{A}^{(K_2,2)}$ and the Khatri-Rao structure of $\mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(L_2,2)} \odot \mathbf{A}^{(3)}$, Algorithm 1 allows us to find $\mathbf{A}^{(1)}$ up to a permutation of its columns and $\mathbf{A}^{(2)}$ up to a permutation and scaling of its columns from Col $(\mathbf{X})$. Taking the Vandermonde structure of $\mathbf{A}^{(K_2,2)}$ into account the generators of $\mathbf{A}^{(2)}$ can be found as follows

$$z_{2,r} = \left(\underline{\mathbf{a}}_r^{(K_2,2)}\right)^\dagger \overline{\mathbf{a}}_r^{(K_2,2)}, \quad r \in \{1, \ldots, R\},$$

Finally, $\mathbf{A}^{(3)}$ can be determined via $\mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)^\dagger \mathbf{X}_{(1)}$.

Lemma III.1 implies that the rank assumption holds generically if (26) is satisfied. ∎

In general, Algorithm 1 can be applied if P of the factor matrices of the CPD are Vandermonde structured under the condition stated in the following corollary.

Corollary III.5. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R. Assume that the factor matrices $\boldsymbol{A}^{(n)}$, $n \in \{1, \ldots, P\}$, are Vandermonde matrices such that the CPD of $\mathcal{X}$ has the matrix representation*

$$\boldsymbol{X} = \left(\bigodot_{p=1}^{P} \boldsymbol{A}^{(K_p,p)} \odot \bigodot_{q=P+1}^{Q} \boldsymbol{A}^{(q)}\right)\left(\bigodot_{p=1}^{P} \boldsymbol{A}^{(L_p,p)} \odot \bigodot_{q=Q+1}^{N} \boldsymbol{A}^{(q)}\right)^T$$

with $L_m = I_m - K_m + 1$, $\forall m \in \{1, \ldots, P\}$. Assume also that the generators of $\boldsymbol{A}^{(1)}$ are distinct and that the matrices $\left(\underline{\boldsymbol{A}}^{(K_1,1)} \odot \cdots \odot \boldsymbol{A}^{(K_P,P)} \odot \boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(Q)}\right)$ and $\left(\boldsymbol{A}^{(L_1,1)} \odot \cdots \odot \boldsymbol{A}^{(L_P,P)} \odot \boldsymbol{A}^{(Q+1)} \odot \cdots \odot \boldsymbol{A}^{(N)}\right)$ have full column rank, then the Vandermonde structured CPD is unique. Generically, this is true if

$$\min_{\substack{L_m = I_m - K_m + 1 \\ m \in \{1, \ldots, P\}}} \left((K_1 - 1)\prod_{i=2}^{P} K_i \prod_{j=P+1}^{Q} I_j, \prod_{i=1}^{P} L_i \prod_{k=Q+1}^{N} I_k\right) \geq R\,. \tag{27}$$

Proof: The result follows directly from Theorem III.2 and matrix representation (13). ∎

E. Vandermonde Factor Matrix with Repeated Generators

So far we have only considered the case where all the generators of the Vandermonde matrix $\mathbf{A}^{(1)}$ are distinct, which is the common assumption in the matrix setting. However, in the tensor setting partial computation is possible in the case where not all generators are distinct. Assume that the generators appear in Q groups of size P_1, P_2, …, P_Q, i.e., $\mathbf{A}^{(1)}$ takes the form

$$\mathbf{A}^{(1)} = \left[\mathbf{1}_{P_1}^T \otimes \mathbf{b}_1, \ldots, \mathbf{1}_{P_Q}^T \otimes \mathbf{b}_Q\right], \tag{28}$$

where $\{\mathbf{b}_q\}_{q=1}^Q$ are distinct Vandermonde vectors. Partition $\mathbf{A}^{(3)}$ and $\widehat{\mathbf{A}}^{(3)}$ in accordance with (28) as follows

$$\mathbf{A}^{(2)} = \left[\mathbf{A}_1^{(2)}, \ldots, \mathbf{A}_Q^{(2)}\right], \quad \mathbf{A}_q^{(2)} \in \mathbb{C}^{I_2 \times P_q}$$
$$\mathbf{A}^{(3)} = \left[\mathbf{A}_1^{(3)}, \ldots, \mathbf{A}_Q^{(3)}\right], \quad \mathbf{A}_q^{(3)} \in \mathbb{C}^{I_3 \times P_q}.$$

Assume, despite the multiplicities of the vectors in (28), that $\underline{\mathbf{A}}^{(K_1,1)} \odot \mathbf{A}^{(2)}$ and $\mathbf{A}^{(L_1,1)} \odot \mathbf{A}^{(3)}$ have full column rank, as in Corollary III.3. It is easy to verify that $\mathbf{A}^{(1)}$ can still be obtained up to block-permutation via Algorithm 1. The eigenvalues of $\mathbf{U}_1^\dagger\mathbf{U}_2$ will be recovered with their multiplicities. For isolated eigenvalues also the corresponding eigenvectors, and consequently the corresponding column vectors of $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ will be recovered. For repeated eigenvalues, only the eigenspace can be recovered, and consequently only the column spaces of the corresponding blocks of $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$. In wireless communication, $\mathbf{A}^{(3)}$ often contains the transmitted signals and is the matrix of interest (see next sections). After having separated the signals corresponding to different Vandermonde vectors, the recovery of the individual column vectors of $\mathbf{A}_q^{(3)}$ from Col $\left(\mathbf{A}_q^{(3)}\right)$ may

be based on properties such as statistical independence, constant modulus, finite alphabet, etc.

IV. MIMO Radar

MIMO radar has received quite some attention in recent years due to its superior identifiability properties over the standard single-phased radar [1], [21], [20], [11], [41]. In [25] a tensor-based approach to MIMO radar was proposed for the following configuration.

Assume that the receive and transmit antenna arrays consist of N and M antennas, respectively. Furthermore, assume that there are R targets in the range of interest, that the transmitted pulse signals last L samples per pulse period and that Q pulse periods are used. Then, by ignoring noise terms the observed data for the qth pulse period are [25]

$$\mathbf{X}^{(q)} = \mathbf{B} D_q(\mathbf{C})\, \mathbf{A}^T \mathbf{S} \in \mathbb{C}^{N \times L}, \tag{29}$$

where $\mathbf{A} \in \mathbb{C}^{M \times R}$ and $\mathbf{B} \in \mathbb{C}^{N \times R}$ are the transmit and receive antenna steering matrices, respectively, $\mathbf{S} \in \mathbb{C}^{M \times L}$ is the collected transmitted pulse waveform matrix and

$$\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R] \in \mathbb{C}^{Q \times R}, \quad \mathbf{c}_r = \left[\beta_{1r}, \ldots, \beta_{Qr}\right]^T,$$

are the collected Radar Cross Section (RCS) fading coefficients. The underlying assumptions in [25] are that both the transmit and receive arrays are equipped with a ULA, and that $\mathbf{S}$ is known and has full row rank. In this case the problem of finding $\mathbf{A}$ and $\mathbf{B}$ from (29) reduces to a two-dimensional harmonic retrieval problem, which can be solved by various methods, see [25] for references and details. Another interesting ULA-type configuration can be found in [3], where it is also assumed that both the transmitter and receiver are equipped with a ULA. However, if the transmit array is not ULA structured or if the transmitted waveforms are unknown, such as in passive radar, the approaches in [25], [3] are not valid anymore. We first rewrite (29) as

$$\mathbf{X}_{(1)} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \vdots \\ \mathbf{X}^{(Q)} \end{bmatrix} = (\mathbf{C} \odot \mathbf{B})\, \mathbf{G} \in \mathbb{C}^{QN \times L}, \tag{30}$$

where $\mathbf{G} = \mathbf{A}^T \mathbf{S} \in \mathbb{C}^{R \times L}$. We will now propose methods for finding $\mathbf{B}$ and possibly also $\mathbf{A}$ from (30) that do not require that the transmit antenna array is structured.

A. Arbitrary Transmit and Receive Arrays and Constant RCS Coefficients

If the RCS coefficients remain constant over time, then $(\mathbf{C})_{qr} = \beta_{qr} e^{(q-1)\kappa_r \sqrt{-1}}$, where $\beta_{qr} = \beta_{pr}$, $\forall p, q$, are the constant RCS coefficients and κ_r is the Doppler frequency for the rth target. Thus, if the generators $\{\kappa_r\}$ of the Vandermonde matrix $\mathbf{C}$ are distinct, then due to Corollary III.3 we know that the CPD (30) is generically unique if $\min\left(\left(K_Q - 1\right) N, L_Q L)\right) \geq R$, subject to $K_Q + L_Q = Q + 1$. Furthermore, Algorithm 1 provides a numerical method. Note that in contrast to [25] we did not impose any particular structure on the steering matrices, i.e., the transmit and receive antenna arrays can be chosen freely.

B. Arbitrary Transmit Array, ULA Receive Array and Varying RCS Coefficients

Consider now the case where the RCS coefficients are changing over time such that $(\mathbf{C})_{qr} = \beta_{qr} e^{(q-1)\kappa_r \sqrt{-1}}$. Due to the CPD structure of (30) it is possible to find the factor matrices of $\mathbf{X}_{(1)}$ under mild conditions, as stated in Theorems I.2 and I.3. Such CPDs are usually computed by means of optimization algorithms that are not guaranteed to find the global optimum. We now explain that ULA structure of the receiver yields a more relaxed uniqueness condition and an interesting computational alternative. Assume that the receiver is equipped with a ULA such that

$$\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R], \quad \mathbf{b}_r = \left[1, z_r, z_R^2, \ldots, z_r^{N-1}\right]^T,$$

where $z_r = e^{\frac{2\pi}{\lambda} \sin(\theta_r) \Delta_x \sqrt{-1}}$ in which Δ_x is the displacement along the x-axis between each sensor, θ_r is the direction of arrival angle with respect to the receive array and λ is the carrier wavelength. Then due to Corollary III.3 it is possible to recover the factor matrices from (30) if $\min\left((K_N - 1)\, Q, L_N L)\right) \geq R$ subject to $K_N + L_N = N + 1$. Again, Algorithm 1 provides a numerical method.

V. Polarization Sensitive Array Processing

In this section we develop an identifiability result and an algorithm for blind signal separation in the context of polarization sensitive array processing. It is based on the results obtained in section III and can be understood as an extension of the results obtained in [42] to the underdetermined case, i.e., the number of impinging signals R exceeds the number of sensors J in the receive antenna array.

Based on the signal model proposed in [4] the following blind receiver was proposed in [42]. Assume that R signals $\{s_r(t)\}$ are impinging on a crossed-dipole array. The output of a polarization sensitive array employing J pairs of crossed-dipoles is

$$\mathbb{C}^{2J \times K} \ni \mathbf{Y}_{(1)} = \sum_{r=1}^{R} (\mathbf{a}_r \otimes \mathbf{b}_r)\, \mathbf{s}_r^T, \tag{31}$$

where $\mathbf{s}_r \in \mathbb{C}^K$ is the signal vector lasting K sample periods, $\mathbf{b}_r \in \mathbb{C}^J$ is the steering vector associated with signal r, and

$$\mathbb{C}^2 \ni \mathbf{a}_r = \begin{bmatrix} \cos\theta_r \cos\phi_r\,, & -\sin\phi_r \\ \cos\theta_r \sin\phi_r\,, & \cos\phi_r \end{bmatrix} \begin{bmatrix} \sin\gamma_r e^{\eta_r \sqrt{-1}} \\ \cos\gamma_r \end{bmatrix}$$

is the polarization vector, where θ_r is the azimuth angle, ϕ_r is the elevation angle and the pair (γ_r, η_r) describes the polarization state of the rth impinging signal, respectively. For a detailed discussion of this data model we refer to [4]. The observed data can be considered as a rank-R tensor

$$\mathbb{C}^{2 \times J \times K} \ni \mathcal{Y} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{s}_r\,. \tag{32}$$

In [42] it was assumed that the receiver is equipped with a ULA and the sources can be considered to be in the far-field such that

$$\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R], \quad \mathbf{b}_r = \left[1, z_r, z_r^2, \ldots, z_r^{J-1}\right]^T, \tag{33}$$

where $z_r = e^{\frac{2\pi}{\lambda}\sin(\theta_r)\sin(\phi_r)\Delta_x\sqrt{-1}}$ and Δ_x is the displacement along the x-axis between each sensor.

Let $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_R]$ and $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_R]$, then we expect to have $\mathrm{k}(\mathbf{A}) = \min(2, R)$ and $\mathrm{k}(\mathbf{S}) = \min(K, R)$. In practice, we have $K \geq R \geq 2$ and thus we expect that $\mathrm{k}(\mathbf{A}) = 2$ and $\mathrm{k}(\mathbf{S}) = R$. Equating $\mathbf{A}^{(1)} = \mathbf{B}$ and substitution into (8) with $\mathbf{A}^{(2)} = \mathbf{A}$ and $\mathbf{A}^{(3)} = \mathbf{S}$, and vice-versa, yields the following inequalities

$$\min(2R, J+R) + 2 \geq 2(R+1) \quad \Leftrightarrow \quad J \geq R\,, \tag{34}$$

$$\min(2R, J+2) + R \geq 2(R+1) \quad \Leftrightarrow \quad J \geq R\,. \tag{35}$$

In order to satisfy (34) or (35) we must have that $J \geq R$. Hence, the identifiability result (8) is only applicable in the overdetermined case, i.e., in order to guaranty uniqueness of the CPD (32) the number of impinging signals R is not allowed to exceed the number of antennas J in the ULA.

On the other hand, Theorem III.2 and consequently also Algorithm 1 are still valid in some underdetermined cases. Let us first apply spatial smoothing. Consider the fourth-order tensor $\mathcal{X} \in \mathbb{C}^{2\times L_2\times K_2\times K}$ with $L_2 + K_2 = J + 1$, constructed as follows

$$\mathcal{X}(:, l_2, k_2, :) = \mathcal{Y}(:, l_2 + k_2 - 1, :)\,.$$

We have

$$\begin{aligned}\mathbb{C}^{K_2K\times 2L_2} \ni \mathbf{X} &= \begin{bmatrix} \mathcal{X}(:,1,1,:) & \cdots & \mathcal{X}(:,1,K_2,:) \\ \vdots & \ddots & \vdots \\ \mathcal{X}(:,L_2,1,:) & \cdots & \mathcal{X}(:,L_2,K_2,:) \end{bmatrix}^T \\ &= \left(\mathbf{B}^{(K_2)} \odot \mathbf{S}\right)\left(\mathbf{B}^{(L_2)} \odot \mathbf{A}\right)^T, \end{aligned} \tag{36}$$

in which $\mathbf{B}^{(K_2)} \in \mathbb{C}^{K_2\times R}$ and $\mathbf{B}^{(L_2)} \in \mathbb{C}^{L_2\times R}$ are Vandermonde matrices. Due to Theorem III.2 and the structure of the matrix (36) we know that we can expect to find the factor matrices of the CPD of $\mathcal{Y}$ up to the inherent ambiguities if the condition $\min((K_2-1)K, 2L_2) \geq R$ is satisfied. As an example, let $R = 6$ and $J = 4$, then (14) becomes $\min((K_2-1)K, 2L_2) \geq 6$. By letting $L_2 = 3$ and thereby also $K_2 = 2$ we get $\min(K, 6) \geq 6 \Leftrightarrow K \geq 6$, which is generally true in practice.

In [10] another polarization sensitive array processing problem of the form (32) was considered. It was again assumed that the receiver is equipped with a ULA. The difference with the above is that the polarization vector is 6-dimensional. Again, the authors of [10] resorted to Kruskal's uniqueness condition (4). Also for this problem an improved identifiability result can be obtained from Corollary III.3. Furthermore, Algorithm 1 may be used to compute the CPD.

VI. Blind Separation in Wireless Communication with ULA Receiver

In this section we show how the results derived in this paper can be used for blind separation in the context of wireless communication. Again, the Vandermonde structure is due to the use of a ULA at the receiver side. In subsection VI-A we consider systems employing oversampling such as DS-CDMA. Subsection VI-B considers the application of the results in multiuser uplink cooperative systems.

A. Blind Separation of Oversampled Signals

Consider a DS-CDMA uplink system, where the base-station is equipped with I antennas and R mobile stations are transmitting synchronously over flat fading channels. In DS-CDMA systems the transmitted encoded data y_{jkr} from user r at the jth chip period and kth symbol period are given by $y_{jkr} = h_{jr}s_{kr}$, where h_{jr} denotes the jth chip of the spreading code from user r and s_{kr} the kth transmitted symbol from user r. Assume that a spreading code of length J is employed and the users are transmitting over a flat fading channel with a gain b_{ir} between the ith receive antenna and the rth transmitter. Then the output of the ith antenna is

$$x_{ijk} = \sum_{r=1}^{R} b_{ir}y_{jkr} = \sum_{r=1}^{R} b_{ir}h_{jr}s_{kr}.$$

Let us stack the channel gains, spreading codes and the transmitted symbols in the matrices $\mathbf{B} \in \mathbb{C}^{I\times R}$, $\mathbf{H} \in \mathbb{C}^{J\times R}$ and $\mathbf{S} \in \mathbb{C}^{K\times R}$ such that $(\mathbf{B})_{ir} = b_{ir}$, $(\mathbf{H})_{jr} = h_{jr}$ and $(\mathbf{S})_{kr} = s_{kr}$. The received data can then be stored in a tensor $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ such that [34]

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{b}_r \circ \mathbf{h}_r \circ \mathbf{s}_r\,. \tag{37}$$

Using guard chips, the model still holds when reflections take place in the far field of the array; $\mathbf{h}_r$ then consists of the convolution of the rth spreading sequence and the rth channel impulse response [34].

In [23] the authors additionally considered the problem when the receiver is equipped with a ULA. It was explained that the ULA structure may increase the robustness of the blind signal separation problem. Here we explain that the ULA additionally improves the uniqueness properties and that it also leads to a simple algorithm. If a ULA is employed and the sources can be considered to be in the far-field, then

$$\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R]\,, \quad \mathbf{b}_r = \left[1, z_r, z_R^2, \ldots, z_r^{I-1}\right]^T,$$

where $z_r = e^{\frac{2\pi}{\lambda}\sin(\theta_r)\sin(\phi_r)\Delta_x\sqrt{-1}}$ in which Δ_x is the displacement along the x-axis between each sensor, θ_r is the azimuth angle, ϕ_r is the elevation angle and λ is the speed of light. We now know from Corollary III.3 that the CPD is generically unique if $\min((K_1-1)J, L_1K) \geq R$ subject to $K_1 + L_1 = I + 1$. Moreover, we may compute the factor matrices $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{S}$ by means of Algorithm 1.

B. Blind Separation in Multiuser Uplink Cooperative Systems

Recently a CPD based blind receiver for multiuser uplink cooperative systems has been proposed in [9]. It was assumed that the M users transmit through a flat-fading channel towards a base station which is equipped with a ULA constructed from K uniformly spaced antennas. Furthermore, it was assumed that each user is multiplexed by $R-1$ relays such that R processed versions of each transmitted signal reach the base station. It was also assumed that the output of the kth receive antenna at the rth relay time-slot and the nth symbol period is

$$x_{krn} = \sum_{m=1}^{M} a_{km}b_{rm}s_{nm}\,, \tag{38}$$

where a_{km} is the antenna response associated with the kth antenna and the mth user, s_{nm} is the nth symbol transmitted by user m and b_{rm} is a scalar associated with the rth relay of the transmitted signal from user m depending on the applied relaying protocol, see [9] and references therein.

Relation (38) corresponds to the CPD

$$\mathbb{C}^{K\times R\times N} \ni \mathcal{X} = \sum_{m=1}^{M} \mathbf{a}_m \circ \mathbf{b}_m \circ \mathbf{s}_m \,, \tag{39}$$

where $(\mathbf{a}_m)_k = a_{km}$, $(\mathbf{b}_m)_r = b_{rm}$ and $(\mathbf{s}_m)_n = s_{nm}$. In [9] they resorted to Theorem I.1 which states that the CPD (39) is expected to be unique if

$$\min(K,M) + \min(R,M) + \min(N,M) \geq 2(M+1)\,. \tag{40}$$

On the other hand, from Corollary III.3 we obtain the following uniqueness condition:

$$\min((K_1-1)N, L_1 R) \geq M\,, \tag{41}$$

subject to $K_1 + L_1 = K + 1$. Again, by comparing (40) with (41) a significantly improved uniqueness condition has been obtained for this problem. Furthermore, Algorithm 1 provides us with an efficient computational method.

VII. Blind Separation of MIMO OFDM Signals in the case of Carrier Frequency Offset and Multipath

To the authors' knowledge the first tensor-based blind receiver for an OFDM system was presented in [15]. We first review the result in [15] and thereafter we explain that the receiver can be used under more relaxed conditions than has been suggested.

We consider a MIMO OFDM system where the receiver is equipped with I antennas and each user employs N subcarriers. In the kth transmission frame the symbol vector

$$\mathbf{s}^{(r)}(k) = \left[s_1^{(r)}(k), \ldots, s_N^{(r)}(k)\right]^T \in \mathbb{C}^N$$

is transmitted by user r. The symbol vector $\mathbf{s}^{(r)}(k)$ is loaded onto each subcarrier and a Cyclic Prefix (CP) is added so that the transmitted data block becomes

$$\mathbf{x}^{(r)}(k) = \mathbf{T}_{\text{CP}}\mathbf{F}^H\mathbf{s}^{(r)}(k) \in \mathbb{C}^{N+D}\,,$$

where $\mathbf{F} \in \mathbb{C}^{N\times N}$ is the DFT matrix defined by $(\mathbf{F})_{mn} = \omega_N^{(m-1)(n-1)}$ with $\omega_N = \frac{1}{\sqrt{N}} e^{-\frac{2\pi\sqrt{-1}}{N}}$, and where $\mathbf{T}_{\text{CP}} = \left[\mathbf{I}_N(1:D,:)^T, \mathbf{I}_N^T\right]^T \in \mathbb{C}^{(N+D)\times N}$ is the CP encoding matrix, with D exceeding the maximum delay spread of the communication channel. Before transmission, a parallel to serial conversion takes place such that the output of the ith receive antenna at transmission frame k and at time instant t is [15]

$$u_i^{(k)}(t) \triangleq \mathbf{u}_i(k(N+D)+t) = \sum_{r=1}^{R} e^{\phi_r t\sqrt{-1}} \sum_{l_{ir}=0}^{L_{ir}} h_i^{(r)}(l_{ir}) x^{(r)}(t-l_{ir})\,,$$

where R is the total number of users, ϕ_r is the normalized carrier frequency offset associated with user r, caused by Doppler effects or synchronization errors, and additive noise has been ignored for notational convenience. For identifiability purposes we assume that $|\phi_r| < \frac{\pi}{N}$. Stack the observed data in the vector

$$\mathbf{u}_i^{(k)} = \left[u_i^{(k)}(1), \ldots, u_i^{(k)}(N+D)\right]^T \in \mathbb{C}^{N+D}.$$

After receiving the transmitted data we decode them as

$$\mathbf{y}_i^{(k)} = \mathbf{T}_{\text{CP}^{-1}}\mathbf{u}_i^{(k)} \in \mathbb{C}^N, \tag{42}$$

in which $\mathbf{T}_{\text{CP}^{-1}} = [\mathbf{0}_{N,D}\,, \mathbf{I}_N] \in \mathbb{C}^{N\times(N+D)}$ is the CP decoding matrix. Due to the CP it can be shown that (42) is equal to [15]:

$$\mathbb{C}^N \ni \mathbf{y}_i^{(k)} = \mathbf{P}^{(r)}\mathbf{F}^H D_i\left(\mathbf{H}^{(r)}\right)\mathbf{s}^{(r)}(k) e^{\phi_r((k+1)(N+D)-N)\sqrt{-1}}\,, \tag{43}$$

where

$$\begin{aligned}
\mathbf{P}^{(r)} &= D_1\left(\left[1, e^{\phi_r\sqrt{-1}}, \ldots, e^{(N-1)\phi_r\sqrt{-1}}\right]^T\right) \in \mathbb{C}^{N\times N}\\
\mathbf{H}^{(r)} &= \left[\mathbf{h}_1^{(r)}, \ldots, \mathbf{h}_N^{(r)}\right] \in \mathbb{C}^{I\times N}, \quad \mathbf{h}_n^{(r)} = \left[h_{1n}^{(r)}, h_{2n}^{(r)}, \ldots, h_{In}^{(r)},\right]^T
\end{aligned}$$

in which $h_{in}^{(r)} = \sum_{l_{ir}=0}^{L_{ir}} h_i^{(r)}(l_{ir}) e^{\frac{\pi\cdot l_{ir}\cdot n\cdot\sqrt{-1}}{N}}$ is the frequency response of the nth subcarrier associated with the ith receive antenna. In [15] it was assumed that $\mathbf{P}^{(r)}$ and $\mathbf{H}^{(r)}$ remain constant over K transmission blocks. Denoting

$$\begin{aligned}
\mathbf{Y}_i &= \left[\mathbf{y}_i(1), \ldots, \mathbf{y}_i(K)\right] \in \mathbb{C}^{N\times K}\\
\mathbf{S}^{(r)} &= \left[\mathbf{s}^{(r)}(1), \ldots, \mathbf{s}^{(r)}(K)\right] \in \mathbb{C}^{N\times K}\\
\mathbf{Q}^{(r)} &= D_1\Big(\Big[e^{\phi_r(2D+N)\sqrt{-1}}, e^{\phi_r(3D+2N)\sqrt{-1}},\\
&\qquad \ldots, e^{\phi_r((K+1)(N+D)-N)\sqrt{-1}}\Big]^T\Big) \in \mathbb{C}^{K\times K},
\end{aligned}$$

we obtain the CPD

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_I \end{bmatrix} = \sum_{r=1}^{R}\left(\mathbf{H}^{(r)} \odot \mathbf{P}^{(r)}\mathbf{F}^H\right)\left(\mathbf{Q}^{(r)}\mathbf{S}^{(r)}\right)^T = \left(\mathbf{H} \odot \widetilde{\mathbf{P}}\right)\widetilde{\mathbf{S}}^T, \tag{44}$$

where

$$\begin{aligned}
\mathbf{H} &= \left[\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(R)}\right] \in \mathbb{C}^{I\times NR}\\
\widetilde{\mathbf{P}} &= \left[\mathbf{P}^{(1)}\mathbf{F}^H, \ldots, \mathbf{P}^{(R)}\mathbf{F}^H\right] \in \mathbb{C}^{N\times NR}\\
\widetilde{\mathbf{S}} &= \left[\mathbf{Q}^{(1)T}\mathbf{S}^{(1)T}, \ldots, \mathbf{Q}^{(R)T}\mathbf{S}^{(R)T}\right] \in \mathbb{C}^{K\times NR}.
\end{aligned}$$

In [15] the authors resorted to Kruskal's identifiability condition (4) which states that the CPD (44) is unique if

$$\mathrm{k}(\mathbf{H}) + \mathrm{k}\left(\widetilde{\mathbf{P}}\right) + \mathrm{k}\left(\widetilde{\mathbf{S}}\right) \geq 2NR + 2\,. \tag{45}$$

In practice we expect that (45) reduces to

$$\min(I, NR) + N + \min(K, NR) \geq 2NR + 2\,. \tag{46}$$

In this derivation the structure of $\widetilde{\mathbf{P}}$ was ignored and consequently the very restrictive identifiability condition (45) was obtained. Note that condition (46) is only useful when N is small, which may not be the case in practice. More recently, in [29] it has been observed that $\widetilde{\mathbf{P}}$ is in fact a Vandermonde matrix. Indeed, let $z_r = e^{\phi_r\sqrt{-1}}$ and $\rho_N = \omega_N^*$, then

$$\mathbf{P}^{(r)}\mathbf{F}^H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_r & z_r\rho_N & \cdots & z_r\rho_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_r^{N-1} & (z_r\rho_N)^{N-1} & \cdots & (z_r\rho_N^{N-1})^{N-1} \end{bmatrix}. \tag{47}$$

The matrix in (47) is a Vandermonde matrix with generators $\{z_r\rho_N^n\}_{n=0}^{N-1}$. By simultaneously exploiting the Vandermonde structure of $\widetilde{\mathbf{P}}$ and the Khatri-Rao product structure of $\mathbf{H} \odot \widetilde{\mathbf{P}}$, Corollary III.3 tells us that the CPD (44) is unique if

$$\underline{\widetilde{\mathbf{P}}}^{(K_1)} \odot \mathbf{H} \text{ and } \widetilde{\mathbf{P}}^{(L_1)} \odot \widetilde{\mathbf{S}} \text{ have full column rank} \quad (48)$$

subject to $K_1 + L_1 = N + 1$ in which the Vandermonde matrices $\widetilde{\mathbf{P}}^{(K_1)}$ and $\widetilde{\mathbf{P}}^{(L_1)}$ are obtained by spatial smoothing. In practice, we expect that (48) yields the condition

$$\min\left((K_1 - 1)I, L_1 K\right) \geq NR \text{ subject to } K_1 + L_1 = N + 1\,. \quad (49)$$

By comparing (46) and (49) it is clear that the identifiability condition has been significantly relaxed.

Algorithm 1 does not take into account the relationship between the columns in (47). An optimization method that does, is presented in [15].

VIII. Simulations

The applications discussed in this paper essentially lead to the same computation. We illustrate the performance of the algorithms for the polarization sensitive array processing problem described in section V and the problem of separating oversampled signals described in subsection VI-A. Let $\mathbf{Y}_{(1)} \in \mathbb{C}^{IJ \times K}$ be the noise-free tensor and let $\mathbf{X}_{(1)} = \mathbf{Y}_{(1)} + \beta \mathbf{N}_{(1)}$, where $\beta \in \mathbb{R}$ and $\mathbf{N}_{(1)} \in \mathbb{C}^{IJ \times K}$ is a noise term with elements randomly drawn from a Gaussian distribution with zero mean and unit variance.

The Signal to Noise Ratio (SNR) is measured as

$$\text{SNR [dB]} = 10 \log\left(\left\|\mathbf{Y}_{(1)}\right\|_F^2 / \left\|\beta \mathbf{N}_{(1)}\right\|_F^2\right).$$

In the case $I_n \geq R$, the distance between the factor matrix $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$ and its estimate $\widehat{\mathbf{A}}^{(n)}$, is measured as

$$\mathrm{P}\left(\mathbf{A}^{(n)}\right) = \min_{\Pi\Lambda} \left\|\mathbf{A}^{(n)} - \widehat{\mathbf{A}}^{(n)} \Pi \Lambda\right\|_F / \left\|\mathbf{A}^{(n)}\right\|_F\,,$$

where Π denotes a permutation matrix and Λ denotes a diagonal matrix. In order to find Π and Λ we use the greedy least squares column matching algorithm proposed in [34].

We compare Algorithm 1 with the classical ALS method which ignores the Vandermonde structure. Let $f\left(\widehat{\mathbf{X}}_{(1)}^{(k)}\right) = \left\|\mathbf{X}_{(1)} - \widehat{\mathbf{X}}_{(1)}^{(k)}\right\|_F$, where $\widehat{\mathbf{X}}_{(1)}^{(k)}$ denotes the estimated tensor at iteration k. We decide that ALS method has converged when $\left|f\left(\widehat{\mathbf{X}}_{(1)}^{(k)}\right) - f\left(\widehat{\mathbf{X}}_{(1)}^{(k+1)}\right)\right| < 1e{-}8$ or when the number of iterations exceeds 2000. In the ALS method the factor matrices are initialized randomly or by means of the Generalized EVD (GEVD) of $\mathbf{X}^{(\cdot\cdot 1)}$ and $\mathbf{X}^{(\cdot\cdot 2)}$ if possible. When only one random initialization is used, the ALS method will be referred to as ALS-CP. When the best out of ten random initializations is used, the ALS method will be referred to as ALS-CP-10. When the ALS method is initialized by means of the GEVD, then it will be referred to as ALS-CP-GEVD. Algorithm 1 will be referred to as CP-VDM. When the CP-VDM is followed by at most 200 ALS refinement iterations it will be referred to as CP-VDM-ALS.

A. Polarization Sensitive Array Processing

The polarization parameters γ_r and η_r are randomly drawn from a Gaussian distribution with zero mean and unit variance while the azimuth and elevation angles θ_r and ϕ_r are randomly drawn from an uniform distribution with support $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The elements of the symbol matrix are randomly drawn from a QPSK constellation.

a) Case 1: $I < R$ and $J, K > R$: The model parameters are $I = 2$, $J = 6$, $K = 100$ and $R = 5$. For this configuration Kruskal's condition (5) is satisfied. Hence, the CPD is unique even when the Vandermonde structure is not taken into account. The mean $\mathrm{P}(\mathbf{S})$ values over 100 trials for varying SNR can be seen in figure 1. We notice that CP-VDM-ALS yields the best performance, and that the ALS refinement only marginally improves the result. We also notice that the ALS method is sensitive w.r.t. its initialization.

b) Case 2: $I, J < R$ and $K > R$: The model parameters are $I = 2$, $J = 4$, $K = 100$ and $R = 5$. In this configuration CPD is not unique by itself. Also, initialization by GEVD is not possible. On the other hand, Corollary III.3 tells us that the Vandermonde structured CPD is unique. The mean $\mathrm{P}(\mathbf{S})$ values over 100 trials for varying SNR can be seen in figure 2. We notice that CP-VDM and CP-VDM-ALS work well while ALS-CP and ALS-CP-10 indeed fail. CP-VDM and CP-VDM-ALS perform about the same.

B. Blind Separation of DS-CDMA Signals

The azimuth and elevation angles θ_r and ϕ_r are randomly drawn from a uniform distribution with support $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Due to possible interchip interference the entries of the spreading code matrix $\mathbf{H}$ are randomly drawn from a Gaussian distribution with zero mean and unit variance. Finally, the elements of the symbol matrix are randomly drawn from a QPSK constellation.

c) Case 3: $J < R$ and $I, K > R$: The model parameters are $I = 3$, $J = 6$, $K = 100$ and $R = 5$. For this configuration Kruskal's condition (5) is satisfied. Hence, the CPD is unique even when the Vandermonde structure is not taken into account. The mean $\mathrm{P}(\mathbf{S})$ values over 100 trials for varying SNR can be seen in figure 3. We notice that below 20 dB SNR the CP-VDM-ALS, ALS-CP-10 and ALS-CP-GEVD methods perform about the same, while the non-iterative CP-VDM performs slightly worse. The pure CPD algorithms perform better than in the simulations in subsection VIII-A because the CPD structure is stronger, due to the fact that the tensor has more slices. Above 20 dB SNR CP-VDM-ALS and CP-VDM perform about the same and better than ALS-CP-10 and ALS-CP-GEVD. Recall that the ALS-CP-10 and ALS-CP-GEVD methods are significantly more computationally expensive than the CP-VDM-ALS and CP-VDM methods. In conclusion, the best method for this problem is CP-VDM-ALS.

d) Case 4: $I, J < R$ and $K > R$: The model parameters are $I = 3$, $J = 4$, $K = 100$ and $R = 8$. In this configuration CPD is not unique by itself. Also, initialization by GEVD is not possible. On the other hand, Corollary III.3 tells us that this Vandermonde structured CPD is unique. The mean $\mathrm{P}(\mathbf{S})$ values over 100 trials for varying SNR can be seen in figure 4. We notice that CP-VDM and CP-VDM-ALS work well while ALS-CP and ALS-CP-10 fail as expected. In this experiment CP-VDM and CP-VDM-ALS perform about the same.

IX. Conclusion

Many problems in signal processing can be formulated as the computation of a CPD with a Vandermonde factor matrix. We first provided a new uniqueness condition which is more relaxed than the uniqueness conditions that are available for the unconstrained CPD. We explained that under this condition, the tensor rank can be estimated as the rank of a matrix, while for the unconstrained CPD the determination of tensor rank is in general NP-hard. Second, we presented an algorithm for computing the factor matrices under the new uniqueness condition. This algorithm essentially relies on a matrix EVD and is guaranteed to find the decomposition if it is exact. For optimization-based algorithms for the unconstrained CPD there is no such guarantee.

In the second part of the paper we illustrated how the presented results can be used in sensor array processing. More precisely, we showed how they can be applied in MIMO radar. We also illustrated how they could be used in underdetermined blind signal separation based on polarization sensitive arrays. Furthermore, we also illustrated the usefulness of the presented results in blind signal separation in wireless communication. In particular, we considered multiuser uplink cooperative systems and systems using oversampling such as DS-CDMA. We also showed how the results can be used to obtain identifiability results for MIMO OFDM systems using cyclic prefixing and suffering from carrier frequency offset.

In the third part of the paper simulations confirmed the practical interest of the proposed algorithm. Particularly in the highly underdetermined case ($I, J < R$) the method showed superior performance.

In this paper we have limited ourselves to problems in which case blind signal separation can be achieved through CPD. In a follow-up paper we will extend the results to array processing problems in the case of coherent or incoherent channels with small delay spread.

References

[1] D. W. Bliss and K. W. Forsythe, "Multiple-Input Multiple Output (MIMO) Radar and Imaging: Degrees of Freedom and Resolution," in *Proc. Asilomar*, 2003, pp. 54–59.

[2] J. D. Carroll and J. J. Chang, "Analysis of Individual Differences in Multidimensional Scaling via N-Way Generalization of Eckart-Young Decomposition," *Psychometrika*, vol. 35, no. 3, pp. 283–319, 1970.

[3] C. Chen and P. P. Vaidyanathan, "MIMO Radar Space-Time Adaptive Processing using Prolate Spheroidal Wave Functions," *IEEE Trans. Signal Process.*, vol. 56, no. 2, pp. 623–635, Feb. 2008.

[4] R. T. Compton, "The Tripole Antenna: An Adaptive Array with Full Polarization Flexibility," *IEEE Trans. Antennas Propagat.*, no. 6, pp. 944–952, Nov. 1981.

[5] L. De Lathauwer, "A Short Introduction to Tensor-Based Methods for Factor Analysis and Blind Source Separation," in *Proc. of the 7th International Symposium on Image and Signal Processing and Analysis (ISPA 2011)*, 4-6 Sep., Dubrovnik, Croatia, 2011, pp. 558–563.

[6] ——, "A Link Between the Canonical Decomposition in Multilinear Algebra and Simultaneous Matrix Diagonalization," *SIAM J. Matrix Anal. Appl.*, vol. 28, no. 3, pp. 642–666, 2006.

[7] I. Domanov and L. De Lathauwer, "On the Uniqueness of the Canonical Polyadic Decomposition – Part I: Basic Results and Uniqueness of One Factor Matrix," ESAT-SISTA, KU Leuven, Belgium, Tech. Rep. 12-66, 2012.

[8] ——, "On the Uniqueness of the Canonical Polyadic Decomposition – Part II: Overall Uniqueness," ESAT-SISTA, KU Leuven, Belgium, Tech. Rep. 12-72, 2012.

[9] C. A. R. Fernandes, A. L. F. de Almeida, and D. B. da Costa, "Unified Tensor Modeling for Blind Receivers in Multiuser Uplink Cooperative Systems," *IEEE Signal Processing Letters*, vol. 19, no. 5, pp. 247–250, May 2012.

[10] X. Guo, S. Miron, and D. Brie, "Three-way Array Analysis on Polarization Signals for Direction-Finding and Blind Source Separation," in *IAR 2007*, Nov. 15-16, Grenoble, France, 2007.

[11] A. Haimovich, R. S. Blum, and L. J. Cimini Jr, "MIMO Radar with Widely Separated Antennas," *IEEE Signal Process. Mag.*, vol. 25, no. 1, pp. 116–129, Jan. 2008.

[12] R. A. Harshman, "Foundations of the Parafac Procedure: Models and Conditions for an Explanatory Multimodal Factor Analysis," *UCLA Working Papers in Phonetics*, vol. 16, pp. 1–84, 1970.

[13] J. Håstad, "Tensor Rank is NP-complete," *J. Algorithms*, vol. 11, no. 4, pp. 644–654, 1990.

[14] C. J. Hillar and L.-H. Lim, "Most Tensor Problems are NP-Hard," *ACM*, vol. 9, no. 4, 2012.

[15] T. Jiang and N. D. Sidiropoulos, "A Direct Blind Receiver for SIMO and MIMO OFDM Systems Subject to Unknown Frequency Offset and Multipath," in *4th IEEE Workshop on Signal Processing Advances in Wireless Communications*, June 15-18, Rome, Italy, 2003, pp. 358–362.

[16] ——, "Kruskal's Permutation Lemma and the Identification of CANDECOMP/PARAFAC and Bilinear Model with Constant Modulus Constraints," *IEEE Trans. Signal Process.*, vol. 52, no. 9, pp. 2625–2636, Sep. 2004.

[17] T. Jiang, N. D. Sidiropoulos, and J. M. F. Ten Berge, "Almost-Sure Identifiability of Multidimensional Harmonic Retrieval," *IEEE Trans. Signal Process.*, vol. 49, no. 9, pp. 1849–1859, Sep. 2001.

[18] H. A. L. Kiers, "Towards a Standardized Notation and Terminology in Multiway Analysis," *J. Chemometrics*, vol. 14, no. 3, pp. 105–122, 2000.

[19] J. B. Kruskal, "Three-way Arrays: Rank and Uniqueness of Trilinear Decompositions, with Applications to Arithmetic Complexity and Statistics," *Linear Algebra Appl.*, vol. 18, pp. 95–138, 1977.

[20] J. Li and P. Stoica, "MIMO Radar with Colocated Antennas," *IEEE Signal Process. Mag.*, vol. 24, no. 5, pp. 106–114, Sep. 2007.

[21] J. Li, P. Stoica, L. Xu, and W. Roberts, "On Parameter Identifiability of MIMO Radar," *IEEE Signal Process. Lett.*, vol. 14, no. 12, pp. 968–971, Dec. 2007.

[22] J. Liu and X. Liu, "An Eigenvector-Based Approach for Multidimensional Frequency Estimation with Improved Identifiability," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 2074–2086, Dec. 2006.

[23] X. Liu and N. D. Sidiropoulos, "Cramér-Rao Lower Bounds for Low-Rank Decomposition of Multidimensional Arrays," *IEEE Trans. Signal Process.*, vol. 49, no. 9, pp. 2074–2086, Sep. 2001.

[24] ——, "Almost Sure Identifiability of Multidimensional Constant Modulus Harmonic Retrieval," *IEEE Trans. Signal Process.*, vol. 50, no. 9, pp. 2366–2368, Sep. 2002.

[25] D. Nion and N. D. Sidiropoulos, "Tensor Algebra and Multidimensional Harmonic Retrieval in Signal Processing for MIMO Radar," *IEEE Trans. Signal Processing*, vol. 58, no. 11, pp. 5693–5705, Nov. 2010.

[26] J. M. Papy, L. De Lathauwer, and S. Van Huffel, "Exponential Data Fitting using Multilinear Algebra: The Single-Channel and Multi-Channel Case," *Numer. Linear Algebra Appl.*, vol. 12, pp. 809–826, 2005.

[27] ——, "Exponential Data Fitting using Multilinear Algebra: The Decimative Case," *J. Chemometrics*, vol. 23, no. 7-8, pp. 341–351, 2009.

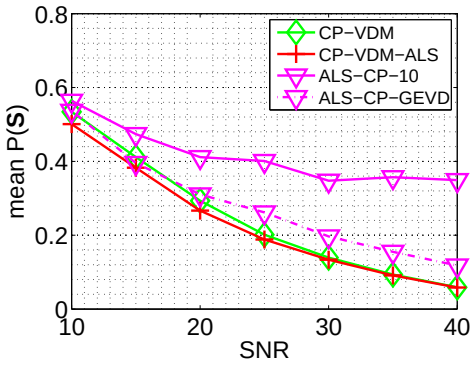
[28] R. Roy and T. Kailath, "Estimation of Signal Parameters via Rotational Invariance Techniques," *IEEE Trans. ASSP*, vol. 32, no. 7, pp. 984–995, Jul. 1989.

[29] A. Santra and K. Hari, "Novel Subspace Methods for Blind Estimation of Multiple CFOs in MIMO-OFDM Systems," in *Proc. ICSP 2010*, Oct 24- 28, Beijing, China, 2010.

[30] T. J. Shan, M. Wax, and T. Kailath, "On Spatial Smoothing for Direction-On-Arrival of Coherent Signals," *IEEE Trans. ASSP*, vol. 33, no. 8, pp. 806–811, Aug. 1985.

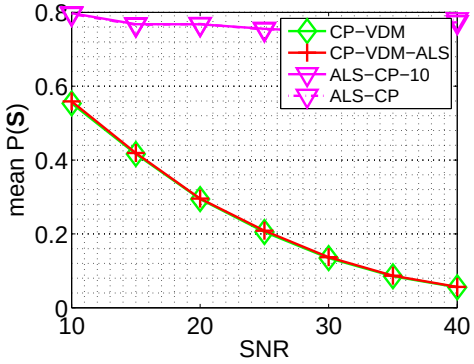
[31] N. D. Sidiropoulos, "Generalizing Carathéodory's Uniqueness of Harmonic Parameterization to N Dimensions," *IEEE Trans. Inf. Theory*, vol. 47, no. 4, pp. 1687–1690, May 2001.

[32] N. D. Sidiropoulos and R. Bro, "On the Uniqueness of Multilinear Decomposition of *N*-way Arrays," *J. Chemometrics*, vol. 14, pp. 229–239, 2000.
[33] N. D. Sidiropoulos, R. Bro, and G. B. Giannakis, "Parallel Factor Analysis in Sensor Array Processing," *IEEE Trans. Signal Processing*, vol. 48, no. 8, pp. 2377–2388, Aug. 2000.
[34] N. D. Sidiropoulos, G. B. Giannakis, and R. Bro, "Blind PARAFAC Receivers for DS-CDMA Systems," *IEEE Trans. Signal Processing*, vol. 48, no. 3, pp. 810–823, Mar. 2000.
[35] N. D. Sidiropoulos and X. Liu, "Identifiability Results for Blind Beamforming in Incoherent Multipath with Small Delay Spread," *IEEE Trans. Signal Processing*, vol. 49, no. 1, pp. 228–236, Jan. 2001.
[36] L. Sorber, M. Van Barel, and L. De Lathauwer, "Optimization-based Algorithms for Tensor Decompositions : Canonical Polyadic Decomposition, Decomposition in rank-(L_r, L_r, 1) Terms and a New Generalization," ESAT-SISTA, KU Leuven, Belgium, Tech. Rep. 12-37.
[37] M. Sørensen and L. De Lathauwer, "Tensor Decompositions with a Vandermonde Factor and Applications in Signal Processing," in *Proc. Asilomar*, Pacific Grove, California, USA, Nov. 4-7, 2012.
[38] M. Sørensen, L. De Lathauwer, P. Comon, S. Icart, and L. Deneire, "Canonical Polyadic Decomposition with a Columnwise Orthonormal Factor Matrix," *SIAM J. Matrix Anal. Appl.*, vol. 33, no. 4, pp. 1190–1213, 2012.
[39] A. Stegeman and N. D. Sidiropoulos, "On Kruskal's Uniqueness Condition for the Candecomp/Parafac Decomposition," *Linear Algebra Appl.*, vol. 420, pp. 540–552, 2007.
[40] A.-J. Van Der Veen, "Algebraic Methods for Deterministic Blind Beamforming," *Proc. IEEE*, vol. 10, pp. 1987–2008, 1998.
[41] H. Yan, J. Li, and G. Liao, "Multitarget Identification and Localization Using Bistatic Radar Systems," *EURASIP Journal on Advances in Signal Processing*, 2008, article ID 283483.
[42] X. Zhang and D. Xu, "Blind PARAFAC Signal Detection for Polarization Sensitive Array," *EURASIP Journal on Advances in Signal Processing*, 2007, article ID 12025.



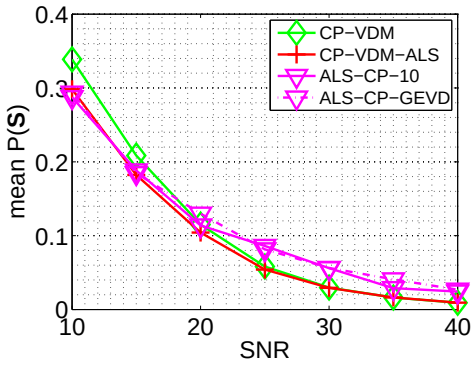
(a) Mean P (**S**).

Fig. 1. Mean P (**S**) values over 100 trials while SNR is varying from 10 to 40 dB, case 1.



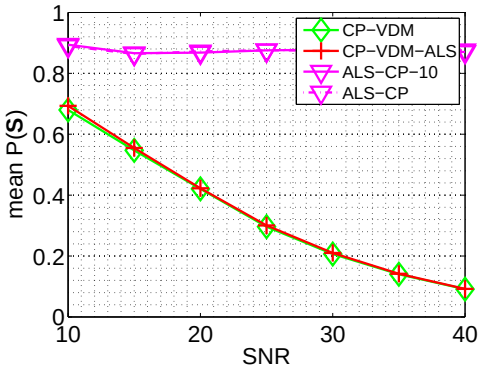
(a) Mean P (**S**).

Fig. 2. Mean P (**S**) values over 100 trials while SNR is varying from 10 to 40 dB, case 2.



(a) Mean P (**S**).

Fig. 3. Mean P (**S**) values over 100 trials while SNR is varying from 10 to 40 dB, case 3.



(a) Mean P (**S**).

Fig. 4. Mean P (**S**) values over 100 trials while SNR is varying from 10 to 40 dB, case 4.

Acknowledgment

Research supported by: (1) Research Council KU Leuven: GOA-Ambiorics, GOA-MaNet, CoE EF/05/006 Optimization in Engineering (OPTEC), CIF1, STRT1/08/23, (2) F.W.O.: (a) project G.0427.10N, (b) Research Communities ICCoS, ANMMM and MLDM, (3) the Belgian Federal Science Policy Office: IUAP P7 (DYSCO II, Dynamical systems, control and optimization, 2012-2017), (4) EU: ERNSI.

Mikael Sørensen received the Master's degree from Aalborg University, Denmark, and the Ph.D. degree from University of Nice, France, in 2006 and 2010, respectively, both in electrical engineering. Since 2010 he has been a Postdoctoral Fellow with the KU Leuven, Belgium, and with the KU Leuven Campus Kortrijk, Kortrijk, Belgium. His research interests include applied linear and multilinear algebra, blind source separation and system identification, and signal processing for wireless communication and sensor array processing.

Lieven De Lathauwer received the Master's degree in electromechanical engineering and the Ph.D. degree in applied sciences from KU Leuven, Belgium, in 1992 and 1997, respectively. He is currently with the KU Leuven, Belgium, and with the KU Leuven Campus Kortrijk, Kortrijk, Belgium. His research interests include linear and multilinear algebra, statistical signal and array processing, higher order statistics, independent component analysis, identification, blind identification, and equalization.

Dr. De Lathauwer is an Associate Editor of the SIAM Journal on Matrix Analysis and Applications and has been an Associate Editor for IEEE Transactions on Signal Processing.