Comparison of the performance of matrix and tensor based multi-channel harmonic analysis

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Abstract

We consider the decomposition of a signal in exponential components in the multi-channel case. The information about the signal is provided as a set of simultaneous signals with identical poles but different complex amplitudes. The goal is to estimate the poles, given only samples of the signal. Matrix-based methods are well-known for the estimation of the unknown parameters (the frequencies and the damping factors). In Papy *et al.* (2005) and Papy (2005) corresponding tensor-based algorithms are proposed.

Comparing the matrix-based and the tensor-based methods for the case of closely spaced poles, we observe that neither of the algorithms is always better than the other. We look for factors that affect the performance of the tensor method, in order to see in which cases it is better to use it instead of the matrix-based method.

Keywords: signal processing, exponentially damped sinusoids, parameter estimation, multi-linear algebra, higher-order tensors

1. Introduction

We consider the decomposition of a signal as a sum of K complex damped or undamped sinusoids. Given N samples x_n, $n = 0, \ldots, N-1$,

$$x_n = \sum_{k=1}^{K} a_k \exp\{j\varphi_k\} \exp\{(-\alpha_k + 2j\pi\nu_k)\, t_n\} = \sum_{k=1}^{K} c_k z_k^n,$$

the goal is to accurately estimate the parameters (the amplitudes a_k, the phases φ_k, the damping factors α_k, and the frequencies ν_k, or more general, the complex amplitudes $c_k = a_k \exp\{j\varphi_k\}$ and the poles $z_k = \exp\{(-\alpha_k + 2j\pi\nu_k)\, \Delta t\}$, where Δt is the sampling time interval and $t_n = n\Delta t$).

For the single-channel, multi-channel, and decimative case, matrix-based methods are well-known for the estimation of the unknown parameters. In Papy (2005) corresponding tensor-based algorithms are proposed. Of interest for this report is the case of closely spaced poles. It has been shown by simulations (see Papy (2005)) that in the decimative case the tensor-based method is more robust against noise than its matrix-based counterpart. Our goal is to compare the two methods in the multi-channel case, where the information about the signal is provided as a set of Q simultaneous signals in different channels in the presence of noise $e_n^{(q)}$. These signals have identical poles but different complex amplitudes, i.e., for $n = 0, \ldots, N-1$, $q = 1, \ldots, Q$,

$$x_n^{(q)} = \sum_{k=1}^{K} a_k^{(q)} \exp\{j\varphi_k^{(q)}\} \exp\{(-\alpha_k + 2j\pi\nu_k)\, t_n\} + e_n^{(q)} = \sum_{k=1}^{K} c_k^{(q)} z_k^n + e_n^{(q)}\,. \qquad (1.1)$$

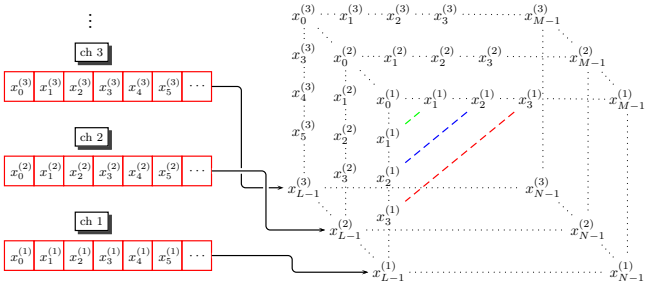
2. Theoretical background

Third-order tensors are generalizations of vectors (first order) and matrices (second order) to third order. They are complex structures, which elements are referred to by three indices. The mode-n ($n = 1, 2, 3$) vectors of a tensor are its columns, rows, etc., i.e., the set of vectors, obtained by varying the n-th index, while keeping the other indices fixed. The dimension of the vector space spanned by the mode-n vectors is called the mode-n rank. It is a generalization of the column and row rank of a matrix. Contrary to the matrix case, different mode-n ranks are not necessarily equal to each other.

In the tensor-based algorithms for parameter estimation, the low-rank approximation of a tensor (see De Lathauwer *et al.* (2000)) plays an essential role. Specific values of the mode-n ranks of the approximation can be chosen, for which the performance of the tensor method is optimal. In the multi-channel case, the main steps of the tensor algorithm are:

HO-HTLSstack Algorithm (Papy *et al.* (2005))

1) Construct a tensor from the sample data, arranged in Hankel matrices, as follows



2) Compute the best rank-(R_1, R_2, R_3) approximation $\hat{A}$ of the tensor as in De Lathauwer *et al.* (2000):

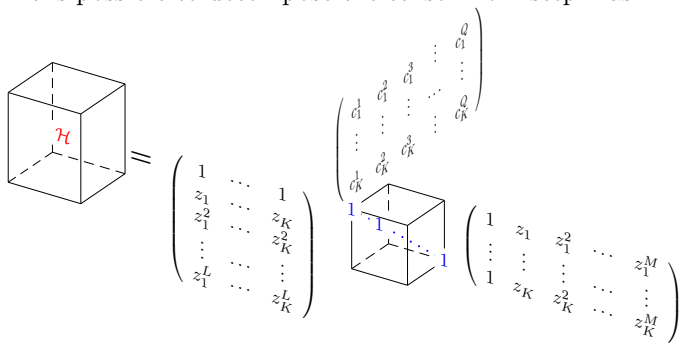
$$\hat{A} = B \times_1 U^{(1)} \times_2 U^{(2)} \times_3 U^{(3)}.$$

3) Compute the eigenvalues λ_k of $\bar{Z}$: $U^{(1)\uparrow} \approx U_{\downarrow}^{(1)} \bar{Z}$, where $U^{(1)\uparrow}$ (resp. $U_{\downarrow}^{(1)}$) is derived form $U^{(1)}$ by deleting it's first (resp. last) row.

4) Estimate the damping factors $\hat{\alpha}_k$ and frequencies $\hat{\nu}_k$ from

$$\lambda_k = \exp\{(-\hat{\alpha}_k + 2\pi j \hat{\nu}_k)\, t_n\}.$$

Remark. It is possible to decompose the tensor from step 1 as



This decomposition will be used later, while explaining the behavior of the tensor method for certain values of the parameters.

3. Simulations

Taking advantage of the tensor structure, the mode-3 rank R_3 of the low-rank approximation of the tensor (see step 2 in the HO-HTLSstack Algorithm) can be reduced more than the other two ranks. Simulations show that if the matrix of the complex amplitudes is ill-conditioned this is reasonable, otherwise not. The following example (taken from Morren *et al.* (2003)) is considered: $\nu_1 = 0.2, \nu_2 = 0.205$, $\alpha_1 = \alpha_2 = 0$, $\varphi_k^{(q)} = 0$, $a_k^{(q)}$ form an ill-conditioned matrix with condition number 100, $t_n = n, k = 1, 2$, $q = 1, \ldots, 13$, $n = 0, \ldots, 24$. Complex, circularly symmetric white Gaussian noise is present. Results of Monte-Carlo simulations of 100 independent runs are shown in Fig. 1 (RRMSE is the relative root mean squared error). It can be seen that in this case the tensor method with reduced mode-3 rank outperforms the matrix method, which has similar behavior to the behavior of the tensor method with equal ranks.

Introducing a mild damping factor improves the performance of the methods. Performing similar simulations as above but with $\alpha_1 = 0, \alpha_2 = 0.4$ we can see (Fig. 2) that all algorithms work even for higher levels of noise. The best method is still the tensor method with reduced mode-3 rank. This observation can be explained by examining the condition number of the Vandermonde matrices of the poles (see Remark on page 2). Fig. 3 shows that the condition number of the Vandermonde matrices is improved when introducing the damping factor. The horizontal line shows the true value of the condition number.

4. Conclusions

In the tensor-based algorithms, the low-rank approximation of a tensor (see De Lathauwer *et al.* (2000)) plays an essential role. Specific values of the mode-n ranks can be chosen, for which the performance of the tensor method is optimal. This technique we use when the matrix of the amplitudes is ill-conditioned. In this case, taking lower value for the mode-3 rank than for the mode-1 and mode-2 ranks improves the performance of the tensor method and it may give better results than the matrix method. Another factor that improves the performance of the tensor method is mild differences in damping. This fact can be explained by examining the Vandermonde matrix of the poles. Introducing mild damping factors in the model improves the condition number of the Vandermonde matrix and thus improves the overall performance of the tensor method.

REFERENCES

De Lathauwer, L., De Moor, B. & Vandewalle, J. 2000 On the Best Rank-1 and Rank-$(R_1, R_2, \ldots, R_N)$ Approximation of Higher-Order Tensors. *SIAM J. Matrix Anal. Appl.* **21**, 1324–1342.

Morren, G., Lemmerling, Ph. & Van Huffel, S. 2003 Decimative subspace-based parameter estimation technique. *Sig. Proc.* **83**, 1025–1033.

Papy, J.-M., De Lathauwer, L. & Van Huffel, S. 2005 Exponential data fitting using multilinear algebra: The single-channel and the multichannel case. *Num. Lin. Alg. Appl.* **12**, 809–826.

Papy, J.-M. 2005 Subspace-based Exponential Data Fitting Using Linear and Multilinear Algebra. *PhD thesis, K.U.Leuven-ESAT, Belgium.*

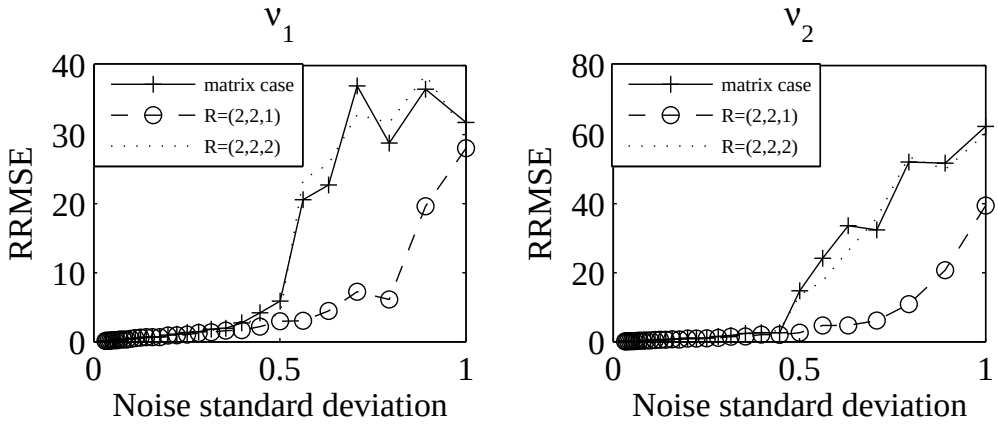


FIGURE 1. Reducing R_3 in step 2 of the algorithm

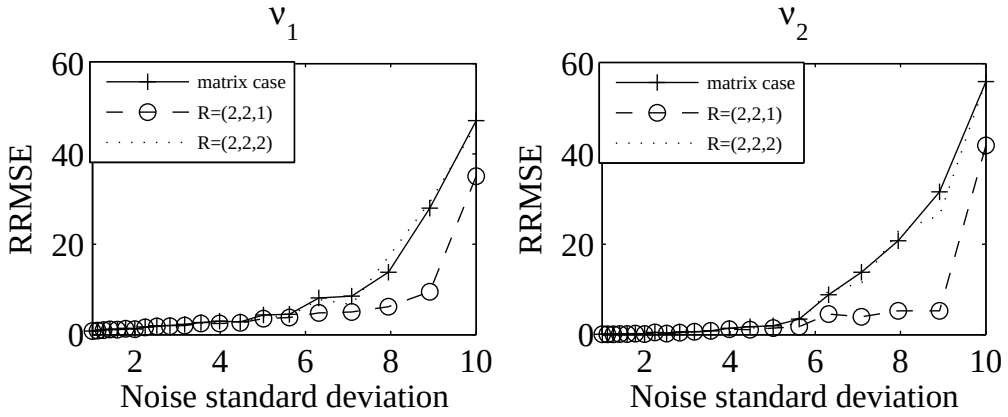
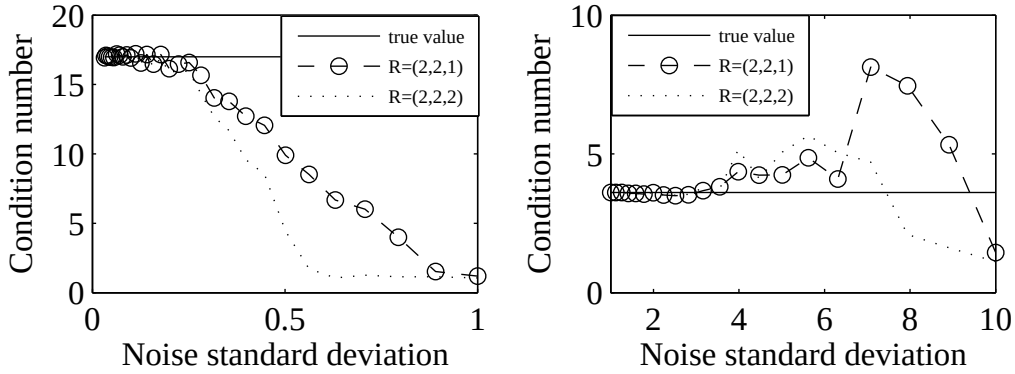


FIGURE 2. Introducing damping factor



(a) without damping (b) with damping factors 0 and 0.4

FIGURE 3. Condition number of the Vandermonde matrix of the poles