A Convex Approximation for Parameter Estimation Involving Parameter-Affine Dynamic Models

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***Abstract*— This paper presents a convex approach for parameter estimation problems (PEPs) involving parameter-affine dynamic systems. By using the available state measurements, the nonconvex PEP is modified such that a convex approximation is obtained. The optimum delivered by this approximation is subsequently used to linearize the original PEP such that a refined solution is obtained. An assessment of the distance between the actual solution and the one delivered by the refined approximation is provided using perturbation analysis. Comparative results using a classical nonlinear parameter estimation procedure and the proposed approach are presented for two benchmark examples where the theoretical assessment of errors in solutions is corroborated numerically.**

I. Introduction

Consider the parameter estimation problem (PEP) involving a parameter-affine nonlinear dynamic model:

$$\min_{p,x(0),\ldots,x(N)} \frac{1}{2}\sum_{k=0}^{N} \|x(k)-\bar{x}(k)\|_Q^2 \tag{1}$$

subject to

$$x(k+1) = \Phi_k(x(k)) + \Gamma_k(x(k))p,\ k\in[0,N-1] \tag{2}$$

$$x(k) \in \mathbb{X}_k, \quad k\in[0,N], \quad p\in\mathbb{P}, \tag{3}$$

where the vectors $p\in\mathbb{R}^{n_p}$ and $x(k)\in\mathbb{R}^{n_x}$ denote model parameters and states, respectively. In order to estimate the value of the vector p, a set of state measurements $\bar{x}(k)\in\mathbb{R}^{n_x}$, $k=0,1,\ldots,N$ with $N+1\geq n_p$, is collected along the interval of interest. Constraints on the parameters and model states are introduced by the sets $\mathbb{P}$ and $\mathbb{X}_k$, respectively.

Notice that any PEP involving Nonlinear Auto-Regressive with Exogenous Input (NARX) models [1] is covered by the formulation presented in (1)-(3), i.e., a problem of the form:

$$\min_{p,y(0),\ldots,y(N)} \sum_{k=0}^{N} \|y(k)-\bar{y}(k)\|_2^2 \tag{4}$$

subject to

$$y(k+m) = \sum_{i=1}^{n_p} p_i f_i\left(y(k+m-1),..,y(k),\right.$$
$$\left. u(k+m-1),..,u(k)\right) \tag{5}$$

$$y(0),\ldots,y(N)\in\mathbb{Y}, \quad p_0,\ldots,p_{n_p}\in\mathbb{P} \tag{6}$$

can be cast in the form presented by defining a state vector $x(k)=[y(0),y(1),\ldots,y(m-1)]^T$.

The PEP (1)-(3) can be reformulated to the compact form:

$$\mathrm{PEP_{NL}}(\bar{\mathbf{x}}): \quad \min_{p,\mathbf{x}} \frac{1}{2}\parallel \mathbf{x}-\bar{\mathbf{x}}\parallel_Q^2, \tag{7}$$

subject to

$$\mathbf{0} = A(\mathbf{x}) - B(\mathbf{x})p - W\mathbf{x}, \tag{8}$$

$$\mathbf{x} \in \mathbb{X}, \quad p\in\mathbb{P}, \tag{9}$$

where $\mathbf{x}=[x(0)^T,x(1)^T,\ldots,x(N)^T]^T$ and $\bar{\mathbf{x}}=[\bar{x}(0)^T,\bar{x}(1)^T,\ldots,\bar{x}(N)^T]^T$ correspond to the discrete time model dynamics and the state measurements, respectively. $A(\mathbf{x})$, $B(\mathbf{x})$ and W are defined as follows:

$$A(\mathbf{x})=\begin{bmatrix}\Phi_0(x(0))\\ \Phi_1(x(1))\\ \vdots\\ \Phi_{N-1}(x(N-1))\end{bmatrix}, B(\mathbf{x})=\begin{bmatrix}-\Gamma_0(x(0))\\ -\Gamma_1(x(1))\\ \vdots\\ -\Gamma_{N-1}(x(N-1))\end{bmatrix},$$

$$W=\begin{bmatrix}0 & \mathrm{I}_{n_x} & 0 & \ldots & 0 & 0\\ 0 & 0 & \mathrm{I}_{n_x} & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 0 & \mathrm{I}_{n_x}\end{bmatrix}, \tag{10}$$

with I_{n_x} denoting the identity matrix of size n_x. The feasible set for the states is redefined as $\mathbb{X}=\mathbb{X}_0\times\mathbb{X}_1\times\ldots\times\mathbb{X}_N$. In the following, the particular structure of (8) in the general nonlinear least squares PEP (7)-(9) is exploited. Additionally, the set described by $\mathbb{X}\times\mathbb{P}$ is assumed to be convex, as it is, e.g., in the case of bound constraints on parameters and states.

The main contributions of this paper are: (1) the new refinement strategy for the convexification approach presented in [2] and (2) the theoretical assessment of the solution errors which are of second order in the size of the perturbation.

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The paper is organized as follows. Section II describes the proposed approach for convexification of this particular class of PEPs. Section III provides an assessment of the solution errors for the presented approach. Section IV shows that the distance between the real solution and the proposed approximation is of second order in the size of the perturbation. Section V exemplifies the proposed method by using two particular dynamic models, the harmonic oscillator and a biological predator-prey model. Conclusions follow in Section VI.

II. The Convex Approach

In order to avoid solving the nonconvex problem (7)-(9), a convex approach is proposed. The method consists of two steps, each one of them involving the solution of a convex problem with the same number of variables as in the original one. The steps are further explained in the following paragraphs.

A. Step 1: Convexification

The nonconvex PEP (7)-(9) can be considerably simplified by using the feasible state measurements as the real solution for the state trajectory, which is equivalent to the following formulation:

$$\mathrm{PEP}_{\mathrm{CVX}}(\bar{\mathbf{x}}): \quad \min_{p,\mathbf{x}} \frac{1}{2} \parallel \mathbf{x}-\bar{\mathbf{x}} \parallel_Q^2, \tag{11}$$

subject to

$$\mathbf{0} = A(\bar{\mathbf{x}}) - B(\bar{\mathbf{x}})p - W\mathbf{x}, \tag{12}$$

$$\mathbf{x} \in \mathbb{X}, \quad p \in \mathbb{P}. \tag{13}$$

Convexity is achieved in the problem (11)-(13) since (1) the cost is quadratic in the states $\mathbf{x}$, (2) the nonlinearity in the model has been removed by introducing the measurement trajectory $\bar{\mathbf{x}}$ as evaluation points in $A(\mathbf{x})$ and $B(\mathbf{x})$, (3) the model is affine in the parameters p and (4) the feasible set $\mathbb{X} \times \mathbb{P}$ is convex. Although the modification in the model could be seen as a linearization around measurements, it differs from that approach since no information of a linearization point for the parameters p is required. For the choice of the matrix W in (10), the described convex modification is equivalent to Prediction Error Methods (PEM) [1]. However, it has been demonstrated that different ways of splitting $A(\mathbf{x})$ and W can provide less biased solutions as illustrated in [2] for a set of numerical examples.

B. Step 2: Refined convex solution

The solution provided by the convex approach in Step 1, $(p^*_{\mathrm{cvx}}(\bar{\mathbf{x}}), \mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}))$, can be refined by linearizing the original PEP (7)-(9) around $(p^*_{\mathrm{cvx}}(\bar{\mathbf{x}}), \mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}))$, resulting in a convex problem of the form

$$\mathrm{PEP}_{\mathrm{CVX-REF}}(\bar{\mathbf{x}}): \quad \min_{p,\mathbf{x}} \frac{1}{2} \parallel \mathbf{x}-\bar{\mathbf{x}} \parallel_Q^2, \tag{14}$$

subject to

$$\mathbf{0} = A_{\mathrm{L}}\mathbf{x} - B_{\mathrm{L}}p + b, \tag{15}$$

$$\mathbf{x} \in \mathbb{X}, \quad p \in \mathbb{P}. \tag{16}$$

where the matrices A_{L}, B_{L} and the vector b are defined as:

$$A_{\mathrm{L}} = \frac{\partial A(\mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}))}{\partial \mathbf{x}} - \frac{\partial B(\mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}))}{\partial \mathbf{x}} p^*_{\mathrm{cvx}}(\bar{\mathbf{x}}) - W, \tag{17}$$

$$B_{\mathrm{L}} = B(\mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}})), \tag{18}$$

$$b = A(\mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}})) - (A_{\mathrm{L}} + W)\mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}) \tag{19}$$

By solving this convex problem a refined solution $(p^*_{\mathrm{CVX-REF}}(\bar{\mathbf{x}}), \mathbf{x}^*_{\mathrm{CVX-REF}}(\bar{\mathbf{x}}))$ is found. This solution is expected to lie closer to the original problem solution $(p^*(\bar{\mathbf{x}}), \mathbf{x}^*(\bar{\mathbf{x}}))$. An assessment of the distance between the actual solution and the one delivered by the presented approach is provided in the following subsection.

Consequently, the proposed algorithm can be summarized in the following steps:

- **Step 1:** Find a solution, $(p^*_{\mathrm{cvx}}(\bar{\mathbf{x}}), \mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}))$, to the modified problem (11)-(13) by a convex optimizer.
- **Step 2:** Linearize the original $\mathrm{PEP}_{\mathrm{NL}}(\bar{\mathbf{x}})$ around $(p^*_{\mathrm{cvx}}(\bar{\mathbf{x}}), \mathbf{x}^*_{\mathrm{cvx}}(\bar{\mathbf{x}}))$ and solve the resulting convex PEP.

III. Assessment of solution error

At this point three problems can be regarded, the original $\mathrm{PEP}_{\mathrm{NL}}(\bar{\mathbf{x}})$, its convex modification, $\mathrm{PEP}_{\mathrm{CVX}}(\bar{\mathbf{x}})$, and the linearization of the original $\mathrm{PEP}_{\mathrm{NL}}(\bar{\mathbf{x}})$ around the solution provided by the convex formulation, $\mathrm{PEP}_{\mathrm{CVX-REF}}(\bar{\mathbf{x}})$. In the following, perturbation analysis is used to investigate how the solution to these problems are related to each other and provide an assessment of the distance between the real solution and the proposed approximation.

For comparison, consider the unperturbed original PEP where a set of noise-free state measurements $\bar{\bar{\mathbf{x}}}$ is obtained and no modelling errors are present. In addition, the following assumptions are introduced:

- A1: The functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are twice continuously differentiable.
- A2: There exist a pair $\bar{\bar{\mathbf{x}}} \in \mathbb{X}$ and $\bar{\bar{p}} \in \mathbb{P}$ such that $0 = A(\bar{\bar{\mathbf{x}}}) - B(\bar{\bar{\mathbf{x}}})\bar{\bar{p}} - W\bar{\bar{\mathbf{x}}}$.
- A3: Both problems, $\mathrm{PEP}_{\mathrm{NL}}(\bar{\bar{\mathbf{x}}})$ and $\mathrm{PEP}_{\mathrm{CVX}}(\bar{\bar{\mathbf{x}}})$, satisfy the strong second order sufficient conditions (SOSC), strict complementarity and constraint regularity [3] at their solution, $(\bar{\bar{\mathbf{x}}}, \bar{\bar{p}})$.

Corollary 1: Under assumptions A2 and A3, the Lagrange multipliers associated to the inequality constraints at the solution $(\bar{\bar{\mathbf{x}}}, \bar{\bar{p}})$ are zero, and none of the inequality constraints is active.

In view of these assumptions, the following lemmata can be formulated:

Lemma 1: If assumption A2 and A3 hold, then $(\bar{\bar{\mathbf{x}}}, \bar{\bar{p}})$ is a global solution to both $\mathrm{PEP}_{\mathrm{NL}}(\bar{\bar{\mathbf{x}}})$ and $\mathrm{PEP}_{\mathrm{CVX}}(\bar{\bar{\mathbf{x}}})$.

Proof: $(\bar{\bar{\mathbf{x}}}, \bar{\bar{p}})$ is feasible and it yields the lowest possible objective value for both optimization problems, i.e., zero. ■

Lemma 2: If A1 to A3 hold, then

$$\|p^*_{\mathrm{cvx}}(\bar{\mathbf{x}}) - \bar{\bar{p}}\| = \mathcal{O}(\|\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}}\|), \tag{20}$$

i.e., the distance between the perturbed convex problem solution and the unperturbed one is a function of the size of the perturbation.

Proof: Notice that in view of Lemma 1, $\bar{\bar{p}}$ is the solution of the convex unperturbed problem as well. Consequently, it is enough to prove that the distance between the unperturbed solution of the convex problem and its perturbed one obeys (20). In order to do so, consider the PEP

$$\min_{p,\mathbf{x}} \frac{1}{2}\|\mathbf{x}-\hat{\mathbf{x}}\|_Q^2 \quad \text{s.t.} \quad A(\hat{\mathbf{x}}) - B(\hat{\mathbf{x}})p - W\mathbf{x} = \mathbf{0} \quad (21)$$

which corresponds to the convex problem for unperturbed ($\hat{\mathbf{x}} = \bar{\bar{\mathbf{x}}}$) and perturbed ($\hat{\mathbf{x}} = \bar{\mathbf{x}}$) measurements. Notice that the inequality constraints can be neglected for $\hat{\mathbf{x}} = \bar{\bar{\mathbf{x}}}$ due to Corollary 1. Additionally, for sufficiently small perturbations over $\hat{\mathbf{x}} = \bar{\bar{\mathbf{x}}}$, the inequality constraints remain inactive, this allows to neglect the inequality constraints when $\hat{\mathbf{x}} = \bar{\mathbf{x}}$. Consequently, the convex formulation is reduced to (21) for the unperturbed measurement set and in a small neighborhood of it.

The Karush-Kuhn-Tucker (KKT) optimality conditions for (21) yields:

$$F(\mathbf{x}^*, p^*, \lambda^*) = \begin{bmatrix} Q & \mathbf{0} & -W^T \\ \mathbf{0} & \mathbf{0} & -B(\hat{\mathbf{x}}) \\ -W & -B(\hat{\mathbf{x}}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(\hat{\mathbf{x}}) \\ p^*(\hat{\mathbf{x}}) \\ \lambda^*(\hat{\mathbf{x}}) \end{bmatrix} - \begin{bmatrix} Q\hat{\mathbf{x}} \\ \mathbf{0} \\ -A(\hat{\mathbf{x}}) \end{bmatrix} = \mathbf{0}. \quad (22)$$

which provides the solution for the perturbed problem $\hat{\mathbf{x}} = \bar{\mathbf{x}}$ and the unperturbed one, $\hat{\mathbf{x}} = \bar{\bar{\mathbf{x}}}$. The change in the solutions given a change in the measurement data is given by (23).

Due to assumptions A1 to A3, the jacobian of $F(\mathbf{x}^*, p^*, \lambda^*)$ is invertible at $(\bar{\bar{\mathbf{x}}}, \bar{\bar{p}}, \bar{\bar{\lambda}})$. Moreover, the perturbed jacobian remains invertible for small changes in the measurement data $\|\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}}\|$. Consequently, the change in the optimal values also depends on the size of the perturbation as can be inferred from the smoothness of the involved functions in (23). ∎

Theorem 1: If Assumptions A1 to A3 are satisfied, then

$$\|p^*(\bar{\mathbf{x}}) - p^*_{\mathrm{CVX-REF}}(\bar{\mathbf{x}})\| = \mathcal{O}(\|\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}}\|^2) \quad (24)$$

holds. Equation (24) gives an assessment of the distance between the original problem solution and the solution provided by the linearization around the minimizer of the convex approximation $p^*_{\mathrm{CVX-REF}}(\bar{\mathbf{x}})$.

IV. Proof to Theorem 1

In order to simplify the notation, the following definitions are introduced:

$$\mathbf{w} = [\mathbf{x}^T, p^T]^T, \quad Q_{\mathbf{w}} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}.$$

Consider the original PEP with the set of perturbed measurements $\bar{\mathbf{x}}$. Following Corollary 1, the original PEP can be formulated as:

$$\mathrm{PEP}_{\mathrm{NL}}(\bar{\mathbf{w}}): \quad \min_{\mathbf{w}} \frac{1}{2}\|\mathbf{w}-\bar{\mathbf{w}}\|^2_{Q_{\mathbf{w}}} \quad (25)$$

subject to

$$g(\mathbf{w}) = 0. \quad (26)$$

Notice that Corollary 1 implies that inequality constraints can be neglected for $\|\bar{\bar{\mathbf{x}}} - \bar{\mathbf{x}}\|$ small enough. The solution of the nonconvex problem (25)-(26) must satisfy the KKT conditions for the quadratic programming subproblem

$$\mathrm{PEP}_{\mathrm{LIN}}(\bar{\mathbf{w}}, \tilde{\mathbf{w}}): \quad \min_{\mathbf{w}} \frac{1}{2}\|\mathbf{w}-\bar{\mathbf{w}}\|^2_{Q_{\mathbf{w}}} \quad (27)$$

subject to

$$g(\tilde{\mathbf{w}}) + \nabla g(\tilde{\mathbf{w}})^T(\mathbf{w} - \tilde{\mathbf{w}}) = 0 \quad (28)$$

around the linearization point $\tilde{\mathbf{w}}$, i.e.,

$$F(\mathbf{w}, \lambda, \bar{\mathbf{w}}, \tilde{\mathbf{w}}) = \begin{bmatrix} Q_{\mathbf{w}}(\mathbf{w} - \bar{\mathbf{w}}) + \nabla g(\tilde{\mathbf{w}})\lambda \\ g(\tilde{\mathbf{w}}) + \nabla g(\tilde{\mathbf{w}})^T(\mathbf{w} - \tilde{\mathbf{w}}) \end{bmatrix} = \mathbf{0}, \quad (29)$$

where λ represents the Lagrange multipliers for the equality constrained problem. These set of equations provide a solution $\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \tilde{\mathbf{w}})$ as a function of the linearization point and the set of measurements.

For sufficiently smooth perturbations, $\|\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}}\|$, and considering assumptions A1 to A3 it is possible to establish the following relations

$$\|\mathbf{w}^*(\bar{\mathbf{w}}) - \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}})\| = \mathcal{O}(\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|^2) \quad (30)$$

and

$$\|\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}}) - \bar{\bar{\mathbf{w}}}\| = \mathcal{O}(\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|) \quad (31)$$

Equation (30) states that the solution provided by the first order predictor $\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}})$ differs from the real solution by $\mathcal{O}(\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|^2)$ as presented in [4, Theorem 3.6 and section 3.4.1]. Equation (31) is a result of perturbation analysis of optimization problems [5] under assumption A1 to A3. Considering (30) and (31), Lemma 1 is proven by showing that the distance between the first order predictor solution $\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}})$ and the solution of the linearization of the problem around the convex problem minimizer $\mathbf{w}_{\mathrm{cvx}}(\bar{\mathbf{w}})$ is of second order in $\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|$, i.e.,

$$\|\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}}) - \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \mathbf{w}_{\mathrm{cvx}})\| = \mathcal{O}(\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|^2) \quad (32)$$

In order to do so, consider the series expansion of the linear predictor solution around the minimizer provided by the convex problem using the perturbed set of measurements,

$$\begin{aligned} \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \mathbf{w}_{\mathrm{cvx}}) &= \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}}) \\ &\quad + \frac{\partial \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \tilde{\mathbf{w}})}{\partial \tilde{\mathbf{w}}} \mid_{\tilde{\mathbf{w}} = \bar{\bar{\mathbf{w}}}} (\mathbf{w}_{\mathrm{cvx}} - \bar{\bar{\mathbf{w}}}) \\ &\quad + \mathcal{O}(\|\mathbf{w}_{\mathrm{cvx}} - \bar{\bar{\mathbf{w}}}\|^2). \end{aligned} \quad (33)$$

The term

$$\frac{\partial \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \tilde{\mathbf{w}})}{\partial \tilde{\mathbf{w}}} \mid_{\tilde{\mathbf{w}} = \bar{\bar{\mathbf{w}}}} (\mathbf{w}_{\mathrm{cvx}} - \bar{\bar{\mathbf{w}}}) \quad (34)$$

is investigated in detail. By evaluating (29) at the solution $(\mathbf{w}^*_{\mathrm{LIN}}, \lambda^*_{\mathrm{LIN}})$,

$$F(\mathbf{w}^*_{\mathrm{LIN}}, \lambda^*_{\mathrm{LIN}}, \bar{\mathbf{w}}, \tilde{\mathbf{w}}) = 0, \quad (35)$$

and applying the implicit function theorem to it, it is possible to obtain an expression for the first factor in (34),

$$\begin{bmatrix} \frac{\partial \mathbf{x}^*(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \\ \frac{\partial p^*(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \\ \frac{\partial \lambda^*(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} Q & \mathbf{0} & -W^T \\ \mathbf{0} & \mathbf{0} & -B(\hat{\mathbf{x}})^T \\ W & -B(\hat{\mathbf{x}}) & \mathbf{0} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{\partial B(\hat{\mathbf{x}})^T}{\partial \hat{\mathbf{x}}} \\ \mathbf{0} & -\frac{\partial B(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(\hat{\mathbf{x}}) \\ p^*(\hat{\mathbf{x}}) \\ \lambda^*(\hat{\mathbf{x}}) \end{bmatrix} + \begin{bmatrix} Q \\ \mathbf{0} \\ \frac{\partial A(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \end{bmatrix} \right) \quad (23)$$

$$\frac{\partial \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \tilde{\mathbf{w}})}{\partial \tilde{\mathbf{w}}} = -\begin{bmatrix} \mathbb{I} & 0 \end{bmatrix} J(\tilde{\mathbf{w}})^{-1} \frac{\partial F}{\partial \tilde{\mathbf{w}}}, \quad (36)$$

with

$$\begin{aligned} J(\tilde{\mathbf{w}}) &= \frac{\partial F(\mathbf{w}^*_{\mathrm{LIN}}, \lambda^*_{\mathrm{LIN}}, \bar{\mathbf{w}}, \tilde{\mathbf{w}})}{\partial (\mathbf{w}^*_{\mathrm{LIN}}, \lambda^*_{\mathrm{LIN}})} \\ &= \begin{bmatrix} Q_{\mathbf{w}} & \nabla g(\tilde{\mathbf{w}}) \\ \nabla g(\tilde{\mathbf{w}})^T & 0 \end{bmatrix} \end{aligned}$$

and

$$\frac{\partial F}{\partial \tilde{\mathbf{w}}} = \begin{bmatrix} \nabla^2 g(\tilde{\mathbf{w}}) \lambda^*_{\mathrm{LIN}} \\ \nabla^2 g(\tilde{\mathbf{w}})(\mathbf{w}^*_{\mathrm{LIN}} - \tilde{\mathbf{w}}) \end{bmatrix}.$$

Equation (36) is obtained by considering that $\frac{\partial \bar{\bar{\mathbf{w}}}}{\partial \bar{\mathbf{w}}} = 0$, i.e., the measurement data does not depend on the linearization point. Assuming A3, and the invertibility of $J(\bar{\bar{\mathbf{w}}})$, at the linearization point ,$\tilde{\mathbf{w}} = \bar{\bar{\mathbf{w}}}$, $J(\bar{\bar{\mathbf{w}}})^{-1}$ and $\nabla^2 g(\bar{\bar{\mathbf{w}}})$ become constants yielding

$$\frac{\partial \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}})}{\partial \tilde{\mathbf{w}}} = \mathcal{O}\left(\left\| \begin{matrix} \lambda^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}}) \\ \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}}) - \bar{\bar{\mathbf{w}}} \end{matrix} \right\| \right) \quad (37)$$

Note that $\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}}) - \bar{\bar{\mathbf{w}}}$ corresponds to the distance between the unperturbed solution $\bar{\bar{\mathbf{w}}}$ and the perturbed one provided by the use of a linear predictor in the constraints. This distance is given by (31) and leads to

$$\frac{\partial \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}})}{\partial \tilde{\mathbf{w}}} = \mathcal{O}(\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|). \quad (38)$$

Hence, combining (38) and Lemma 2 yields

$$\frac{\partial \mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \tilde{\mathbf{w}})}{\partial \tilde{\mathbf{w}}} \Big|_{\tilde{\mathbf{w}} = \bar{\bar{\mathbf{w}}}} (\mathbf{w}_{\mathrm{cvx}} - \bar{\bar{\mathbf{w}}}) = \mathcal{O}(\|\bar{\mathbf{w}} - \bar{\bar{\mathbf{w}}}\|^2) \quad (39)$$

Consequently, by using (39) and Lemma 2 in (33), the solutions $\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \mathbf{w}_{\mathrm{cvx}})$ and $\mathbf{w}^*_{\mathrm{LIN}}(\bar{\mathbf{w}}, \bar{\bar{\mathbf{w}}})$ are identically apart from second order perturbations ■

V. CASE STUDIES

The proposed approach is used to estimate the parameters in two case studies. Comparative results are illustrated for the nonlinear problem, the convex approach in Step 1 and the linearized problem around the convex approach solution in Step 2.

A. The Harmonic Oscillator

Consider the PEP arising from the estimation of the frequency in a linear oscillator. The PEP can be cast as[1]:

$$\min_{p, \mathbf{x}} \frac{1}{2} \sum_{k=0}^{92} \sum_{j=1}^{2} (x_j(k) - \bar{x}_j(k))^2 \quad (40)$$

[1]In order to preserve the affine parameter-dependence, the model is discretized using central differences

subject to

$$\begin{aligned} x_1(k+1) &= x_1(k-1) + 2h x_2(k), & (41) \\ x_2(k+1) &= x_2(k-1) - 2hp x_1(k), & (42) \\ x &\in \mathbb{R}^2,\ p \geq 0 & (43) \end{aligned}$$

A set of measurements has been emulated by simulating the model during 6.5s with a sampling period of $h = 71.4\mathrm{ms}$ and adding colored noise in the bands 0.4 to 0.7Hz and 6.5 to 6.7Hz. Hence, the PEP can be formulated as in (7)-(8). If $A(\mathbf{x})$ and $B(\mathbf{x})$ are affine in $\mathbf{x}$, then, for a fixed p the problem

$$\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{x} - \bar{\mathbf{x}}\|^2_Q, \quad \text{s.t.} \quad A(\mathbf{x}) - B(\mathbf{x})p - W\mathbf{x} = 0 \quad (44)$$

is convex and its global solution can be computed. This allows to generate Figure 1 where the least squares cost is presented. The Nonconvex PEP is modified by the approach in Step 1, leading to

$$\min_{p, \mathbf{x}} \frac{1}{2} \sum_{k=0}^{92} \sum_{j=1}^{2} (x_j(k) - \bar{x}_j(k))^2 \quad (45)$$

subject to

$$\begin{aligned} x_1(k+1) &= x_1(k-1) + 2h \bar{x}_2(k), & (46) \\ x_2(k+1) &= x_2(k-1) - 2hp \bar{x}_1(k), & (47) \\ x &\in \mathbb{R}^2,\ p \geq 0 & (48) \end{aligned}$$

Notice that in this modification the matrix W and $A(x)$ has been splitted in a different way than in (10) as proposed in [2]. Consequently the solution achieved is not the same as when using PEM. Hereafter, the problem is linearized around this solution leading to a minimum lying closer to the original solution as illustrated in Figure 1. Notice that the computational effort in solving one nonconvex optimization problem is replaced by the the cost of solving two convex problems. Additionally, no knowledge about p needs to be used for initialization. Results are summarized in Table I, where the nonlinear optimizer is initialized in the neighborhood of the solution, i.e., $x_{\mathrm{ini}}(0) = [0.5,\ 0.5]$, $p_{\mathrm{ini}} = 0.5\pi^2$. Notice that the norm on the problem minimizers indicates that the refined solution can be used already as a much better approximation to the minimizer than the one provided by the convex problem in Step 1 at the cost of solving one extra convex problem.

B. The Lotka-Volterra Model

Consider a Lotka-volterra model. Equations (50)-(51) describe the time evolution of the population density for two species in a biological system, a predator $x_2(k)$ and its prey $x_1(k)$. The dynamic behavior of this interaction is characterized by the following parameters:

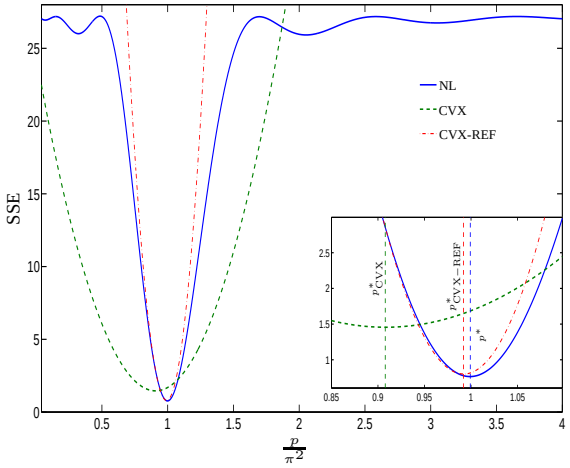

Fig. 1. Least squares costs for the estimation of the natural frequency in a harmonic oscillator. Three costs are illustrated, NL corresponding to the original problem, CVX corresponding to the convex problem (11)-(13) in Step 1 and the cost obtained from the linearization of the original problem around the solution of (11)-(13) in Step 2, CVX-REF.

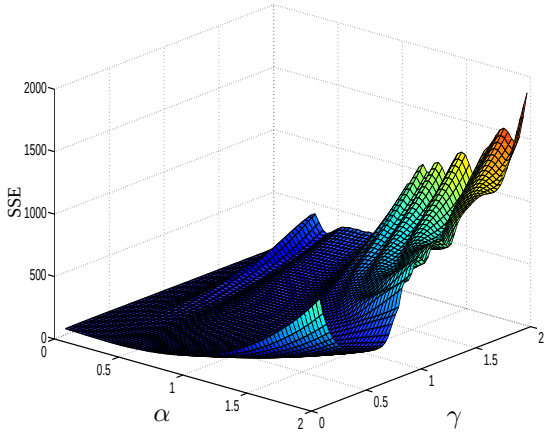

Fig. 2. SSE as a function of the parameters α and γ for the PEP in the Lotka-Volterra model. The parameters β, δ and the initial condition are equal to their original values, i.e., $\beta = 0.5,\ \delta = 0.4,\ [x_1(0), x_2(0)]^T = [1, 0.5]^T$.

TABLE I

OPTIMIZATION RESULTS FOR THE OSCILLATOR PEP.

Parameter	NL	Step 1	Step 2
$x(0)^T$	$[-0.0032,\ 1.02]$	$[0.0011,\ 0.828]$	$[-0.0063,\ 1.021]$
$\frac{p}{\pi^2}$	0.9989	0.9078	0.992
SSE	0.765	1.456	0.8013
$\|P^* - P\|_2$[a]	0	0.92	0.0685
Iterations	19	1	2[b]

[a] P^* corresponds to the optimum given by the nonlinear solver including initial conditions, i.e., $P^* = [x^*(0)^T,\ p^*]^T$.

[b] The procedure to obtain the linearization point (Step 1) is considered as the first iteration.

α: intrinsic rate of prey population increase,
β: predation rate coefficient,
γ: reproduction rate of predators per 1 prey eaten,
δ: predator mortality rate,

leading to a parameter vector $p = [\alpha,\ \beta,\ \gamma,\ \delta]^T$. This model is used as a benchmark for parameter estimation in [6] where only two of the four parameter are estimated. However, in order to make the problem more challenging, here the whole set of parameters is estimated. A set of noisy state measurements is emulated by simulating the model with nominal parameters $p = [0.6,\ 0.5,\ 0.7,\ 0.4]^T$, $x(0) = [1,\ 0.5]^T$ and adding gaussian noise with a variance $\sigma^2 = 0.05$. The PEP can be formulated as:

$$\min_{p,\mathbf{x}} \frac{1}{2} \sum_{k=0}^{200} \sum_{j=1}^{2} \left(x_j(k) - \bar{x}_j(k)\right)^2 \tag{49}$$

subject to

$$\begin{aligned} x_1(k+1) &= x_1(k) + \alpha x_1(k) - \beta x_1(k) x_2(k), && (50)\\ x_2(k+1) &= x_2(k) - \gamma x_2(k) + \delta x_1(k) x_2(k), && (51)\\ 0 &\leq \alpha, \beta, \gamma, \delta \leq 2, && (52) \end{aligned}$$

where $\bar{x}(k)$ represents the noisy state measurements. The contaminated data sequence is used to evaluate the cost (49) with two parameters and the initial condition fixed to the nominal values. Figure 2 shows the nonconvex cost as a function of the varied parameters. Notice that in this case a wrong solution can be easily obtained by an inappropriate initialization of the problem.

The PEP is cast as in (7)-(9). In this example the state vector $\mathbf{x} = [x_1(0), x_2(0), \ldots, x_1(N), x_2(N)]^T$ is optimized along with the four model-parameters using a nonlinear optimizer initialized with a value in the neighborhood of the solution (which in practice would not be available). On the other hand, the PEP is convexified following the approach proposed in Step 1[2], leading to

$$\min_{p,\mathbf{x}} \frac{1}{2} \sum_{k=0}^{200} \sum_{j=1}^{2} \left(x_j(k) - \bar{x}_j(k)\right)^2 \tag{53}$$

subject to

$$\begin{aligned} x_1(k+1) &= x_1(k) + \alpha \bar{x}_1(k) - \beta \bar{x}_1(k) \bar{x}_2(k), && (54)\\ x_2(k+1) &= x_2(k) - \gamma \bar{x}_2(k) + \delta \bar{x}_1(k) \bar{x}_2(k), && (55)\\ 0 &\leq x_1, x_2 \leq 5, && (56)\\ 0 &\leq \alpha, \beta, \gamma, \delta \leq 2. && (57) \end{aligned}$$

Hereafter, the solution to the convex problem, is used to linearize (49)-(52) leading to a linearly constrained least squares problem. Table II summarizes the results obtained by the nonlinear solver, Steps 1 and 2. Notice that the number of iterations corresponds to the number of convex problems solved. In Table II and Figure 3 it is possible to evidence that Step 1 provides a solution lying close to the one provided by the nonlinear optimizer. On the other hand, the refined solution in Step 2 generates a state trajectory that practically

[2] The splitting of the matrices $A(\bar{\mathbf{x}})$ and $W\mathbf{x}$ is different from PEM as can be seen from (54)-(55).

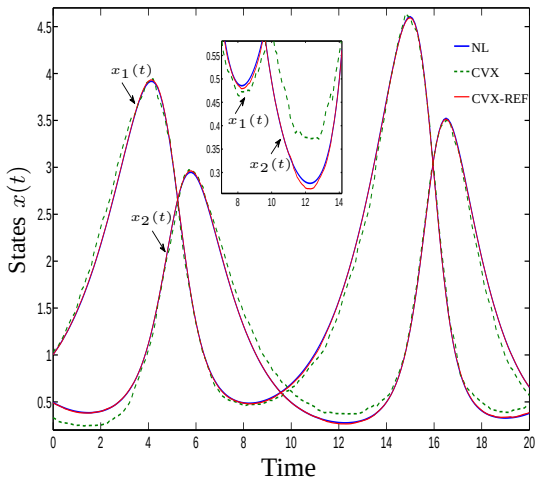

Fig. 3. Time evolution of the identified model using the set of parameters provided by the nonlinear solver, the convex approach and the refined convex approach.

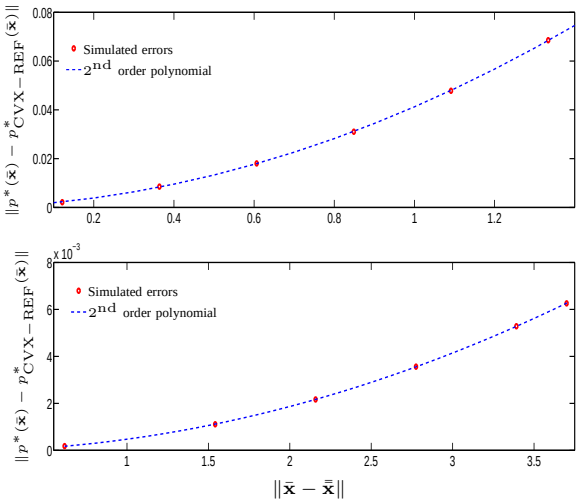

Fig. 4. Errors in the solution provided by the proposed approach as a function of the size of the perturbation for the harmonic oscillator PEP (top) and the Lotka-Volterra PEP (bottom)

TABLE II

OPTIMIZATION RESULTS FOR THE LOTKA-VOLTERRA PEP.

Parameter	NL	Step 1	Step 2
$x(0)^T$	[1.006, 0.493]	[1.029, 0.33]	[1.004, 0.49]
α	0.6032	0.6032	0.6042
β	0.5059	0.5050	0.5064
γ	0.6903	0.6891	0.6901
δ	0.3971	0.3973	0.3971
SSE	8.775	52.77	8.793
$\|P^* - P\|_2$[a]	0	165.2×10^{-3}	4.4×10^{-3}
Iterations	142	1	2[b]

[a] P^* corresponds to the optimum given by the nonlinear solver including initial conditions, i.e., $P^* = [x^*(0)^T, p^{*T}]^T$.

[b] The procedure to obtain the linearization point (Step 1) is considered as the first iteration.

overlaps the one generated by the solution of the PEP_{NL}. Consequently, for this application, solving only two convex problems provides a solution which lies very close to the one provided by the nonlinear optimizer.

In order to corroborate of Theorem 1, six experiments are run for the presented PEPs, each one with a state sequence contaminated with gaussian noise with different variances. The distance between the solution provided by a nonlinear optimizer and the proposed approach in Step 2 is calculated along with the size of the perturbation with respect to the noise-free data. Figure 4 summarizes the results. Clearly the distance between the actual solution $p^*(\bar{\mathbf{x}})$ an the one provided by the refined convex approximation $p^*_{\text{CVX}-\text{REF}}(\bar{\mathbf{x}})$ is of second order in the size of the perturbation. Notice that a second order polynomial accurately fits the data corresponding to the six experiments for both case studies.

VI. CONCLUSION

A convex approach for the solution of parameter estimation problems in nonlinear parameter-affine dynamic models has been presented. The advantages of the method lie in the fact that no initialization of the unknown parameters is required and only two convex problems are solved to approximate the solution of the original PEP. It is demonstrated that the distance between the actual solution and the one provided by the proposed method is of second order in the size of the perturbation. The method relies in the availability of full state measurements, this limitation can be overcome in some applications, e.g., mechanics or electrical systems where acceleration, speed, position, currents and voltages are easily measured or estimated. Two benchmark examples have been studied, showing that the heuristic of the method delivers an approximate minimizer to the original problem and reduces the complexity of the solution procedure by using the power of convex optimization.

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