DECOMPOSITION OF A TENSOR INTO MULTILINEAR RANK-$(M_r, N_r, \cdot)$ TERMS*

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Abstract. We present new generic and deterministic uniqueness results for block term decompositions (BTD). These uniqueness conditions hold under mild assumptions and apply to more general settings than previously known results. We also present an algebraic algorithm for the computation of BTDs. Our algorithm requires no knowledge of the block sizes appearing in the BTD: these block sizes are recovered from the algorithm. Through numerical simulations, we illustrate that, in contrast to competing optimization-based methods, even in noisy settings our algebraic algorithm can successfully recover an underlying BTD without knowledge of block sizes provided the signal-to-noise ratio is sufficiently high. We observe that the algorithm can significantly improve one's ability to successfully recover a BTD when it is used as an algebraic initialization for leading optimization routines. Moreover, only a few optimization iterations are required to successfully converge to the BTD from the algebraic solution.

Key words. multilinear algebra, tensors, block term decomposition, multilinear rank

AMS subject classifications. 15A23, 15A69

1. Introduction.

Tensors are natural objects for studying higher-order data and for modeling nonlinear interactions between input and output data sets. As a consequence, these multiindexed arrays have come to play an important role in many signal processing and machine learning applications [6, 34]. Tensor decompositions generalize matrix decompositions and are powerful tools that enable recovery of underlying components. A particularly exciting feature of many types of tensor decomposition is that, with mild assumptions, the tensor decomposition is (essentially) unique [15, 17, 26]. In particular, unlike in the matrix case, one often does not need to make assumptions on the factors such as orthogonality to have uniqueness in the tensor setting. This essential uniqueness makes tensor decomposition a natural approach for problems such as blind source separation [6, 7, 34].

Two fundamental tensor decompositions are the Tucker/multilinear singular value decomposition (MLSVD) [12, 39] and the canonical polyadic decomposition (CPD) [3, 24]. The MLSVD extends notions of row and column rank to tensors and reveals the principal subspaces associated to a tensor [14]. On the other hand, the CPD expresses a tensor as a minimal sum of rank-1 terms thereby leading to another (different) natural notion of tensor rank. Each of these rank-1 terms can be interpreted as an "elementary component" of the tensor.

In 2008, the MLSVD and CPD were connected through the introduction of the block term decomposition (BTD) [8, 9, 13]. In this decomposition, a tensor is viewed as a sum of elementary block tensors that do not necessarily have rank 1. This

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decomposition is more flexible than CPD since sources of interest may fail to be captured by individual rank-1 terms [5, 10, 11, 20, 31, 32, 33, 38, 41]. We build upon the existing literature by providing uniqueness conditions and algorithms for computing more general types of BTDs. In particular, the BTDs considered in [9] are either $(M, N, \cdot)$ decompositions where all blocks have the same size, or $(L_r, L_r, 1)$ decompositions where blocks can have different sizes, but all blocks have mode-3 rank equal to one. The present article considers $(M_r, N_r, \cdot)$ decompositions where all blocks can have different sizes and can have multilinear rank larger than one in the third mode. This further increases the flexibility of the BTD and, notably, allows one to express a tensor as a sum of rank-1 and block terms. Uniqueness conditions for $(L_r, L_r, 1)$ decompositions are also considered in [18]; however, the parameter ranges in which the uniqueness results of [18] are applicable, are distinct from those of the present article. The differences are summarized in [18, Table 1.1].[1]

Our contributions include generic and deterministic uniqueness conditions for the multilinear rank-$(M_r, N_r, \cdot)$ decomposition that hold under mild assumptions; see Theorems 2.1 and 2.3 and Corollaries 2.4 to 2.6. Furthermore, the results are constructive and lead to an algebraic algorithm for the computation of multilinear rank-$(M_r, N_r, \cdot)$ decompositions, which is presented as Algorithm 2.1. A surprising feature of this algorithm is that the user need not specify the size of the block terms to successfully compute a decomposition. That is, the algorithm can successfully independently recover the parameters M_r and N_r and R which determine the size and number of blocks. The algorithm performs well even in noisy settings and can in many cases (approximately) recover the underlying BTD. The efficacy of the algorithm is illustrated through a variety of numerical simulations in section 3.

1.1. Terminology and problem statement. Throughout the paper $\mathbb{F}$ denotes the field of real or complex numbers; bold-face lower case, bold-face capital case, and capital calligraphic letters refer to vectors, matrices, and tensors, respectively (e.g., $\mathbf{a}_1$, $\mathbf{B}_2$, and $\mathcal{T}$); $\mathbf{O}$ and $\mathbf{0}$ denote a zero matrix and a zero vector, respectively; $\mathbf{I}_n$ and $\mathbf{O}_{m\times n}$ denote the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively; $\mathbf{O}_n$ a is shorthand for $\mathbf{O}_{n\times n}$; $\mathrm{Null}\,(\cdot)$ denotes the null space of a matrix; $r_{\mathbf{A}}$ denotes the rank of a matrix $\mathbf{A}$; "$*$" , "^{T}", and "^{H}" denote the conjugation, transpose, and Hermitian transpose, respectively. The Kronecker product of $\mathbf{A} \in \mathbb{F}^{I\times J}$ and $\mathbf{B} \in \mathbb{F}^{K\times L}$ is denoted by $\otimes$ and defined as

$$\mathbf{A} \otimes \mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1J}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{I1}\mathbf{B} & \dots & a_{IJ}\mathbf{B} \end{bmatrix} \in \mathbb{F}^{IK\times JL}.$$

The Khatri–Rao product of $\mathbf{A} \in \mathbb{F}^{I\times R}$ and $\mathbf{B} \in \mathbb{F}^{J\times R}$ is denoted by $\odot$ and defined as $\mathbf{A} \odot \mathbf{B} := \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \dots & \mathbf{a}_R \otimes \mathbf{b}_R \end{bmatrix} \in \mathbb{F}^{IJ\times R}$. The CPD of tensor $\mathcal{T}$ is denoted by

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r =: [\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!],$$

[1] Table 1.1 in [18] compares to a preprint [19] of the present article.

where $\circ$ is the outer product. For $\mathcal{T} \in \mathbb{F}^{I_1 \times \cdots \times I_{n-1} \times I_n \times I_{n+1} \times \cdots \times I_N}$ and $\mathbf{M} \in \mathbb{F}^{J \times I_n}$, the mode-$n$ tensor-matrix product $\mathcal{T} \bullet_n \mathbf{M} \in \mathbb{F}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N}$ is defined as

$$(\mathcal{T} \bullet_n \mathbf{M})_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} t_{i_1 \cdots i_{n-1} i_n i_{n+1} \cdots i_N} m_{j i_n}.$$

Let tensor $\mathcal{T} \in \mathbb{F}^{I \times J \times K}$ and let $\mathbf{T}_1, \ldots \mathbf{T}_K \in \mathbb{F}^{I \times J}$ denote the frontal slices of $\mathcal{T}$. The matrices

$$\begin{aligned}
\mathbf{T}_{[1;3,2]} &\coloneqq \begin{bmatrix} \mathbf{T}_1 & \cdots & \mathbf{T}_K \end{bmatrix} \in \mathbb{F}^{I \times KJ}, \\
\mathbf{T}_{[2;3,1]} &\coloneqq \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}} & \cdots & \mathbf{T}_K^{\mathrm{T}} \end{bmatrix} \in \mathbb{F}^{J \times KI}, \\
\mathbf{T}_{[3;2,1]} &\coloneqq \begin{bmatrix} \mathrm{vec}(\mathbf{T}_1) & \cdots & \mathrm{vec}(\mathbf{T}_K) \end{bmatrix}^{\mathrm{T}} \in \mathbb{F}^{K \times JI},
\end{aligned}$$

are called the *matrix unfoldings* of $\mathcal{T}$ and the triplet of their ranks

$$(r_{\mathbf{T}_{[1;3,2]}}, r_{\mathbf{T}_{[2;3,1]}}, r_{\mathbf{T}_{[3;2,1]}}),$$

is called the *multilinear (ML) rank* of $\mathcal{T}$. Here, $\mathrm{vec}(\mathbf{T}_j)$ denotes the vector formed by stacking in increasing order the columns of $\mathbf{T}_j$ in a single vector.

A tensor $\mathcal{T}$ has ML rank-$(M, N, \cdot)$ if $r_{\mathbf{T}_{[1;3,2]}} = M$, $r_{\mathbf{T}_{[2;3,1]}} = N$ and $r_{\mathbf{T}_{[3;2,1]}}$ is not specified. It can easily be shown that $\mathcal{T}$ is ML rank-$(M, N, \cdot)$ if and only if its frontal slices can be simultaneously factorized as

$$\mathbf{T}_k = \mathbf{A}\mathbf{D}_k\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D}_k \in \mathbb{F}^{M \times N}, \qquad k = 1, \ldots, K, \tag{1.1}$$

where the factors $\mathbf{A} \in \mathbb{F}^{I \times M}$ and $\mathbf{B} \in \mathbb{F}^{J \times N}$ have full column rank and the matrices

$$\mathbf{D}_{[1;3,2]} \coloneqq [\mathbf{D}_1 \ \ldots \ \mathbf{D}_K] \in \mathbb{F}^{M \times KN} \quad \text{and} \quad \mathbf{D}_{[2;3,1]} \coloneqq [\mathbf{D}_1^{\mathrm{T}} \ \ldots \ \mathbf{D}_K^{\mathrm{T}}] \in \mathbb{F}^{N \times KM}$$

have full row rank. (One can, for instance, take $\mathbf{A}$ and $\mathbf{B}$ equal to the "$\mathbf{U}$" factors in the compact SVDs of $\mathbf{T}_{[1;3,2]}$ and $\mathbf{T}_{[2;3,1]}$, respectively, and set $\mathbf{D}_k = \mathbf{A}^{\mathrm{H}}\mathbf{T}_k\mathbf{B}^*$.) It is worth noting that the factors $\mathbf{A}$, $\mathbf{D}_k$, and $\mathbf{B}$ in (1.1) are not unique.

Moreover, a generic ML rank-$(M, N, \cdot)$ tensor can be parameterized with $(I - M)M + (J - N)N + MNK$ parameters. Indeed, if $\mathbf{X} \in \mathbb{F}^{M \times M}$ and $\mathbf{Y} \in \mathbb{F}^{N \times N}$ are nonsingular matrices, then $\mathbf{T}_1, \ldots, \mathbf{T}_K$ can also be factorized as

$$\mathbf{T}_k = (\mathbf{A}\mathbf{X}) \left(\mathbf{X}^{-1}\mathbf{D}_k\mathbf{Y}^{-\mathrm{T}}\right) (\mathbf{B}\mathbf{Y})^{\mathrm{T}} = \widehat{\mathbf{A}}\widehat{\mathbf{D}}_k\widehat{\mathbf{B}}^{\mathrm{T}}, \qquad k = 1, \ldots, K. \tag{1.2}$$

Assuming that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}_k$ in (1.2) are generic, we can set $\mathbf{X}$ (resp. $\mathbf{Y}$) equal to the inverse of the matrix formed by the first M (resp. N) rows of $\mathbf{A}$ (resp. $\mathbf{B}$), implying that the right-hand side of the K equations

$$\mathbf{T}_k = \widehat{\mathbf{A}}\widehat{\mathbf{D}}_k\widehat{\mathbf{B}}^{\mathrm{T}} = \begin{bmatrix} \mathbf{I}_M \\ * \end{bmatrix} [*] \begin{bmatrix} \mathbf{I}_N \\ * \end{bmatrix}^{\mathrm{T}}, \qquad k = 1, \ldots, K.$$

can be parameterized with $(I - M)M + (J - N)N + MNK$ parameters.

We will write the K simultaneous factorizations in (1.1) as

$$\mathcal{T} = \mathcal{D} \bullet_1 \mathbf{A} \bullet_2 \mathbf{B},$$

where $\mathcal{D}$ denotes an $M \times N \times K$ tensor with frontal slices $\mathbf{D}_1, \ldots, \mathbf{D}_K$.

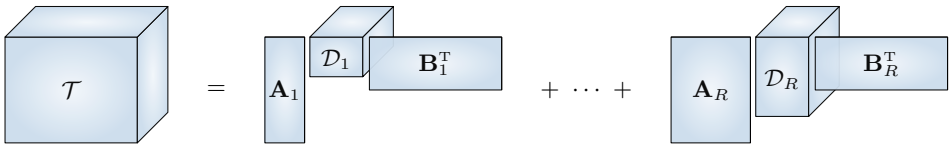

FIG. 1.1. *ML rank-$(M_r, N_r, \cdot)$ decomposition of a third-order tensor.*

In this paper, we study the decomposition of $\mathcal{T} \in \mathbb{F}^{I\times J\times K}$ into a sum of ML rank-$(M_r, N_r, \cdot)$ tensors which is illustrated in Figure 1.1 and defined as:

$$\mathcal{T} = \sum_{r=1}^{R} \mathcal{D}_r \bullet_1 \mathbf{A}_r \bullet_2 \mathbf{B}_r, \qquad \mathcal{D}_r \in \mathbb{F}^{M_r\times N_r\times K}, \quad \mathbf{A}_r \in \mathbb{F}^{I\times M_r}, \quad \mathbf{B}_r \in \mathbb{F}^{J\times N_r}. \tag{1.3}$$

Note that a decomposition of the form (1.3) always exists for some particular choices of $(M_r, N_r, \cdot)$. In particular, if one takes $M_r = N_r = 1$ for all r, then one arrives at a CPD. However, in cases where signals are not captured by individual rank-1 terms, there may exist decompositions of the form (1.3) that have fewer terms than a CPD. For example, individual CPD terms may combine into a single block term that has a rank larger than its maximum mode-n rank. The goal of the present article is to find a "meaningful" BTD that enables recovery of underlying components. In particular, such a BTD should be unique.

DEFINITION 1.1. *An ML rank-$(M_r, N_r, \cdot)$ decomposition of a tensor $\mathcal{T}$ is called* unique *if for any two decompositions of the form (1.3), the block terms in the summation are the same up to a permutation.*

DEFINITION 1.2. *An ML rank-$(M_r, N_r, \cdot)$ decomposition is said to be* generically unique *for tensors of size $I \times J \times K$ if*

$$\mu(\{(\mathbf{A}_1, \dots, \mathbf{A}_R, \mathbf{B}_1, \dots, \mathbf{B}_R, \mathcal{D}_1, \dots, \mathcal{D}_R) : \text{the decomposition of } \mathcal{T} = \sum_r \mathcal{D}_r \bullet_1 \mathbf{A}_r \bullet_2 \mathbf{B}_r \text{ is not unique}\}) = 0,$$

where μ denotes Lebesgue measure on $\mathbb{F}^{I\sum M_r + J\sum N_r + K\sum M_r N_r}$.

We say that $\mathcal{T}$ is a sum of R *generic* ML rank-$(M_r, N_r, \cdot)$ terms if the entries of $\mathbf{A}_r$, $\mathbf{B}_r$, and $\mathcal{D}_r$ in (1.3) are randomly sampled from an absolutely continuous distribution. Generic uniqueness means uniqueness that holds with probability one.

Let

$$\mathbf{A} = [\mathbf{A}_1 \ \dots \ \mathbf{A}_R] \in \mathbb{F}^{I\times\sum M_r} \ \text{ and } \ \mathbf{B} = [\mathbf{B}_1 \ \dots \ \mathbf{B}_R] \in \mathbb{F}^{J\times\sum N_r}. \tag{1.4}$$

Throughout the paper we assume that the matrices

$$\mathbf{A} \ \text{ and } \ \mathbf{B} \ \text{ have full column rank.} \tag{1.5}$$

Assumption (1.5) implies that $\sum M_r \le I$ and $\sum N_r \le J$. This assumption mirrors standard assumptions made in many CPD algorithms and uniqueness results; see, e.g., [16]. Furthermore, the assumption is not restrictive as, in practice, tensors often have at least two dimensions that are larger than the sum of component sizes.

1.2. Previous results. Let $\mathbf{D}_{r,k} \in \mathbb{F}^{M_r \times N_r}$ denote the kth frontal slice of the tensor $\mathcal{D}_r$ and let

$$\mathbf{D}_k = \operatorname{blockdiag}(\mathbf{D}_{1,k}, \dots, \mathbf{D}_{R,k}) \in \mathbb{F}^{\sum M_r \times \sum N_r}, \qquad k = 1, \dots, K \tag{1.6}$$

denote a block-diagonal matrix with the matrices $\mathbf{D}_{1,k}, \dots, \mathbf{D}_{R,k}$ on the diagonal. By (1.1), the decomposition (1.3) can be interpreted as a (possibly nonsymmetric and nonsquare) *joint block diagonalization* (JDB) of the frontal slices of $\mathcal{T}$:

$$\mathbf{T}_k = \mathbf{A}\mathbf{D}_k\mathbf{B}^{\mathrm{T}} = \mathbf{A} \operatorname{blockdiag}(\mathbf{D}_{1,k}, \dots, \mathbf{D}_{R,k})\mathbf{B}^{\mathrm{T}}, \qquad k = 1, \dots, K, \tag{1.7}$$

where $\mathbf{A}$ and $\mathbf{B}$ are defined in (1.4).

We first note that uniqueness and algorithms for (1.7) are also studied in [1]. While the algorithm and uniqueness results presented in [1] are related to our present (deterministic) results, there is an important difference in perspective which leads to a difference in the applicability of the results. In particular, [1] considers *blind* BTDs, and the main uniqueness result of the article, Theorem 3, states that if a tensor has full ML rank and the sum of block sizes in each mode is less than the corresponding mode-n rank, then its *blind* BTD is unique. The present article, on the other hand, provides results that can be used to determine if a particular format of tensor BTD is unique (both deterministically and generically). Considering that tensors can have multiple incompatible BTD formats [22], this is a valuable perspective to consider. In addition, our results relax the assumption that the sum of block sizes in each mode is less than the corresponding mode-n rank that appears in [1]. Our results instead require that the sum of block sizes is less than the corresponding mode-n rank in only two out of three modes. For example, the upcoming Theorem 2.3 can be used to conclude generic uniqueness of a $5 \times 7 \times 6$ tensor with full ML rank as the sum of two block terms with ML ranks $(3, 4, 5)$ and $(2, 3, 6)$.

Aside from this, uniqueness and algorithms for problem (1.7) have been studied in the literature in the following cases:

1. JBD by congruence and $*$-congruence, i.e., the cases $\mathbf{B} = \mathbf{A}$ and $\mathbf{B} = \mathbf{A}^*$, implying the constraint $M_r = N_r$ for all r; see, for instance, [2, 4, 29, 27] and the references therein.
2. All blocks are row vectors, i.e., $M_1 = \cdots = M_R = 1$, implying that the ML rank-$(M_r, N_r, \cdot)$ terms in (1.3) are actually ML rank-$(1, N_r, N_r)$; see, for instance, [9, 10, 18, 22, 30, 37].
3. All blocks are column vectors, i.e., $N_1 = \cdots = N_R = 1$ which follows from 2. by taking the transpose of the identities in (1.7).

Apart from the items 1–3 above, conditions for the generic uniqueness of (1.7) were given in [9] under the additional assumptions that $M_r = N_r$ for all r and $K \geq 3$. In addition to addressing the general case, our upcoming results provide new generic uniqueness conditions in these various special cases.

The nonsymmetric JBD can also be seen as a symmetric coupled JBD using an embedding of a nonsymmetric matrix $\mathbf{A}$ into a symmetric matrix [25, Theorem 7.3.3]:

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}.$$

Using this embedding approach, [2, Theorem 2.5] can be applied to obtain conditions for uniqueness of nonsymmetric JBD that are similar to those presented in our Theorem 2.1. However, the uniqueness result of Theorem 2.1 is strictly more general

than this embedding approach. In particular, there exist matrix tuples exhibiting particular symmetry patterns that have a unique JBD which can be treated using Theorem 2.1, but not using [2, Theorem 2.5]. For example, if one considers the tensor $\mathcal{T} \in \mathbb{R}^{4\times4\times4}$ with frontal slices

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \right),$$

then our results can be used to determine that the (natural) JBD is unique. However, [2, Theorem 2.5] does not capture this tensor. The present result also has a practical advantage since it can be verified numerically working with smaller matrices than one encounters in the embedding approach.

For completeness, one can wonder if [18, Theorem 1.10] can be used with the embedding approach. As it turns out, [18, Theorem 1.10] is never applicable in this context, as it requires generic block terms. The block terms encountered when considering embeddings are necessarily block-antidiagonal and are therefore not generic.

2. Uniqueness result and algorithm.

We now present results on uniqueness and present an algorithm for the computation of ML rank-$(M_r, N_r, \cdot)$ decomposition of tensors in (1.3). Of note is that, in contrast to previous works, our results can be applied in the case where $M_r \neq N_r$ for some or all of the indices r.

2.1. Exact decomposition.

THEOREM 2.1. *Let $\mathcal{T}$ admit the ML rank-$(M_r, N_r, \cdot)$ decomposition (1.3), the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}_1, \ldots, \mathbf{D}_K$ be defined as in (1.4) and (1.6), respectively, and set*

$$\mathbf{M}_{\mathcal{D}} := \begin{bmatrix} \mathbf{D}_1^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{D}_1 \\ \vdots & \vdots \\ \mathbf{D}_K^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{D}_K \end{bmatrix}. \tag{2.1}$$

Assume that

1. $\mathbf{A}$ *and* $\mathbf{B}$ *have full column rank and*
2. $\dim \operatorname{Null}(\mathbf{M}_{\mathcal{D}}) = R$.

Then the ML rank-$(M_r, N_r, \cdot)$ decomposition of $\mathcal{T}$ is unique and can be computed by means of coupled simultaneous eigenvalue decomposition (CS-EVD).

Proof. See section 4. ∎

Under assumption (1.5), the dimension of the null spaces of $\mathbf{M}_{\mathcal{D}}$ and $\mathbf{M}_{\mathcal{T}}$ are always equal, and the relationship between the null spaces of these matrices plays an important role in Algorithm 2.1. Lemma 4.2 explains the connection between these matrices and their null spaces.

Assumption 2 in Theorem 2.1 is equivalent to assuming that the system of matrix equations

$$\mathbf{X}\mathbf{D}_k = \mathbf{D}_k\mathbf{Y}, \qquad k = 1, \ldots, K \tag{2.2}$$

has exactly R linearly independent solutions $(\mathbf{X}, \mathbf{Y}) \in \mathbb{F}^{\sum M_r \times \sum M_r} \times \mathbb{F}^{\sum N_r \times \sum N_r}$. If

$$\begin{aligned} &\mathbf{P}_{I,r} = \operatorname{blockdiag}(\mathbf{O}_{M_1}, \ldots, \mathbf{O}_{M_{r-1}}, \mathbf{I}_{M_r}, \mathbf{O}_{M_{r+1}}, \ldots, \mathbf{O}_{M_R}) \quad \text{and} \\ &\mathbf{P}_{J,r} = \operatorname{blockdiag}(\mathbf{O}_{N_1}, \ldots, \mathbf{O}_{N_{r-1}}, \mathbf{I}_{N_r}, \mathbf{O}_{N_{r+1}}, \ldots, \mathbf{O}_{N_R}), \qquad r = 1, \ldots, R, \end{aligned} \tag{2.3}$$

then $(\mathbf{X}, \mathbf{Y}) = (\mathbf{P}_{I,r}, \mathbf{P}_{J,r})$ is a solution of (2.2), as

$$\mathbf{P}_{I,r}\mathbf{D}_k = \mathbf{D}_k\mathbf{P}_{J,r} = \\ \text{blockdiag}(\mathbf{O}_{M_1\times N_1}, \ldots, \mathbf{O}_{M_{r-1}\times N_{r-1}}, \mathbf{D}_{r,k}, \mathbf{O}_{M_{r+1}\times N_{r+1}}, \ldots, \mathbf{O}_{M_R\times N_R}).$$

Thus, Assumption 2 in Theorem 2.1 intuitively states that the dimension of the subspace of solutions of (2.2) is minimal. We emphasize that the above computation illustrates that if a $\mathcal{T}$ admits an R term BTD, then $\dim \text{Null}\,(\mathbf{M}_\mathcal{T}) \geq R$.

The algebraic procedure for computing ML rank-$(M_r, N_r, \cdot)$ decompositions which stems from Theorem 2.1 is summarized in Algorithm 2.1. By Assumption 1 in Theorem 2.1, the number of columns of $\mathbf{A}$ and $\mathbf{B}$ cannot exceed the number of rows, i.e., $\sum M_r \leq I$ and $\sum N_r \leq J$. To simplify the presentation, we assume in Algorithm 2.1 that $\sum M_r = I$ and $\sum N_r = J$, implying that $\mathbf{A}$ and $\mathbf{B}$ are square nonsingular matrices and that $\mathcal{T} \in \mathbb{F}^{\sum M_r \times \sum N_r \times K}$. Since, by Lemma 4.1 below, $\mathcal{T}$ has ML rank-$(\sum M_r, \sum N_r, \cdot)$, this compression step can always be performed without loss of generality, e.g., via the multilinear singular value decomposition [12]. Following the above discussion regarding Assumption 2, our algorithm results in a BTD where the number of block terms is equal to $\dim \text{Null}\,(\mathbf{M}_\mathcal{T})$. Moreover, the structure of $\text{Null}\,(\mathbf{M}_\mathcal{T})$ determines the BTD that is found by our algorithm.

Note that Algorithm 2.1 also recovers the number of blocks, R, as well as the dimensions of the blocks, i.e., the values M_r and N_r. In particular, the ML rank profile is determined by eigenvalue multiplicities of matrices formed from a basis for $\text{Null}\,(\mathbf{M}_\mathcal{T})$. More precisely, in steps 1–4 of Algorithm 2.1, we construct matrices $\mathbf{U}_{I,1}, \ldots, \mathbf{U}_{I,R} \in \mathbb{F}^{I\times I}$ and $\mathbf{U}_{J,1}, \ldots, \mathbf{U}_{J,R} \in \mathbb{F}^{J\times J}$ that can be simultaneously diagonalized by $\mathbf{A}$ and $\mathbf{B}^{-\mathrm{T}}$, respectively. In step 5 of Algorithm 2.1, we use the fact that for each r, the spectra of $\mathbf{U}_{I,r}$ and $\mathbf{U}_{J,r}$ relate to each other as in (2.5)–(2.6). Thus, if the values $\alpha_1, \ldots, \alpha_R$ are generic, then the distinct eigenvalues of the matrices $\mathbf{F} = \alpha_1\mathbf{U}_{I,1} + \cdots + \alpha_R\mathbf{U}_{I,R}$ and $\mathbf{G} = \alpha_1\mathbf{U}_{J,1} + \cdots + \alpha_R\mathbf{U}_{J,R}$ are the same and their multiplicities are $M_1, \ldots, M_R$ and $N_1, \ldots, N_R$ respectively. Hence, the values M_r, N_r and the corresponding blocks $\mathbf{A}_r$, $\mathbf{B}_r$ can be recovered from the EVDs of $\mathbf{F}$ and $\mathbf{G}$.

The following corollary extends the results on uniqueness and computation of the ML rank-$(1, N_r, N_r)$ decomposition in [9, 10, 37].

COROLLARY 2.2. *Let $\mathcal{T}$ admit a ML rank-$(1, N_r, N_r)$ decomposition and let the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}_1, \ldots, \mathbf{D}_K$ be defined as in (1.4) (with $I = R$) and (1.6), respectively. Also, define*

$$\mathbf{C}_r \coloneqq [\mathbf{D}_{r,1}^{\mathrm{T}} \ \ldots \ \mathbf{D}_{r,K}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{F}^{K\times N_r}, \qquad r = 1, \ldots, R. \tag{2.7}$$

Assume that

1. $\mathbf{A}$ *and* $\mathbf{B}$ *have full column rank and*
2. $r_{[\mathbf{C}_i \ \mathbf{C}_j]} \geq \max(N_i, N_j) + 1$ *for all* $1 \leq i < j \leq R$.

Then the ML rank-$(1, N_r, N_r)$ decomposition of $\mathcal{T}$ is unique and can be computed by means of CS-EVD.

Proof. See section 4. ∎

The following theorem presents inequalities that relate tensor dimensions to the size of summands appearing in a ML rank-$(M_r, N_r, \cdot)$ decomposition. If satisfied, these inequalities guarantee that Assumptions 1 and 2 in Theorem 2.1 hold for generic matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}_1, \ldots, \mathbf{D}_K$.

Algorithm 2.1 Computation of BTD (1.3) by CS-EVD (see Theorem 2.1)

Require: tensor $\mathcal{T} \in \mathbb{F}^{I\times J\times K}$ such that $I = \sum M_r$ and $J = \sum N_r$

1: Define the IJK-by-$(I^2 + J^2)$ matrix $\mathbf{M}_\mathcal{T}$ as

$$\mathbf{M}_\mathcal{T} = \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{T}_1 \\ \vdots & \vdots \\ \mathbf{T}_K^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{T}_K \end{bmatrix} \tag{2.4}$$

2: Compute $\mathbf{u}_r = [u_{1,r} \ \dots \ u_{I^2+J^2,r}]^{\mathrm{T}}$, $r = 1, \dots, R$, that form a basis of $\mathrm{Null}\,(\mathbf{M}_\mathcal{T})$
3: For $r = 1, \dots, R$, reshape $[u_{1,r} \ \dots \ u_{I^2,r}]^{\mathrm{T}}$ into the $I \times I$ matrix $\mathbf{U}_{I,r}$
4: For $r = 1, \dots, R$, reshape $[u_{I^2+1,r} \ \dots \ u_{I^2+J^2,r}]^{\mathrm{T}}$ into the $J \times J$ matrix $\mathbf{U}_{J,r}$
5: Solve CS-EVD problem

$$\mathbf{U}_{I,r} = \mathbf{A}\,\mathrm{blockdiag}(g_{1r}\mathbf{I}_{M_1}, \dots, g_{Rr}\mathbf{I}_{M_R})\mathbf{A}^{-1}, \tag{2.5}$$

$$\mathbf{U}_{J,r} = \mathbf{B}^{-\mathrm{T}}\,\mathrm{blockdiag}(g_{1r}\mathbf{I}_{N_1}, \dots, g_{Rr}\mathbf{I}_{N_R})\mathbf{B}^{\mathrm{T}}, \quad r = 1, \dots, R \tag{2.6}$$

for the matrices $\mathbf{A}$, $\mathbf{B}$ and for the values $M_1, \dots, M_R$, $N_1, \dots, N_R$
6: Solve (1.7) for the matrices $\mathbf{D}_1, \dots, \mathbf{D}_K$
7: For $r = 1, \dots, R$, stack $\mathbf{D}_{r,1}, \dots, \mathbf{D}_{r,K}$ into the $M_r \times N_r \times K$ tensor $\mathcal{D}_r$
8: **return** $\mathbf{A}_r \in \mathbb{F}^{I\times M_r}$, $\mathbf{B}_r \in \mathbb{F}^{J\times N_r}$ and $\mathcal{D}_r \in \mathbb{F}^{M_r\times N_r\times K}$ such that (1.3) holds

THEOREM 2.3. *Let $\mathcal{T}$ be a sum of R generic ML rank-$(M_r, N_r, \cdot)$ terms. Assume*

$$I \geq \sum M_r, \quad J \geq \sum N_r, \quad K \geq \max\left(\max_{r=1,\dots,R}\left\lceil\frac{M_r}{N_r}\right\rceil + \max_{r=1,\dots,R}\left\lceil\frac{N_r}{M_r}\right\rceil, 3\right), \tag{2.8}$$

where $\lceil x\rceil$ denotes the smallest integer not less than x. Then, the decomposition of $\mathcal{T}$ into a sum of ML rank-$(M_r, N_r, \cdot)$ terms is unique and can be computed by means of CS-EVD.

Proof. See section 5. ∎

The following corollaries easily follow from Theorem 2.3.

COROLLARY 2.4. *Let $\mathcal{T}$ be a sum of $R_1 \geq 1$ generic ML rank-$(1, N_r, N_r)$ terms and $R_2 \geq 1$ generic ML rank-$(M_r, 1, M_r)$ terms. Assume that*

$$I \geq R_1 + \sum M_r, \qquad J \geq R_2 + \sum N_r, \qquad \text{and} \qquad K \geq \max M_r + \max N_r.$$

Then, the decomposition of $\mathcal{T}$ into a sum of R_1 ML rank-$(1, N_r, N_r)$ terms and R_2 ML rank-$(M_r, 1, M_r)$ terms is unique and can be computed by means of CS-EVD.

COROLLARY 2.5. *Let $\mathcal{T}$ be a sum of R generic ML rank-$(1, N_r, N_r)$ terms. Assume that*

$$I \geq R, \qquad J \geq \sum N_r, \qquad \text{and} \qquad K \geq \max N_r + 1. \tag{2.9}$$

Then the decomposition of $\mathcal{T}$ into a sum of ML rank-$(1, N_r, N_r)$ terms is unique and can be computed by means of CS-EVD.

COROLLARY 2.6. *Let $\mathcal{T}$ be a sum of R generic ML rank-$(M, N, \cdot)$ terms. Assume that*

$$I \geq RM, \qquad J \geq RN, \qquad K \geq \max\left(\left\lceil\frac{M}{N}\right\rceil + \left\lceil\frac{N}{M}\right\rceil, 3\right).$$

Then the decomposition of $\mathcal{T}$ into a sum of ML rank-$(M, N, \cdot)$ terms is unique and can be computed by means of CS-EVD.

The bounds in Corollaries 2.4 to 2.6 are new and guarantee that a generic ML rank $(M_r, N_r, \cdot)$ decomposition can be computed by Algorithm 2.1. The bound on K in (2.9) complements the bounds for generic uniqueness of the ML rank-$(1, N, N)$ decomposition in [9, Section 4] and is more relaxed than the bound for generic uniqueness of ML rank-$(1, N_r, N_r)$ decomposition in [30, Theorem 1.7 (E)].

In the remaining part of this section, we apply Theorem 2.1 to (generic) constrained JBD problems. Table 2.1 summarizes the various constraints that we consider. Uniqueness of the constrained JBD problem means uniqueness of the corresponding constrained ML rank-(N_r, N_r, K) decomposition. Generic uniqueness of the constrained JBD problem can be defined similarly as in Definition 1.2. By way of example, we resort to the following definition of generic uniqueness of the symmetric JBD problem.

TABLE 2.1
Some examples of constrained JBD problems.

Type of JBD problem	Constraints in (1.7)	$\mathbb{F}$
JBD by congruence	$\mathbf{B} = \mathbf{A}$	$\mathbb{R}$ or $\mathbb{C}$
symmetric JBD	$\mathbf{B} = \mathbf{A}$ and $\mathbf{D}_k = \mathbf{D}_k^{\mathrm{T}}$, $k = 1, \ldots, K$	$\mathbb{R}$ or $\mathbb{C}$
JBD by $*$-congruence	$\mathbf{B} = \mathbf{A}^*$	$\mathbb{C}$
Hermitian JBD	$\mathbf{B} = \mathbf{A}^*$ and $\mathbf{D}_k = \mathbf{D}_k^{\mathrm{H}}$, $k = 1, \ldots, K$	$\mathbb{C}$

DEFINITION 2.7. *Let L_r denote a linear bijection between $\mathbb{F}^{K\frac{N_r(N_r+1)}{2}}$ and the subspace of $N_r \times N_r \times K$ tensors whose frontal slices are symmetric matrices. Let also μ be Lebesgue measure on $\mathbb{F}^{I\sum N_r + K\sum \frac{N_r(N_r+1)}{2}}$. We say a symmetric JBD with block size (N_r, N_r, K) is generically unique for tensors of size $I \times I \times K$ if*

$$\mu(\{(\mathbf{A}_1, \ldots, \mathbf{A}_R, \mathbf{d}_1, \ldots, \mathbf{d}_R) : \text{the constrained decomposition of } \mathcal{T} = \sum_{r=1}^{R} L_r(\mathbf{d}_r) \cdot_1 \mathbf{A}_r \cdot_2 \mathbf{A}_r \quad \text{is not unique}\}) = 0.$$

Theorem 2.3 guarantees that if $K \geq 3$ and the diagonal blocks $\mathbf{D}_{r,k}$ in the JBD problem (1.7) are square and generic, then the decomposition is unique and can be computed by means of CS-EVD. The following theorem states that these conditions remain valid for all of the constrained JBD problems in Table 2.1 as well and that the computation can be done using an (uncoupled) S-EVD.

THEOREM 2.8. *Fix I, K, and R and N_r for $r = 1, \ldots, R$. Assume that $I \geq \sum N_r$, and $K \geq 3$. Then, for tensors of size $I \times I \times K$, solutions of each of the constrained JBD problems in Table 2.1 with block size (N_r, N_r, K) are generically unique and can be computed by means of S-EVD.*

Steps 1–4 of Algorithm 2.1 together with a modified version of step 5 constitute the algebraic procedure related to Theorem 2.8. In step 5, because of the constraints $\mathbf{B} = \mathbf{A}$ and $\mathbf{B} = \mathbf{A}^*$, the CS-EVD problem (2.5)–(2.6) turns into an S-EVD problem. It can be shown that in the case of the JBD by congruence problem the proposed reduction to S-EVD essentially coincides with the one in [2, Subsection 2.3].

Proof. See section 5. ∎

2.2. Implementation. The mathematical description of Algorithm 2.1 leaves room for several implementation decisions. We now discuss an algebraic version that does not depend on optimization and estimates all parameters when decomposing a (noisy) tensor.

First, a standard multilinear compression step is performed. Let $\mathcal{S} \in \mathbb{F}^{I_1 \times I_2 \times I_3}$ have a decomposition into ML-rank-$(M_r, N_r, \cdot)$ terms with $I_1 \geq I = \sum_{r=1}^{R} M_r$, $I_2 \geq J = \sum_{r=1}^{R} N_r$ and $I_3 \geq K$, and let $\tilde{\mathcal{S}} = \mathcal{S} + \mathcal{N}$ be a measurement of $\mathcal{S}$ where perturbation $\mathcal{N}$ is due to noise or model errors. While $\mathcal{S}$ has ML rank (I, J, K) by the assumptions in Theorem 2.1, $\tilde{\mathcal{S}}$ can have a larger ML rank because of $\mathcal{N}$, but can be approximated by a tensor of ML rank (I, J, K):

$$\tilde{\mathcal{S}} \approx \mathcal{T} \cdot_1 \mathbf{U} \cdot_2 \mathbf{V} \cdot_3 \mathbf{W}. \tag{2.10}$$

This approximation is obtained using standard low ML rank approximation algorithms (e.g., truncated MLSVD) and the resulting core tensor $\mathcal{T} \in \mathbb{F}^{I \times J \times K}$ is used as input for Algorithm 2.1. If the ML rank (I, J, K) is not known a priori, the mode-n ranks I, J, and K can be estimated from the ML singular values $\sigma_{i_n}^{(n)}$ of $\tilde{\mathcal{S}}$, $i_n = 1, \ldots, I_n$, $n = 1, 2, 3$, using any (matrix) rank estimation procedure designed for noisy data. Here, we take $I = I_n - r_1$ were r_1 is the largest value such that $\operatorname{Var}\left(\{\sigma_k^{(1)}\}_{k=I_1-r_1+1}^{I_1}\right) \leq \theta$ which works well in the case of independently and identically distributed (i.i.d.) Gaussian noise. The mode-n ranks J and K are estimated similarly.

Second, the matrix $\mathbf{M}_{\mathcal{T}}$ is constructed using (2.4) and a basis for $\operatorname{Null}(\mathbf{M}_{\mathcal{T}})$ is computed via the SVD by selecting the right singular vectors corresponding to the R smallest singular values which are exactly zero in the noiseless, exact case, and small otherwise. In the latter case, any rank estimation strategy can be used, e.g., the variance-based strategy outlined above. Note that the smallest singular value $\sigma_{I^2+J^2}$ of $\mathbf{M}_{\mathcal{T}}$ is ignored in the computation of the variance as it is always zero since the vectorized identity matrix is trivially an element of $\operatorname{Null}(\mathbf{M}_{\mathcal{T}})$ for any tensor $\mathcal{T}$.

Third, the CS-EVD can be computed using algebraic or optimization-based techniques. For the experiments, an algebraic version using the generalized eigenvalue decomposition (GEVD) is implemented. Stacking $\mathbf{U}_{I,r}$, $r = 1, \ldots, R$, as the frontal slices of a tensor $\mathcal{X} \in \mathbb{F}^{I \times I \times R}$ and $\mathbf{U}_{J,r}$, $r = 1, \ldots, R$, as the frontal slices of a tensor $\mathcal{Y} \in \mathbb{F}^{J \times J \times R}$, the rank-$I$ and rank-J CPD of $\mathcal{X} = [\![\mathbf{A}, \mathbf{A}^{-\mathrm{T}}, \mathbf{G}_1]\!]$ and $\mathcal{Y} = [\![\mathbf{B}^{-\mathrm{T}}, \mathbf{B}, \mathbf{G}_2]\!]$, resp., are computed independently using a GEVD-based CPD algorithm [28] to obtain $\mathbf{A}$ and $\mathbf{B}$[2]. For clarity, we emphasize that $\mathbf{G}_1 \in \mathbb{F}^{R \times I}$ and $\mathbf{G}_2 \in \mathbb{F}^{R \times J}$ are matrices whose columns all consist of the same R vectors. That is, $\mathbf{G}_1 = \mathbf{C}\mathbf{P}_1$ and $\mathbf{G}_2 = \mathbf{C}\mathbf{P}_2$, where $\mathbf{C} \in \mathbb{F}^{R \times R}$ contains R linearly independent columns and $\mathbf{P}_1 \in \{0, 1\}^{R \times I}$ and $\mathbf{P}_2 \in \{0, 1\}^{R \times J}$ are binary matrices with exactly one 1 in each column, replicating the rth column $\mathbf{c}_r$ M_r, resp., N_r times. Also, note that the order of the columns in $\mathbf{A}$ and $\mathbf{B}$ is arbitrary and does not necessarily match the partitioning in (1.4). As the algebraic algorithm does not enforce the inverse constraint, and does not take the repeated columns in $\mathbf{G}_1$ and $\mathbf{G}_2$ into account (see (2.5) and (2.6)), $\mathbf{G}_1$ and $\mathbf{G}_2$ are estimated separately from the least squares systems to improve accuracy:

$$(\mathbf{A}^{-\mathrm{T}} \odot \mathbf{A})\mathbf{G}_1 = \mathbf{X}_{[1,2;3]}, \quad \text{and} \quad (\mathbf{B} \odot \mathbf{B}^{-\mathrm{T}})\mathbf{G}_2 = \mathbf{Y}_{[1,2;3]}. \tag{2.11}$$

[2]Note that it is possible to take a linear combination of the frontal slices $\mathbf{U}_{I,r}$ ($\mathbf{U}_{J,r}$) and estimate $\mathbf{A}$ ($\mathbf{B}$) using the eigenvalue decomposition. As this linear combination is typically not a symmetric matrix, care must be taken to obtain real eigenvectors if the tensor is real. Moreover, this approach was less accurate than the GEVD technique in the experiments.

From the experiments in section 3, we will see that the accuracy of the overall decomposition into ML-rank-$(M_r, N_r, \cdot)$ terms can be improved by computing the CPDs more accurately, e.g., using the generalized eigenspace decomposition [21], or by following the GEVD by an optimization-based refinement step, e.g., the Gauss–Newton algorithm in `cpd_nls` [35, 40].

Fourth, to estimate M_r and N_r, the columns of $\mathbf{G}_1$ and $\mathbf{G}_2$, resp., are jointly clustered into R groups with M_r columns from $\mathbf{G}_1$ and N_r from $\mathbf{G}_2$. To achieve this, one can for example use an alternating k-means algorithm.

Fifth, using the clustering results, the columns of $\mathbf{A}$ and $\mathbf{B}$ can be ordered to match the partitioning in (1.4) and the core tensors $\mathcal{D}_r$ are computed by solving

$$\begin{bmatrix} \mathbf{B}_1 \otimes \mathbf{A}_1 & \dots & \mathbf{B}_R \otimes \mathbf{A}_R \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\mathbf{D}_{1,k}) \\ \vdots \\ \operatorname{vec}(\mathbf{D}_{R,k}) \end{bmatrix} = \operatorname{vec}(\mathbf{T}_k), \quad k = 1, \dots, K, \tag{2.12}$$

in a least-squares sense. This procedure is more accurate in a noisy setting than bringing $\mathbf{A}$ and $\mathbf{B}$ to the left-hand side in (1.7) as the off-diagonal blocks of the matrix $\operatorname{blockdiag}(\mathbf{D}_{1,k}, \dots, \mathbf{D}_{R,k})$ are forced to be zero.

Finally, we undo the compression and return the factors $\mathbf{UA}$ and $\mathbf{VB}$, and the core tensors $\mathcal{D}_r \bullet_3 \mathbf{W}$.

3. Simulations.

In this section, we present numerical simulations that illustrate the performance of Algorithm 2.1.

3.1. Recovering a decomposition without prior knowledge.

The proposed CS-EVD algorithm can recover a decomposition into multilinear rank $(M_r, N_r, \cdot)$ terms without prior knowledge, as shown using the following small-scale experiment. In this experiment, tensors $\mathcal{T} \in \mathbb{C}^{9\times 8\times 5}$ that can be written as the sum of $R = 3$ terms with multilinear ranks $(2, 3, \cdot)$, $(4, 2, \cdot)$ and $(3, 3, \cdot)$, are constructed; hence $M = \begin{bmatrix} 2, 4, 3 \end{bmatrix}$ and $N = \begin{bmatrix} 3, 2, 3 \end{bmatrix}$. The matrices $\mathbf{A} \in \mathbb{C}^{9\times 9}$ and $\mathbf{B} \in \mathbb{C}^{8\times 8}$ have entries of which the real and imaginary parts are drawn independently from a standard normal distribution. If the condition numbers $\operatorname{cond}(\mathbf{A})$ or $\operatorname{cond}(\mathbf{B})$ are larger than 10, new matrices generated. The entries of the core tensors $\mathcal{D}_r \in \mathbb{C}^{M_r \times N_r \times 5}$ are similarly drawn from a normal distribution. The tensor $\mathcal{T}$ can be perturbed by a Gaussian i.i.d. noise tensor $\mathcal{N}$ which is scaled such that the signal-to-noise ratio (SNR) $20 \log_{10}(||\mathcal{T}|| \,/\, ||\mathcal{N}||)$ has a specified value (in dB). This results in a noisy observation $\tilde{\mathcal{T}} = \mathcal{T} + \mathcal{N}$ of the tensor $\mathcal{T}$. We use the proposed algebraic version of CS-EVD that uses the GEVD and estimates all parameters unless stated otherwise; see subsection 2.2. The performance is compared to standard optimization routines: the alternating least squares algorithm for the $(M_r, N_r, \cdot)$ decomposition [13, type-2 BTD], denoted by ALS, and the Gauss–Newton implementation `btd_nls` [36, 40] of the general BTD [9], denoted by NLS. For the optimization algorithms, 25 random initializations are generated per experiment; see `btd_rnd` [40]. We generate 100 random tensors $\tilde{\mathcal{T}}$ and test the performance for three noise levels and the noiseless case.

In contrast to optimization methods, CS-EVD does not require estimates for the parameters R, M_r and N_r in the exact case. In Table 3.1, we see that these parameters can be estimated even under noisy conditions provided the SNR is sufficiently high, and that if R is estimated correctly, then the values M_r and N_r are also correct. For the experiments where the SNR is 20 dB, the dimension of $\operatorname{Null}(\mathbf{M}_{\mathcal{T}})$ is often estimated incorrectly as the singular value gap $\sigma_{I^2+J^2-R}/\sigma_{I^2+J^2-R+1}$ is at most 1.76, which is too small for the rank estimation procedure. As this rank-estimation step

TABLE 3.1

Starting without knowledge of the values of R and $\{M_r, N_r\}_{r=1}^R$, the proposed CS-EVD algorithm can recover the correct number of terms and their sizes provided the SNR is sufficiently high. Furthermore, for all tensors in the experiment, we see that if R was estimated correctly, then all values of M_r and N_r were also estimated correctly.

successful estimation (/100 experiments)			
	SNR		
estimate of	20 dB	40 dB	80 dB
R	36	100	100
$\{M_r, N_r\}_{r=1}^R$	36	100	100

TABLE 3.2

The proposed CS-EVD algorithm successfully recovers the decomposition, unless the tensor is highly perturbed by noise. When the solution is used as initialization for an optimization algorithm, almost all tensors are recovered if R is estimated correctly. This shows that accuracy is the main barrier to success (as opposed to the ability to correctly estimate R). An experiment is considered successful if the relative term error $E \leq \tau$ with $\tau = 0.25, 0.2, 0.002$ for each column, resp.

success rate (/100 experiments)				
		SNR		
	known	20 dB	40 dB	80 dB
CS-EVD	–	9	99	100
	R	10	99	100
CS-EVD as init for NLS	–	34	100	100
	R	94	100	100
best (of 25) random init for NLS	$\{M_r, N_r\}_{r=1}^R$	99	100	100
best (of 25) random init for ALS	$\{M_r, N_r\}_{r=1}^R$	99	100	100

is not the core of the algorithm, we assume R is given for the remainder of this experiment; this choice only influences the success rate in the case the SNR is 20 dB.

Next, we investigate whether the correct underlying decomposition can be recovered by CS-EVD when only I and J are known. We say that a decomposition $\mathcal{T} = \sum_r \mathcal{D}_r \bullet_1 \mathbf{A}_r \bullet_2 \mathbf{B}_r$ is successfully recovered if all parameters R and $\{M_r, N_r\}_{r=1}^R$ are correctly estimated and the maximal relative term error E for the computed decomposition $\hat{\mathcal{D}}_r \bullet_1 \hat{\mathbf{A}}_r \bullet_2 \hat{\mathbf{B}}_r$ satisfies

$$E := \max_r \frac{\left|\left|\mathcal{D}_r \bullet_1 \mathbf{A}_r \bullet_2 \mathbf{B}_r - \hat{\mathcal{D}}_r \bullet_1 \hat{\mathbf{A}}_r \bullet_2 \hat{\mathbf{B}}_r\right|\right|}{||\mathcal{D}_r \bullet_1 \mathbf{A}_r \bullet_2 \mathbf{B}_r||} \leq \tau, \tag{3.1}$$

where τ is a specified parameter. See Table 3.2 for the values of τ that are used.

From Table 3.2, we conclude that CS-EVD can indeed recover the decomposition if the SNR is sufficiently high. However, if the CS-EVD solution is provided as the initialization for NLS, the success rate improves significantly. Hence, even for low SNR, CS-EVD provides a good initialization if R is estimated correctly. The optimization methods recover the solution almost always (after using up to 25 initializations), but, as can be seen in Figure 3.1, the algorithms successfully recover the decomposition for only a third of the initializations.

In particular, for each of 100 randomly generated $(\mathbf{A}, \mathbf{B}, \{\mathcal{D}_r\}_{r=1}^3)$, Figure 3.1

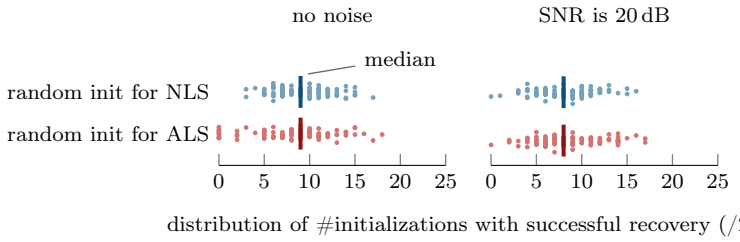

FIG. 3.1. *The correct solution is found by the optimization-based algorithms only for approximately one in three random initializations (median). Note that for some experiments, the correct solution is found for only a few or even no random initializations. Each dot represents one out of 100 experiments and is randomly vertically shifted to improve the visualization.*

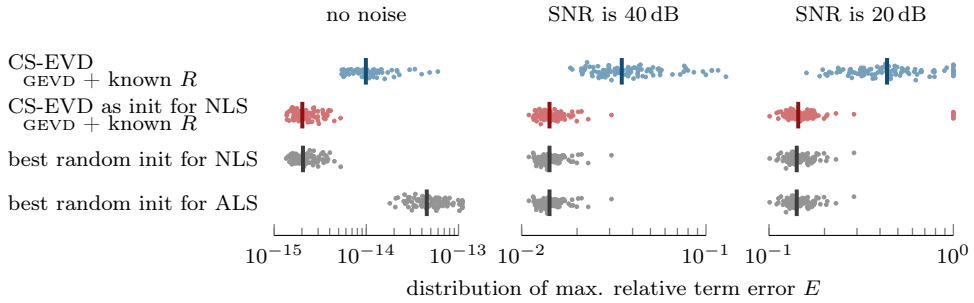

FIG. 3.2. *The accuracy of the decomposition computed using the CS-EVD method is slightly lower than when optimization-based methods such as NLS or ALS are used. However, CS-EVD provides a good initialization and uses only R as prior knowledge. (Not knowing R only changes the success rate for the experiment with an SNR of 20 dB, as R is estimated correctly for the other cases; see Table 3.1.) Each dot represents one out of 100 experiments and is randomly vertically shifted to improve the visualization; the bar represents the median.*

plots the number of initializations (out of 25) for ALS and NLS that successfully recover the decomposition of $\mathcal{T} = \sum_r \mathcal{D}_r \bullet_1 \mathbf{A}_r \bullet_2 \mathbf{B}_r$ (using the definition of successful recovery in (3.1)). By looking at the median, we observe that only 8 out of 25 initializations lead to successful recovery even in the noiseless case. On the other hand, CS-EVD serves as an extremely reliable initialization once the SNR is sufficiently high. Moreover, these optimization methods require an initial guess and therefore the underlying parameters M_r and N_r need to be known.

Figure 3.2 shows that the accuracy of CS-EVD is not optimal which negatively affects the success rate. However, for both the CS-EVD and the best random initialization, NLS converges to the same solution. Finally, we investigate whether the CS-EVD algorithm can be improved by providing the values of R and $\{M_r, N_r\}_{r=1}^R$, or by using the extra (internal) refinement step using `cpd_nls`; see subsection 2.2. In Figure 3.3, we see that extra information does not improve the accuracy, but the extra refinement step does. Hence, refining the solution of CS-EVD using NLS is beneficial.

3.2. A practical, larger-scale example. In practice, the goal is often to decompose a noisy tensor $\tilde{\mathcal{S}} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with $I_1 \geq I$, $I_2 \geq J$ and $I_3 \geq K$, into an unknown sum of simple rank-1 terms and more complex multilinear rank $(L_r, L_r, 1)$ or $(M_r, N_r, \cdot)$ terms. In this experiment, we investigate the ability of CS-EVD to find the correct decomposition when changing the known parameters: I and J, R, or all

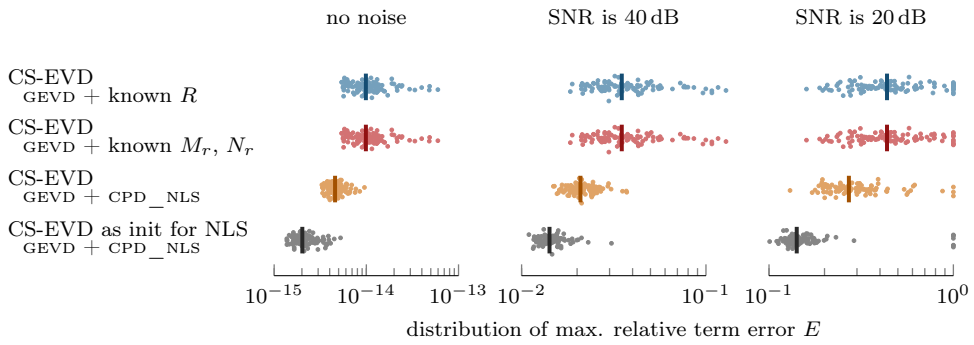

FIG. 3.3. *While the default algebraic version of the proposed CS-EVD algorithm is reasonably accurate, the accuracy can still be improved by replacing the GEVD algorithm used within. Note that knowing the correct parameters M_r and N_r, $r = 1, \ldots, R$ does not improve accuracy; however, it does improve the success rate as seen in Table 3.2. The accuracy can be improved by using the computed result as initialization for an optimization-based algorithm, e.g., based on NLS. Each dot represents one out of 100 experiments and is randomly vertically shifted to improve the visualization; the bar represents the median.*

TABLE 3.3

The proposed CS-EVD algorithm recovers the correct solution for most (10 dB) or almost all (40 dB) experiments, while the optimization routines, which require prior information, have a much lower success rate. Even if the optimization method finds the correct solution for a given experiment, the number of initializations for which this happens is at most 3 out of 100. For each SNR value, a total of 100 experiments are run and in each experiment, 100 random initializations are tried. The number of experiments that have at least k out of 100 random initializations that converge to the correct solution, as well as the total number of successful experiments are shown.

	SNR = 10 dB				SNR = 40 dB			
	k				k			
	1	2	3	# successful	1	2	3	# successful
CS-EVD as init for ALS	–	–	–	83	–	–	–	98
random init for ALS	49	8	1	49	42	8	1	42
random init for NLS	30	5	1	30	21	2	0	21

values M_r and N_r. As we will show, the performance depends on the effectiveness of the (matrix) rank estimation procedures. Although rank estimation is a core step in the algorithm, methods for accurate rank estimation are not the focus of this work.

A similar strategy as in subsection 3.1 is used to generate a noiseless tensor $\mathcal{T}$, but now with $M = [2, 4, 3, 5, 3, 1, 1]$, $N = [3, 2, 3, 2, 1, 1, 1]$ and $K = 50$; hence $I = \sum_r M_r = 19$ and $J = \sum_r N_r = 13$. Next, the tensor $\mathcal{S} \in \mathbb{R}^{100 \times 100 \times 50}$ is constructed as $\mathcal{S} = \mathcal{T} \cdot_1 \mathbf{U} \cdot_2 \mathbf{V}$ in which $\mathbf{U} \in \mathbb{C}^{100 \times I}$ and $\mathbf{V} \in \mathbb{C}^{100 \times J}$ are random matrices with orthonormal columns. Finally, the tensor $\tilde{\mathcal{S}}$ is obtained by perturbing $\mathcal{S}$ by noise such that the SNR is a given value. For these experiments, the results of the GEVDs in (2.5) and (2.6) are refined using a few iterations of Gauss–Newton; see subsection 2.2. If stated explicitly, the algebraic result is further refined using ALS.

The problem studied in this section is a hard problem for optimization algorithms, as, in the case where the SNR is 10 dB, ALS does not converge to the correct solution for any of 100 random initializations for 51 out of 100 experiments, as can be seen in Table 3.3. Similarly, NLS does not converge to the correct solution for any of 100

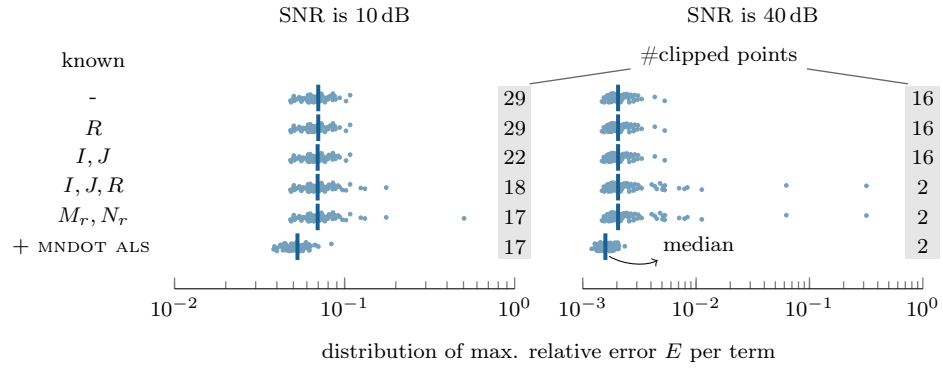

FIG. 3.4. *The knowledge of additional information does not improve the accuracy, only the success rate; see Figure 3.5. An additional refinement step using an optimization-based algorithm such as ALS can improve the accuracy slightly. Results shown for a decomposition of a noisy* $100 \times 100 \times 50$ *tensor into* $R = 7$ *multilinear rank* $(M_r, N_r, \cdot)$ *terms with* $M = [2, 4, 3, 5, 3, 1, 1]$ *and* $N = [3, 2, 3, 2, 1, 1, 1]$*, hence* $I = 19$ *and* $J = 13$*. Each dot represents one out of 100 experiments and is randomly vertically shifted to improve the visualization; the same shifts are used for all plots. Relative errors larger than 1 are clipped.*

random initializations for 70 out of 100 experiments. Moreover, when investigating the 49 (30) experiments where the correct solution is found by ALS (NLS), we see that very few of the 100 initializations used for each tensor result in the correct solution. Indeed, in exactly one case a total of three out of 100 initializations resulted in this correct solution; in all other cases, only one or two initializations were successful. In contrast, the CS-EVD method finds the correct solution for 83 and 98 experiments (for an SNR of 10 dB, resp. 40 dB), even without knowledge of the block size parameters $\{M_r, N_r\}_{r=1}^{R}$ which, in contrast, are required by ALS and NLS.

In terms of accuracy, it is clear from Figure 3.4 that knowing more parameters does not lower the error. Again, using a refinement step does result in a lower error. The improvement is limited in this case though: for the 10 dB experiment the median error reduces from 0.070 if no parameters are known to 0.053 if the result is refined.

A prerequisite for a successful decomposition is that the multilinear rank, the null space dimension R, and the block size parameters M_r and N_r are all estimated correctly. Although knowing parameters does not lower the error, we see from Figure 3.5 that knowledge of the block size parameters does improve the success rate. In particular, Figure 3.5 illustrates that if the multilinear rank is estimated correctly, the main reason for failure is that the dimension of the null space is estimated wrongly. The null space is not clearly defined as the singular value gap is too small: Let σ_r be the rth smallest singular value, then in almost all failed cases, $\sigma_{R+1} \leq 2\sigma_R$. If all parameters are known, and therefore the correct null space is used, the CS-EVD algorithm is still successful for smaller singular value gaps.

Finally, for reference, we give indicative median timings for the 10 dB experiment[3]. The CS-EVD approach only requires 1.91 s which includes 0.05 s for MLSVD compression, 0.28 s for the computation of the GEVD, 1.28 s for the (internal) refinement step with `cpd_nls` and clustering, and 0.30 s for refinement of the CS-EVD solution using ALS. In contrast, ALS and NLS require 7.30 s and 79.18 s, respectively,

[3]The experiments are performed on a server with an Intel Xeon Gold 6248 CPU and 68 GB of RAM, running Rocky Linux 8.6, Matlab 2021b, and Tensorlab 3.0.

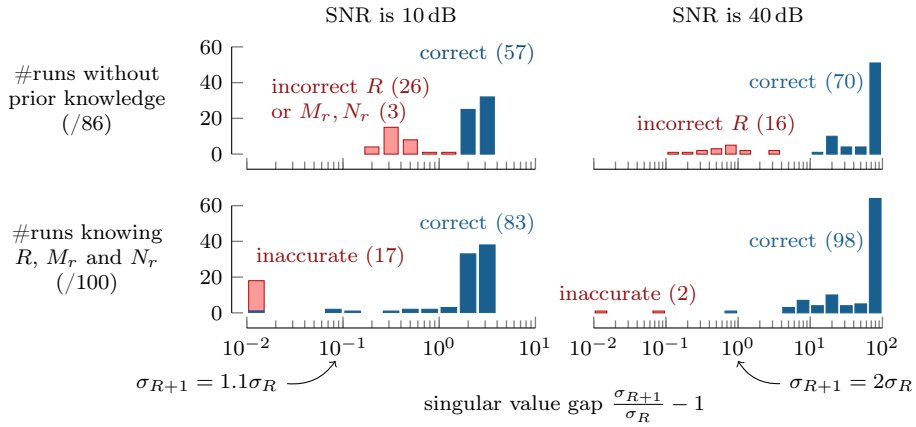

FIG. 3.5. *Most failures are due to an insufficiently large singular value gap, which causes* Null $(\mathbf{M}_{\mathcal{T}})$ *to be estimated incorrectly. Note that* σ_r *denotes the* r*th* smallest *singular value in this plot. For each experiment, the reason why the decomposition is not found is indicated. Results shown for a decomposition of a noisy* $100 \times 100 \times 50$ *tensor into* $R = 7$ *multilinear rank* $(M_r, N_r, \cdot)$ *terms with* $M = [2, 4, 3, 5, 3, 1, 1]$ *and* $N = [3, 2, 3, 2, 1, 1, 1]$*. In the top row, 14 experiments are excluded as* I *and* J *are estimated incorrectly.*

per random initialization. We emphasize that in most cases, either one or none of the 100 random initializations tried for ALS and NLS resulted in a successful decomposition. Thus, the average total time required to obtain a successful decomposition using ALS or NLS with random initializations is significantly longer. Hence, while the CS-EVD method not only does not require parameter estimates or an initialization, it is also fast, even for larger tensors.

4. Proofs of Theorem 2.1 and Corollary 2.2. We need the following lemmas.

LEMMA 4.1. *Let the conditions of Theorem 2.1 hold and let* $\mathbf{U}_1$ *and* $\mathbf{U}_2$ *denote the "*$\mathbf{U}$*" factors in the compact SVDs of* $\mathbf{T}_{[1;3,2]}$ *and* $\mathbf{T}_{[2;3,1]}$*, respectively. Then*

1. *the tensor* $\mathcal{T} \in \mathbb{F}^{I \times J \times K}$ *has ML rank-*$(\sum M_r, \sum N_r, \cdot)$*;*
2. *the "compressed" matrices* $\mathbf{A}^c := \mathbf{U}_1^{\mathrm{H}}\mathbf{A}$ *and* $\mathbf{B}^c := \mathbf{U}_2^{\mathrm{H}}\mathbf{B}$ *are square and nonsingular;*
3. $\mathbf{A} = \mathbf{U}_1\mathbf{A}^c$ *and* $\mathbf{B} = \mathbf{U}_2\mathbf{B}^c$*;*
4. *the "compressed" tensor* $\mathcal{T}^c := \mathcal{T} \cdot_1 \mathbf{U}_1^{\mathrm{H}} \cdot_2 \mathbf{U}_2^{\mathrm{H}} \in \mathbb{F}^{\sum M_r \times \sum N_r \times K}$ *has ML rank-*$(\sum M_r, \sum N_r, \cdot)$*;*
5. *tensor* $\mathcal{T}^c$ *admits the ML rank-*$(M_r, N_r, \cdot)$ *decomposition*

$$\mathcal{T}^c = \sum_{r=1}^{R} \mathcal{D}_r \cdot_1 (\mathbf{U}_1^{\mathrm{H}}\mathbf{A}_r) \cdot_2 (\mathbf{U}_2^{\mathrm{H}}\mathbf{B}_r),$$

where the tensors $\mathcal{D}_r \in \mathbb{F}^{M_r \times N_r \times K}$ *are the same as in decomposition (1.3).*

Proof. Lemma 4.1 is a canonical multilinear orthonormal compression. The details of the proof are given in the supplementary materials. ∎

LEMMA 4.2. *Let* $I = \sum M_r$*,* $J = \sum N_r$*,* $\mathbf{A}$ *and* $\mathbf{B}$ *be nonsingular, and let the matrices* $\mathbf{M}_{\mathcal{D}}$ *and* $\mathbf{M}_{\mathcal{T}}$ *be as defined in (2.1) and (2.4), respectively. Then*

1. *the matrices* $\mathbf{M}_{\mathcal{D}}$ *and* $\mathbf{M}_{\mathcal{T}}$ *are related by*
$$\mathbf{M}_{\mathcal{T}} = \operatorname{blockdiag}(\mathbf{B}\otimes\mathbf{A},\dots,\mathbf{B}\otimes\mathbf{A})\cdot\mathbf{M}_{\mathcal{D}}\cdot\begin{bmatrix}\mathbf{A}^{\mathrm{T}}\otimes\mathbf{A}^{-1} & \mathbf{O}\\ \mathbf{O} & \mathbf{B}^{-1}\otimes\mathbf{B}^{\mathrm{T}}\end{bmatrix};$$
2. *the matrices* $\mathbf{X}\in\mathbb{F}^{I\times I}$ *and* $\mathbf{Y}\in\mathbb{F}^{J\times J}$ *satisfy the equations* $\mathbf{X}\mathbf{D}_k = \mathbf{D}_k\mathbf{Y}$, $k=1,\dots,K$, *if and only if* $\mathbf{M}_{\mathcal{D}}\begin{bmatrix}\operatorname{vec}(\mathbf{X})\\ \operatorname{vec}(\mathbf{Y})\end{bmatrix} = \mathbf{0}$;
3. *the matrices* $\mathbf{X}\in\mathbb{F}^{I\times I}$ *and* $\mathbf{Y}\in\mathbb{F}^{J\times J}$ *satisfy the equations* $\mathbf{X}\mathbf{T}_k = \mathbf{T}_k\mathbf{Y}$, $k=1,\dots,K$, *if and only if* $\mathbf{M}_{\mathcal{T}}\begin{bmatrix}\operatorname{vec}(\mathbf{X})\\ \operatorname{vec}(\mathbf{Y})\end{bmatrix} = \mathbf{0}$;
4. $\operatorname{Null}(\mathbf{M}_{\mathcal{D}}) \supseteq \operatorname{span}\left\{\begin{bmatrix}\operatorname{vec}(\mathbf{P}_{I,r})\\ \operatorname{vec}(\mathbf{P}_{J,r})\end{bmatrix} : r=1,\dots,R\right\}$, *where the projectors* $\mathbf{P}_{I,r}$ *and* $\mathbf{P}_{J,r}$ *are defined in* (2.3);
5. $\operatorname{Null}(\mathbf{M}_{\mathcal{T}}) = \begin{bmatrix}\mathbf{A}^{-\mathrm{T}}\otimes\mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{B}\otimes\mathbf{B}^{-\mathrm{T}}\end{bmatrix}\operatorname{Null}(\mathbf{M}_{\mathcal{D}})$;
6. $\dim\operatorname{Null}(\mathbf{M}_{\mathcal{T}}) = \dim\operatorname{Null}(\mathbf{M}_{\mathcal{D}}) \geq R$;
7. *if* $\dim\operatorname{Null}(\mathbf{M}_{\mathcal{D}}) = R$, *then the matrices*
$$[\mathbf{A}\mathbf{D}_1\mathbf{B}^{\mathrm{T}}\ \dots\ \mathbf{A}\mathbf{D}_K\mathbf{B}^{\mathrm{T}}]\in\mathbb{F}^{I\times JK}\quad\text{and}\quad[\mathbf{B}\mathbf{D}_1^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\ \dots\ \mathbf{B}\mathbf{D}_K^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}]\in\mathbb{F}^{J\times IK}$$
have full row rank.

Proof. 1. is straightforward from (1.7), (2.1), and (2.4); 2.–4. are trivial; 5. follows from 1.; 6. follows from 5. and 4.; 7. follows from Lemma 4.1 item 1. and the identities

$$\begin{aligned}\mathbf{T}_{[1;3,2]} &= \mathbf{A}\mathbf{D}_{[1;3,2]}\operatorname{blockdiag}(\mathbf{B}^{\mathrm{T}},\dots,\mathbf{B}^{\mathrm{T}}),\\ \mathbf{T}_{[2;3,1]} &= \mathbf{B}\mathbf{D}_{[2;3,1]}\operatorname{blockdiag}(\mathbf{A}^{\mathrm{T}},\dots,\mathbf{A}^{\mathrm{T}}).\end{aligned}\tag{4.1}$$

∎

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 4.1, without loss of generality (w.l.o.g.), we can assume that $I=\sum M_r$ and $J=\sum N_r$.

1. First, we show that decomposition (1.3) can be computed algebraically. From assumption 2. and Lemma 4.2 4., 6. we have that
$$\operatorname{Null}(\mathbf{M}_{\mathcal{D}}) = \operatorname{span}\left\{\begin{bmatrix}\operatorname{vec}(\mathbf{P}_{I,r})\\ \operatorname{vec}(\mathbf{P}_{J,r})\end{bmatrix} : r=1,\dots,R\right\}\quad\text{and}\quad\dim\operatorname{Null}(\mathbf{M}_{\mathcal{T}}) = R.$$
Hence, by Lemma 4.2 5.,
$$\operatorname{Null}(\mathbf{M}_{\mathcal{T}}) = \begin{bmatrix}\mathbf{A}^{-\mathrm{T}}\otimes\mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{B}\otimes\mathbf{B}^{-\mathrm{T}}\end{bmatrix}\operatorname{span}\left\{\begin{bmatrix}\operatorname{vec}(\mathbf{P}_{I,r})\\ \operatorname{vec}(\mathbf{P}_{J,r})\end{bmatrix} : r=1,\dots,R\right\}.\tag{4.2}$$
Let the columns of $\mathbf{U}\in\mathbb{F}^{(I^2+J^2)\times R}$ form a basis of $\operatorname{Null}(\mathbf{M}_{\mathcal{T}})$. We represent $\mathbf{U}$ as
$$\mathbf{U} = \begin{bmatrix}\operatorname{vec}(\mathbf{U}_{I,1}) & \dots & \operatorname{vec}(\mathbf{U}_{I,R})\\ \operatorname{vec}(\mathbf{U}_{J,1}) & \dots & \operatorname{vec}(\mathbf{U}_{J,R})\end{bmatrix},\tag{4.3}$$
where $\mathbf{U}_{I,r}\in\mathbb{F}^{I\times I}$ and $\mathbf{U}_{J,r}\in\mathbb{F}^{J\times J}$, $r=1,\dots,R$. From (4.2) and (4.3), it follows that there exists a unique nonsingular $R\times R$ matrix $\mathbf{G}$ such that
$$\begin{aligned}\begin{bmatrix}\operatorname{vec}(\mathbf{U}_{I,1}) & \dots & \operatorname{vec}(\mathbf{U}_{I,R})\end{bmatrix} &= \left(\mathbf{A}^{-\mathrm{T}}\otimes\mathbf{A}\right)[\operatorname{vec}(\mathbf{P}_{I,1})\ \dots\ \operatorname{vec}(\mathbf{P}_{I,R})]\mathbf{G},\\ \begin{bmatrix}\operatorname{vec}(\mathbf{U}_{J,1}) & \dots & \operatorname{vec}(\mathbf{U}_{J,R})\end{bmatrix} &= \left(\mathbf{B}\otimes\mathbf{B}^{-\mathrm{T}}\right)[\operatorname{vec}(\mathbf{P}_{J,1})\ \dots\ \operatorname{vec}(\mathbf{P}_{J,R})]\mathbf{G}.\end{aligned}$$

Hence, (2.5)–(2.6) hold. Since the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ are nonsingular, it follows that the matrices $\mathbf{U}_{I,1},\dots,\mathbf{U}_{I,R}$ are linearly independent. Thus, we can recover $\mathbf{A}$ from (2.5). Similarly, we can recover $\mathbf{B}$ from (2.6).

2. Now, we show that decomposition (1.3) is unique. Assume that (1.3) holds for $\mathbf{A}_r$, $\mathbf{B}_r$, and $\mathbf{D}_r$ replaced by $\widetilde{\mathbf{A}}_r$, $\widetilde{\mathbf{B}}_r$, and $\widetilde{\mathbf{D}}_r$. To prove the uniqueness it is sufficient to show that $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ and $\mathbf{M}_{\widetilde{\mathcal{D}}}$ satisfy Assumptions 1 and 2, i.e., that $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ are nonsingular and $\dim \operatorname{Null}\left(\mathbf{M}_{\widetilde{\mathcal{D}}}\right) = R$. In particular, this will imply from the first part of the proof that $\widetilde{\mathbf{A}}_r$, $\widetilde{\mathbf{B}}_r$, and $\widetilde{\mathbf{D}}_r$ can be computed using the same algebraic procedure as $\mathbf{A}_r$, $\mathbf{B}_r$, and $\mathbf{D}_r$. It is clear that the output of this algebraic procedure is unique up to permutations and scaling.

Now, by (1.7),

$$
\begin{aligned}
[\mathbf{A}\mathbf{D}_1\mathbf{B}^{\mathrm{T}} \ \dots \ \mathbf{A}\mathbf{D}_K\mathbf{B}^{\mathrm{T}}] &= [\mathbf{T}_1 \ \dots \ \mathbf{T}_K] \\
&= [\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}_1\widetilde{\mathbf{B}}^{\mathrm{T}} \ \dots \ \widetilde{\mathbf{A}}\widetilde{\mathbf{D}}_K\widetilde{\mathbf{B}}^{\mathrm{T}}] \qquad (4.4)\\
&= \widetilde{\mathbf{A}}[\widetilde{\mathbf{D}}_1\widetilde{\mathbf{B}}^{\mathrm{T}} \ \dots \ \widetilde{\mathbf{D}}_K\widetilde{\mathbf{B}}^{\mathrm{T}}], \\
[\mathbf{B}\mathbf{D}_1^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \ \dots \ \mathbf{B}\mathbf{D}_K^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}] &= [\mathbf{T}_1^{\mathrm{T}} \ \dots \ \mathbf{T}_K^{\mathrm{T}}] \\
&= [\widetilde{\mathbf{B}}\widetilde{\mathbf{D}}_1^{\mathrm{T}}\widetilde{\mathbf{A}}^{\mathrm{T}} \ \dots \ \widetilde{\mathbf{B}}\widetilde{\mathbf{D}}_K^{\mathrm{T}}\widetilde{\mathbf{A}}^{\mathrm{T}}] \qquad (4.5)\\
&= \widetilde{\mathbf{B}}[\widetilde{\mathbf{D}}_1^{\mathrm{T}}\widetilde{\mathbf{A}}^{\mathrm{T}} \ \dots \ \widetilde{\mathbf{D}}_K^{\mathrm{T}}\widetilde{\mathbf{A}}^{\mathrm{T}}].
\end{aligned}
$$

Since, by Lemma 4.2 item 7., the matrices in the left-hand side of (4.4) and (4.5) have full row rank, it follows that the matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ are nonsingular. Since, by assumption 2., $\dim \operatorname{Null}\left(\mathbf{M}_{\mathcal{D}}\right) = R$, and, by Lemma 4.2 5.,

$$
\operatorname{Null}\left(\mathbf{M}_{\mathcal{T}}\right) = \begin{bmatrix} \mathbf{A}^{-\mathrm{T}} \otimes \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \otimes \mathbf{B}^{-\mathrm{T}} \end{bmatrix} \operatorname{Null}\left(\mathbf{M}_{\mathcal{D}}\right) = \qquad (4.6)
$$

$$
\begin{bmatrix} \widetilde{\mathbf{A}}^{-\mathrm{T}} \otimes \widetilde{\mathbf{A}} & \mathbf{O} \\ \mathbf{O} & \widetilde{\mathbf{B}} \otimes \widetilde{\mathbf{B}}^{-\mathrm{T}} \end{bmatrix} \operatorname{Null}\left(\mathbf{M}_{\widetilde{\mathcal{D}}}\right),
$$

it follows that $\dim \operatorname{Null}\left(\mathbf{M}_{\widetilde{\mathcal{D}}}\right) = R$. ∎

Proof of Corollary 2.2. It is sufficient to prove that assumption 2 in Corollary 2.2 implies assumption 2 in Theorem 2.1. Thus, we need to show that the subspace of solutions of (2.2) is spanned by the trivial solutions $(\mathbf{X},\mathbf{Y}) = (\mathbf{P}_{I,r},\mathbf{P}_{J,r})$, $r = 1,\dots,R$, where $\mathbf{P}_{I,r}$ and $\mathbf{P}_{J,r}$ are defined in (2.3) and $M_1 = \dots = M_R = 1$.

Let $\mathbf{X} = (x_{ij})_{i,j=1}^{R}$ and let $\mathbf{Y} = (\mathbf{Y}_{ij})_{i,j=1}^{R}$ consist of the blocks $\mathbf{Y}_{ij} \in \mathbb{F}^{N_i \times N_j}$. Since the matrices $\mathbf{D}_k$ are block-diagonal, we can rewrite (2.2) as

$$
x_{ij}\mathbf{D}_{j,k} = \mathbf{D}_{i,k}\mathbf{Y}_{ij}, \qquad 1 \le i,j \le R, \qquad k = 1,\dots,K,
$$

which, by (2.7), is equivalent to

$$
x_{ij}\mathbf{C}_j = \mathbf{C}_i\mathbf{Y}_{ij}, \qquad 1 \le i,j \le R. \tag{4.6}
$$

We need to show that

$$
\mathbf{Y}_{ii} = x_{ii}\mathbf{I}_{N_i}, \qquad 1 \le i \le R, \tag{4.7}
$$

$$
\mathbf{Y}_{ij} = \mathbf{O}, \quad x_{ij} = 0, \qquad 1 \le i \ne j \le R. \tag{4.8}
$$

(i) We first prove (4.7). By (4.6),

$$
\mathbf{C}_i(x_{ii}\mathbf{I}_{N_i} - \mathbf{Y}_{ii}) = \mathbf{O}, \qquad 1 \le i \le R. \tag{4.9}
$$

Since the $1 \times N_i \times K$ tensor $\mathcal{D}_i$ in (1.3) is ML rank-$(1, N_i, N_i)$ and the second matrix unfolding of $\mathcal{D}_i$ coincides with $\mathbf{C}_i^{\mathrm{T}}$, it follows that $\mathbf{C}_i^{\mathrm{T}}$ has full row rank or that $\mathbf{C}_i$ has full column rank. Hence (4.7) follows from (4.9).

(ii) We next prove (4.8). It is clear that (4.6) can be rewritten as

$$\mathbf{O} = [\mathbf{C}_i \;\; \mathbf{C}_j] \begin{bmatrix} \mathbf{Y}_{ij} \\ x_{ij}\mathbf{I}_{N_j} \end{bmatrix}. \tag{4.10}$$

Assume to the contrary that $x_{ij} \neq 0$. Then from (4.10), Sylvester's rank inequality, and Assumption 2 it follows that

$$0 \geq r_{[\mathbf{C}_i \;\; \mathbf{C}_j]} + r_{[\mathbf{Y}_{ij}^{\mathrm{T}} \;\; x_{ij}\mathbf{I}_{N_j}]^{\mathrm{T}}} - (N_i + N_j) \geq \max(N_i, N_j) + 1 + N_j - (N_i + N_j) \geq 1,$$

which is a contradiction. Thus, $x_{ij} = 0$. Hence, by (4.6), $\mathbf{C}_i\mathbf{Y}_{ij} = \mathbf{O}$. Since $\mathbf{C}_i$ has full column rank, it follows that $\mathbf{Y}_{ij} = \mathbf{O}$. ∎

5. Proofs of Theorem 2.3 and Theorem 2.8. We need the following two lemmas.

LEMMA 5.1. *Let*

$$K = \max\left(\left\lceil \frac{M}{N} \right\rceil + 1, \left\lceil \frac{N}{M} \right\rceil + 1, 3\right). \tag{5.1}$$

Then there exist matrices $\mathbf{E}_1, \ldots, \mathbf{E}_K \in \mathbb{F}^{M \times N}$ *such that the system of matrix equations*

$$\mathbf{X}\mathbf{E}_k = \mathbf{E}_k\mathbf{Y}, \qquad \mathbf{X} \in \mathbb{F}^{M \times M}, \;\; \mathbf{Y} \in \mathbb{F}^{N \times N} \qquad\qquad k = 1, \ldots, K \tag{5.2}$$

has only the trivial solutions $\mathbf{X} = c\mathbf{I}_M$ *and* $\mathbf{Y} = c\mathbf{I}_N$*, where* $c \in \mathbb{F}$*.*

Proof. It is clear that by taking the transpose of (5.2) we obtain an equivalent problem of the form (5.2) in which M and N are switched. Thus, w.l.o.g., we can always assume that $M \geq N$. We consider three cases: $M = N$, $N < M \leq 2N$, and $2N < M$.

Case $M = N$. By assumption (5.1), $K = 3$. Let $\mathbf{a} \in \mathbb{F}^N$ be a vector with distinct entries and let $\mathbf{D} \in \mathbb{F}^{N \times N}$ be a matrix with nonzero entries. We show that if $\mathbf{E}_1 = \mathbf{I}_N$, $\mathbf{E}_2 = \operatorname{diag}(\mathbf{a})$, and $\mathbf{E}_3 = \mathbf{D}$, then (5.2) has only trivial solutions.

The equation $\mathbf{X}\mathbf{E}_1 = \mathbf{E}_1\mathbf{Y}$ implies that $\mathbf{X} = \mathbf{Y}$. Hence,

$$\mathbf{Y}\operatorname{diag}(\mathbf{a}) = \mathbf{Y}\mathbf{E}_2 = \mathbf{X}\mathbf{E}_2 = \mathbf{E}_2\mathbf{Y} = \operatorname{diag}(\mathbf{a})\mathbf{Y}.$$

Since the entries of $\mathbf{a}$ are distinct, it follows that $\mathbf{Y}$ is a diagonal matrix, $\mathbf{Y} = \operatorname{diag}(y_1, \ldots, y_N)$. Finally, since the entries of $\mathbf{E}_3 = \mathbf{D}$ are nonzero, the equation

$$\operatorname{diag}(y_1, \ldots, y_N)\mathbf{D} = \mathbf{Y}\mathbf{E}_3 = \mathbf{X}\mathbf{E}_3 = \mathbf{E}_3\mathbf{Y} = \mathbf{D}\operatorname{diag}(y_1, \ldots, y_N)$$

implies that $y_1 = \cdots = y_N =: c$. Thus, $\mathbf{X} = \mathbf{Y} = c\mathbf{I}_N$.

Case $N < M \leq 2N$. By assumption (5.1), $K = 3$. Let $\mathbf{a} \in \mathbb{F}^N$ be a vector with distinct entries, $\mathbf{D} \in \mathbb{F}^{(M-N) \times N}$ be a full-row-rank matrix with nonzero entries, and let $\widehat{\mathbf{I}}_{M-N}$ denote the matrix formed by the first $M - N$ rows of $\mathbf{I}_N$. We show that if

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_N \\ \mathbf{O} \end{bmatrix}, \qquad \mathbf{E}_2 = \begin{bmatrix} \mathbf{O} \\ \mathbf{D} \end{bmatrix}, \qquad \mathbf{E}_3 = \begin{bmatrix} \operatorname{diag}(\mathbf{a}) \\ \widehat{\mathbf{I}}_{M-N} \end{bmatrix},$$

then (5.2) has only trivial solutions. Let $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_3 \\ \mathbf{X}_2 & \mathbf{X}_4 \end{bmatrix}$ with $\mathbf{X}_1 \in \mathbb{F}^{N\times N}$, $\mathbf{X}_2, \mathbf{X}_3^{\mathrm{T}} \in \mathbb{F}^{(M-N)\times N}$ and $\mathbf{X}_4 \in \mathbb{F}^{(M-N)\times(M-N)}$. Then the equations $\mathbf{X}\mathbf{E}_1 = \mathbf{E}_1\mathbf{Y}$, $\mathbf{X}\mathbf{E}_2 = \mathbf{E}_2\mathbf{Y}$, and $\mathbf{X}\mathbf{E}_3 = \mathbf{E}_3\mathbf{Y}$ imply

$$\mathbf{X}_1 = \mathbf{Y} \quad \text{and} \quad \mathbf{X}_2 = \mathbf{O}, \tag{5.3}$$
$$\mathbf{X}_3\mathbf{D} = \mathbf{O} \quad \text{and} \quad \mathbf{X}_4\mathbf{D} = \mathbf{D}\mathbf{Y},$$

$$\mathbf{X}_1 \operatorname{diag}(\mathbf{a}) + \mathbf{X}_3\widehat{\mathbf{I}}_{M-N} = \operatorname{diag}(\mathbf{a})\mathbf{Y} \quad \text{and} \quad \mathbf{X}_2 \operatorname{diag}(\mathbf{a}) + \mathbf{X}_4\widehat{\mathbf{I}}_{M-N} = \widehat{\mathbf{I}}_{M-N}\mathbf{Y}, \tag{5.4}$$

respectively. Since $\mathbf{X}_3\mathbf{D} = \mathbf{O}$ and $\mathbf{D}$ has full row rank, it follows that $\mathbf{X}_3 = \mathbf{O}$. Hence, by (5.3) and (5.4),

$$\mathbf{Y} \operatorname{diag}(\mathbf{a}) = \mathbf{X}_1 \operatorname{diag}(\mathbf{a}) + \mathbf{X}_3\widehat{\mathbf{I}}_{M-N} = \operatorname{diag}(\mathbf{a})\mathbf{Y}.$$

Since all entries of $\mathbf{a}$ are distinct, it follows that $\mathbf{Y}$ is a diagonal matrix, $\mathbf{Y} = \operatorname{diag}(y_1, \dots, y_N)$. From the second identities in (5.3) and (5.4) it follows that $\mathbf{X}_4 = \operatorname{diag}(y_1, \dots, y_{M-N})$. Finally, since the entries of $\mathbf{D}$ are nonzero, it follows from $\mathbf{X}_4\mathbf{D} = \mathbf{D}\mathbf{Y}$ that $y_1 = \cdots = y_N =: c$. Hence, $\mathbf{Y} = c\mathbf{I}_N$ and $\mathbf{X} = \operatorname{blockdiag}(\mathbf{X}_1, \mathbf{X}_4) = c\mathbf{I}_M$.

Case $2N < M$. By assumption (5.1), $K = \lceil \frac{M}{N} \rceil + 1$. Let $r = M - (K-2)N$. Then $0 < r \leq N$. We define $\mathbf{E}_1, \dots, \mathbf{E}_{K-2} \in \mathbb{F}^{M\times N}$ and $\widehat{\mathbf{E}}_{K-1} \in \mathbb{F}^{M\times r}$ as blocks in the partition $\mathbf{I}_M = [\mathbf{E}_1 \ \dots \ \mathbf{E}_{K-2} \ \widehat{\mathbf{E}}_{K-1}]$ and set $\mathbf{E}_{K-1} = [\widehat{\mathbf{E}}_{K-1} \ \mathbf{O}]$, where the zero matrix has $N-r$ columns. We also set $\mathbf{E}_k = \begin{bmatrix} \operatorname{diag}(\mathbf{a}) \\ \mathbf{D} \\ \mathbf{O} \end{bmatrix}$, where $\mathbf{a} \in \mathbb{F}^N$ is a vector with distinct entries and $\mathbf{D} \in \mathbb{F}^{N\times N}$ is a matrix with nonzero entries. We show that for such choice of the matrices $\mathbf{E}_k$, system (5.2) has only the trivial solution. Following a similar argument as in the previous case, from the first $K-1$ equations in (5.2) it follows that $\mathbf{X} = \operatorname{blockdiag}(\mathbf{Y}, \dots, \mathbf{Y}, \widehat{\mathbf{Y}})$. Here $\hat{\mathbf{Y}}$ denotes the upper $r \times r$ principal submatrix of $\mathbf{Y}$. That is, if one writes $\mathbf{Y} \in \mathbb{F}^{N\times N}$ with respect to the orthogonal decomposition $\mathbb{F}^N = \mathbb{F}^r \oplus \mathbb{F}^{N-r}$, then one has

$$\mathbf{Y} = \begin{bmatrix} \hat{\mathbf{Y}} & * \\ * & * \end{bmatrix}.$$

The remaining equation $\mathbf{X}\mathbf{E}_K = \mathbf{E}_K\mathbf{Y}$ in (5.2) implies that $\mathbf{Y}\operatorname{diag}(\mathbf{a}) = \operatorname{diag}(\mathbf{a})\mathbf{Y}$ and $\mathbf{Y}\mathbf{D} = \mathbf{D}\mathbf{Y}$. Now, in exactly the same way as "*Case* $M = N$" one can obtain that $\mathbf{Y} = c\mathbf{I}_N$ for some $c \in \mathbb{F}$. Hence, $\mathbf{X} = c\mathbf{I}_M$. ∎

LEMMA 5.2. *Let*

$$K = \lceil M_2/N_2 \rceil + \lceil N_1/M_1 \rceil. \tag{5.5}$$

Then there exist matrices $\mathbf{E}_1, \dots, \mathbf{E}_K \in \mathbb{F}^{M_2\times N_2}$ *and* $\mathbf{F}_1, \dots, \mathbf{F}_K \in \mathbb{F}^{M_1\times N_1}$ *such that the system of matrix equations*

$$\mathbf{X}\mathbf{E}_k = \mathbf{F}_k\mathbf{Y}, \qquad \mathbf{X} \in \mathbb{F}^{M_1\times M_2}, \ \mathbf{Y} \in \mathbb{F}^{N_1\times N_2} \qquad k = 1, \dots, K \tag{5.6}$$

has only the zero solution.

Proof. We consider four cases.

Case $M_1 > N_1$ *and* $M_2 < N_2$. In this case, (5.5) simplifies to $K = 2$. Define

$$\mathbf{E}_1 = \mathbf{O}, \qquad \mathbf{E}_2 = [\mathbf{I}_{M_2} \ \mathbf{O}], \qquad \mathbf{F}_1 = \begin{bmatrix} \mathbf{I}_{N_1} \\ \mathbf{O} \end{bmatrix}, \qquad \mathbf{E}_2 = \mathbf{O}.$$

Then the equations $\mathbf{X}\mathbf{E}_1 = \mathbf{F}_1\mathbf{Y}$ and $\mathbf{X}\mathbf{E}_2 = \mathbf{F}_2\mathbf{Y}$ imply that $\mathbf{Y} = \mathbf{O}$ and $\mathbf{X} = \mathbf{O}$, respectively.

Case $M_1 > N_1$ and $M_2 \geq N_2$. Simplifying (5.5) gives $K = \lceil M_2/N_2 \rceil + 1$. Let $\mathbf{E}_1, \dots, \mathbf{E}_{K-1}$ be matrices such that $[\mathbf{E}_1 \ \dots \mathbf{E}_{K-1}]$ has full row rank, let $\mathbf{E}_K$ be an arbitrary matrix, $\mathbf{F}_1, \dots, \mathbf{F}_{K-1}$ be zero matrices, and $\mathbf{F}_K = \begin{bmatrix} \mathbf{I}_{N_1} \\ \mathbf{O} \end{bmatrix}$. Then, from the first $K-1$ equations in (5.6) it follows that $\mathbf{X}[\mathbf{E}_1 \ \dots \ \mathbf{E}_{K-1}] = \mathbf{O}$. Hence, $\mathbf{X} = \mathbf{O}$. Now the remaining equation $\mathbf{X}\mathbf{E}_K = \mathbf{F}_K\mathbf{Y}$ implies that $\mathbf{Y} = \mathbf{O}$.

Case $M_1 \leq N_1$ and $M_2 < N_2$. In this case, the problem can be reduced to the setting used in the previous case by taking the transpose of the equations in (5.6).

Case $M_1 \leq N_1$ and $M_2 \geq N_2$. By (5.5), $K = \lceil M_2/N_2 \rceil + \lceil N_1/M_1 \rceil$. Let $\mathbf{E}_1, \dots, \mathbf{E}_{\lceil M_2/N_2 \rceil}$ be matrices such that $[\mathbf{E}_1 \ \dots \ \mathbf{E}_{\lceil M_2/N_2 \rceil}]$ has full row rank and let $\mathbf{E}_{\lceil M_2/N_2 \rceil + 1} = \mathbf{E}_K = \mathbf{O}$. Let also $\mathbf{F}_1 = \cdots = \mathbf{F}_{\lceil M_2/N_2 \rceil} = \mathbf{O}$ and $\mathbf{F}_{\lceil M_2/N_2 \rceil + 1}, \dots, \mathbf{F}_K$ be matrices such that $[\mathbf{F}_{\lceil M_1/N_2 \rceil + 1}^{\mathrm{T}} \ \dots \ \mathbf{F}_K^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank. Then from the first $\lceil M_2/N_2 \rceil$ and the last $\lceil N_1/M_1 \rceil$ equations in (5.6) it follows that

$$\mathbf{X}[\mathbf{E}_1 \ \dots \ \mathbf{E}_{\lceil M_2/N_2 \rceil}] = \mathbf{O} \qquad \text{and} \qquad \mathbf{O} = [\mathbf{F}_{\lceil M_1/N_2 \rceil + 1}^{\mathrm{T}} \ \dots \ \mathbf{F}_K^{\mathrm{T}}]^{\mathrm{T}}\mathbf{Y},$$

respectively. Hence, $\mathbf{X} = \mathbf{O}$ and $\mathbf{Y} = \mathbf{O}$. ∎

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. It is sufficient to show that Assumptions 1 and 2 in Theorem 2.1 hold for generic matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}_1, \dots, \mathbf{D}_K$. The first two inequalities in (2.8) express the fact that the number of columns of $\mathbf{A}$ and $\mathbf{B}$ cannot exceed the number of rows. Since $\mathbf{A}$ and $\mathbf{B}$ are generic, this means that Assumption 1 in Theorem 2.1 holds. To prove that Assumption 2 in Theorem 2.1 also holds it is sufficient to show that the subspace of solutions of (2.2) has dimension R. In the remaining part of the proof, we show that for generic matrices $\mathbf{D}_1, \dots, \mathbf{D}_K$ the subspace of solutions of (2.2) is spanned by $(\mathbf{P}_{I,1}, \mathbf{P}_{J,1}), \dots, (\mathbf{P}_{I,R}, \mathbf{P}_{J,R})$, where the projectors $\mathbf{P}_{I,r}$, $\mathbf{P}_{J,r}$ are defined in (2.3).

Let $\mathbf{X} = (\mathbf{X}_{ij})_{i,j=1}^{R}$ and $\mathbf{Y} = (\mathbf{Y}_{ij})_{i,j=1}^{R}$ consist of the blocks $\mathbf{X}_{ij} \in \mathbb{F}^{M_i \times M_j}$ and $\mathbf{Y}_{ij} \in \mathbb{F}^{N_i \times N_j}$, respectively. Since the matrices $\mathbf{D}_k$ are block-diagonal, we can rewrite (2.2) in terms of the blocks $\mathbf{X}_{ij}$ and $\mathbf{Y}_{ij}$ as

$$\mathbf{X}_{ij}\mathbf{D}_{j,k} = \mathbf{D}_{i,k}\mathbf{Y}_{ij}, \qquad k = 1, \dots, K, \tag{5.7}$$

where $i, j \in \{1, \dots, R\}$. Define the subsets

$$\Omega_{ij}, \ \Omega_i \subseteq \left(\mathbb{F}^{M_1 \times N_1} \times \cdots \times \mathbb{F}^{M_R \times N_R}\right) \times \cdots \times \left(\mathbb{F}^{M_1 \times N_1} \times \cdots \times \mathbb{F}^{M_R \times N_R}\right)$$

as follows. A tuple $(\mathbf{D}_{1,1}, \dots, \mathbf{D}_{R,1}, \dots, \mathbf{D}_{1,K}, \dots, \mathbf{D}_{R,K})$ is an element of Ω_{ij} if and only if the only solution to

$$\mathbf{X}\mathbf{D}_{j,k} = \mathbf{D}_{i,k}\mathbf{Y}, \qquad k = 1, \dots, K, \quad \text{with} \quad \mathbf{X} \in \mathbb{F}^{M_i \times M_j} \quad \text{and} \quad \mathbf{Y} = \mathbb{F}^{N_i \times N_j}$$

is $\mathbf{X} = \mathbf{O}_{M_i \times M_j}$ and $\mathbf{Y} = \mathbf{O}_{N_i \times N_j}$. Similarly, a tuple $(\mathbf{D}_{1,1}, \dots, \mathbf{D}_{R,K})$ is an element of Ω_i if and only if the only solutions to

$$\mathbf{X}\mathbf{D}_{j,k} = \mathbf{D}_{i,k}\mathbf{Y}, \qquad k = 1, \dots, K, \quad \text{with} \quad \mathbf{X} \in \mathbb{F}^{M_i \times M_i} \quad \text{and} \quad \mathbf{Y} \in \mathbb{F}^{N_i \times N_i}$$

are of the form $\mathbf{X} = c\mathbf{I}_{M_i}$ and $\mathbf{Y} = c\mathbf{I}_{N_i}$ for some $c \in \mathbb{F}$.

To prove that $(\mathbf{X}, \mathbf{Y})$ belongs to the linear subspace spanned by $(\mathbf{P}_{I,1}, \mathbf{P}_{J,1})$, $\ldots$, $(\mathbf{P}_{I,R}, \mathbf{P}_{J,R})$ we have to show that $(\cap\Omega_{ij}) \bigcap (\cap\Omega_i)$ is a set of full measure. In particular, since the $\mathbf{D}_1, \ldots, \mathbf{D}_K$ are generic, this will then imply that

$$(\mathbf{D}_{1,1}, \ldots, \mathbf{D}_{R,1}, \ldots, \mathbf{D}_{1,K}, \ldots, \mathbf{D}_{R,K}) \in (\cap\Omega_{ij}) \bigcap (\cap\Omega_i). \tag{5.8}$$

Intuitively, elements of Ω_{ij} and Ω_i are tuples $(\mathbf{D}_{1,1}, \ldots, \mathbf{D}_{R,1}, \ldots, \mathbf{D}_{1,K}, \ldots, \mathbf{D}_{R,K})$ such that (5.7) has only trivial solutions for the chosen $\mathbf{D}_{i,k}$ and $\mathbf{D}_{j,k}$. Thus, (5.8) will imply that the set of solutions (2.2) is the span of $(\mathbf{P}_{I,1}, \mathbf{P}_{J,1}), \ldots, (\mathbf{P}_{I,R}, \mathbf{P}_{J,R})$.

Since the intersection of a finite number of sets of full measure is also a set of full measure it is sufficient to prove that all sets in the intersection have full measure. This can be done as follows.

1) Let $i \neq j$. W.l.o.g. we can assume that $i = 1$ and $j = 2$. Then, by (5.7),

$$\mathbf{X}_{12}\mathbf{D}_{2,k} = \mathbf{D}_{1,k}\mathbf{Y}_{12}, \qquad k = 1, \ldots, K. \tag{5.9}$$

It is clear that system (5.9) has only the zero solution if and only if the matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{D}_{2,1}^{\mathrm{T}} \otimes \mathbf{I}_{M_1} & -\mathbf{I}_{N_2} \otimes \mathbf{D}_{1,1} \\ & \vdots \\ \mathbf{D}_{2,K}^{\mathrm{T}} \otimes \mathbf{I}_{M_1} & -\mathbf{I}_{N_2} \otimes \mathbf{D}_{1,K} \end{bmatrix}$$

has full column rank. Hence, from Lemma 5.2 and assumption (2.8) it follows that there exist matrices $\mathbf{D}_{1,1}, \ldots, \mathbf{D}_{1,K}$ and $\mathbf{D}_{2,1} \ldots \mathbf{D}_{2,K}$ such that $\mathbf{M}$ has full column rank. Now, we assume that $\mathbf{D}_{1,1}, \ldots, \mathbf{D}_{1,K}$ and $\mathbf{D}_{2,1} \ldots \mathbf{D}_{2,K}$ in (5.9) are generic. Then, by [15, Lemma 6.3], $\mathbf{M}$ has full column rank, implying that system (5.9) has only the zero solution. Hence, by Fubini's theorem [23, Theorem C, p.148], Ω_{12} is a set of full measure.

2) Let $i = j$. W.l.o.g., we can assume that $i = 1$. We have to prove that the system

$$\mathbf{X}_{11}\mathbf{D}_{1,k} = \mathbf{D}_{1,k}\mathbf{Y}_{11}, \qquad k = 1, \ldots, K, \tag{5.10}$$

has only the trivial solution $\mathbf{X}_{11} = c\mathbf{I}_{M_1}$ and $\mathbf{Y}_{11} = c\mathbf{I}_{N_1}$, where $c \in \mathbb{F}$. It is clear that system (5.10) has only the trivial solution if and only if the dimension of the null space of the matrix

$$\mathbf{M}' = \begin{bmatrix} \mathbf{D}_{1,1}^{\mathrm{T}} \otimes \mathbf{I}_{M_1} & -\mathbf{I}_{N_1} \otimes \mathbf{D}_{1,1} \\ & \vdots \\ \mathbf{D}_{1,K}^{\mathrm{T}} \otimes \mathbf{I}_{M_1} & -\mathbf{I}_{N_1} \otimes \mathbf{D}_{1,1} \end{bmatrix}$$

is one. Hence, from Lemma 5.1 and Assumption (2.8) it follows that there exist matrices $\mathbf{D}_{1,1}, \ldots, \mathbf{D}_{1,K}$ such that $M_1^2 + N_1^2 - 1$ columns of $\mathbf{M}'$ are linearly independent. Now, we assume that $\mathbf{D}_{1,1}, \ldots, \mathbf{D}_{1,K}$ are generic. Then, by [15, Lemma 6.3], the same $M_1^2 + N_1^2 - 1$ columns of $\mathbf{M}$ are linearly independent, implying that system (5.10) has only the trivial solution. Hence, by Fubini's theorem [23, Theorem C, p.148], Ω_{11} is a set of full measure. ∎

Proof of Theorem 2.8. We show that Assumptions 1 and 2 in Theorem 2.1 hold for generic matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}_1, \ldots, \mathbf{D}_K$, where $\mathbf{B} = \mathbf{A}$ or $\mathbf{B} = \mathbf{A}^*$ and $\mathbf{D}_1, \ldots, \mathbf{D}_K$ may be symmetric, Hermitian or unconstrained.

For Assumption 1, since $\mathbf{A}$ is generic and $I \geq \sum N_r$, it follows that $\mathbf{A}$ has full column rank, hence $\mathbf{B}$ has full column rank. To show that Assumption 2 holds,

w.l.o.g., we assume that $K = 3$. We proceed as in the proof of Theorem 2.3. First, we show that the subspace of the solutions of (2.2) has dimension R for a specific choice of $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$. Let $\mathbf{D}_1 = \mathbf{I}_{\sum N_r}$, $\mathbf{D}_2$ be a diagonal matrix with distinct real values on the main diagonal and let $\mathbf{D}_3 = \text{blockdiag}(\mathbf{D}_{1,3}, \ldots, \mathbf{D}_{R,3})$, where the entries of $\mathbf{D}_{1,3}, \ldots, \mathbf{D}_{R,3}$ are nonzero. Then $\mathbf{X}\mathbf{D}_1 = \mathbf{D}_1\mathbf{Y}$ implies that $\mathbf{X} = \mathbf{Y}$. From $\mathbf{X}\mathbf{D}_2 = \mathbf{D}_2\mathbf{Y} = \mathbf{D}_2\mathbf{X}$, it follows that $\mathbf{X} = \text{diag}(x_1, \ldots, x_{\sum N_r})$. Finally, from $\text{diag}(x_1, \ldots, x_{\sum N_r})\mathbf{D}_3 = \mathbf{X}\mathbf{D}_3 = \mathbf{D}_3\mathbf{Y} = \mathbf{D}_3 \,\text{diag}(x_1, \ldots, x_{\sum N_r})$ it follows that $\mathbf{X} = \mathbf{Y} = \text{blockdiag}(c_1\mathbf{I}_{N_1}, \ldots, c_R\mathbf{I}_{N_R})$ for some $c_1, \ldots, c_R \in \mathbb{F}$.

Thus, Assumption 2 in Theorem 2.1 holds for a specific choice of $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$. Hence some $2(\sum N_r)^2 - R$ columns of $\mathbf{M}_{\mathcal{D}}$ are linearly independent. From [15, Lemma 6.3] it follows that if $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$ are generic symmetric, Hermitian or unconstrained matrices, then the same $2(\sum N_r)^2 - R$ columns of $\mathbf{M}_{\mathcal{D}}$ are linearly independent, implying that $\dim \text{Null}\,(\mathbf{M}_{\mathcal{D}}) = R$. ∎

6. Conclusion. We presented new deterministic and generic conditions for the uniqueness of an ML rank-$(M_r, N_r, \cdot)$ decomposition. Compared to previous uniqueness conditions for block term decompositions, our results have the important feature that the blocks appearing in the decomposition can both be of different sizes and need not have mode-3 rank 1 thereby allowing our results to be used in significantly more general situations.

In addition, we presented the algebraic CS-EVD algorithm for the computation of ML rank-$(M_r, N_r, \cdot)$ decompositions. One appealing aspect of this algorithm is that there is no need to specify the parameters M_r, N_r, and R which determine the size and number of blocks in the decomposition. Indeed, provided the signal-to-noise ratio is sufficiently high, our algorithm can independently recover these block sizes.

In numerical simulations, we saw that our CS-EVD algorithm can successfully (approximately) recover an underlying BTD from a noisy tensor. We observed that the algorithm performs competitively in terms of accuracy on small-scale problems with leading optimization routines for computing ML rank-$(M_r, N_r, \cdot)$ decompositions. On harder, large-scale problems, we see that the CS-EVD algorithm can be used as an algebraic initialization for optimization-based methods and that doing so significantly improves the accuracy of the final decomposition when compared to optimization alone. Furthermore, we see that only a small number of optimization iterations are needed to refine CS-EVD-based initializations.

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