Robust Synthesis of Constrained Linear State Feedback using LMIs and Polyhedral Invariant Sets

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***Abstract*— This paper deals with the synthesis of linear feedback controllers for constrained discrete-time linear systems with polytopic uncertainty description. This problem was initially tackled using ellipsoidal robust invariant sets in the paper by Kothare (Kothare *et al.*, Automatica, 1996) leading to an LMI-based optimization problem. The contribution of this paper is twofold. First the aforementioned method is extended to also take into account mixed state/input constraints and cross-terms between states and inputs in the objective function. Secondly, polyhedral invariant sets (Pluymers *et al.*, ACC, 2005) are introduced in the synthesis algorithm in order to eliminate conservative constraint handling. This results in a provably more optimal feedback controller. The resulting method can either be used on-line in a receding horizon fashion to obtain a robust constrained controller or off-line in order to obtain a feedback controller with a guaranteed feasible region for use in dual-mode or other robust model-based predictive control algorithms. A numerical example is given to illustrate the new algorithm.**

I. INTRODUCTION

Model-based predictive control (MPC) is an optimization-based control paradigm that has found widespread adoption in practical applications due to the fact that constraints on the inputs and states can be explicitly taken into account in the controller formulation. Some maturity has been achieved with respect to MPC for linear models, but robustness aspects [4], [9], [11] and nonlinear MPC are still active research areas. An overview can be found in [1], [5], [10].

This paper combines and improves two complimentary algorithms in order to obtain a computationally tractable control algorithm with improved control characteristics. On the one hand the robust MPC method introduced by Kothare [3] is used. This algorithm is able to construct linear robustly stabilizing feedback controllers for LPV systems subject to constraints, but has the disadvantage that it uses ellipsoidal invariant sets. This can lead to conservative constraint handling, even when the algorithm is applied in a receding horizon fashion, i.e. when a feedback law is recomputed at each time instant. On the other hand the algorithm recently introduced in [6], [7] is used to obtain an improved characterization of the feasible region of the resulting controller. This was previously impossible since no algorithm existed to efficiently compute polyhedral invariant sets for LPV systems. The downside of the latter algorithm is that it does not allow the simultaneous construction of a controller and the corresponding polyhedral invariant set, as does the algorithm discussed in [3].

The contribution of this paper is twofold. First the method of [3] is extended to also include mixed state/input constraints and cross-terms between states and inputs in the quadratic objective function. Secondly, the use of polyhedral invariant sets is introduced in the controller synthesis to assess the conservativeness of the resulting controller and to iteratively recompute the linear feedback law in order to improve the constraint handling. The resulting algorithm consists of the sequential solution of several SDPs.

This paper is organized as follows. Section II first describes the problem formulation and introduces some necessary notation, after which Section III summarizes the results of [3]. Section IV then discusses the first contribution regarding the extension towards mixed state/input constraints and cost terms. Section V discusses the second contribution consisting of the use of polyhedral invariant sets in the controller synthesis. Section VI provides a numerical example illustrating the results discussed in the previous sections.

Throughout this paper MATLAB notation is used to denote rows, columns and elements of matrices, i.e. $A_{[i,:]}$ denotes the i-th row of matrix A and $A_{[i,j]}$ denotes the ij-th entry of matrix A. The notation $[A_1;\ A_2]$ will be used to denote stacked matrices $[A_1^{\mathrm{T}}\ A_2^{\mathrm{T}}]^{\mathrm{T}}$. Vector sets will be denoted using calligraphic upper case symbols $(\mathcal{A}, \mathcal{B}, \ldots)$. Ellipsoidal and polyhedral invariant sets are respectively denoted as $\mathcal{E}$ and $\mathcal{P}$. Matrix sets will be denoted with Greek upper case symbols, e.g. Ω. Scalar inequality will be denoted as $a > b$, while matrix inequalities will be denoted as $A \succ B$, indicating that $A - B$ is positive definite.

II. PROBLEM FORMULATION

This paper considers Linear Parameter-Varying (LPV) systems with polytopic uncertainty description

$$x_{k+1} = A(k)x_k + B(k)u_k, \qquad k \geq 0, \tag{1}$$

with

$$[A(k)\ B(k)] \in \Omega \equiv \mathrm{Co}\{[A_1\ B_1], \ldots, [A_p\ B_p]\}, \quad k \geq 0. \tag{2}$$

$x_k \in \mathbb{R}^{n_x}, u_k \in \mathbb{R}^{n_u}$ respectively denote the state and the input of the system at discrete time k, with n_x, n_u respectively denoting the number of states and inputs. $A(k) \in \mathbb{R}^{n_x \times n_x}, B(k) \in \mathbb{R}^{n_x \times n_u}$ are the system matrices at time k and $\Omega \subset \mathbb{R}^{n_x \times (n_x + n_u)}$ denotes the uncertainty polytope of the LPV system. The matrices $A_i, B_i, i = 1, \ldots, p$ represent the vertices of the uncertainty polytope, with $p \in \mathbb{N}\backslash\{0\}$ denoting the number of vertices describing Ω. It is important to note that the matrices $A(k), B(k)$ are time-varying, but their value is not necessarily known beforehand, so this information cannot be taken into account in the controller design.

The system is subject to state and input constraints

$$x_k \in \mathcal{X} \equiv \{x | A_x x \leq 1_{\mathrm{v}}\}, \qquad \forall k \geq 0, \tag{3a}$$

$$u_k \in \mathcal{U} \equiv \{u | A_u u \leq 1_{\mathrm{v}}\}, \qquad \forall k \geq 0, \tag{3b}$$

where 1_{v} in each equation denotes a column matrix of appropriate length containing only 1's. In later sections also mixed state and input constraints will be considered :

$$[x_k; u_k] \in \mathcal{Y} \equiv \{y | A_{xu} y \leq 1_{\mathrm{v}}\}, \qquad \forall k \geq 0. \tag{4}$$

The aim is to find a linear feedback controller

$$u_k = -Kx_k, \qquad k \geq 0, \tag{5}$$

that robustly asymptotically stabilizes (1)-(2) without violating constraints (3) and/or (4) for a given initial state $\bar{x} \in \mathbb{R}^n_x$. Optimality of the controller is defined using the following cost function :

$$J(x_0) \equiv \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q & N \\ N^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \tag{6}$$

with $0 \prec Q = Q^{\mathrm{T}} \in \mathbb{R}^{n_x \times n_x}, 0 \prec R = R^{\mathrm{T}} \in \mathbb{R}^{n_u \times n_u}, N \in \mathbb{R}^{n_x \times n_u}$ respectively denoting state, input and mixed state/input cost weighting matrices.

The problem discussed in this paper can more formally be summarized as follows :

Problem 1 (P1): *Given a system* (1)-(2) *subject to constraints* (3)-(4)*, an optimality criterion defined as* (6) *and an initial state $\bar{x} \in \mathbb{R}^{n_x}$, find a feedback gain K such that the controller* (5) *results in a minimal worst-case (over all possible trajectories starting from the initial state $\bar{x}$) control cost* (6) *and without violating constraints* (3)-(4) *for any of the possible trajectories.*

The initial state $\bar{x}$ can be a state chosen off-line by the user in order to obtain a feedback controller with a desired feasible region. Alternatively P1 can be solved on-line at each time instant k, where $\bar{x}$ is then chosen as the current state measurement x_k.

The following section describes the solution to a relaxation of the above problem, i.e. the solution for the case where no constraints of the form (4) are present, where the cost matrix $N = 0$ and where constraint satisfaction is guaranteed by means of an ellipsoidal invariant set that is a level set of a quadratic Lyapunov function for the closed-loop system.

III. ROBUST MPC USING LINEAR MATRIX INEQUALITIES

This section summarizes the results presented in [3], where the following relaxation of Problem P1 is solved exactly using an SDP based formulation :

Problem 2 (P2): *Given a system* (1)-(2) *subject to constraints* (3)*, an optimality criterion defined as* (6) *with $N = 0$ and an initial state $\bar{x} \in \mathbb{R}^{n_x}$, find a feedback gain K such that*

- *the worst-case cost function $V(\bar{x}) = \bar{x}^{\mathrm{T}} P \bar{x} \geq J(\bar{x})$ is minimal with $0 \prec P = P^{\mathrm{T}} \in \mathbb{R}^{n_x \times n_x}$ satisfying the Lyapunov inequality*

$$P - (A_i - B_i K)^{\mathrm{T}} P (A_i - B_i K) \succ Q + K^{\mathrm{T}} R K, \quad i = 1, \ldots, p, \tag{7}$$

- *the given initial state $\bar{x}$ lies within a feasible invariant ellipsoid $\mathcal{E}$ of the form*

$$\mathcal{E} \equiv \{x | x^{\mathrm{T}} P x \leq \gamma\}, \tag{8}$$

with $\gamma > 0$.

Before formulating the solution to P2, we formally define the concept of feasible invariant sets.

Definition 1 (Robust Positive Invariance): *Given an LPV system* (1)-(2) *and a controller* (5)*, then a set $\mathcal{S} \in \mathbb{R}^{n_x}$ is a robust positive invariant set if*

$$(A - BK)x \in \mathcal{S}, \quad \forall x \in \mathcal{S}, \forall [A\ B] \in \Omega. \tag{9}$$

Definition 2 (Feasibility): *A set $\mathcal{S}$ is feasible with respect to constraints* (3)-(4) *if*

$$x \in \mathcal{X}, \quad -Kx \in \mathcal{U}, \quad [x; -Kx] \in \mathcal{Y}, \qquad \forall x \in \mathcal{S}. \tag{10}$$

In what follows we will refer to robust positive invariant sets as invariant sets, without explicitly mentioning the terms robust and positive.

As shown in [3] the following algorithm now solves Problem P2 exactly :

Algorithm 1 ([3], applied off-line): *Given a system* (1)-(2) *subject to constraints* (3)*, an optimality criterion defined as* (6) *with $N = 0$ and an initial state $\bar{x} \in \mathbb{R}^{n_x}$, solve the following optimization problem :*

$$\min_{\gamma, Y, Z = Z^{\mathrm{T}} \succ 0} \gamma, \tag{11a}$$

subject to

$$\begin{bmatrix} 1 & * \\ \bar{x} & Z \end{bmatrix} \succ 0, \tag{11b}$$

$$\begin{bmatrix} Z & * & * & * \\ Q^{\frac{1}{2}}Z & \gamma I & * & * \\ R^{\frac{1}{2}}Y & 0 & \gamma I & * \\ AZ+BY & 0 & 0 & Z \end{bmatrix} \succ 0, \tag{11c}$$

$$\begin{bmatrix} Z & * \\ (A_u)_{[j,:]}Y & 1 \end{bmatrix} \succ 0, \quad j = 1, \ldots, m_u, \tag{11d}$$

$$\begin{bmatrix} Z & * \\ (A_x)_{[j,:]}Z & 1 \end{bmatrix} \succ 0, \quad j = 1, \ldots, m_x, \tag{11e}$$

where m_u and m_x denote the number of rows in matrices A_u and A_x respectively. Asterisks are used to denote the corresponding transpose of the lower block part of symmetric matrices. The optimal solutions to this optimization problem are denoted as γ^o, Y^o, Z^o. The feedback matrix K, the closed-loop Lyapunov function $V(x) = x^T P x$ and invariant ellipsoid $\mathcal{E}$ defined as

$$K = -Y^o (Z^o)^{-1}, \tag{11f}$$

$$P = \gamma^o (Z^o)^{-1}, \tag{11g}$$

$$\mathcal{E} = \{x | x^T (Z^o)^{-1} x \leq 1\}, \tag{11h}$$

are optimal solutions to Problem P2.

Remark 1: LMI (11d) is a slight generalization of the input constraint LMI derived in [3] in order to deal with input constraints of the form (3b). LMI (11e) imposes that the invariant ellipsoid $\mathcal{E}$ should lie within the state constraints and is a slight simplification of the state constraint LMI derived in [3] that only imposed that the one-step contraction of $\mathcal{E}$ (i.e. the set within all states at the next time step are guaranteed to lie) should lie within the state constraints.

The following theorem summarizes the stability and feasibility proofs given in [3] :

Theorem 1: If (11) is feasible then the resulting feedback controller (5) with feedback gain (11f) is robustly asymptotically stable and satisfies constraints (3) for the initial state $\bar{x}$.

Proof : Since (11c) is an LMI representation of (7) it is clear that $V(x) = x^T P x$ is a valid Lyapunov function that guarantees asymptotic stability of the resulting closed-loop system. Furthermore, since $\mathcal{E}$ by construction is a level set of $V(x)$, it is clear that $\mathcal{E}$ is robustly invariant, i.e.

$$x_{k+1} \in \mathcal{E}, \quad \forall x_k \in \mathcal{E}, \forall [A(k)\ B(k)] \in \Omega, k \geq 0. \tag{12}$$

Therefore, because (11b) imposes that $x_0 = \bar{x} \in \mathcal{E}$, it is proven by induction on k that $x_k \in \mathcal{E}, \forall k \geq 0$. Since (11e) and (11d) imply that $\mathcal{E} \subset \mathcal{X}$ and $-Kx \in \mathcal{U}, \forall x \in \mathcal{E}$ respectively, it is clear that $\mathcal{E}$ is a feasible set and therefore constraints (3) are satisfied $\forall k \geq 0$. This shows that K results in guaranteed constraint satisfaction for the initial state $\bar{x}$, which then concludes the proof. □

Algorithm 1 can be used to synthesize robust linear feedback controllers with a desired feasibility regions (i.e. invariant ellipsoids $\mathcal{E}$) or can be applied on-line to obtain a robust constrained controller, resulting in the following algorithm :

Algorithm 2 ([3], applied on-line): *Given a system* (1)-(2) *subject to constraints* (3)*, an optimality criterion defined as* (6) *with $N = 0$, solve at each time instant $k \geq 0$ the optimization problem* (11) *with $\bar{x} = x_k$, where x_k is the current state measurement and apply $u_k = -Kx_k$ to the system.*

Algorithm 2 has similar feasibility and stability properties as Algorithm 1. Proofs are omitted due to space constraints. We refer to [3] for details.

IV. MIXED STATE/INPUT COST AND CONSTRAINTS

This section eliminates the assumptions of P2 that $N = 0$ and that no mixed constraints (4) are present and shows how these more general situations still lead to an LMI-based optimization problem.

A. Mixed state/input cost terms

In order to allow for $N \neq 0$ the Lyapunov inequality (7) needs to be adapted and reformulated in an LMI similar to (11c). By substitution of $u_k = -Kx_k$, the objective function (6) can be rewritten as

$$J(x_0) \equiv \sum_{k=0}^{\infty} x_k^T \begin{bmatrix} I \\ -K \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} I \\ -K \end{bmatrix} x_k,$$

which then results in the following Lyapunov inequalities :

$$P - (A_i - B_i K)^T P (A_i - B_i K) \succ \begin{bmatrix} I \\ -K \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} I \\ -K \end{bmatrix}, \quad i = 1, \ldots, p. \tag{13}$$

After substitution of $K = -YZ^{-1}$ and $P = \gamma Z^{-1}$, left and right multiplication with $Z = Z^T \succ 0$ and division by $\gamma > 0$, these inequalities become

$$Z - (A_i Z + B_i Y)^T Z^{-1} (A_i Z + B_i Y) \succ \frac{1}{\gamma} \begin{bmatrix} Z \\ Y \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} Z \\ Y \end{bmatrix}, \quad i = 1, \ldots, p. \tag{14}$$

By applying the Schur complement this can be formulated as the LMI

$$\begin{bmatrix} Z & * & * \\ A_i Z + B_i Y & Z & * \\ Q_{\text{xu}}^{\frac{1}{2}} \begin{bmatrix} Z \\ Y \end{bmatrix} & 0 & \gamma I \end{bmatrix} \succ 0, \quad i = 1, \ldots, p, \tag{15}$$

with $Q_{\text{xu}} = \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix}$. This LMI replaces (11c) in case $N \neq 0$. If $N = 0$, both LMIs are easily shown to be equivalent.

B. Mixed state/input constraints

The aim of this subsection is to reformulate (4) into sufficient LMI conditions in the optimization variables γ, Y, Z. First, we rewrite (4) as

$$A_{xu,x} x_k + A_{xu,u} u_k \leq 1_v, \qquad k \geq 0,$$

where $A_{xu} = [A_{xu,x}\ A_{xu,u}]$. After substitution of $u_k = -Kx_k$ this can be rewritten as

$$[A_{xu,x}\ - A_{xu,u}K]x_k \leq 1_v, \qquad k \geq 0,$$

which, for each row of A_{xu} separately, is satisfied if :

$$\max_{z\in\mathcal{E}} \|[(A_{xu,x})_{[j,:]}\ - A_{xu,u}K]z\| \leq 1, \quad j = 1,\ldots,m_{xu},$$

with m_{xu} denoting the number of rows in A_{xu}. This is equivalent with

$$\bar{\sigma}([(A_{xu,x})_{[j,:]}\ - A_{xu,u}K]Z^{\frac{1}{2}}) \leq 1, \quad j = 1,\ldots,m_{xu},$$

with $\bar{\sigma}(\cdot)$ denoting the largest singular value, which (similar to the derivation of the state constraint LMI in [3]) is satisfied if

$$\begin{bmatrix} Z & * \\ (A_{xu})_{[j,:]}\begin{bmatrix} Z \\ Y \end{bmatrix} & 1 \end{bmatrix} \succ 0, \quad j = 1,\ldots,m_{xu}. \quad (16)$$

This LMI can be added as an additional constraint to (11) in case constraints of the form (4) are present.

V. CONTROLLER SYNTHESIS USING POLYHEDRAL INVARIANT SETS

This section discusses how polyhedral invariant sets can be integrated in the synthesis process to reduce conservative constraint handling and obtain more optimal controllers. This development in turn can be used for obtaining dual-mode [4] or scheduled [12] MPC algorithms with improved constraint handling. Section V-A describes how polyhedral invariant sets can be constructed for LPV systems by summarizing the results from [6], [7]. Section V-B then explains how these invariant sets can be integrated with the algorithms described in Section III.

A. Polyhedral Invariant Sets for LPV Systems

Recently an algorithm was introduced [6], [7] that allows the construction of polyhedral invariant sets for LPV systems with polytopic uncertainty description. The algorithm iteratively adds constraints to the imposed constraint set $\mathcal{X}$ until robust invariance is obtained. The algorithm can be summarized as follows :

Algorithm 3 (Invariant Sets for LPV systems): Given an LPV system (1)-(2), a controller (5) and constraints (3)-(4), construct an invariant set $\mathcal{P} = \{x|A_{\mathcal{P}}x \leq 1_{\mathrm{v}}\}$ using the following steps :

- Set $A_{\mathcal{P}} := T_0 = [A_x; -A_uK; A_{xu,x} - A_{xu,u}K]$ and $i := 1$.
- Iterate until all constraints $T_{i-1}(A_1 - B_1K) \leq 1_{\mathrm{v}}, \ldots, T_{i-1}(A_p - B_pK) \leq 1_{\mathrm{v}}$ are redundant with respect to $A_{\mathcal{P}}x \leq 1_{\mathrm{v}}$:
  - Set $T_i = [T_{i-1}(A_1 - B_1K); \ldots; T_{i-1}(A_p - B_pK)]$.
  - Remove all constraints from T_i that are redundant with respect to $[A_{\mathcal{P}}; T_i]$.
  - Set $A_{\mathcal{P}} = [A_{\mathcal{P}}; T_i]$.
  - Remove all redundant constraints from $A_{\mathcal{P}}$.

Return $\mathcal{P} = \{x|A_{\mathcal{P}}x \leq 1_{\mathrm{v}}\}$.

The following properties are proven in [6], [7] and are repeated here without proof :

- Algorithm 3 terminates in a finite number of iterations if the closed-loop system (1)-(2),(5) is quadratically stable, i.e. if there exists a quadratic Lyapunov function. This is the case if K is calculated using Algorithm 1.
- The set $\mathcal{P}$ is feasible with respect to (3)-(4) and robust positive invariant with respect to the closed-loop dynamics of (1)-(2),(5).
- The set $\mathcal{P}$ is the Maximal Admissible Set (MAS, [2]), i.e. another set $\mathcal{P}'$ can only be a feasible invariant set if $\mathcal{P}' \subset \mathcal{P}$. This implies that also for any ellipsoidal feasible invariant set $\mathcal{E}$ the property $\mathcal{E} \subset \mathcal{P}$ holds.

Every iteration of Algorithm 3 consists of solving a set of Linear Programs (LP's) to determine whether or not new constraints are redundant with respect to the current set $\mathcal{P}$. Due to the fact that only non-redundant constraints are added and that redundant constraints are regularly removed, a large efficiency increase is obtained compared to a more naive approach. As a result, for lower-dimensional systems, the expected exponential increase of the number of constraints is avoided. Algorithm 3 can therefore be considered to be an efficient way to compute polyhedral invariant sets for lower-dimensional systems.

For higher-dimensional systems one can revert to computing reduced-complexity invariant inner approximations of the MAS by using the results from [8].

B. Controller Synthesis

In this section we combine the results discussed in Sections III and V. Two algorithms will be formulated :

- An algorithm that consists of first applying Algorithm 1 after which a polyhedral invariant set is computed using Algorithm 3 resulting in an exact characterization of the feasible region of the closed-loop system.
- An algorithm that consists of iteratively recomputing K (using Algorithm 1) and $\mathcal{P}$ (using Algorithm 3) in order to increase the optimality of K subject to the feasibility requirement $\bar{x} \in \mathcal{P}$.

The first algorithm obviously results in the same controller as Algorithm 1, but returns a more exact characterization of the feasible region, whereas the second algorithm exploits the improved characterization of the feasible region to improve optimality of the controller.

Algorithm 4: *Given a system* (1)-(2) *subject to constraints* (3)-(4)*, an optimality criterion defined as* (6) *with* $N = 0$ *and an initial state* $\bar{x} \in \mathbb{R}^{n_x}$*, perform the following steps :*

- *Apply Algorithm 1 to obtain a feedback gain* K *and a Lyapunov function* $V(x) = x^{\mathrm{T}}Px$*.*
- *Apply Algorithm 3 to obtain the MAS* $\mathcal{P}$ *for the closed-loop system* (1)-(2),(5) *subject to constraints* (3)-(4)*.*

Although Algorithm 4 is rather straightforward, there are a few interesting points to make :

- Since by construction $\bar{x} \in \mathcal{E}$, it is also guaranteed that $\bar{x} \in \mathcal{P}$,

- The Lyapunov function $V(x) = x^{\mathrm{T}}Px$, that is proven in [3] to be valid within $\mathcal{E}$ is also valid for all states $x \in \mathcal{P}$, since all the imposed constraints are satisfied for trajectories starting from such states.
- An upper bound to the control cost $J(\bar{x})$ is given by γ^{o}. This is due to the fact that in the optimum equation (11b) is satisfied with equality, i.e. $\gamma^{\mathrm{o}} = \bar{x}^{\mathrm{T}}P\bar{x} \equiv V(\bar{x})$. Since $V(x)$ is an upper bound to the worst case value of $J(x)$, γ serves as an upper bound to the control cost of trajectories starting from the initial value $\bar{x}$.

Based on these observations we can now formulate an algorithm that iteratively applies Algorithm 4 in order to find an optimal feedback gain K over all feedback gains for which $\bar{x} \in \mathcal{P}$ (instead of $\bar{x} \in \mathcal{E}$ in case of Algorithm 1).

Algorithm 5: *Given a system* (1)-(2) *subject to constraints* (3)-(4)*, an optimality criterion defined as* (6) *with* $N = 0$ *and an initial state* $\bar{x} \in \mathbb{R}^{n_x}$ *such that* (11) *is feasible, solve the following optimization problem :*

$$\min_{c} \quad \gamma(c), \tag{17a}$$

$$\text{s.t.} \quad \bar{x} \in \mathcal{P}_{K(c)}, \tag{17b}$$

where $\gamma(c)$ *and* $K(c)$ *are the values obtained with Algorithm 1 for relaxed constraints* $\mathcal{X}', \mathcal{U}', \mathcal{Y}'$ *:*

$$\mathcal{X}' = c\mathcal{X}, \qquad \mathcal{U}' = c\mathcal{U}, \qquad \mathcal{Y}' = c\mathcal{Y}, \tag{18}$$

with c *a positive scalar.* $\mathcal{P}_{K(c)}$ *is the MAS for the closed-loop system* (1)-(2),(5) *(with* $K = K(c)$*) subject to constraints* (3),(4)*. Return* $K(c^{\mathrm{o}})$ *and* $\mathcal{P}_{K(c^{\mathrm{o}})}$*, with* c^{o} *denoting the optimal solution of* (17)*.*

Optimization problem (17) is a scalar optimization problem with a monotonously decreasing objective function $\gamma(c)$. Therefore the problem is reduced to finding the largest value of c for which (17b) is still satisfied. Since in typical situations the set $\mathcal{C} \triangleq \{c|\bar{x} \in \mathcal{P}_{K(c)}\}$ is convex, one can solve optimization problem (17) by means of interval reduction techniques, e.g. bisection search. In degenerate cases where $\mathcal{C}$ is not convex, one can still easily find a feasible solution due to the following lemma.

Lemma 1: Optimization problem (17) is feasible for $c = 1$.

Proof : One can see that for $c = 1$ the obtained values for $\gamma(c), K(c), \mathcal{P}_{K(c)}$ are identical to the values $\gamma, K, \mathcal{P}$ obtained with Algorithm 4. Since $\bar{x} \subset \mathcal{P} = \mathcal{P}_{K(c)}$ this implies that (17b) is satisfied for $c = 1$. □.

Lemma 1 indicates that an interval reduction method initialized with the interval $[1, \bar{c}], \bar{c} > 1$ will always find a feasible solution to (5). Standard methods can find sufficiently accurate solutions ($\sim 10^{-10}$) in 10 to 20 iterations, with each iteration consisting of the computation of $K(c)$ and the corresponding $\mathcal{P}_{K(c)}$.

Theorem 2: Consider the optimal value $\gamma(c^{\mathrm{o}})$ of Algorithm 5 and the optimal value γ^{o} of Algorithm 1, then the following property holds :

$$\gamma(c^{\mathrm{o}}) \leq \gamma^{\mathrm{o}}. \tag{19}$$

Proof : Since $c = 1$ is always a feasible solution to (17) by virtue of Lemma 1 and $\gamma(1) \equiv \gamma^{\mathrm{o}}$ the theorem is trivially proven. □

Theorem 3: The feedback gain $K(c^{\mathrm{o}})$ obtained with Algorithm 5 robustly asymptotically stabilizes (1)-(2) and satisfies constraints (3)-(4) for trajectories starting from initial state $\bar{x}$.

Proof : Since $K(c^{\mathrm{o}})$ is essentially calculated using Algorithm 1, it robustly asymptotically stabilizes (1)-(2) with Lyapunov function $V(x) = x^{\mathrm{T}}P(c^{\mathrm{o}})x$, with $P(c^{\mathrm{o}})$ defined in a similar way as $K(c^{\mathrm{o}})$. Since $\bar{x} \in \mathcal{P}(c^{\mathrm{o}})$ and $\mathcal{P}(c^{\mathrm{o}})$ is a robust feasible invariant set with respect to the closed-loop system (1)-(2),(5) (with $K = K(c^{\mathrm{o}})$) subject to constraints (3),(4), it is also guaranteed by construction that all trajectories starting from $\bar{x}$ satisfy the imposed constraints. □

Similar to Algorithm 2, it is possible to apply Algorithms 4 and 5 on-line in a receding-horizon fashion. We will refer to these algorithms as Algorithms 4b and 5b.

VI. EXAMPLE

We consider a numerical example describing a double integrator with polytopic model uncertainty described by

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{20a}$$

$$A_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}. \tag{20b}$$

The system is subject to constraints (3)-(4) defined as

$$A_x = [0.01I; \ -0.1I], \tag{21a}$$

$$A_u = [1; \ -2], \tag{21b}$$

$$A_{xu} = [0.1 \ \ 0 \ \ -2]. \tag{21c}$$

The control objective (6) is defined as

$$Q = I, \qquad R = 0.01, \qquad N = [0.05; 0]. \tag{22}$$

Figure 1 depicts resulting invariant sets and trajectories corresponding to controllers computed using Algorithms 4 and 5. Algorithm 4 computes identical controllers for symmetrically positioned initial states, although the imposed constraints are non-symmetrical, which illustrates that it cannot efficiently deal with this setting. The depicted polyhedral invariant sets (dashed) also show that the initial state in some cases lies well within the feasible region, which indicates that the feedback controller will not reach any of the imposed constraints for this initial state. Algorithm 5 results in controllers whose feasible region exactly contain the imposed initial states. Table I indicates that this improved constraint handling leads to more optimal controllers at the expense of an increased computation time.

Figure 2 shows trajectories using Algorithms 2 and 5b. The system behavior was chosen to be alternating between $[A_1 \ B_1]$ and $[A_2 \ B_2]$. Both algorithms lead to stable behavior and satisfy all imposed constraints, including the mixed state/input constraint. Algorithm 5b leads to more complex control behavior and has non-conservative constraint handling, as can be verified in Figure 2. This leads to increased

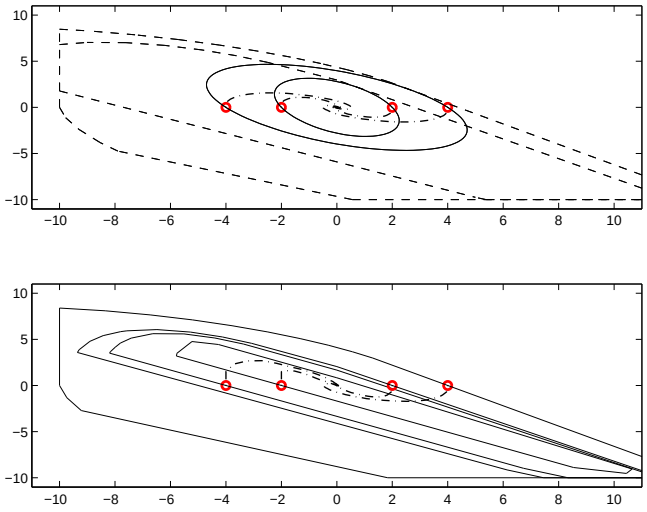

Fig. 1. **Top :** Ellipsoidal (solid) and polyhedral (dashed) invariant sets and trajectories (dash-dotted) corresponding to feedback controllers computed using Algorithm 4 for different initial state $\bar{x}$ (depicted as circles) for the LPV system defined by (20) subject to constraints defined by (21) and for cost matrices (22). **Bottom :** Polyhedral invariant sets (solid) and trajectories (dash-dotted) corresponding to feedback controllers computed using Algorithm 5.

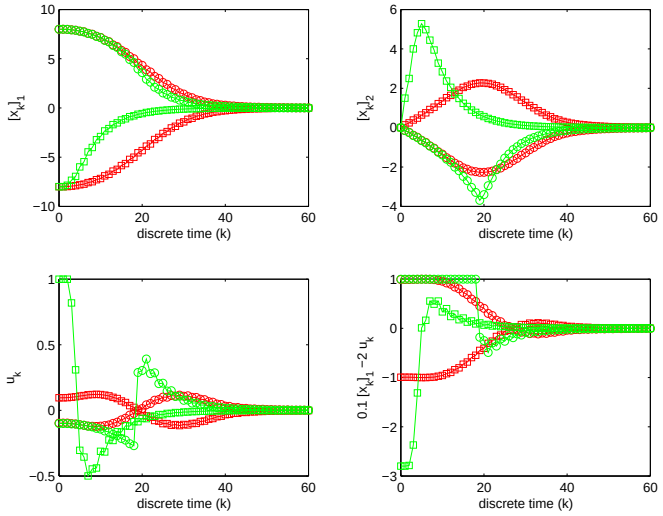

Fig. 2. Input and state trajectories resulting from Algorithms 2 (red) and 5b (green) for initial states $[-8;0]$ (squares) and $[8;0]$ (circles). The same system, constraints and cost matrices as in Figure 1 were used. The lower right subfigure illustrates that the imposed mixed state/input constraint is successfully taken into account.

optimality, with Algorithm 2 resulting in a control cost of 1150.4 for both initial states and Algorithm 5b leading to a control cost of 606.3 and 1104.5 for initial state $[-8;0]$ and $[8;0]$ respectively.

VII. CONCLUSION AND FUTURE WORK

This paper extends the results of [3] in two ways. First the algorithm is extended to also deal with mixed state/input constraints and cost terms. Secondly the algorithm for computing polyhedral invariant sets for LPV systems introduced in [7] is combined with this method to improve the constraint handling in the controller synthesis. The obtained algorithms can be applied either off-line to compute robustly stabilizing linear feedback controllers with guaranteed feasibility or can be applied on-line in a receding horizon fashion.

Future work consists of more tightly integrating the polyhedral invariant sets into the controller synthesis and extending this formulation to gain-scheduled controller synthesis for certain classes of nonlinear systems.

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$\bar{x}$	$[-4;0]$	$[-2;0]$	$[2;0]$	$[4;0]$	T
γ for Algorithm 4	282.78	58.70	58.70	282.78	0.41s
γ for Algorithm 5	207.70	48.41	53.82	270.58	101s

TABLE I

UPPER BOUNDS FOR THE TOTAL CONTROL COST (6) AND FOR DIFFERENT VALUES OF $\bar{x}$ AND AVERAGE COMPUTATION TIME USING TWO DIFFERENT ALGORITHMS FOR COMPUTING THE FEEDBACK CONTROLLER. THE SAME SYSTEM, CONSTRAINTS AND COST MATRICES AS IN FIGURE 1 WERE USED.

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