ON BEST LOW RANK APPROXIMATION OF POSITIVE DEFINITE TENSORS*

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Abstract. Tensors, or multiindexed arrays, play an important role in fields such as machine learning and signal processing. These higher-order generalizations of matrices allow for preservation of higher-order structure present in data, and low rank decompositions of tensors allow for recovery of underlying information. In many cases, the underlying tensor of interest is positive (semi)definite, i.e. the homogeneous polynomial associated to the tensor has nonnegative evaluation on all inputs.

An archetypal problem is that one has a noisy measurement of some low rank signal tensor of interest. This measurement itself does not have low rank, so one must compute a best low rank approximation of the measured tensor to recover component information. As it turns out, the set of tensors of rank less than or equal to R is in general not closed when $R > 1$, and, as a consequence, best low rank approximations can fail to exist. In the case that a best low rank approximation does not exist, near optimal low rank approximations suffer numerical issues and cannot be used to reliably approximate underlying component information. As a consequence, existence guarantees for best low rank tensor approximations are of great practical and theoretical interest.

This article develops deterministic guarantees for the existence of best low rank approximations of tensors which are positive semidefinite. In particular, we show that the set of low rank positive semidefinite tensors is relatively closed as a subset of the set of positive semidefinite tensors. We use this fact to give a deterministic bound which may be used to guarantee the existence of a best low rank approximation of a noisy low rank positive semidefinite tensor. Furthermore, we show that this condition guarantees uniqueness of the canonical polyadic decomposition for best low rank approximations. In addition, for order three tensors, we prove that our bound is sharp and that it can be computed using semidefinite programming.

The main results of the article are illustrated through numerical experiments. These numerical experiments show that our bound is highly predictive of numerical issues when attempting to recover underlying component information from a noisy low rank positive semidefinite tensor.

Key words. tensors, canonical polyadic decomposition, best low rank approximation, border rank, positive definite, multilinear algebra, multidimensional arrays

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1. Introduction.

Higher-order tensors serve as natural models for data in a wide variety of applications in signal processing and machine learning [45, 6]. These multi-indexed arrays can be used to capture higher-order information present in data and decompositions of tensors enable recovery of underlying component information [8]. In many such applications, the tensors which arise will naturally be positive (semi)definite [25, 37], i.e., the associated homogeneous polynomial has nonnegative evaluation on all inputs. For example, collections of positive semidefinite tensors in the form of higher-order statistics play a key role in blind source separation [2, 36, 54, 9]. In addition, it is common to have applications, e.g. diffusion tensor imaging [41, 1], where the output of a particular non-linear function is positive for any input. Such

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functions can be effectively modelled using positive definite tensors.

Tensors in applications frequently have approximately low-rank with respect to some decomposition. One decomposition of great interest is the canonical polyadic decomposition (CPD) which expresses a tensor as a sum of rank-1 components. This important decomposition has the distinctive property that, with mild assumptions, the CPD of a low-rank tensor is essentially unique [14, 29, 46]. The essential uniqueness of CPD makes it an indispensable tool in applications where one wishes to extract component information from a signal or data set with little prior knowledge [27, 11].

As real-world tensors only have approximately low rank, to extract component information from a signal or data set of interest, one must compute a *low-rank approximation* of the measured tensor. Though a natural problem, the best low-rank approximation problem for CPD is, strictly speaking, not well-posed. Indeed, a best rank R approximation of a given tensor may not exist if $R > 1$ [30, 12]. In the case that a best rank R approximation does not exist, it is known that near optimal rank R approximations must exhibit "diverging components" [28, 49].

In this article, we study the best low rank approximation problem for collections of tensors which are positive (semi)definite. In particular, we give deterministic conditions which guarantee the existence of a best low rank approximation of a measured tensor $\mathcal{T} + \mathcal{E}$. Here $\mathcal{T}$ is a low-rank positive (semi)definite tensor of interest and $\mathcal{E}$ is noise. Roughly speaking, our main result, Theorem 2.2, states that $\mathcal{T} + \mathcal{E}$ has a best low-rank approximation provided that the magnitude of $\mathcal{E}$ is bounded above by half of the smallest nonzero eigenvalue (in the sense of [32, 37]) of $\mathcal{T}$. This result is obtained as a consequence of Theorem 2.1 which shows that the set of low rank positive definite tensors is closed as a subset of the set of positive definite tensors. In Section 5, we illustrate our results in numerical experiments. Here we observe that the existence bound of Theorem 2.2 is highly predictive of errors in the recovery component information from a measured tensor.

1.1. Connection to the literature on existence of best low rank approximation. While the best low-rank approximation is ill-posed in the general setting, there are several known existence results in the context of constrained tensor decompositions. For example, it has been shown [34, Theorem 25] that best low rank approximations exist subject to coherence constraints on the factors of the decomposition. In particular, this result implies that if one restricts to considering factor matrices which have orthogonal columns, then best low rank approximations exist, see also [24, 47]. In addition, it has been shown that best approximations exist if one requires the factor matrices to have nonnegative entries [33, 42]. Intuitively, factor matrices which exhibit diverging components do not have bounded coherence and cannot be nonnegative. As such, these constraints prevent diverging components and thereby guarantee existence of best constrained approximations.

Taking a different approach, [17] considers existence of unconstrained best low rank approximations for tensors of order three. Here the CPD is connected to a "joint generalized eigenvalue decomposition" and perturbation theoretic bounds for this decomposition are used to provide deterministic guarantees for the existence of best low rank approximations. The present article is a middle ground between these two approaches. Instead of considering constraints on the decomposition, we place constraints on the tensor $\mathcal{T}$ and leave the decomposition of the best low rank approximation of $\mathcal{T} + \mathcal{E}$ unconstrained.

Furthermore, this article builds on the novel "engineering" perspective on well-posedness of best low rank approximation which was introduced by [17]: Best low

rank approximation is well-posed when applied to a measured tensor $\mathcal{T}+\mathcal{E}$ so long as the signal-to-noise ratio is sufficiently high and the underlying signal $\mathcal{T}$ is sufficiently well-posed. Certainly if the measured tensor $\mathcal{T}+\mathcal{E}$ arises in a context where the underlying signal is positive definite, then a best low rank approximation which is used to model $\mathcal{T}$ should also be positive definite. That is, to obtain meaningful results in this context, the signal-to-noise ratio and conditioning of $\mathcal{T}$ should be good enough so that near optimal low rank approximations to $\mathcal{T}+\mathcal{E}$ are positive definite. It is this setting in which our results guarantee the existence of a best low rank approximation.

In addition to the above results, it has recently been shown that a unique best rank R approximation exists with probability one when working over $\mathbb{C}$ [43]. In contrast, when considering CPD over $\mathbb{R}$, non-existence of a best low rank approximation occurs with positive probability [48, 51]. We also mention a connection to [31, Theorem 1] where it is shown that, for even orders, the set of tensors which can be "completely decomposed" into a sum of R or less terms forms a closed set. This result is related to our upcoming Theorem 2.1. However, we do not assume that the tensor at hand is completely decomposable and we consider both even and odd orders.

1.2. Organization. The remainder of the article is organized as follows. Section 2 introduces our notation and presents our main results. Section 3 contains a collection of lemmas which are used in the proofs of our main results. Section 4 gives the proofs of our main results. Finally, Section 5 presents numerical experiments.

2. Notation and main results. We now introduce our notation and definitions and give formal statements of our main results. We denote scalars, vectors, matrices, and tensors by lower case (m), bold lower case ($\mathbf{m}$), bold upper case ($\mathbf{M}$), and calligraphic script ($\mathcal{M}$), respectively. Given a matrix $\mathbf{M}\in\mathbb{R}^{I_1\times I_2}$, let $\mathbf{M}^{\mathrm{T}}$ denote the transpose of $\mathbf{M}$, and if $\mathbf{M}$ is invertible, let $\mathbf{M}^{-1}$ denote the inverse of $\mathbf{M}$. In addition we let the collection $\{\mathbf{e}_j\}_{j=1}^{J}$ denote the standard basis in $\mathbb{R}^J$ and we let $\mathbf{I}_J$ denote the $J\times J$ identity matrix.

In this article, a **tensor** $\mathcal{T}\in\mathbb{R}^{I_1\times I_2\times\cdots\times I_L}$ is a multiindexed array of size $I_1\times I_2\times\cdots\times I_L$ with entries in $\mathbb{R}$. The integer L is called the **order** of $\mathcal{T}$. We let $\langle\cdot,\cdot\rangle$ denote the usual inner product on $\mathbb{R}^{I_1\times I_2\times\cdots\times I_L}$. The Frobenius norm of $\mathcal{T}$, denoted $\|\mathcal{T}\|_{\mathrm{F}}$, is defined by $\|\mathcal{T}\|_{\mathrm{F}}=\sqrt{\langle\mathcal{T},\mathcal{T}\rangle}$.

A **mode-ℓ** fiber of a tensor is a vector obtained by fixing all indices but the ℓth. The **mode-ℓ** unfolding of $\mathcal{T}$, denoted $\mathbf{T}_{[\ell]}\in\mathbb{R}^{I_\ell\times\Pi_{j\neq\ell}\ I_j}$, is an $I_\ell\times\Pi_{j\neq\ell}\ I_j$ matrix whose columns are equal to the mode-ℓ fibers of $\mathcal{T}$. The range of the unfolding $\mathbf{T}_{[1]}$ often plays an important role in our results. We call this subspace the **fiber space**[1] of $\mathcal{T}$ and denote it by $\mathrm{fib}(\mathcal{T})$.

We restrict our attention to tensors of order $L=N+1$ where N is an even integer and $I_1=I_2=\cdots=I_N\triangleq I$. We write this space of tensors as $(\mathbb{R}^I)^{\otimes N}\otimes\mathbb{R}^J$. We emphasize that we allow $J=1$ in which case one can naturally identify $(\mathbb{R}^I)^{\otimes N}\otimes\mathbb{R}^1=(\mathbb{R}^I)^{\otimes N}$. Thus our main results apply both to tensors of order N and order $N+1$.

We frequently view an element of $(\mathbb{R}^I)^{\otimes N}\otimes\mathbb{R}^J$ as a collection of J elements of $(\mathbb{R}^I)^{\otimes N}$. Given a tensor $\mathcal{T}\in(\mathbb{R}^I)^{\otimes N}\otimes\mathbb{R}^J$, we call a tensor which is obtained by fixing the $(N+1)$th index of $\mathcal{T}$ equal to j and letting all other indices run free a **(order N) frontal slice** of $\mathcal{T}$. We denote the jth frontal slice of $\mathcal{T}$ by $\mathcal{T}_{(j)}$.

[1] As will soon be discussed, we typically consider tensors which are (partially) symmetric. Thus it makes no difference to consider the range of the mode-1 unfolding rather than the range of some other (appropriate) unfolding. This justifies calling this subspace the fiber space of $\mathcal{T}$.

2.1. Symmetric and positive definite tensors. Say a tensor in $(\mathbb{R}^I)^{\otimes N}$ is **symmetric** if it is invariant under any permutation of indices. We say an element of $(\mathbb{R}^I)^{\otimes N} \otimes \mathbb{R}^J$ has **symmetric frontal slices (SFS)** if all of its frontal slices are symmetric. We let $S^N\mathbb{R}^I$ denote the subspace of symmetric tensors in $(\mathbb{R}^I)^{\otimes N}$ and we let $(S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ denote the subspace of tensors with SFS in $(\mathbb{R}^I)^{\otimes N} \otimes \mathbb{R}^J$. Also see [4, 15, 16] for discussion of SFS tensors.

Given a tensor $\mathcal{T} \otimes (\mathbb{R}^I)^{\otimes N}$, we define the quantity $\lambda_{\min}(\mathcal{T})$ by

$$\lambda_{\min}(\mathcal{T}) := \min_{\|\mathbf{u}\|_2=1} \langle \mathcal{T}, \mathbf{u}^{\otimes N} \rangle.$$

Here $\mathbf{u}^{\otimes N}$ denotes the symmetric tensor $\mathbf{u} \otimes \mathbf{u} \otimes \cdots \otimes \mathbf{u} \in S^N\mathbb{R}^I$. Additionally, given a subspace $\mathfrak{S} \subseteq \mathbb{R}^I$, we define $\lambda_{\min}(\mathcal{T}|_{\mathfrak{S}})$ by

$$\lambda_{\min}(\mathcal{T}|_{\mathfrak{S}}) := \min_{\substack{\mathbf{u}\in\mathfrak{S} \\ \|\mathbf{u}\|_2=1}} \langle \mathcal{T}, \mathbf{u}^{\otimes N} \rangle.$$

Say a tensor $\mathcal{T} \in (\mathbb{R}^I)^{\otimes N}$ is **positive (semi)definite (PD and PSD, resp.)**, denoted $\mathcal{T} \succ 0$ (resp. $\mathcal{T} \succeq 0$), if $\mathcal{T}$ is symmetric and $\lambda_{\min}(\mathcal{T}) > 0$ (resp. $\lambda_{\min}(\mathcal{T}) \geq 0$). In addition, for a given subspace $\mathfrak{S} \subseteq \mathbb{R}^I$, we say the **restriction of $\mathcal{T}$ to $\mathfrak{S}$ is positive (semi)definite**, denoted $\mathcal{T}|_{\mathfrak{S}} \succ 0$ (resp. $\mathcal{T}|_{\mathfrak{S}} \succeq 0$), if $\mathcal{T}$ is symmetric and $\lambda_{\min}(\mathcal{T}|_{\mathfrak{S}}) > 0$ (resp. $\lambda_{\min}(\mathcal{T}|_{\mathfrak{S}}) \geq 0$). For clarity we emphasize that we do not require $\mathcal{T}$ to be symmetric for $\lambda_{\min}(\mathcal{T})$ or $\lambda_{\min}(\mathcal{T}|_{\mathfrak{S}})$ to be defined, though we do require $\mathcal{T}$ to be symmetric for $\mathcal{T}$ to be PD. A straightforward computation shows that N must be even if $\mathcal{T}$ is PD.

The quantity $\lambda_{\min}(\mathcal{T})$ is equal to the smallest eigenvalue of the symmetrization[2] of $\mathcal{T}$, see e.g., [32, 37]. Connections between PD tensors and tensor eigenvalues of this type have been studied by a number of authors, e.g., see [53, 40].

2.2. Tensor rank and the canonical polyadic decomposition. Given vectors $\mathbf{v}^\ell \in \mathbb{R}^{I_\ell}$ for $\ell = 1, \ldots, L$, define the **outer product** of these vectors, denoted $\mathbf{v}^1 \otimes \mathbf{v}^2 \otimes \cdots \otimes \mathbf{v}^L$, to be the $I_1 \times I_2 \times \cdots \times I_L$ tensor with $i_1, i_2, \ldots, i_L$ entry equal to

$$\mathbf{v}^1(i_1)\mathbf{v}^2(i_2)\ldots\mathbf{v}^L(i_L).$$

Here $\mathbf{v}^\ell(i_\ell)$ denotes the i_ℓth entry of $\mathbf{v}^\ell$. A nonzero tensor which can be expressed as the outer product of a collection of vectors is called a **rank one tensor**.

Given any tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_L}$, there exists some collection of vector tuples $\{(\mathbf{v}_r^1, \ldots, \mathbf{v}_r^L)\}_{r=1}^R \subset \mathbb{R}^{I_1} \times \cdots \times \mathbb{R}^{I_L}$ such that

$$\mathcal{T} = \sum_{r=1}^R \mathbf{v}_r^1 \otimes \cdots \otimes \mathbf{v}_r^L. \tag{2.1}$$

If R is as small as possible, then $\mathcal{T}$ is said to have **rank** R and the decomposition in equation (2.1) is called a **canonical polyadic decomposition** (CPD) of $\mathcal{T}$. In the case that $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$, one may alternatively decompose $\mathcal{T}$ as

$$\mathcal{T} = \sum_{r=1}^R (\mathbf{u}_r)^{\otimes N} \otimes \mathbf{v}_r \tag{2.2}$$

[2]The symmetrization of a tensor $\mathcal{T} \in (\mathbb{R}^I)^{\otimes N}$ is the unique symmetric tensor $\mathrm{sym}(\mathcal{T}) \in (\mathbb{R}^I)^{\otimes N}$ which satisfies $\langle \mathcal{T}, \mathbf{u}^{\otimes N} \rangle = \langle \mathrm{sym}(\mathcal{T}), \mathbf{u}^{\otimes N} \rangle$ for all $\mathbf{u} \in \mathbb{R}^I$.

where each $\mathbf{u}_r \in \mathbb{R}^I$ and $\mathbf{v}_r \in \mathbb{R}^J$. If R is as small as possible in this decomposition, then we say $\mathcal{T}$ has **symmetric frontal slice (SFS) rank** R. When $J = 1$, our notion of SFS rank coincides with the classical notion of symmetric rank, e.g., see [7].

In stark contrast to the matrix case, the set of tensors of rank less than or equal to R is, in general, not a closed set [30, 12]. This is closely related to nonexistence of best low rank approximations. For example, there exists a rank 3 tensor which is a limit of rank 2 tensors, e.g, see [12, 30]. Such a tensor cannot have a best rank 2 approximation. Say a $\mathcal{T}$ tensor has **border rank** R if $\mathcal{T}$ is a limit of rank R tensors and there is no integer $M < R$ such that $\mathcal{T}$ is a limit of rank M tensors. Similarly, for $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$, we say $\mathcal{T}$ has **border SFS rank** R if $\mathcal{T}$ is a limit of SFS rank R tensors and there is no integer $M < R$ such that $\mathcal{T}$ is a limit of SFS rank M tensors.

We warn the reader rank is not necessarily equal to SFS rank for general tensors [44]. However, these notions coincide for the low SFS rank tensors considered in the article, see Lemma 3.3. Still, there is an important yet subtle issue which must be taken into account: It is not known in general if low border SFS rank tensors have border rank equal to border SFS rank. Properly accounting for this subtlety motivates our decision to consider both these notions of rank.

Given a decomposition of a tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_L}$ as in equation (2.1), let $\mathbf{U}^{(\ell)} \in \mathbb{R}^{I_\ell \times R}$ be the matrix whose rth column is the vector $\mathbf{u}_r^\ell$. Write

$$\mathcal{T} = \left[\!\!\left[\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \ldots, \mathbf{U}^{(L)} \right]\!\!\right]$$

as a compact notation for the CPD of $\mathcal{T}$. The matrices $\mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(L)}$ are called the **factor matrices** of $\mathcal{T}$. Say a CPD $\mathcal{T} = \left[\!\!\left[\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \ldots, \mathbf{U}^{(L)} \right]\!\!\right]$ is **unique** if for any other CPD $\mathcal{T} = [\![\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(L)}]\!]$, there exists a permutation matrix Π and invertible diagonal matrices $\mathbf{D}_1, \ldots \mathbf{D}_L$ such that

$$\mathbf{U}^{(\ell)}\mathbf{D}_\ell\Pi = \mathbf{V}^{(\ell)} \qquad \text{for all } \ell = 1, \ldots, L \qquad \text{and} \qquad \Pi_{\ell=1}^{L}\mathbf{D}_\ell = \mathbf{I}.$$

Finally, for a tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_L}$, define the **spectral norm** $\|\mathcal{T}\|_{\text{sp}}$ of $\mathcal{T}$ by

$$\|\mathcal{T}\|_{\text{sp}} = \max_{\substack{\mathbf{v}^\ell \in \mathbb{R}^{I_\ell} \\ \|\mathbf{v}^\ell\| = 1}} \langle \mathcal{T}, \mathbf{v}^1 \otimes \mathbf{v}^2 \otimes \cdots \otimes \mathbf{v}^L \rangle.$$

The spectral norm of $\mathcal{T}$ is equal to the Frobenius norm of a best rank-1 approximation of $\mathcal{T}$ and to the largest singular value of $\mathcal{T}$. See [19, 20, 21, 39] for further discussion.

2.3. Main results. We now state our main results. Our first main result, Theorem 2.1, shows that the set of low rank tensors that have a linear combination of frontal slices that is PD is relatively closed as a subset of the set of tensors that have a linear combination of frontal slices that is PD.

THEOREM 2.1. *Let $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ be a tensor with multilinear rank $(R, \ldots, R, K)$ and with border SFS rank at most R. Assume there is vector $\mathbf{w} \in \mathbb{R}^J$ such that $(\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}})|_{\text{fib}(\mathcal{T})} \succ 0$. Then $\mathcal{T}$ has rank and SFS rank equal to R with CPD $\mathcal{T} = [\![\mathbf{U}, \ldots, \mathbf{U}, \mathbf{V}]\!]$ where $\mathbf{U} \in \mathbb{R}^{I \times R}$ with rank R and $\mathbf{V} \in \mathbb{R}^{J \times R}$ and where all entries of $\mathbf{w}^{\mathrm{T}}\mathbf{V}$ are positive. Furthermore, if either $N \geq 4$ or $\mathbf{V}$ does not have a pair of columns which is collinear, then the CPD of $\mathcal{T}$ is unique.*

Proof. See Section 4.1. ∎

Our second main result, Theorem 2.2, uses Theorem 2.1 to produce a deterministic bound which guarantees the existence of a best low rank approximation of a measured tensor which has a PD linear combination of frontal slices.

THEOREM 2.2. *Let $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ and assume $\mathcal{T}$ has SFS rank R. Also let $\mathcal{E} \in (\mathbb{R}^I)^{\otimes N} \otimes \mathbb{R}^J$ and let $\|\cdot\|$ be a norm on $(\mathbb{R}^I)^{\otimes N} \otimes \mathbb{R}^J$ which satisfies*[3]

$$\|\mathcal{E}\|_{\mathrm{sp}} \leq \|\mathcal{E}\| \qquad \text{for all } \mathcal{E} \in (\mathbb{R}^I)^{\otimes N} \otimes \mathbb{R}^J. \tag{2.3}$$

If there exists an R dimensional subspace $\mathfrak{S} \subseteq \mathbb{R}^I$ such that

$$\|\mathcal{E}\| < \max_{\|\mathbf{w}\|=1} \lambda_{\min}\Big(\big((\mathcal{T}+\mathcal{E}) \cdot_{N+1} \mathbf{w}^{\mathrm{T}}\big)|_{\mathfrak{S}}\Big), \tag{2.4}$$

then $\mathcal{T}+\mathcal{E}$ has a best SFS rank R approximation in the $\|\cdot\|$ norm. Furthermore, if $\hat{\mathcal{T}} = [\![\hat{\mathbf{U}}, \ldots, \hat{\mathbf{U}}, \hat{\mathbf{V}}]\!]$ is a best SFS rank R approximation of $\mathcal{T}+\mathcal{E}$ and either $N \geq 4$ or $\hat{\mathbf{V}}$ does not have a pair of columns which is collinear, then the CPD of $\hat{\mathcal{T}}$ is unique.

As a consequence, if one sets

$$\lambda_* := \max_{\|\mathbf{w}\|=1} \lambda_{\min}\Big(\big(\mathcal{T} \cdot \mathbf{w}^{\mathrm{T}}\big)|_{\mathrm{fib}(\mathcal{T})}\Big),$$

then every tensor in an open $\|\cdot\|$ norm ball of radius $\lambda_*/2$ centered at $\mathcal{T}$ has a best SFS rank R approximation in the $\|\cdot\|$ norm. In other words, if $\mathcal{S}$ is a tensor satisfying $\|\mathcal{S}-\mathcal{T}\| < \lambda_*/2$, then $\mathcal{S}$ has a best SFS rank R approximation.

Proof. See Section 4.2. ∎

In practice, Theorem 2.2 may be used to attempt to guarantee that a measured tensor $\mathcal{T}+\mathcal{E} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ has a best rank R approximation. To do this, one can simply compute any (not necessarily optimal) rank R approximation $\hat{\mathcal{T}}$ to $\mathcal{T}+\mathcal{E}$, then check if there is an R dimensional subspace $\mathfrak{S} \subseteq \mathbb{R}^I$ such that

$$\|\mathcal{T}+\mathcal{E}-\hat{\mathcal{T}}\| < \max_{\|\mathbf{w}\|=1} \lambda_{\min}\Big(\big((\mathcal{T}+\mathcal{E})\cdot_{N+1}\mathbf{w}^{\mathrm{T}}\big)|_{\mathfrak{S}}\Big) = \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \min_{\substack{\mathbf{s}\in\mathfrak{S} \\ \|\mathbf{s}\|=1}} (\mathcal{T}+\mathcal{E})\cdot_1\mathbf{s}^{\mathrm{T}}\cdot_2\ldots\cdot_N\mathbf{s}^{\mathrm{T}}\cdot_{N+1}\mathbf{w}^{\mathrm{T}}.$$

A natural choice of subspace $\mathfrak{S}$ to use is the R dimensional dominant subspace of $\mathrm{fib}(\mathcal{T})$, i.e., the span of the R principal left singular vectors of an unfolding $(\mathbf{T}+\mathbf{E})_{[n]}$ for some $n \leq N$. For general N, the quantity appearing on the right-hand side of the above inequality is NP-hard to compute [26]. However, various authors have studied methods related to computing this quantity, e.g., see [5, 38]. Furthermore, if $N = 2$, then, for a fixed subspace $\mathfrak{S}$, the quantity can be computed via semidefinite programming, see the upcoming Theorem 4.1.

A common approach for best low rank approximation in practice is to first compute a low multilinear rank approximation $\tilde{\mathcal{T}}$ to the measured tensor $\mathcal{T}+\mathcal{E}$. Thus, a natural use of Theorem 2.2 is to attempt to guarantee that $\tilde{\mathcal{T}}$ has a best rank R approximation. One advantage of this approach is that the distance from $\mathcal{T}+\mathcal{E}$ to a rank R approximation of $\mathcal{T}+\mathcal{E}$ will typically be much larger than the distance from $\tilde{\mathcal{T}}$ to a rank R approximation of $\tilde{\mathcal{T}}$. Indeed, signal enhancement/noise suppression is a standard use of low multilinear rank approximations for both tensors and matrices.

Finally we briefly discuss some cases in which Theorem 2.2 can be applied. Of course, if one knows that the tensor $\mathcal{T}$ has a linear combination of frontal slices which is PD, then the quantity λ_* appearing in Theorem 2.2 is strictly positive. Still, there exist tensors $\mathcal{T}$ with symmetric frontal slices for which λ_* is not positive in which case no conclusion can be made. On the other hand, if $J \geq R$ and $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ is a generic rank R tensor, i.e., $\mathcal{T}$ has generic factor matrices, then a straightforward computation shows that λ_* is strictly positive.

[3] For example, the Frobenius norm and spectral norm both satisfy inequality (2.3).

2.4. Main result for $N = 2$. We now present two results which address the computability and sharpness of our bound in the special case $N = 2$. This case is of particular interest as it is directly related to an important class of independent component analysis problems, e.g., see [2, 36, 54]. We first show that the quantity appearing on the right-hand side of inequality (2.4) is computable when $N = 2$.

THEOREM 2.3. *Let $\mathcal{T} + \mathcal{E} \in (S^2\mathbb{R}^I) \otimes \mathbb{R}^J$ and let $\mathfrak{S} \subseteq \mathbb{R}^I$ be an R dimensional subspace. Set*

$$\lambda_* = \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}\Big(\big((\mathcal{T}+\mathcal{E}) \cdot_3 \mathbf{w}^{\mathrm{T}}\big)|_{\mathfrak{S}}\Big),$$

and assume $\lambda_* \geq 0$. Then λ_* is computable via semidefinite programming.

Proof. The semidefinite program which may be used to compute λ_* is given as Theorem 4.1 in Section 4.3. ∎

We mention that while λ_* is computable for a fixed subspace $\mathfrak{S}$, the largest bound provided by Theorem 2.2 is in fact

$$\max_{\substack{\mathfrak{S}\subseteq\mathbb{R}^I \\ \dim(\mathfrak{S})=R}} \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}\Big(\big((\mathcal{T}+\mathcal{E}) \cdot_{N+1} \mathbf{w}^{\mathrm{T}}\big)|_{\mathfrak{S}}\Big).$$

Even if $N = 2$, unless $R = I$, this optimization is non-convex. One may of course attempt to directly optimize this quantity, e.g., using an alternating scheme. However, as previously mentioned, a natural choice of subspace $\mathfrak{S}$ to use is the span of the R principal left singular vectors of the unfolding $(\mathbf{T} + \mathbf{E})_{[1]}$.

Finally, Theorem 2.4 shows that when $N = 2$, Theorem 2.2 is sharp.

THEOREM 2.4. *Let $\mathcal{T} \in (S^2\mathbb{R}^I) \otimes \mathbb{R}^J$ be any tensor and set*

$$\lambda_* = \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{v}\|=1}} \lambda_{\min}\Big((\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T})}\Big) \tag{2.5}$$

Then there exists a tensor $\mathcal{E}_*$ with $\|\mathcal{E}_*\|_{\mathrm{sp}} = \max\{\lambda_*, 0\}$ such that the tensor $\mathcal{T} + \mathcal{E}_*$ does not have a linear combination of frontal slices which is PD on $\mathrm{fib}(\mathcal{T})$.

Furthermore, if $J = 2$ and $\mathcal{T}$ has rank R and $\lambda_* \geq 0$, then $\mathcal{E}_*$ can be chosen so that $\|\mathcal{E}_*\|_{\mathrm{sp}} = \lambda_*$ and so that any open neighborhood centered at $\mathcal{T} + \mathcal{E}_*$ contains a tensor with border rank R and rank strictly greater than R. That is, any open neighborhood containing $\mathcal{T} + \mathcal{E}_*$ contains a tensor that does not have a best rank R approximation.

Proof. See Section 4.4. ∎

We also note that, since there are standard notions of Hermitian and PD matrices in $\mathbb{C}^{I\times I}$, our results in the $N = 2$ case can easily be extended to the complex setting. These extensions and their proofs are straightforward, so we have restricted our discussion to the real setting so as to simplify exposition. Statements of our results over the complexes when $N = 2$ are given in the supplementary materials.

3. Basic results on low rank PD tensors. In this section we collect several lemmas which will be useful in the proofs of our main results. Readers who are not interested in the technical details of the proofs may proceed immediately to Section 5 for numerical experiments which illustrate our results.

Our first lemma, Lemma 3.1, establishes the general form for the CPD of a low rank tensor which is PD on the range of one of its unfoldings. We show that such a tensor is a (positive coefficient) sum of PSD rank-1 tensors.

LEMMA 3.1. *Let $\mathcal{T} \in S^N\mathbb{R}^I$ be a symmetric tensor where N is even and assume that $\mathcal{T}$ has multilinear rank $(R, \dots, R)$ and that $\mathcal{T}|_{\mathrm{fib}(\mathcal{T})} \succ 0$. If $\mathcal{T}$ has rank at most R, then both the rank and the symmetric rank of $\mathcal{T}$ are equal to R and there exists a collection of linearly independent vectors $\{\mathbf{u}_r\}_{r=1}^R \subset \mathbb{R}^I$ such that*

$$\mathcal{T} = \sum_{r=1}^{R} (\mathbf{u}_r)^{\otimes N}. \tag{3.1}$$

Furthermore, the decomposition of $\mathcal{T}$ in equation (3.1) is unique if $N \geq 4$.

Lemma 3.1 serves as a clear generalization of the matrix case to the tensor setting. Indeed, a rank R positive semidefinite matrix $\mathbf{M} \in \mathbb{R}^{I \times I}$ can be written as

$$\mathbf{M} = \sum_{r=1}^{R} (\mathbf{u}_r)^{\otimes 2}$$

where the set $\{\mathbf{u}_r\}$ is linearly independent. Our lemma shows that this naturally extends to PSD tensors with rank R and multilinear rank $(R, \dots, R)$, and that issues regarding rank versus symmetric rank, e.g., see [44], do not occur for these tensors.

Proof. By assumption $\mathcal{T}$ has multilinear rank $(R, \dots, R)$ and rank at most R so $\mathcal{T}$ has rank R. Using [18, Theorem 1.1] then shows that $\mathcal{T}$ has symmetric rank R.

Now since $\mathcal{T}$ has symmetric rank R, there exist vectors $\{\mathbf{u}_r\}_{r=1}^R \subset \mathbb{R}^I$ such that

$$\mathcal{T} = \sum_{r=1}^{R} \delta_r (\mathbf{u}_r)^{\otimes N}$$

where $\delta_r = \pm 1$ for each r. The linear independence of the collection $\{\mathbf{u}_r\}_{r=1}^R$ is immediate from the assumption that $\mathcal{T}$ has multilinear rank $(R, \dots, R)$.

It remains to prove that $\delta_r = 1$ for each r. Our assumptions imply that $\mathrm{fib}(\mathcal{T})$ has dimension R. Therefore, for each fixed index r_0 there is a unit vector $\mathbf{w}_{r_0} \in \mathrm{fib}(\mathcal{T})$ such that $\langle \mathbf{u}_{r_0}, \mathbf{w}_{r_0} \rangle \neq 0$ but $\langle \mathbf{u}_r, \mathbf{w}_{r_0} \rangle = 0$ for each $r \neq r_0$. Using the assumption $\mathcal{T}|_{\mathrm{fib}(\mathcal{T})} \succ 0$ then implies

$$\langle \mathcal{T}, (\mathbf{w}_{r_0})^{\otimes N} \rangle = \sum_{r=1}^{R} \delta_r \langle \mathbf{u}_r, \mathbf{w}_{r_0} \rangle^N = \delta_{r_0} \langle \mathbf{u}_{r_0}, \mathbf{w}_{r_0} \rangle^N > 0.$$

Since N is even and $\langle \mathbf{u}_{r_0}, \mathbf{w}_{r_0} \rangle \neq 0$ we have $\langle \mathbf{u}_{r_0}, \mathbf{w}_{r_0} \rangle^N > 0$. It follows that $\delta_{r_0} = 1$.

The fact that the CPD of $\mathcal{T}$ is unique when $N \geq 4$ is a straightforward consequence of a well know result of Harshman [23]. ∎

We next show that, if a tensor is PD on some subspace of $\mathbb{R}^I$, then the tensor is also positive on the range of one of its unfoldings.

LEMMA 3.2. *Let $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$. Assume $\mathcal{T}$ has border SFS rank R and that there is an R dimensional subspace $\mathfrak{S} \subseteq \mathbb{R}^I$ and a vector $\mathbf{w}_* \in \mathbb{R}^J$ such that*

$$(\mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}})|_{\mathfrak{S}} \succ 0.$$

Then $\mathcal{T}$ has multilinear rank $(R, \dots, R, K)$ for some $K \leq R$ and

$$\mathrm{fib}(\mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}) = \mathrm{fib}(\mathcal{T}) \qquad \text{and} \qquad (\mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T})} \succ 0.$$

Proof. Set $\mathfrak{T} = \mathrm{fib}(\mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}})$. A straightforward computation shows that $\mathfrak{T} \subseteq \mathrm{fib}(\mathcal{T})$. Now, let $\{\mathbf{s}_r\}_{r=1}^R$ be a basis for $\mathfrak{S}$ and for each r write $\mathbf{s}_r = \mathbf{u}_r \oplus \mathbf{v}_r$ where $\mathbf{u}_r \in \mathfrak{T}$ and $\mathbf{v}_r \in \mathfrak{T}^\perp$. We first show that the collection $\{\mathbf{u}_r\}_{r=1}^R$ is linearly independent. Assume towards a contradiction that $\{\mathbf{u}_r\}_{r=1}^R$ is not linearly independent, then there exists some nonzero vector $\boldsymbol{\alpha} \in \mathbb{R}^R$ such that $\sum_{r=1}^R \alpha_r \mathbf{u}_r = \mathbf{0}$. However, this implies

$$\sum_{r=1}^R \alpha_r \mathbf{s}_r = \left(\sum_{r=1}^R \alpha_r \mathbf{u}_r\right) \oplus \left(\sum_{r=1}^R \alpha_r \mathbf{v}_r\right) = \sum_{r=1}^R \alpha_r \mathbf{v}_r \in \mathfrak{T}^\perp,$$

and hence

$$\left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \left(\sum_{r=1}^R \alpha_r \mathbf{s}_r\right)^{\otimes N} \right\rangle = \left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \left(\sum_{r=1}^R \alpha_r \mathbf{v}_r\right)^{\otimes N} \right\rangle = 0.$$

Further, $\sum_{r=1}^R \alpha_r \mathbf{s}_r \neq \mathbf{0}$ since $\{\mathbf{s}_r\}_{r=1}^R$ is a basis for $\mathfrak{S}$, so this is a contradiction. Thus $\{\mathbf{u}_r\}_{r=1}^R$ is linearly independent and $\mathfrak{T}$ has dimension at least R.

Since the border SFS rank of $\mathcal{T}$ upper bounds the dimension of $\mathrm{fib}(\mathcal{T})$ and since $\mathfrak{T} \subseteq \mathrm{fib}(\mathcal{T})$, the subspace $\mathfrak{T}$ must have dimension R. It follows that $\{\mathbf{u}_r\}_{r=1}^R$ forms a basis for this space. Furthermore, since $\mathfrak{T}$ has dimension R and $\mathrm{fib}(\mathcal{T})$ has dimension at most R, the containment $\mathfrak{T} \subseteq \mathrm{fib}(\mathcal{T})$ implies that $\mathfrak{T} = \mathrm{fib}(\mathcal{T})$. In addition, since $\mathcal{T}$ has SFS, it follows that $\mathcal{T}$ has multilinear rank $(R, \ldots, R, K)$.

Now, given any nonzero vector $\mathbf{u} \in \mathfrak{T}$, there exists a nonzero $\boldsymbol{\gamma} \in \mathbb{R}^R$ such that $\sum_{r=1}^R \gamma_r \mathbf{u}_r = \mathbf{u}$. We claim that

$$\left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{u}^{\otimes N} \right\rangle = \left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{s}^{\otimes N} \right\rangle \tag{3.2}$$

where $\mathbf{s} = \sum_{r=1}^R \gamma_r \mathbf{s}_r$. To see this, set $\mathbf{v} = \sum_{r=1}^R \gamma_r \mathbf{v}_r$ and consider the expansion

$$\left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{s}^{\otimes N} \right\rangle = \left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, (\mathbf{u} \oplus \mathbf{v})^{\otimes N} \right\rangle = \sum_{n=0}^N \binom{N}{n} \left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{u}^{\otimes n} \otimes \mathbf{v}^{\otimes (N-n)} \right\rangle.$$

Note that $\mathbf{v} \in \mathfrak{T}^\perp$ together with the symmetry of $\mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}$ implies

$$\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{u}^{\otimes n} \otimes \mathbf{v}^{\otimes (N-n)} \rangle = 0$$

whenever $n < N$, from which equation (3.2) follows. Finally, note that $\mathbf{s} = \mathbf{u} \oplus \mathbf{v}$ is nonzero since $\mathbf{u}$ is nonzero. We have already shown that $\mathcal{T}|_{\mathfrak{S}} \succ 0$, so equation (3.2) together with the fact that $\mathbf{s}$ is nonzero implies $\left\langle \mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{u}^{\otimes N} \right\rangle > 0$ and moreover that $(\mathcal{T} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T})} \succ 0$. ∎

We now give the general form for the CPD of a low rank tensor that has a linear combination of frontal slices that is PD.

LEMMA 3.3. *Let $\mathcal{T} \in (S^N \mathbb{R}^I) \otimes \mathbb{R}^J$ with multilinear rank $(R, \ldots, R, K)$ and assume there is a vector $\mathbf{w} \in \mathbb{R}^J$ such that $(\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T})} \succ 0$. Then $\mathcal{T}$ has rank R if and only if $\mathcal{T}$ has SFS rank R. In addition, if $\mathcal{T}$ has rank R, then $\mathcal{T}$ decomposes as $\mathcal{T} = [\![\mathbf{U}, \ldots, \mathbf{U}, \mathbf{V}]\!]$ where $\mathbf{U} \in \mathbb{R}^{I \times R}$ with rank R and $\mathbf{V} \in \mathbb{R}^{J \times R}$ and where all entries of $\mathbf{w}^{\mathrm{T}}\mathbf{V}$ are positive. Furthermore, if either $N \geq 4$ or $\mathbf{V}$ does not have a pair of columns which is collinear, then the CPD of $\mathcal{T}$ is unique.*

Proof. First note that if $\mathcal{T}$ has SFS rank R, then $\mathcal{T}$ has rank less than or equal to R. However, by assumption $\mathcal{T}$ has multilinear rank $(R, \ldots, R, K)$ from which it follows that the rank of $\mathcal{T}$ is at least R. Thus $\mathcal{T}$ has rank R if $\mathcal{T}$ has SFS rank R.

Now assume that $\mathcal{T}$ has rank R. To show that $\mathcal{T}$ has SFS rank R, we will show that $\mathcal{T}$ decomposes as in the statement of the lemma. To this end write $\mathcal{T} = [\![\mathbf{U}_1, \dots, \mathbf{U}_N, \mathbf{U}_{N+1}]\!]$. By assumption $(\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T})} \succ 0$, which implies the entries of $\mathbf{w}^{\mathrm{T}}\mathbf{U}_{N+1}$ must be nonzero as otherwise $\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}}$ would have rank strictly less than R and this would imply that there is a vector $\mathbf{x} \in \mathrm{fib}(\mathcal{T})$ such that $\langle \mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}}, \mathbf{x}^{\otimes N} \rangle = 0$. Therefore, we may WLOG scale the factor matrices $\mathbf{U}_1, \dots, \mathbf{U}_{N+1}$ such that the entries of the vector $\mathbf{w}^{\mathrm{T}}\mathbf{U}_{N+1}$ are all equal to 1. From this we have $\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}} = [\![\mathbf{U}_1, \dots, \mathbf{U}_N]\!]$. We now consider two cases.

First suppose $N \geq 4$. In this case $\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}}$ is a symmetric tensor that has rank at most R and satisfies $(\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T})} \succ 0$. It follows from Lemma 3.2 that $(\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}})|_{\mathrm{fib}(\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}})} \succ 0$, so we may use Lemma 3.1 to conclude that $\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}}$ has symmetric rank R and uniquely decomposes as

$$\mathcal{T} \cdot_{N+1} \mathbf{w}^{\mathrm{T}} = [\![\mathbf{U}, \dots, \mathbf{U}]\!]$$

for some rank R matrix $\mathbf{U} \in \mathbb{R}^{I \times R}$. We conclude that up to scaling and a common permutation of columns one has $\mathbf{U}_i = \mathbf{U}$ for each i and that $\mathcal{T}$ has the decomposition

$$\mathcal{T} = [\![\mathbf{U}, \dots, \mathbf{U}, \mathbf{U}_{N+1}]\!].$$

The fact that the CPD of $\mathcal{T}$ is unique when $\mathcal{T}$ when $N \geq 4$ follows quickly from the fact that $\mathbf{U}$ has rank R, e.g., see [23].

Now assume that $N = 2$ and hence $N + 1 = 3$. In this case, set $\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}$ is a positive semidefinite matrix and therefore has a unique positive semidefinite square root. Let $\mathbf{M}$ denote the Moore-Penrose pseudo inverse of the positive semidefinite square root of $\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}$ and set $\mathcal{A} = \mathcal{T} \cdot_1 \mathbf{M} \cdot_2 \mathbf{M}$. Then $\mathcal{A}$ has rank R and symmetric frontal slices, and moreover, $\mathcal{A} \cdot_3 \mathbf{w}^{\mathrm{T}}$ is a rank R symmetric matrix with all eigenvalues equal to 0 or 1. Since $\mathcal{T}$ has rank R and symmetric frontal slices, a routine argument then shows that for each j, the null space of $\mathbf{A}_j$ contains the null space of $\mathcal{A} \cdot_3 \mathbf{w}^{\mathrm{T}}$ for all $j = 1, \dots, J$ and, moreover, that the frontal slices of $\mathcal{A}$ satisfy $\mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0}$ for all $i, j = 1, \dots, J$. It follows that there is an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{Q}\mathbf{A}_j\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{M}\mathbf{T}_j(\mathbf{Q}\mathbf{M})^{\mathrm{T}} = \mathbf{\Lambda}_j \oplus \mathbf{0}_{I-R} \qquad \text{for all} \qquad j = 1, \dots, J,$$

where each $\mathbf{\Lambda}_j$ is a diagonal $R \times R$ matrix. It is then straightforward to show that $\mathcal{T}$ has SFS rank R decomposition $\mathcal{T} = [\![\mathbf{U}, \mathbf{U}, \mathbf{V}]\!]$ where $\mathbf{U} \in \mathbb{R}^{I \times R}$ has rank R. The uniqueness of the CPD if $\mathbf{V}$ does not have collinear columns again follows from [23]. ∎

4. Proofs of main results. We now prove the main results of the article.

4.1. Proof of Theorem 2.1. We begin by proving Theorem 2.1. The intuitive idea behind the proof is that the positivity of the rank-1 terms in a positive semidefinite slice mix prevents the occurrence of diverging components when a sequence of tensors $\{\mathcal{T}^\ell\} \subset (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ converges to a tensor $\mathcal{T} \in (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ which has a positive semidefinite slice mix. Indeed, if two of these positive semidefinite rank one terms diverged, then there must be some unit vector $\mathbf{v} \in \mathbb{R}^I$ such that $\sup\langle \mathcal{T}^\ell, \mathbf{v}^{\otimes N} \rangle = \infty$. However, this cannot be the case since $\lim\langle \mathcal{T}^\ell, \mathbf{v}^{\otimes N} \rangle = \langle \mathcal{T}, \mathbf{v}^{\otimes N} \rangle$.

Proof. First note that if $\mathcal{T}$ has border SFS rank strictly less than R, then $\mathcal{T}$ cannot have multilinear rank $(R, \dots, R, K)$. We conclude that $\mathcal{T}$ has border SFS rank R. Next, by an appropriate orthogonal transformation of $\mathbb{R}^J$ we can WLOG assume that $\mathbf{w} = \mathbf{e}_1$ hence $\mathcal{T} \cdot_{N+1} \mathbf{e}_1^{\mathrm{T}} = \mathcal{T}_{(1)}$ and $\mathcal{T}_{(1)}|_{\mathrm{fib}(\mathcal{T})} \succ 0$. Let $\{\mathcal{T}^\ell\}_{\ell=1}^\infty \subset (S^N\mathbb{R}^I) \otimes \mathbb{R}^J$ be a sequence of SFS rank R tensors such that $\lim \mathcal{T}^\ell = \mathcal{T}$. For each ℓ write

$$\mathcal{T}^\ell = [\![\mathbf{U}^\ell, \mathbf{U}^\ell, \dots, \mathbf{U}^\ell, \mathbf{V}^\ell]\!]$$

where $\mathbf{U}^\ell \in \mathbb{R}^{I\times R}$ and where $\mathbf{V}^\ell \in \mathbb{R}^{J\times R}$ is normalized so that each entry of its first row is equal to $1, 0$, or -1. Our immediate goal is to show that (by passing to a subsequence if necessary) the sequence $\{\mathbf{U}^\ell\}_{\ell=1}^\infty$ converges to a rank R matrix.

Note that $\mathcal{T}_{(1)}|_{\mathrm{fib}(\mathcal{T})} \succ 0$ implies $\mathcal{T}^\ell_{(1)}|_{\mathrm{fib}(\mathcal{T})} \succ 0$ for large enough ℓ. To simplify exposition we assume $\mathcal{T}^\ell_{(1)}|_{\mathrm{fib}(\mathcal{T})} \succ 0$ for all ℓ. It follows from Lemma 3.3 that $\mathcal{T}^\ell$ has rank R for all ℓ. Furthermore we have

$$\mathcal{T}^\ell_{(1)} = \sum_{r=1}^R v^\ell_{1j}(\mathbf{u}^\ell_r)^{\otimes N} \succ 0 \tag{4.1}$$

where the set $\{\mathbf{u}^\ell_r\}_{r=1}^R$ is linearly independent for each ℓ. It follows that each $\mathcal{T}^\ell_{(1)}$ has symmetric rank R for each R. Additionally, following the argument in the proof of Lemma 3.1 shows that $v^\ell_{1j} > 0$ for each ℓ and j. It follows from our normalization of the $\mathbf{V}^\ell$ that $v^\ell_{1j} = 1$ for each ℓ and j, and hence

$$\mathcal{T}^\ell_{(1)} = \sum_{r=1}^R (\mathbf{u}^\ell_r)^{\otimes N} \qquad \text{for all } \ell.$$

To show that there exists a subsequence of the $\mathbf{U}^\ell$ which converges to some limit, it is sufficient to show that the $\mathbf{U}^\ell$ have bounded norm, and to show that the $\mathbf{U}^\ell$ have bounded norm, it is sufficient to show that the sequence $\{\|\mathbf{u}^\ell_{r_0}\|\}_{\ell=1}^\infty$ is bounded for each $r_0 = 1, \ldots, R$. Since the $\mathcal{T}^\ell_{(1)}$ converge to $\mathcal{T}_{(1)}$, there must exist some finite constant $C < \infty$ such that $\sup_\ell \|\mathcal{T}^\ell_{(1)}\|_{\mathrm{sp}} \le C$. Thus for each $r_0 = 1, \ldots, R$ we have

$$C \ge \sup_\ell \|\mathcal{T}^\ell_{(1)}\|_{\mathrm{sp}} \ge \sup_\ell \left\langle \mathcal{T}^\ell_{(1)}, \left(\frac{\mathbf{u}^\ell_{r_0}}{\|\mathbf{u}^\ell_{r_0}\|}\right)^{\otimes N} \right\rangle = \sup_\ell \sum_{r=1}^R \left\langle \mathbf{u}^\ell_r, \left(\frac{\mathbf{u}^\ell_{r_0}}{\|\mathbf{u}^\ell_{r_0}\|}\right) \right\rangle^N \ge \sup_\ell \|\mathbf{u}^\ell_{r_0}\|^N$$

It follows that $\sup_\ell \|\mathbf{U}^\ell\| < \infty$ so by passing to a subsequence if necessary we can assume there is some matrix $\mathbf{U} \in \mathbb{R}^{R\times R}$ such that $\mathbf{U} = \lim \mathbf{U}^\ell$.

We now show that $\mathbf{U}$ has rank R. By assumption $\mathrm{fib}(\mathcal{T})$ has dimension R so if $\mathbf{U}$ has rank less than R, then there must exist a unit vector $\mathbf{y} \in \mathrm{fib}(\mathcal{T})$ such that $\mathbf{y}^{\mathrm{T}}\mathbf{U} = \mathbf{0}$. In this case we would have

$$\lim \langle \mathbf{u}^\ell_r, \mathbf{y}\rangle = \langle \mathbf{u}_r, \mathbf{y}\rangle = 0 \qquad \text{for each } r = 1, \ldots, R.$$

But then we would have

$$\langle \mathcal{T}^\ell_{(1)}, \mathbf{y}^{\otimes N}\rangle = \lim_\ell \langle \mathcal{T}^\ell_{(1)}, \mathbf{y}^{\otimes N}\rangle = \lim \sum_{r=1}^R \langle \mathbf{u}^\ell_r, \mathbf{y}\rangle^{N-1} = \sum_{r=1}^R (\lim \langle \mathbf{u}^\ell_r, \mathbf{y}\rangle)^N = 0,$$

a contradiction to the assumption that $\mathcal{T}^\ell_{(1)}|_{\mathrm{fib}(\mathcal{T})} \succ 0$. Therefore $\mathbf{U}$ has rank R.

Now momentarily assume that the $\mathbf{V}^\ell$ have bounded norm. Then, again by passing to a subsequence if necessary, the $\mathbf{V}^\ell$ converge to some matrix $\mathbf{V}$. Having already shown $\lim \mathbf{U}^\ell = \mathbf{U}$ exists, we would then have

$$\mathcal{T} = \lim \mathcal{T}^\ell = \lim [\![\mathbf{U}^\ell, \ldots, \mathbf{U}^\ell, \mathbf{V}^\ell]\!] = [\![\mathbf{U}, \ldots, \mathbf{U}, \mathbf{V}]\!],$$

which would show that $\mathcal{T}$ has SFS rank R. An application of Lemma 3.3 would then show that $\mathcal{T}$ has rank R and that, if either $N \ge 4$ or $\mathbf{V}$ does not have a pair of

columns which is collinear, then the CPD of $\mathcal{T}$ is unique. Thus we only need show that the $\mathbf{V}^\ell$ have bounded norm.

To this end, suppose towards a contradiction that the $\mathbf{V}^\ell$ do not have bounded norm. Then the $\mathbf{V}^\ell$ must have some common entry, say the (i_*, j_*)th, which does not stay bounded in modulus. That is, we must have $\sup_\ell |v^\ell_{i_* j_*}| = \infty$. Since each $\mathbf{U}^\ell$ has rank R, for each ℓ there exists a unit vector $\mathbf{w}^\ell_{j_*} \in \operatorname{ran} \mathbf{U}^\ell$ which satisfies

$$\langle \mathbf{w}^\ell_{j_*}, \mathbf{u}^\ell_r \rangle = 0$$

whenever $r \neq j_*$ and $\langle \mathbf{w}^\ell_{j_*}, \mathbf{u}^\ell_{j_*} \rangle \neq 0$. Furthermore, since the $\mathbf{U}^\ell$ converge to $\mathbf{U}$ and since $\mathbf{U}$ has rank R, the ranges of the $\mathbf{U}^\ell$ converge to the range of $\mathbf{U}$. It follows that

$$\liminf_\ell |\langle \mathbf{w}^\ell_{j_*}, \mathbf{u}^\ell_{j_*} \rangle| > 0.$$

We then have

(4.2)

$$\begin{aligned}\|\mathcal{T}\|_{\mathrm{sp}} \geq \|\mathcal{T}_{(i_*)}\|_{\mathrm{sp}} = \lim \|\mathcal{T}^\ell_{(i_*)}\|_{\mathrm{sp}} &\geq \sup_\ell \left| \langle \mathcal{T}^\ell_{(i_*)}, (\mathbf{w}^\ell_{j_*})^{\otimes N} \rangle \right| = \sup_\ell \left| \sum_{r=1}^R \langle v^\ell_{i_* r} (\mathbf{u}^\ell_r)^{\otimes N}, (\mathbf{w}^\ell_{j_*})^{\otimes N} \rangle \right| \\ &= \sup_\ell \left| \sum_{r=1}^R v^\ell_{i_* r} \langle \mathbf{u}^\ell_r, \mathbf{w}^\ell_{j_*} \rangle^{\otimes N} \right| = \sup_\ell \left| v^\ell_{i_* j_*} \right| \left| \langle \mathbf{u}^\ell_{j_*}, \mathbf{w}^\ell_{j_*} \rangle \right|^N.\end{aligned}$$

However, since $\sup_\ell |v^\ell_{i_* j_*}| = \infty$ while $\liminf_\ell |\langle \mathbf{w}^\ell_{j_*}, \mathbf{u}^\ell_{j_*} \rangle| > 0$, we must have

$$\sup_\ell \left| v^\ell_{i_* j_*} \right| \left| \langle \mathbf{u}^\ell_{j_*}, \mathbf{w}^\ell_{j_*} \rangle \right|^N = \infty,$$

a contradiction to equation (4.2). We conclude that the $\mathbf{V}^\ell$ have bounded norm. ∎

4.2. Proof of Theorem 2.2. The proof of Theorem 2.2 follows a simple strategy. We first show that if $\mathcal{T}$ and $\mathcal{E}$ satisfy equation (2.4), then any best border SFS rank at most R approximation of $\mathcal{T} + \mathcal{E}$ has a linear combination of frontal slices which is positive semidefinite. It then follows from Theorem 2.1 that any best border SFS rank at most R approximation of $\mathcal{T} + \mathcal{E}$ has (SFS) rank equal to R, hence is a best rank R approximation of $\mathcal{T} + \mathcal{E}$.

Proof. Since the set of tensors with border SFS rank at most R is closed, $\mathcal{T} + \mathcal{E}$ is guaranteed to have a (not necessarily unique) best approximation among the set of border rank at most R tensors, which we denote by $\hat{\mathcal{T}}$. Write $\hat{\mathcal{T}} = \mathcal{T} + \mathcal{E} + \hat{\mathcal{E}}$ and note that one must have $\|\hat{\mathcal{E}}\| \leq \|\mathcal{E}\|$, as otherwise $\mathcal{T}$ would be a better approximation to $\mathcal{T} + \mathcal{E}$ among the set of border SFS rank at most R tensors than $\hat{\mathcal{T}}$.

Now, let $\mathbf{w}_* \in \mathbb{R}^J$ be a unit vector which maximizes the right-hand of inequality (2.4) and let $\mathbf{s} \in \mathfrak{S}$ be any unit vector. Then we have

$$\begin{aligned}\langle \hat{\mathcal{T}} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{s}^{\otimes N} \rangle &= \langle (\mathcal{T} + \mathcal{E}) \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{s}^{\otimes N} \rangle + \langle \hat{\mathcal{E}} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}, \mathbf{s}^{\otimes N} \rangle \\ &\geq \lambda_{\min}\Big(\big((\mathcal{T} + \mathcal{E}) \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}} \big)|_{\mathfrak{S}} \Big) - \|\hat{\mathcal{E}}\|_{\mathrm{sp}} \\ &\geq \lambda_{\min}\Big(\big((\mathcal{T} + \mathcal{E}) \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}} \big)|_{\mathfrak{S}} \Big) - \|\hat{\mathcal{E}}\| \qquad > \quad 0,\end{aligned}$$

from which it follows that $\hat{\mathcal{T}} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}}$ is PD on $\mathfrak{S}$. It follows that we may use Lemma 3.2 to conclude that

$$(\hat{\mathcal{T}} \cdot_{N+1} \mathbf{w}_*^{\mathrm{T}})|_{\mathfrak{fib}(\mathcal{T})} \succ 0.$$

Using Theorem 2.1 then shows that $\hat{\mathcal{T}}$ decomposes as $[\![\hat{\mathbf{U}}, \ldots, \hat{\mathbf{U}}, \hat{\mathbf{V}}]\!]$ and has both rank and SFS rank equal to R and that, if either $N \geq 4$ or $\hat{\mathbf{V}}$ does not have a pair of columns which is collinear, then the CPD of $\hat{\mathcal{T}}$ is unique. We conclude that $\hat{\mathcal{T}}$ is a best SFS rank R approximation to $\mathcal{T} + \mathcal{E}$. ∎

4.3. Computation of bounds via semidefinite programming. We now illustrate how one may use semidefinite programming to compute the quantity appearing on the right-hand side of equation (2.4) when $N = 2$. More precisely, for a tensor $\mathcal{T} \in (S^2\mathbb{R}^I) \otimes \mathbb{R}^J$ and R dimensional subspace $\mathfrak{S} \subset \mathbb{R}^I$, we show how to compute

$$\max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}\Big((\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}})|_{\mathfrak{S}}\Big)$$

which appears in Theorem 2.2. To accomplish this, let $\mathbf{S} \in \mathbb{R}^{I\times R}$ be a matrix whose columns form an orthonormal basis for $\mathfrak{S} \subseteq \mathbb{R}^I$ and set $\mathcal{S} = \mathcal{T} \cdot_1 \mathbf{S}^{\mathrm{T}} \cdot_2 \mathbf{S}^{\mathrm{T}} \in (S^2\mathbb{R}^R) \otimes \mathbb{R}^J$. A straightforward computation then shows that

$$\max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}\Big((\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}})|_{\mathfrak{S}}\Big) = \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}(\mathcal{S} \cdot_3 \mathbf{w}^{\mathrm{T}}).$$

Thus it is sufficient to show that one can compute the quantity on the right hand side of the above equality.

THEOREM 4.1. *Let $\mathcal{T} \in (S^2\mathbb{R}^R) \otimes \mathbb{R}^J$ and let $\{\mathbf{e}_r\}_{r=1}^{R+1}$ be the standard orthonormal basis in $\mathbb{R}^{R+1}$. For $r = 1, \ldots, R$ set*

$$\mathbf{M}_{(r)} = \mathbf{e}_1\mathbf{e}_{r+1}^{\mathrm{T}} + \mathbf{e}_{r+1}\mathbf{e}_1^{\mathrm{T}}.$$

Also let $\mathcal{M} \in \mathbb{R}^{(R+1)\times(R+1)\times R}$ be the tensor whose rth frontal slice is $\mathbf{M}_{(r)}$ and set

$$\lambda_* = \max_{\substack{\mathbf{w}\in\mathbb{R}^R \\ \|\mathbf{w}\|=1}} \lambda_{\min}(\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}).$$

Then we have

$$\begin{aligned} \max\{\lambda_*, 0\} \;=\; & \max_{c\in\mathbb{R},\mathbf{w}\in\mathbb{R}^R} \quad c \\ & \text{s.t.} \;\; -c\mathbf{I}_R + \mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}} \succeq 0 \quad \text{and} \quad \mathbf{I}_{R+1} + \mathcal{M} \cdot_3 \mathbf{w}^{\mathrm{T}} \succeq 0. \end{aligned} \tag{4.3}$$

In words, computing λ_* is equivalent to solving the semidefinite program above.

Proof. First assume that $\lambda_* \leq 0$. Then $\mathcal{T}$ does not have a linear combination of frontal slices which is PD, so

$$-c\mathbf{I}_R + \mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}} \succeq 0$$

can only be satisfied if $c \leq 0$. Taking $c = 0$ and $\mathbf{w} = \mathbf{0}$ then maximizes the SDP.

Now assume $\lambda_* > 0$ and note that the feasible set of the linear matrix inequality

$$\mathbf{I}_{R+1} + \mathcal{M} \cdot_3 \mathbf{w}^{\mathrm{T}} \succeq 0$$

is the unit 2-norm ball in $\mathbb{R}^R$. Let $(c_*, \mathbf{w}_*)$ denote an optimizer of the SDP on the right-hand side of equation (4.3). A straightforward check shows that $\lambda_* > 0$

implies $c_* > 0$. Furthermore, since $(c_*/\|\mathbf{w}_*\|_2, \mathbf{w}_*/\|\mathbf{w}_*\|)$ also satisfies the linear matrix inequality and since $\|\mathbf{w}_*\| \leq 1$, we must have $\|\mathbf{w}_*\| = 1$. Combining these observations shows that, in the case $\lambda_* > 0$, our SDP is equivalent to

$$\begin{aligned} &\max_{c\in\mathbb{R},\mathbf{w}\in\mathbb{R}^R,\|\mathbf{w}\|_2=1} && c \\ &\text{s.t.} && -c\mathbf{I}_R + \mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}} \succeq 0. \end{aligned} \tag{4.4}$$

For a fixed unit vector $\mathbf{w}$, the largest c such that $-c\mathbf{I}_R + \mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}} \succeq 0$ is equal to the smallest eigenvalue of $\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}$. Thus solving equation (4.4) is equivalent to maximizing the smallest eigenvalue of $\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}$ over $\mathbf{w} \in \mathbb{R}^R$ with $\|\mathbf{w}\|_2 = 1$. ∎

4.4. Sharpness of Theorem 2.2. In this section we prove Theorem 2.4 which illustrates that, when $N = 2$, Theorem 2.2 is sharp. The upcoming proof makes uses of the notion of generalized eigenvalues and eigenvectors of a matrix pencil $(\mathbf{M}_1, \mathbf{M}_2) \in (\mathbb{R}^{I\times I})^2$. Say a nonzero vector $\mathbf{v}$ is a **generalized eigenvector** of $(\mathbf{M}_1, \mathbf{M}_2)$ if there exists a nonzero vector $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ and a nonzero vector $\mathbf{y} \in \mathbb{R}^I$ such that

$$\mathbf{M}_1\mathbf{v} = \gamma_1\mathbf{y} \qquad \text{and} \qquad \mathbf{M}_2\mathbf{v} = \gamma_2\mathbf{y} \tag{4.5}$$

We call the one dimensional subspace $\operatorname{span}((\gamma_1, \gamma_2)) \subset \mathbb{R}^2$ the **generalized eigenvalue** of $(\mathbf{M}_1, \mathbf{M}_2)$ corresponding to $\mathbf{v}$. See [50, 22] for further details.

Borrowing the perspective of [17] on "joint" generalized eigenvalues and eigenvectors of a tensor, we mention that one can naturally identify an $I \times I \times 2$ tensor $\mathcal{M}$ with the pencil of $I \times I$ matrices $(\mathbf{M}_1, \mathbf{M}_2) \in (\mathbb{R}^{I\times I})^2$, i.e. $\mathcal{M} \cong (\mathbf{M}_1, \mathbf{M}_2)$. With this in mind, we call the generalized eigenvalues and eigenvectors of the matrix pencil $(\mathbf{M}_1, \mathbf{M}_2)$ the generalized eigenvalues and eigenvectors of $\mathcal{M}$.

Proof. Let $\mathcal{T}$ and λ_* be as in the statement of the theorem. Following the discussion before Theorem 4.1, if $\mathbf{S} \in \mathbb{R}^{I\times R}$ is an isometry mapping onto $\operatorname{fib}(\mathcal{T})$, then

$$\max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}\Big((\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}})|_{\operatorname{fib}(\mathcal{T})}\Big) = \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}(\mathcal{T} \cdot_1 \mathbf{S}^{\mathrm{T}} \cdot_2 \mathbf{S}^{\mathrm{T}} \cdot_3 \mathbf{w}^{\mathrm{T}}).$$

Therefore, we can WLOG assume that $I = R$ and $\operatorname{fib}(\mathcal{T}) = \mathbb{R}^R$. Furthermore, by an appropriate unitary transform in each mode of $(S^2\mathbb{R}^R) \otimes \mathbb{R}^J$ we can WLOG assume that $\lambda_* = \lambda_{\min}(\mathbf{T}_{(1)})$ and that $\mathbf{T}_{(1)}$ has the form

$$\mathbf{T}_{(1)} = \begin{pmatrix} \lambda_*\mathbf{I}_n & 0 \\ 0 & \mathbf{\Lambda} \end{pmatrix}$$

where $\mathbf{\Lambda}$ is a PD diagonal matrix of size $R - n$ and where $\lambda_{\min}(\mathbf{\Lambda}) > \lambda_*$. That is, $\mathbf{\Lambda} - \lambda_*\mathbf{I}_{R-n} \succ 0$. Set

$$\mathcal{E}_* = -\lambda_* \sum_{i=1}^{n} \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_1.$$

We will show that $\mathcal{T} + \mathcal{E}_*$ does not have a PD linear combination of frontal slices.

Suppose towards a contradiction that there is a PD linear combination of the frontal slices of $\mathcal{T} + \mathcal{E}_*$. That is, there exists some linear combination of the matrices

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{pmatrix}, \mathbf{T}_{(2)}, \mathbf{T}_{(3)}, \ldots, \mathbf{T}_{(J)}$$

which is PD. Thus there is a unit vector $(\alpha_2, \alpha_3, \ldots, \alpha_J) \in \mathbb{R}^{J-1}$ such that if one sets

$$\mathbf{P} = \sum_{j=2}^{J} \alpha_j \mathbf{T}_{(j)} \qquad \text{and partitions} \qquad \mathbf{P} = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{21}^{\mathrm{T}} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

with respect to the decomposition $\mathbb{R}^R = \mathbb{R}^n \oplus \mathbb{R}^{R-n}$, then one has that $\mathbf{P}_{11}$ is strictly PD. From this point, a straightforward argument using the Schur complement and routine calculus techniques shows that there is a $\delta > 0$ such that

$$\lambda_{\min}\left(\sqrt{1-\delta^2}\mathbf{T}_{(1)} + \delta\mathbf{P}\right) = \lambda_{\min}\left(\sqrt{1-\delta^2}\mathbf{T}_{(1)} + \delta\sum_{j=2}^{J}\alpha_j\mathbf{T}_{(j)}\right) > \lambda_*. \tag{4.6}$$

Since $(\alpha_2, \ldots, \alpha_J)$ was taken to be a unit vector, the vector $\boldsymbol{\alpha}_*$ defined by

$$\boldsymbol{\alpha}_* := (\sqrt{1-\delta^2}, \delta\alpha_2, \ldots, \delta\alpha_J) \in \mathbb{R}^J$$

is also a unit vector. This shows that equation (4.6) contradicts the definition of λ_*. Thus $\mathcal{T} + \mathcal{E}_*$ cannot have a linear combination of frontal slices which is PD.

Now assume that $J = 2$. We will show that, if $\mathcal{E}$ is defined as above, then any open neighborhood around $\mathcal{T} + \mathcal{E}_*$ contains a tensor which has rank strictly greater than border rank. To prove this we use the fact that the rank and border rank of a tensor of size $R \times R \times 2$ can be determined from its generalized eigenvalues. In particular, as a consequence of [48, Lemma 1], any open neighborhood that contains a $R \times R \times 2$ tensor that has a generalized eigenvalue with algebraic multiplicity at least two must contain a tensor with rank strictly greater than border rank.

From before we have

$$(\mathbf{T} + \mathbf{E}_*)_{(1)} = \begin{pmatrix} \mathbf{0}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda} \end{pmatrix}.$$

Set $\mathbf{P} = (\mathbf{T} + \mathbf{E}_*)_{(2)} = \mathbf{T}_{(2)}$ and write

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{21}^{\mathrm{T}} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

with respect to the decomposition $\mathbb{R}^R = \mathbb{R}^n \oplus \mathbb{R}^{R-n}$. We will consider several cases.

First suppose $n \geq 2$. In this case $\operatorname{span}((1,0))$ is a generalized eigenvalue of $\mathcal{T} + \mathcal{E}_*$ with corresponding generalized eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Thus $\mathcal{T} + \mathcal{E}_*$ has a generalized eigenvalue with algebraic multiplicity at least 2, and any open set containing $\mathcal{T} + \mathcal{E}_*$ contains a tensor with rank strictly greater than border rank.

Now assume that $n = 1$. The first part of the proof shows that no linear combination of $\mathbf{P}_{11} \in \mathbb{R}$ can be PD from which it follows that $\mathbf{P}_{11} = 0$. Next note that $((\mathbf{T}+\mathbf{E}_*)_{(1)}, \mathbf{P})$ has a generalized eigenvalue with algebraic multiplicity at least two if and only if $((\mathbf{T}+\mathbf{E}_*)_{(1)}, \mathbf{P}+\alpha(\mathbf{T}+\mathbf{E}_*)_{(1)})$ has a generalized eigenvalue with algebraic multiplicity at least two for some $\alpha \in \mathbb{R}$. Since $\boldsymbol{\Lambda}$ is PD, we can choose α so that $\mathbf{P}_{22} + \alpha\boldsymbol{\Lambda}$ is PD. It follows that we can without loss of generality assume $\mathbf{P}_{22}$ is PD.

We now consider two subcases of the $n = 1$ case. First suppose that $\mathbf{P}$ is not invertible. If $\mathbf{P}\mathbf{e}_1 = \mathbf{0}$, then we must have $\mathbf{P}_{21} = \mathbf{0}$. In this case, the frontal slices of $\mathcal{T} + \mathcal{E}$ have block decomposition

$$((\mathbf{T} + \mathbf{E}_*)_{(1)}, \mathbf{P}) = \left(\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda} \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{22} \end{pmatrix}\right).$$

It is then straightforward to show that any open neighborhood of $\mathcal{T} + \mathcal{E}_*$ contains a tensor with a generalized eigenvalue with algebraic multiplicity at least 2. On the other hand, if $\mathbf{P}\mathbf{e}_1 \neq \mathbf{0}$ then there is some vector $\mathbf{v}$ such that $\mathbf{P}\mathbf{v} = \mathbf{0}$ and such that the set $\{\mathbf{e}_1, \mathbf{v}\}$ is linearly independent. In this case $\operatorname{span}((1,0))$ is a generalized eigenvalue of $\mathcal{T} + \mathcal{E}_*$ for which both $\mathbf{e}_1$ and $\mathbf{v}$ are corresponding generalized eigenvectors, and hence $\mathcal{T} + \mathcal{E}_*$ itself has a generalized eigenvalue with algebraic multiplicity at least 2. We conclude that in either of these subcases, any open neighborhood of $\mathcal{T} + \mathcal{E}_*$ contains a tensor with rank strictly greater than border rank.

Now assume that $\mathbf{P}$ is invertible. In this case, [35, Theorem 2.1 (ii)] shows that $\mathbf{P}_{21} \neq \mathbf{0}$ and that $\mathbf{P}^{-1}$ has block decomposition

$$\begin{pmatrix} * & * \\ * & \mathbf{P}_{22}^{-1} - \mathbf{P}_{22}^{-1}\mathbf{P}_{21}(\mathbf{P}_{21}^{\mathrm{T}}\mathbf{P}_{22}^{-1}\mathbf{P}_{21})^{-1}\mathbf{P}_{21}^{\mathrm{T}}\mathbf{P}_{22}^{-1} \end{pmatrix},$$

hence

$$\mathbf{P}^{-1}(\mathbf{T} + \mathbf{E}_*)_{(1)} = \begin{pmatrix} \mathbf{0} & * \\ \mathbf{0} & (\mathbf{P}_{22}^{-1} - \mathbf{P}_{22}^{-1}\mathbf{P}_{21}(\mathbf{P}_{21}^{\mathrm{T}}\mathbf{P}_{22}^{-1}\mathbf{P}_{21})^{-1}\mathbf{P}_{21}^{\mathrm{T}}\mathbf{P}_{22}^{-1})\mathbf{\Lambda} \end{pmatrix}.$$

It follows that

$$\mathbf{P}^{-1}(\mathbf{T} + \mathbf{E}_*)_{(1)} \begin{pmatrix} 0 \\ \mathbf{\Lambda}^{-1}\mathbf{P}_{21} \end{pmatrix} = \mathbf{0} \qquad \text{and} \qquad \mathbf{P}^{-1}(\mathbf{T} + \mathbf{E}_*)_{(1)}\mathbf{e}_1 = \mathbf{0},$$

where $\mathbf{\Lambda}^{-1}\mathbf{P}_{21} \neq \mathbf{0}$. Thus $\mathcal{T} + \mathcal{E}_*$ again has a generalized eigenvalue with algebraic multiplicity two in this case, and hence any open neighborhood containing $\mathcal{T} + \mathcal{E}_*$ must contain a tensor with rank strictly greater than border rank. ∎

5. Numerical Experiments. This section contains numerical results which illustrate the existence bound of Theorem 2.2. We first apply our bound to second-order blind identification (SOBI) as proposed in [2]. Next we examine the average bound obtained in synthetic data. Finally we examine the sharpness of our bound.

5.1. PD existence vs. SOBI. We first perform the SOBI experiment outlined in the Tensorlab demos [13]. More precisely, we consider an instantaneous linear mixture $\mathbf{x}[k] \in \mathbb{R}^{10}$ of three stationary uncorrelated sources $\mathbf{s}[k] \in \mathbb{R}^3$, i.e.,

$$\mathbf{x}[k] = \mathbf{M}\mathbf{s}[k] + \mathbf{n}[k].$$

Here $\mathbf{M} \in \mathbb{R}^{10\times 3}$ contains the mixing coefficients and $\mathbf{n}[k]$ is additive noise. In our experiment, we generate the temporally coherent sources $\mathbf{s}[k]$ by passing $K = 10^3, 10^4, 10^5$, or 10^6 i.i.d. unit-variance zero-mean Gaussian samples through a finite impulse response filter of length $L = 20$ which has random coefficients. We then form a tensor $\mathcal{C}_x^{(2)}$ whose frontal slices are computed by the sample averaging

$$\mathcal{C}_x^{(2)}(:,:,\tau) = \frac{1}{2(K-1-\tau)} \sum_{k=1}^{K-\tau} \mathbf{x}[k]\mathbf{x}[k+\tau]^{\mathrm{T}} + \mathbf{x}[k+\tau]\mathbf{x}[k]^{\mathrm{T}}$$

where $\tau = 1, \ldots, 20$. Here the frontal slices are symmetrized so that we may apply the SDP of Theorem 4.1. In this setup, the filter coefficients and the entries of $\mathbf{M}$ and $\mathbf{n}[k]$ are all i.i.d. drawn from the Gaussian distribution.

In the noiseless case (and with infinitely many samples), the mixing matrix $\mathbf{M}$ can be exactly recovered by computing a rank 3 CPD of $\mathcal{C}_x^{(2)}$ provided that the CPD

of $\mathcal{C}_x^{(2)}$ is unique. Thus we may approximate $\mathbf{M}$ in the noisy case by computing a CPD of a rank 3 approximation of $\mathcal{C}_x^{(2)}$.

In our experiment, for various choices of SNR and for each choice of $K = 10^3, 10^4, 10^5$, and 10^6, we generate $\mathbf{M}, \mathbf{s}[k]$, and $\mathbf{n}[k]$ and compute the corresponding tensor $\mathcal{C}_x^{(2)}$ as described above. We then compute (the core of) a (normalized[4]) multilinear rank $(3,3,3)$ approximation $\tilde{\mathcal{C}}_x^{(2)}$ of $\mathcal{C}_x^{(2)}$ and use Theorem 4.1 to compute our bound λ_* which we will use to attempt guarantee the existence of a best rank 3 approximation of $\tilde{\mathcal{C}}_x^{(2)}$. In addition we compute a rank 3 approximation $\mathcal{S}_x^{(2)}$ of $\tilde{\mathcal{C}}_x^{(2)}$ and check if the spectral distance[5] between these tensors is less than λ_*, i.e., we check if $\|\tilde{\mathcal{C}}_x^{(2)} - \mathcal{S}_x^{(2)}\|_{\text{sp}} < \lambda_*$. If this is the case, then we may use Theorem 2.2 to guarantee that $\tilde{\mathcal{C}}_x^{(2)}$ has a best rank 3 approximation. In addition, we compute the relative mixing matrix error between the true mixing matrix $\mathbf{M}$ and the approximate mixing matrix $\hat{\mathbf{M}}$ which is recovered from the CPD of the rank 3 approximation $\mathcal{S}_x^{(2)}$. Here the relative mixing matrix error is defined as $\|\mathbf{M} - \hat{\mathbf{M}}\|_{\text{F}}/\|\mathbf{M}\|_{\text{F}}$ where the columns of $\hat{\mathbf{M}}$ have been optimally scaled and permuted to match those of $\mathbf{M}$.

For each choice of SNR, this basic experiment is repeated a total of 500 times and for each choice of $T = 10^3, 10^4, 10^5$, and 10^6 we record the median value of λ_*, the median spectral distance between $\tilde{\mathcal{C}}_x^{(2)}$ and $\mathcal{S}_x^{(2)}$ and the median relative mixing matrix error. In addition we record the proportion of the 500 triples $\mathbf{M}, \mathbf{s}[k], \mathbf{n}[k]$ for which we were able to guarantee existence of a best rank 3 approximation of $\tilde{\mathcal{C}}_x^{(2)}$.

The results of this experiment are plotted in Figure 1. Here we see that the our bound λ_* is highly predictive of one's ability to accurately recover the mixing matrix $\mathbf{M}$. We observe that one can typically accurately recover $\mathbf{M}$ when one is able to guarantee the existence of a best approximation of rank 3 approximation $\mathcal{S}_x^{(2)}$. On the other hand, the approximation of $\mathbf{M}$ is typically highly inaccurate when one is unable to guarantee the existence of a best rank 3 approximation of $\mathcal{S}_x^{(2)}$.

5.2. PD Bound vs. JGEVD bound. We next compare the PD existence bound of Theorem 2.2 to the existence bound for general (not necessarily PD) tensors presented in [17, Theorem 2.7]. More precisely, in the following experiment we randomly generate real rank I tensors with SFS of size $I \times I \times J$ and normalize the tensors to have spectral norm one. For each tensor $\mathcal{T}$ we use Theorem 4.1 to compute the radius of a spectral norm ball centered at $\mathcal{T}$ in which best rank I approximations are guaranteed to exist. In addition, we use the joint generalized eigenvector (JGEV) based bound of [17, Theorem 2.7] to compute the radius of a Frobenius norm ball centered at $\mathcal{T}$ in which best rank I approximations are guaranteed to exist. For each choice of I and J, a total of 20 tensors are generated, and the mean of the PD bound as well as the mean of the JGEV bound are reported. The choices of I considered are $I = 2$ through 10, and the choices of J are either $J = I$ or $J = 2$.

We use two different methods generating the factor matrices of the random $\mathcal{T}$. Let $\mathcal{T} = [\![\mathbf{U}, \mathbf{U}, \mathbf{V}]\!]$ be a SFS tensor. In the left-hand image of Figure 2, the entries of $\mathbf{U}$ are i.i.d. sampled from the standard normal distribution and the entries of $\mathbf{V}$ are

[4]The tensor $\tilde{\mathcal{C}}_x^{(2)}$ is normalized to have unit spectral norm so the results from the runs are more directly comparable. This normalization has no effect on recovery the mixing matrix $\mathbf{M}$.

[5]We approximate the spectral distance between $\tilde{\mathcal{C}}_x^{(2)}$ and $\mathcal{S}_x^{(2)}$ by using Tensorlab [52] to compute a near optimal rank 1 approximation of the residual $\tilde{\mathcal{C}}_x^{(2)} - \mathcal{S}_x^{(2)}$. We compute a total of eleven rank one approximations, one computed using a dominant singular vector initialization and ten computed using random initialization, then report the Frobenius norm of the best of these approximations.

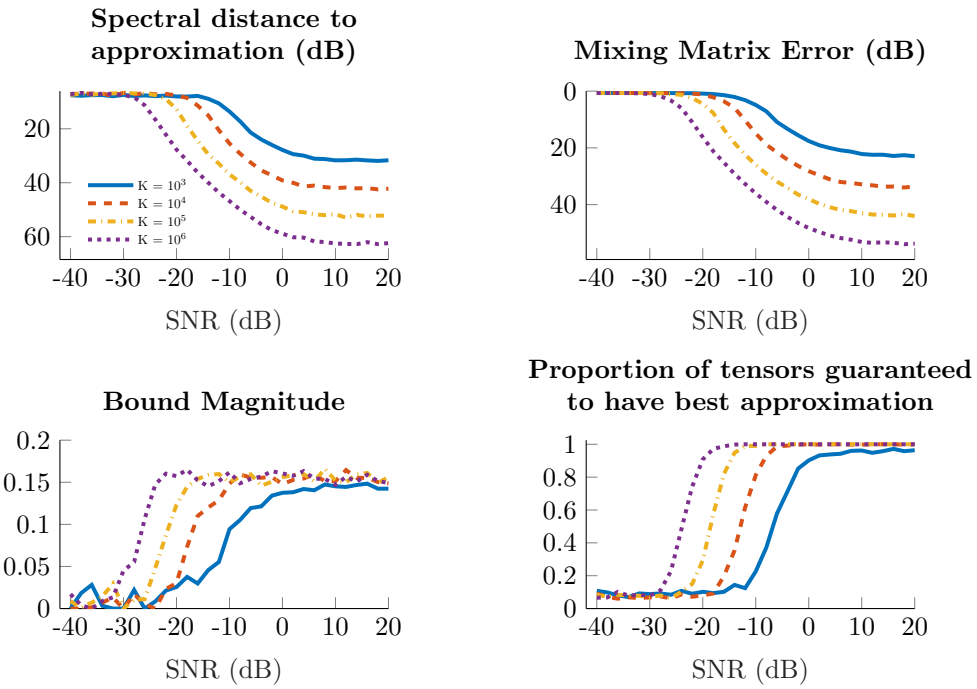


FIG. 1. *Magnitude of existence bound, distance to best approximation, mixing matrix error, and proportion of tensors guaranteed to have a best approximation. There is a close link in ability to guarantee existence of a best low rank approximation and mixing matrix error.*

i.i.d. sampled from the uniform distribution on $[0, 1]$. Meanwhile, in the right-hand image of Figure 2, the entries of both $\mathbf{U}$ and $\mathbf{V}$ are all independently sampled from the standard normal distribution. The key difference between these choices is that, if $\mathbf{V}$ has entries i.i.d. sampled from the uniform distribution on $[0, 1]$, then all frontal slices of $\mathcal{T}$ will be PD. On the other hand, if the entries of $\mathbf{V}$ are drawn from the standard normal distribution, then $\mathcal{T}$ may fail to have a PD frontal slice.

We see that the bound obtained using Theorem 2.2 significantly outperforms the bound of [17, Theorem 2.7] except the $J = 2$ setting in the right-hand image of Figure 2. This is due to the fact that, once R becomes large, it becomes unlikely that the span of a pair of random $I \times I$ symmetric matrices contains a PD matrix. In this case, the right-hand side of inequality (2.4) is not positive and Theorem 2.2 cannot be applied. In particular, when $I \geq 8$ and $J = 2$, none of the tensors we generated whose factor $\mathbf{V}$ was sampled from a Gaussian distribution had a PD slice mix.

We emphasize that not only is the value of the PD bound typically significantly larger than the JGEV bound, but also the PD bound uses the spectral norm while the JGEV bound uses the Frobenius norm. The Frobenius norm is always greater than or equal to the spectral norm of a tensor, and typically the Frobenius norm will be much larger than the spectral norm. Thus, a spectral norm ball of radius δ is much larger than a Frobenius norm ball of radius δ. As a consequence, the gap between these bounds is even larger than the figures may suggest.

5.3. Sharpness of PD bound. In our last experiment we examine the sharpness of the existence bound of Theorem 2.2 for tensors of order three. To accomplish this, we illustrate the effect of the perturbation $\mathcal{E}_*$ described in Theorem 2.4 on a tensor. More precisely, in the following experiment we fix I and J and randomly generate

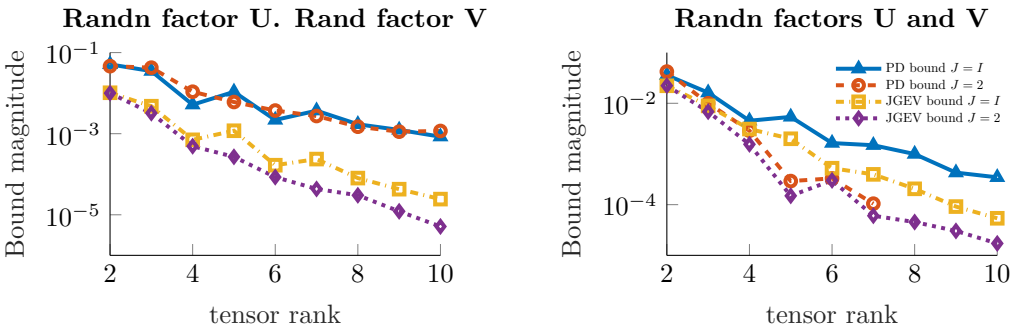


FIG. 2. *Existence bounds computed using Theorem 2.2 and [17, Theorem 2.7]. The PD bound is typically significantly larger than the JGEV bound, particularly when the entries of* $\mathbf{V}$ *are sampled from the uniform distribution on* $[0, 1]$. *However, when the entries of* $\mathbf{V}$ *are sampled from the standard normal distribution and* $J = 2$ *and* I *is large, a PD bound cannot be computed as the initial tensor does not have a linear combination of frontal slices which is PD. This is why the line showing the PD bound for* $J = 2$ *in the right hand image stops at* $R = 7$.

a rank I tensor of the form $\mathcal{T} = [\![\mathbf{U}, \mathbf{U}, \mathbf{V}]\!] \in (S^2\mathbb{R}^I) \otimes \mathbb{R}^J$ by i.i.d. sampling entries of $\mathbf{U} \in \mathbb{R}^{I\times I}$ from the standard normal distribution and i.i.d. sampling entries of $\mathbf{V} \in \mathbb{R}^{J\times I}$ from the uniform distribution on $[0, 1]$. We then compute the perturbation $\mathcal{E}_*$ described in Theorem 2.4 such that $\mathcal{T} + \mathcal{E}_*$ does not have a linear combination of frontal slices which is PD and such that

$$\|\mathcal{E}_*\|_{\text{sp}} = \max_{\substack{\mathbf{w}\in\mathbb{R}^J \\ \|\mathbf{w}\|=1}} \lambda_{\min}(\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}).$$

To assess the effect of the perturbation $\mathcal{E}_*$ on the ability to recover the CPD of $\mathcal{T}$, for various choices of $\alpha \in [0, 2]$, we set $\mathcal{T}(\alpha) = \mathcal{T} + \alpha\mathcal{E}_*$ and compute a rank R approximation $\hat{\mathcal{T}}(\alpha) = [\![\mathbf{U}(\alpha), \mathbf{U}(\alpha), \mathbf{V}(\alpha)]\!]$ to $\mathcal{T}(\alpha)$. Here we normalize the columns of $\mathbf{U}$ and $\mathbf{V}$ so that $\|\mathbf{u}_i\|_2 = \|\mathbf{v}_i\|_2$ for all $i = 1, \ldots, I$ where $\mathbf{u}_i$ and $\mathbf{v}_i$ denote the ith columns of $\mathbf{U}$ and $\mathbf{V}$, respectively.

We examine three questions:

1. Does $\mathcal{T}(\alpha)$ have a best rank R approximation?
2. Does $\hat{\mathcal{T}}(\alpha)$ have a unique CPD?
3. Does the CPD of $\hat{\mathcal{T}}(\alpha)$ closely approximate the CPD of $\mathcal{T}$?

To address question 1, we look for the presence of diverging components in the CPD of $\hat{\mathcal{T}}(\alpha)$. If diverging components occur, then, from our normalization of the columns of $\mathbf{U}(\alpha)$ and $\mathbf{V}(\alpha)$, there must be columns of $\mathbf{U}(\alpha)$ and $\mathbf{V}(\alpha)$ whose norms go to infinity.[6] Additionally, commonly when diverging components are observed, there is a pair of columns which become (anti-)parallel. Thus to search for diverging components, we record the maximum norm of a column of $\mathbf{U}(\alpha)$, and the minimum angle between a pair of columns of $\mathbf{U}(\alpha)$ or of $\mathbf{V}(\alpha)$.

In the case $\mathcal{T} + \alpha\mathcal{E}_* \in (S^2\mathbb{R}^I) \otimes \mathbb{R}^2$, it is known [48] that if (the core of the multilinear singular value decomposition (MLSVD), see [10], of) $\mathcal{T}+\alpha\mathcal{E}_*$ has repeated or complex generalized eigenvalues, then any open neighborhood centered at $\mathcal{T}+\alpha\mathcal{E}_*$ contains a tensor which does not have a best rank I approximation. For more general tensors $\mathcal{T} + \alpha\mathcal{E}_* \in (S^2\mathbb{R}^I) \otimes \mathbb{R}^J$ with $J \geq 2$, the presence of repeated or complex generalized eigenvalues in the first two slices of the MLSVD of $\mathcal{T}+\alpha\mathcal{E}_*$ is still indicative

[6] Note that the normalization of the columns also enforces that column norms can go to infinity if and only if the norm of the corresponding rank one term goes to infinity.

of potential numerical issues when attempting to compute a best rank I approximation of $\mathcal{T} + \alpha\mathcal{E}_*$. Indeed, a popular strategy for CPD computation is to first algebraically compute a CPD approximation by finding the generalized eigenvectors of the first two slices[7] of the MLSVD of $\mathcal{T} + \alpha\mathcal{E}_*$. However, this strategy does not produce a very meaningful rank I initialization if the generalized eigenvalues of the first two MLSVD slices are either repeated or complex.

Question 2 is addressed (in part) by examining the minimum angle of between the columns the factors as having a nonzero angle between all pairs of columns of $\mathbf{U}(\alpha)$ and of $\mathbf{V}(\alpha)$ is a necessary (though not sufficient) condition for uniqueness. To address question 3, we record the relative factor matrix error between the true CPD $\mathcal{T} = [\![\mathbf{U}, \mathbf{U}, \mathbf{V}]\!]$ and the approximate CPD $[\![\mathbf{U}(\alpha), \mathbf{U}(\alpha), \mathbf{V}(\alpha)]\!]$. Here the relative factor matrix error is defined as

$$\max\left(\frac{\|\mathbf{U} - \mathbf{U}(\alpha)\|_{\mathrm{F}}}{\|\mathbf{U}\|_{\mathrm{F}}}, \frac{\|\mathbf{V} - \mathbf{V}(\alpha)\|_{\mathrm{F}}}{\|\mathbf{V}\|_{\mathrm{F}}}\right),$$

where columns of $\mathbf{U}(\alpha)$ and $\mathbf{V}(\alpha)$ and have been optimally permuted and scaled to match the columns of $\mathbf{U}$ and $\mathbf{V}$, respectively.

We compute the approximation $\hat{\mathcal{T}}(\alpha) = [\![\mathbf{U}(\alpha), \mathbf{U}(\alpha), \mathbf{V}(\alpha)]\!]$ of $\mathcal{T}$ by computing a total of four optimization based CPD approximations of $\mathcal{T} + \alpha\mathcal{E}_*$, one using a GEVD based initialization[8] and three using random initialization. These approximations are computed using the "cpd" function[9] available in Tensorlab [52]. The CPD which minimizes the spectral norm to $\mathcal{T} + \alpha\mathcal{E}_*$ is taken to be $\hat{\mathcal{T}}(\alpha)$.

For each choice of I and J, we repeat the above experiment $10,000$ times, and, for each choice of α, we report the $0.01, 0.05, 0.25$, and 0.5 quantiles of the minimum observed column angles, in addition to the $0.99, 0.95, 0.75$, and 0.5 quantiles of the maximum column norms and factor matrix error. In addition, for each fixed α we record the proportion of the $10,000$ tensors $\mathcal{T} + \alpha\mathcal{E}_*$ for which the first two MLSVD slices have either complex or (approximately[10]) repeated generalized eigenvalues.

5.3.1. $I = 4$ **and** $J = 4$**.** The results of the above experiment with $I = J = 4$ and $10,000$ trials are plotted in Figure 3. In this setting, Theorem 2.4 does not guarantee that $\mathcal{T} + \mathcal{E}_*$ does not have a best rank 4 approximation. Figure 3 suggests that $\mathcal{T} + \mathcal{E}_*$ can indeed have a best rank 4 approximation as most quantiles remain well conditioned for α close to 1. However, for a small number of randomly generated tensors, we observe that the approximate CPD $[\![\mathbf{U}(\alpha), \mathbf{U}(\alpha), \mathbf{V}(\alpha)]\!]$ of $\mathcal{T} + \alpha\mathcal{E}_*$ encounters significant numerical issues when $\alpha \geq 1$. This suggests that there are tensors $\mathcal{T}$ for which $\mathcal{T} + \mathcal{E}_*$ does not have a best rank 4 approximation.

Figure 3 illustrates that, for some of the tensors $\mathcal{T}$ generated in this experiment, one can no longer reliably recover the CPD of $\mathcal{T}$ from the approximate CPD of $\mathcal{T} + \alpha\mathcal{E}_*$ when $\alpha \geq 1$. For $\alpha \geq 1.5$, the CPD of $\mathcal{T}$ is not reliably recovered for over twenty-five percent of our randomly generated tensors. The left-hand images in this figure

[7] One can use any subpencil of $\mathcal{T} + \alpha\mathcal{E}_*$ to compute an algebraic CPD initialization via a generalized eigenvalue decomposition. Using the first two slices of the MLSVD is a popular choice as it can help address some (but not all) numerical issues, e.g., see [3].

[8] If the first two slices of the MLSVD of $\mathcal{T} + \alpha\mathcal{E}_*$ have repeated generalized eigenvalues, then we replace the GEVD initialization with an additional random initialization. This is done to avoid numerical issues when computing GEVD based initializations from ill-conditioned matrix pencils.

[9] We use Tensorlab to enforce that the computed decomposition is partially symmetric. Aside from this, we use the default parameter settings of the "cpd" function.

[10] In this experiment, we consider generalized eigenvalues to be approximately equal if the chordal distance, e.g., see [50], between them is less than 10^{-5}.

suggest that diverging components do occur for some tensors when computing a rank I approximation of $\mathcal{T}+\alpha\mathcal{E}_*$ once $\alpha \geq 1$. Furthermore, the bottom right image shows that, when $\alpha \geq 1$, the first two slices of the MLSVD of $\mathcal{T}+\alpha\mathcal{E}_*$ typically have repeated or complex generalized eigenvalues. This further suggests that there can be numerical challenges in computing a best approximation even if a best approximation exists.

It is natural to wonder if the tensors $\mathcal{T}$ which encounter the most significant numerical issues in computing approximations of $\mathcal{T}+\alpha\mathcal{E}_*$ for $\alpha \geq 1$ have worse than average conditioning for $\alpha < 1$. We individually examined several of these tensors and surprisingly found that they were typically among the best conditioned tensors when $\alpha < 1$.

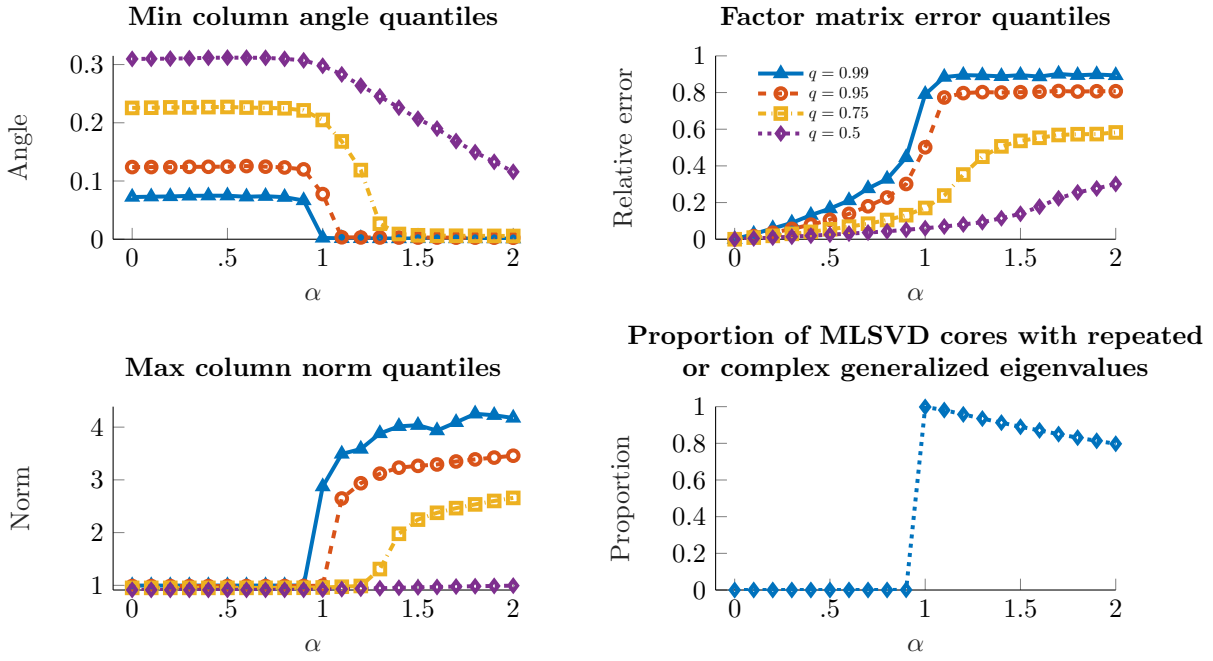


FIG. 3. *Quality of recovery of the CPD of the rank 4 tensor* $\mathcal{T} \in (S^2\mathbb{R}^4)\otimes\mathbb{R}^2$ *from a rank 4 approximation of* $\mathcal{T}+\alpha\mathcal{E}_*$ *for* $\alpha \in [0,2]$. *Here* $\mathcal{E}_*$ *is as described in Theorem 2.4. Significant numerical issues are encountered for many tensors when* $\alpha \geq 1$.

5.3.2. $I = 4$ **and** $J = 2$**.** The results for this experiment with $I = 4$ and $J = 2$ and $10,000$ trials are plotted in Figure 4. Here, Theorem 2.4 does guarantee that any neighborhood of $\mathcal{T}+\mathcal{E}_*$ contains a tensor which does not have a best rank 4 approximation. Figure 4 indeed illustrates that computing a rank 4 approximation of $\mathcal{T}+\alpha\mathcal{E}_*$ is highly ill-conditioned when $\alpha \geq 1$.

6. Conclusion. We presented a deterministic bound for the existence of a best low rank approximation of a SFS tensor that has a linear combination of frontal slices that is positive (semi)definite. We derived this bound by proving that the set of low rank SFS tensors that have a linear combination of frontal slices that is PD is relatively closed as a subset of the set of SFS tensors which have a linear combination of frontal slices that is PD. Furthermore we showed that, for tensors in $(S^2\mathbb{R}^I)\otimes\mathbb{R}^J$, our bound is sharp and that it is computable using semidefinite programming. Our results were illustrated on a variety of numerical experiments. The experiments showed that our bound is highly predictive of numerical errors that can occur when one attempts to

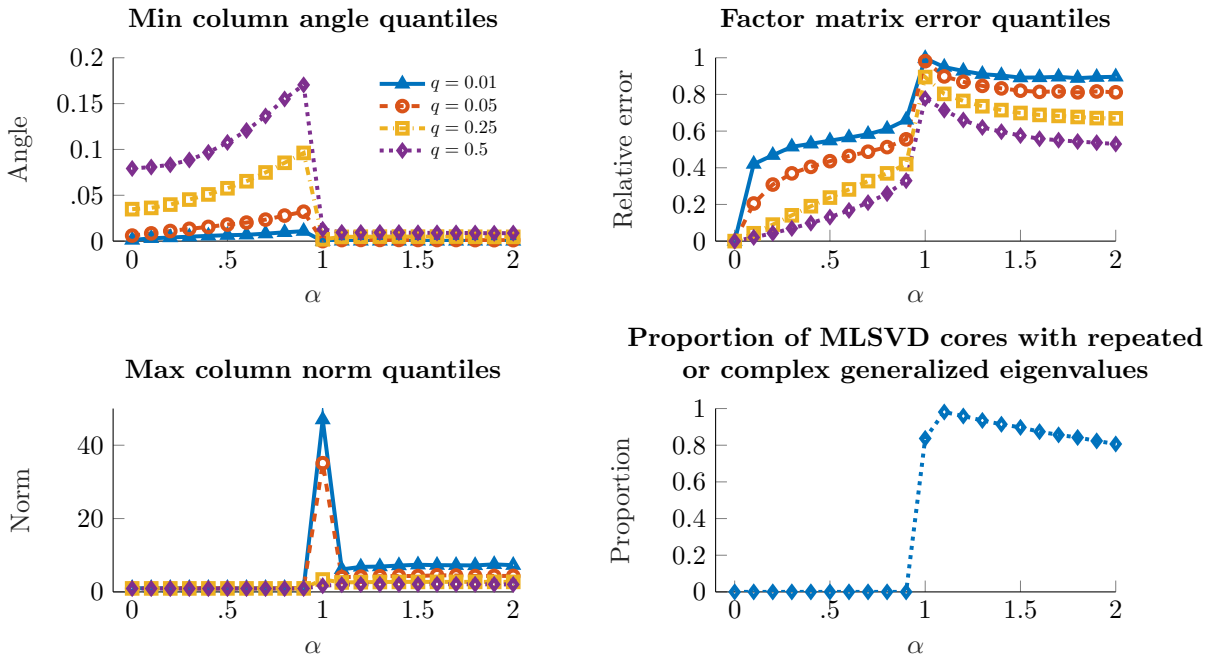


FIG. 4. *Quality of recovery of the CPD of the rank 4 tensor* $\mathcal{T} \in (S^2\mathbb{R}^4) \otimes \mathbb{R}^2$ *from a rank 4 approximation of* $\mathcal{T} + \alpha\mathcal{E}_*$ *for* $\alpha \in [0, 2]$. *Here* $\mathcal{E}_*$ *is as described in Theorem 2.4. Significant numerical issues are encountered for all tensors when* $\alpha \geq 1$.

recover the factors of a low rank SFS tensor $\mathcal{T}$ from a noisy tensor $\mathcal{T} + \mathcal{E}$.

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