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Uniqueness result and algebraic algorithm for decomposition into multilinear rank-$(M_r, N_r, \cdot)$ terms and joint block diagonalization

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***Abstract*—The block term decomposition (BTD) is a highly flexible tensor decomposition and can be viewed as a unification of the canonical polyadic and multilinear singular value decompositions. The BTD expresses a tensor as a sum of rank-one and block (CPD/MLSVD-like) terms. As a subproblem, the BTD captures joint block diagonalization (JBD) of a collection of matrices. This makes the BTD well-suited to a large number of signal processing tasks including blind source separation, hyperspectral unmixing, and source localization. We present novel, mild uniqueness conditions for the BTD and JBD, and give a new algebraic algorithm. A key feature of the algorithm is that it does not require knowledge of the block sizes appearing in the BTD. The efficacy of the algorithm is demonstrated through a blind source separation experiment.**

***Index Terms*—multilinear algebra, tensors, block term decomposition, joint block diagonalization, uniqueness, algorithm**

I. Introduction

Tensors are natural objects for representing higher-order data and for modeling nonlinear interactions between input and output data. These multiindexed arrays have come to play an important role in many signal processing and machine learning applications [1]. Tensor decompositions generalize matrix decompositions and are powerful tools that enable recovery of underlying components. Furthermore, collections of matrices can naturally be viewed as third-order tensors. In this way, tensor decompositions generalize important *joint* matrix decompositions, such as joint matrix diagonalization.

Many types of tensor decomposition are (essentially) unique under mild conditions [2]–[4]. This is in stark contrast to the matrix case where matrix decompositions are only unique under strong assumptions such as orthogonality or triangularity of the factors. This essential uniqueness makes tensor decomposition a natural approach for problems such as blind source separation [1], [5].

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We study the block term decomposition (BTD) of a tensor which was introduced in 2008 [6]–[8] and can intuitively be viewed as a unification of the multilinear singular value decomposition (MLSVD) and the canonical polyadic decomposition (CPD). The MLSVD extends notions of row and column rank to tensors and reveals the principal subspaces associated with a tensor [9]. On the other hand, the CPD expresses a tensor as a minimal sum of rank-1 terms thereby leading to another (different) natural notion of tensor rank. The BTD connects these decompositions by expressing a tensor as a summation of block (MLSVD-like) terms, some of which may be rank-1 (as in CPD). Furthermore, the BTD is more flexible than the CPD since sources of interest may fail to be captured by individual rank-1 terms [10]–[16]. See [17] for additional references.

The BTD generalizes the problem of joint block matrix diagonalization (JBD). If one views a third-order tensor as a collection of matrices and places appropriate constraints, e.g., symmetry, on the BTD factors, then the constrained BTD is equivalent to a JBD of the corresponding matrices. JBD has itself been applied to various problems including parameter estimation [18], source localization [19], and blind source separation [20]–[22]; see Section V for a detailed discussion.

We present new uniqueness conditions for the BTD and consequently for JBD. We show that under (mild) assumptions, a BTD can be computed algebraically by first finding the null space of a particular matrix and then performing a coupled simultaneous eigenvalue decomposition (CS-EVD). Under the conditions, the null space and CS-EVD have (essentially) unique solutions, hence the BTD is (essentially) unique, which is exploited in the new algebraic algorithm. A key feature of this algorithm is that, in contrast to optimization-based methods, the block sizes and the number of block terms appearing in the BTD are not required as inputs: these parameters are recovered by the algorithm. Because of the connection between the BTD and JBD, our BTD uniqueness conditions lead to new generic uniqueness conditions for JBDs.

II. Terminology and problem statement

Throughout the paper, $\mathbb{F}$ denotes the field of real or complex numbers; bold-face lower case, bold-face capital case, and capital calligraphic letters refer to vectors, matrices, and tensors, respectively (e.g., $\mathbf{a}_1$, $\mathbf{B}_2$, and $\mathcal{T}$); $\mathbf{I}_n$ denotes the $n \times n$ identity matrix; $\mathrm{Null}\,(\cdot)$ denotes the null space of a matrix; $r_{\mathbf{A}}$ denotes the rank of a matrix $\mathbf{A}$; "$*$", "^{T}", and "^{H}" denote the conjugate, transpose, and Hermitian transpose, respectively.

Let $\mathcal{T} \in \mathbb{F}^{I \times J \times K}$ be a tensor and let $\mathbf{T}_1, \ldots \mathbf{T}_K \in \mathbb{F}^{I \times J}$ denote the frontal slices of $\mathcal{T}$. The matrices

$$\begin{aligned}
\mathbf{T}_{[1;3,2]} &= [\mathbf{T}_1 \ \ldots \ \mathbf{T}_K] \in \mathbb{F}^{I \times KJ}, \\
\mathbf{T}_{[2;3,1]} &= [\mathbf{T}_1^{\mathrm{T}} \ \ldots \ \mathbf{T}_K^{\mathrm{T}}] \in \mathbb{F}^{J \times KI}, \\
\mathbf{T}_{[3;2,1]} &= [\mathrm{vec}(\mathbf{T}_1) \ \ldots \ \mathrm{vec}(\mathbf{T}_K)]^{\mathrm{T}} \in \mathbb{F}^{K \times JI},
\end{aligned}$$

are *matrix unfoldings* of $\mathcal{T}$ and the triplet of their ranks $(r_{\mathbf{T}_{[1;3,2]}}, r_{\mathbf{T}_{[2;3,1]}}, r_{\mathbf{T}_{[3;2,1]}})$ is called the *multilinear (ML) rank* of $\mathcal{T}$. Here, $\mathrm{vec}(\mathbf{T}_j)$ denotes the vector formed by stacking in increasing order the columns of $\mathbf{T}_j$ in a single vector.

A tensor $\mathcal{T}$ has ML rank-$(M, N, \cdot)$ if $r_{\mathbf{T}_{[1;3,2]}} = M$, $r_{\mathbf{T}_{[2;3,1]}} = N$ and $r_{\mathbf{T}_{[3;2,1]}}$ is not specified. It can be shown that $\mathcal{T}$ is ML rank-$(M, N, \cdot)$ if and only if its frontal slices can be simultaneously factorized as

$$\mathbf{T}_k = \mathbf{A}\mathbf{D}_k\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D}_k \in \mathbb{F}^{M \times N}, \qquad k = 1, \ldots, K, \quad (1)$$

where the factors $\mathbf{A} \in \mathbb{F}^{I \times M}$ and $\mathbf{B} \in \mathbb{F}^{J \times N}$ have full column rank and the matrices

$$\mathbf{D}_{[1;3,2]} \in \mathbb{F}^{M \times KN} \quad \text{and} \quad \mathbf{D}_{[2;3,1]} \in \mathbb{F}^{N \times KM}$$

have full row rank. (One can, for instance, take $\mathbf{A}$ and $\mathbf{B}$ equal to the "$\mathbf{U}$" factor in the compact SVDs of $\mathbf{T}_{[1;3,2]}$ and $\mathbf{T}_{[2;3,1]}$, respectively, and set $\mathbf{D}_k = \mathbf{A}^{\mathrm{H}}\mathbf{T}_k\mathbf{B}^{*}$.) Note that the factors $\mathbf{A}$, $\mathbf{D}_k$, and $\mathbf{B}$ in Eq. (1) are not unique.

We write the K simultaneous factorizations in Eq. (1) as

$$\mathcal{T} = \mathcal{D} \cdot_1 \mathbf{A} \cdot_2 \mathbf{B},$$

where $\mathcal{D}$ is an $M \times N \times K$ tensor with frontal slices $\mathbf{D}_1, \ldots, \mathbf{D}_K$, and $\cdot_n$ is the mode-n tensor-matrix product.

A BTD of a tensor $\mathcal{T}$ can be described as a decomposition of $\mathcal{T}$ into a sum of ML rank-$(M_r, N_r, \cdot)$ tensors. In particular, the ML rank-$(M_r, N_r, \cdot)$ decomposition of a $\mathcal{T}$ is defined as:

$$\begin{aligned}
\mathcal{T} = \sum_{r=1}^{R} \mathcal{D}_r \cdot_1 \mathbf{A}_r \cdot_2 \mathbf{B}_r, \qquad & \mathbf{A}_r \in \mathbb{F}^{I \times M_r}, \\
\mathcal{D}_r \in \mathbb{F}^{M_r \times N_r \times K}, \qquad & \mathbf{B}_r \in \mathbb{F}^{J \times N_r}.
\end{aligned} \quad (2)$$

An ML rank-$(M_r, N_r, \cdot)$ decomposition of a tensor $\mathcal{T}$ is called (essentially) *unique* if for any two decompositions of the form Eq. (2) one can be obtained from the other by a permutation of summands. This implies that the *subspaces* $\mathrm{col}(\mathbf{A}_r)$ and $\mathrm{col}(\mathbf{B}_r)$ are unique, $r = 1, \ldots, R$.

Throughout the paper, we assume that the matrices

$$\begin{aligned}
\mathbf{A} &= [\mathbf{A}_1 \ \ldots \ \mathbf{A}_R] \in \mathbb{F}^{I \times \sum M_r}, \quad \text{and} \\
\mathbf{B} &= [\mathbf{B}_1 \ \ldots \ \mathbf{B}_R] \in \mathbb{F}^{J \times \sum N_r},
\end{aligned} \quad (3)$$

have full column rank. Additionally, let $\mathbf{D}_{r,k} \in \mathbb{F}^{M_r \times N_r}$ denote the kth frontal slice of the tensor $\mathcal{D}_r$ and for $k = 1, \ldots, K$, let

$$\mathbf{D}_k = \mathrm{blockdiag}(\mathbf{D}_{1,k}, \ldots, \mathbf{D}_{R,k}) \in \mathbb{F}^{\sum M_r \times \sum N_r} \quad (4)$$

denote a block-diagonal matrix with matrices $\mathbf{D}_{1,k}, \ldots, \mathbf{D}_{R,k}$ on the diagonal.

III. Uniqueness result and algorithm

We now present a novel condition for the uniqueness of the BTD. In contrast to previous works, our result can be applied in the case where $M_r \neq N_r$ for some or all of the indices r.

Theorem 1. *Let $\mathcal{T}$ admit the ML rank-$(M_r, N_r, \cdot)$ decomposition Eq. (2), the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}_1, \ldots, \mathbf{D}_K$ be defined as in Eq. (3) and Eq. (4), respectively, and set*

$$\mathbf{M}_{\mathcal{D}} := \begin{bmatrix} \mathbf{D}_1^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{D}_1 \\ \vdots & \vdots \\ \mathbf{D}_K^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{D}_K \end{bmatrix}. \quad (5)$$

Assume that

1) $\mathbf{A}$ *and* $\mathbf{B}$ *have full column rank and*
2) $\dim \mathrm{Null}\,(\mathbf{M}_{\mathcal{D}}) = R$.

Then the ML rank-$(M_r, N_r, \cdot)$ decomposition of $\mathcal{T}$ is unique and can be computed by means of coupled simultaneous eigenvalue decomposition (CS-EVD).

Proof. The full proof is given in [17]. □

Note that $\mathrm{Null}\,(\mathbf{M}_{\mathcal{D}}) \geq R$ for any choice of $\mathcal{D}$. This follows from the fact that the system of matrix equations

$$\mathbf{X}\mathbf{D}_k = \mathbf{D}_k\mathbf{Y} \in \mathbb{F}^{M \times N}, \qquad k = 1, \ldots, K \quad (6)$$

has at least R linearly independent solutions, which in turn follows from Eq. (4). Therefore, Assumption 2 intuitively states that $\mathrm{Null}\,(\mathbf{M}_{\mathcal{D}})$ is minimal.

IV. Algebraic algorithm

The algebraic procedure for computing a BTD which stems from Theorem 1 is summarized in Algorithm 1. To simplify exposition, Theorem 1 and Algorithm 1 are stated for the "square" case where $\sum M_r = I$ and $\sum N_r = J$. The more general situation of $\sum M_r \leq I$ and $\sum N_r \leq J$ can be handled by standard preprocessing, e.g., by orthogonal compression using MLSVD.

We emphasize that Algorithm 1 also recovers the number R and the dimensions of the blocks, i.e., the values M_r and N_r. Indeed, in steps 1–4 of Algorithm 1, we construct matrices $\mathbf{U}_{I,1}, \ldots, \mathbf{U}_{I,R} \in \mathbb{F}^{I \times I}$ and $\mathbf{U}_{J,1}, \ldots, \mathbf{U}_{J,R} \in \mathbb{F}^{J \times J}$ that can be simultaneously diagonalized by $\mathbf{A}$ and $\mathbf{B}^{-\mathrm{T}}$, respectively. For a random choice of scalars $\alpha_1, \ldots, \alpha_R$ the parameters $M_1, \ldots, M_R$ and $N_1, \ldots, N_R$ can be obtained by determining the multiplicities of the eigenvalues of the matrices $\mathbf{F} = \alpha_1\mathbf{U}_{I,1} + \cdots + \alpha_R\mathbf{U}_{I,R}$ and $\mathbf{G} = \alpha_1\mathbf{U}_{J,1} + \cdots + \alpha_R\mathbf{U}_{J,R}$. Indeed, $\mathbf{F}$ and $\mathbf{G}$ each have the same R distinct eigenvalues $\lambda_1, \ldots, \lambda_R$ which occur with multiplicities $M_1, \ldots, M_R$ and $N_1, \ldots, N_R$, respectively.

Algorithm 1 Computation of BTD (2) by CS-EVD.
Require: $\mathcal{T} \in \mathbb{F}^{I \times J \times K}$ such that $I = \sum M_r$ and $J = \sum N_r$.

1: Define the IJK-by-$(I^2 + J^2)$ matrix $\mathbf{M}_{\mathcal{T}}$ as

$$\mathbf{M}_{\mathcal{T}} = \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{T}_1 \\ \vdots & \vdots \\ \mathbf{T}_K^{\mathrm{T}} \otimes \mathbf{I}_I & -\mathbf{I}_J \otimes \mathbf{T}_K \end{bmatrix}. \tag{7}$$

2: Compute $\mathbf{u}_r = [u_{1,r} \ \ldots \ u_{I^2+J^2,r}]^{\mathrm{T}}$, $r = 1, \ldots, R$, that form a basis of $\mathrm{Null}\,(\mathbf{M}_{\mathcal{T}})$.
3: For $r = 1, \ldots, R$, reshape $[u_{1,r} \ \ldots \ u_{I^2,r}]^{\mathrm{T}}$ into the $I \times I$ matrix $\mathbf{U}_{I,r}$ and reshape $[u_{I^2+1,r} \ \ldots \ u_{I^2+J^2,r}]^{\mathrm{T}}$ into the $J \times J$ matrix $\mathbf{U}_{J,r}$.
4: For $r = 1, \ldots, R$, solve, e.g., algebraically using a GEVD-based approach, the CS-EVD problem

$$\mathbf{U}_{I,r} = \mathbf{A}\,\mathrm{blockdiag}(g_{1r}\mathbf{I}_{M_1}, \ldots, g_{Rr}\mathbf{I}_{M_R})\mathbf{A}^{-1}, \tag{8}$$
$$\mathbf{U}_{J,r} = \mathbf{B}^{-\mathrm{T}}\,\mathrm{blockdiag}(g_{1r}\mathbf{I}_{N_1}, \ldots, g_{Rr}\mathbf{I}_{N_R})\mathbf{B}^{T}, \tag{9}$$

with respect to the matrices $\mathbf{A}$, $\mathbf{B}$ and the values $M_1, \ldots, M_R$, $N_1, \ldots, N_R$.
5: Given $\mathbf{A}, \mathbf{B}$, and $\{\mathbf{T}_k\}_{k=1}^K$, solve the set of linear equations

$$\mathbf{T}_k = \mathbf{A}\mathbf{D}_k\mathbf{B}^{\mathrm{T}} = \mathbf{A}\,\mathrm{blockdiag}(\mathbf{D}_{1,k}, \ldots, \mathbf{D}_{R,k})\mathbf{B}^{\mathrm{T}} \tag{10}$$

for $\mathbf{D}_1, \ldots, \mathbf{D}_K$.
6: For $r = 1, \ldots, R$, stack $\mathbf{D}_{r,1}, \ldots, \mathbf{D}_{r,K}$ into the $M_r \times N_r \times K$ tensor $\mathcal{D}_r$.
7: **return** $\mathbf{A}_r \in \mathbb{F}^{I \times M_r}$, $\mathbf{B}_r \in \mathbb{F}^{J \times N_r}$ and $\mathcal{D}_r \in \mathbb{F}^{M_r \times N_r \times K}$ such that Eq. (2) holds.

The term "CS"-EVD is used since Eqs. (8) and (9) involve a *simultaneous* EVD of R matrices with the same eigenvectors, which is related to the CPD problem [1], [4], [23]. Also, the eigenvalues in Eqs. (8) and (9) are shared so the problems are *coupled*. In practice, this can, e.g., be implemented by solving both simultaneous EVDs separately, followed by clustering and matching the eigenvalues. See [17] for more details.

V. Joint Block Diagonalization

By Eq. (10), the decomposition Eq. (2) can be interpreted as a (possibly nonsymmetric and nonsquare) *joint block diagonalization* (JDB) of the frontal slices of $\mathcal{T}$. Uniqueness and algorithms for JBDs have been studied in the literature, e.g., see [24]–[29], in the following cases:

1) JBD by congruence and $*$-congruence, i.e., the cases $\mathbf{B} = \mathbf{A}$ and $\mathbf{B} = \mathbf{A}^*$, where $M_r = N_r$ for all r; see, for instance, [30]–[32] and the references therein.
2) All blocks are row vectors, i.e., $M_1 = \cdots = M_R = 1$, implying that the ML rank-$(M_r, N_r, \cdot)$ terms in Eq. (2) are ML rank-$(1, N_r, N_r)$; see, e.g., [7], [10], [33]–[36].
3) All blocks are column vectors, i.e., $N_1 = \cdots = N_R = 1$ which follows from 2) by taking the transpose of the identities in Eq. (10).

Apart from the items 1–3 above, conditions for the generic uniqueness of Eq. (10) were given in [7] under the additional assumptions that $M_r = N_r$ for all r and $K \geq 3$. In addition to addressing the general case, our results provide new generic uniqueness conditions in these various special cases; see Table I for a summary of the various constraints.

TABLE I
Some examples of constrained JBD problems

Type of JBD problem	Constraints in Eq. (10)	$\mathbb{F}$
JBD by congruence	$\mathbf{B} = \mathbf{A}$	$\mathbb{R}$ or $\mathbb{C}$
symmetric JBD	$\mathbf{B} = \mathbf{A}$ and $\mathbf{D}_k = \mathbf{D}_k^{\mathrm{T}}$ $\forall\, k$	$\mathbb{R}$ or $\mathbb{C}$
JBD by $*$-congruence	$\mathbf{B} = \mathbf{A}^*$	$\mathbb{C}$
Hermitian JBD	$\mathbf{B} = \mathbf{A}^*$ and $\mathbf{D}_k = \mathbf{D}_k^{\mathrm{H}}$ $\forall\, k$	$\mathbb{C}$

Uniqueness of the constrained JBD problem means uniqueness of the corresponding constrained ML rank-(N_r, N_r, K) decomposition. We say a constrained JBD, i.e., a constrained ML rank-(N_r, N_r, K) decomposition, is generically unique if the decomposition is unique for generic choices of matrices $\mathbf{A}_1, \ldots, \mathbf{A}_R$ of size $I \times N_r$ and of tensors $\mathcal{D}_1, \ldots, \mathcal{D}_R$ of size $N_r \times N_r \times K$ with symmetric frontal slices.

Theorem 2. *Fix I, K, R and N_r for $r = 1, \ldots, R$. Assume that $I \geq \sum N_r$, and $K \geq 3$. Then, for tensors of size $I \times I \times K$, solutions of each of the constrained JBD problems in Table I with block size (N_r, N_r, K) are generically unique and can be computed by means of S-EVD.*

Proof. The full proof is given in [17]. □

Steps 1–4 of Algorithm 1 together with a modified version of step 5 constitute the algebraic procedure related to Theorem 2. In step 5, because of the constraints $\mathbf{B} = \mathbf{A}$ and $\mathbf{B} = \mathbf{A}^*$, the CS-EVD problem Eq. (8)–(9) turns into an S-EVD problem, i.e., Eqs. (8) and (9) become identical. In the case of JBD by congruence, the proposed reduction to S-EVD essentially coincides with the one in [30, Subsection 2.3].

VI. Experiments

The benefits of the proposed algebraic method are illustrated for a DS-CDMA application. Following the blind receiver model in [37], we perform blind separation of a convolutive mixture of signals in the near field, which results in a multipath problem where intersymbol interference occurs. Assume that R users transmit I symbols $s_i^{(r)}$ that impinge via N_r discrete paths on J receiver antennas, and assume that the length of the unknown spreading code is K. The ith measured symbol at the jth antenna for the kth chip can be written as

$$y_{ijk} = \sum_{r=1}^{R} \sum_{n_r=1}^{N_r} a_{jn_r}^{(r)} \sum_{m=1}^{M_r} h_{k+(m-1)K}^{(r,n_r)} s_{i-m+1}^{(r)}, \tag{11}$$

where $\mathbf{A}^{(r)} \in \mathbb{R}^{J \times N_r}$ is the part of the mixing matrix corresponding to user r, $\mathbf{h}^{(r,n_r)} \in \mathbb{R}^{M_r K}$ contains the samples from the spreading waveform $h_r(t - \tau_{n_r})$ for transmitter r and unknown path delay τ_{n_r}, and M_r is the maximal channel length per user. The matricized vectors $\mathbf{h}^{(r,n_r)}$ are collected in a tensor $\mathcal{H}^{(r)} \in \mathbb{R}^{M_r \times N_r \times K}$. The shifted symbols $s_{i-m_r+1}^{(r)}$

are collected in a Toeplitz matrix $\mathbf{S}^{(r)} \in \mathbb{C}^{I \times M_r}$; $s^{(r)}_{i-m_r+1} = 0$ for $i < m_r$. The signal part $\mathcal{Y} \in \mathbb{C}^{I \times J \times K}$ of the tensor of received data $\tilde{\mathcal{Y}} = \mathcal{Y} + \mathcal{N}$ then admits an ML rank-$(M_r, N_r, \cdot)$ decomposition [37]:

$$\mathcal{Y} = \sum_{r=1}^{R} \mathcal{H}^{(r)} \cdot_1 \mathbf{S}^{(r)} \cdot_2 \mathbf{A}^{(r)}, \tag{12}$$

and $\mathcal{N}$ is an unknown noise tensor.

As an experiment, we assume $R = 4$ users transmit $I = 500$ random 4-QAM symbols. The number of receiver antennas is $J = 10$ and the spreading-code length is $K = 6$. The channel length for each user is $M = [5, 3, 3, 4]$ and the number of paths per user is $N = [2, 3, 2, 3]$. The entries of $\mathcal{H}^{(r)}$ and $\mathbf{A}^{(r)}$ are drawn independently from a normal distribution. The tensor $\mathcal{Y}$ is finally perturbed by Gaussian i.i.d. noise $\mathcal{N}$ such that the signal-to-noise ratio (SNR) $\frac{||\mathcal{Y}||}{||\mathcal{N}||}$ is fixed.

Following the generic uniqueness results in Section III, the noiseless model is unique with probability one. Note that, in contrast to [38], $J \geq \sum_r N_r$ is required as the Toeplitz structure is not exploited.

We compare the symbol error rate averaged over all R users for the following scenarios:

- CS-EVD with only $I = \sum_r M_r$, $J = \sum_r N_r$ and R known,
- CS-EVD with all M_r, N_r and R known, and a refinement step using alternating least squares (ALS),
- ALS with random initialization with all M_r, N_r and R known.

Implementation details on the CS-EVD algorithm can be found in [17]; for the computation of the CS-EVD itself, the generalized eigenspace decomposition algorithm for a CPD is used [23]. For the ALS method [8], we report for each experiment the error for the 'median' and 'best' random initial guess (out of ten), i.e., the initialization that resulted in the median/best error per experiment. Note that all parameters need to be known beforehand for ALS, while the CS-EVD method estimates most parameters. The quality of the estimates of the values $I = \sum_r M_r$, $J = \sum_r N_r$ and R depends mostly on the rank detection techniques for noisy matrices, which are not under test here; hence they are assumed to be known.

As no extra structure is imposed, the estimated matrices $\hat{\mathbf{S}}_r$ are a random mixture of the Toeplitz matrices $\mathbf{S}^{(r)}$, i.e., $\hat{\mathbf{S}}^{(r)} = \mathbf{S}^{(r)}\mathbf{M}^{(r)}$, where $\mathbf{M}^{(r)}$ is a mixing matrix. To recover the Toeplitz structure, blind deconvolution is performed as described in [39], [40]. After this blind deconvolution step, the scale and phase ambiguity of the noisy estimated symbols $\hat{s}^{(r)}_i$ is fixed by setting the modulus to one and rotating over the angle that provides the best match to the original symbols $s^{(r)}_i$. Finally, the average symbol error rate is computed as

$$\text{symbol error rate} = \frac{1}{IR} \sum_{r=1}^{R} \sum_{i=1}^{I} |\hat{s}^{(r)}_i - s^{(r)}_i|. \tag{13}$$

In Figure 1, we see that CS-EVD can accurately estimate the original symbols even if the filter lengths M_r and the number of paths N_r are unknown: if the SNR is at least 10 dB, the median performance is perfect recovery; for lower SNR the accuracy deteriorates which is expected as we do not impose prior knowledge such as the Toeplitz structure. If all parameters are known (as is the case for the optimization methods) and refinement is used, the performance of CS-EVD improves slightly. When using ALS from a random initialization, we see that both the 'median' and 'best' initial guess per experiment have a higher median error, hence highlighting the need for many initializations. For lower SNR, ALS performs better but with median error rates over 23% for the best initialization, the results are not good either.

Finally, we can see that the CS-EVD method is fast.[1] The CS-EVD algorithm requires 0.14 s per run on average (1.30 s if refinement is included). ALS requires 4.87 s per initialization, which is already significantly slower. Taking into account that multiple initializations are needed, using the CS-EVD result as an initialization significantly speeds up the computation.

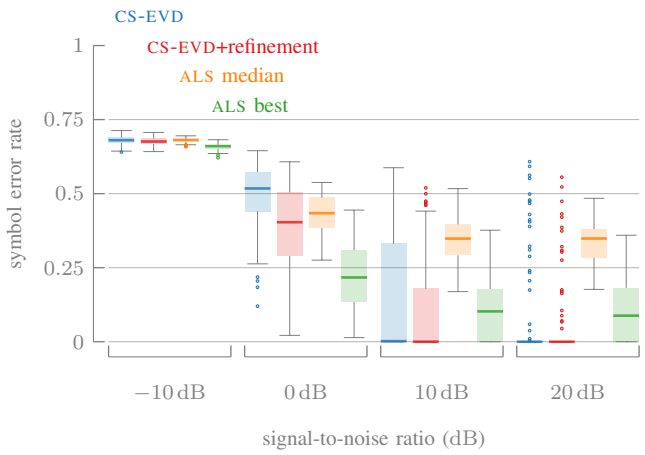

Fig. 1. The algebraic CS-EVD algorithm outperforms the ALS method in almost all cases, even if the latter is initialized ten times. Only for low SNR and when the result for the best initialization is used, ALS performs slightly better. The boxplot reports the results of 100 experiments. The ALS method is initialized ten times per experiment; the errors for the 'median' and 'best' initialization per experiment are reported.

VII. Conclusion

This article contributed new generic and deterministic uniqueness conditions for BTDs and JBDs which hold under mild conditions. Furthermore, a novel algebraic algorithm for the computation of BTDs and JBDs was presented. Among the algorithm's strengths is its ability to recover the number and sizes of blocks appearing in the decomposition, i.e., the parameters, R, M_r, and N_r. The efficacy of the proposed methods was illustrated through numerical experiments. In particular, our algebraic algorithm was shown to improve both numerical reliability and computation time when compared to ALS in a DS-CDMA setting.

[1]The experiments are performed on a laptop running Ubuntu 20.04, Matlab 2019b and Tensorlab 3.0 [41]. The laptop has an Intel(R) Xeon(R) E-2176M CPU running at 2.70 GHz and 32 GB of RAM.

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