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A dual decomposition approach to partial crosstalk cancellation in a multi-user xDSL environment[1]

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[1]This report is available by anonymous ftp from *ftp.esat.kuleuven.ac.be* in the directory *pub/sista/jvangorp/reports/05-201_DDCancellation.pdf*

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Abstract

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A dual decomposition approach to partial crosstalk cancellation in a multi-user xDSL environment

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Index Terms

dual decomposition, partial crosstalk cancellation, xDSL, crosstalk, multi-user signal coordination, resource allocation

I. Introduction

CROSSTALK, which is typically 10-15 dB larger than the background noise, is a major source of performance degradation in xDSL systems. One strategy for dealing with this crosstalk is crosstalk cancellation. Several crosstalk cancellation schemes have been proposed. Linear pre- and post filtering [2] [3] requires coordination at both the transmitters and receivers. Successive interference cancellation or precompensation [4] [5] can be used if there is only coordination available at the receivers or transmitters respectively. For this level of coordination, it is shown in [6] [7] that a simple linear zero-forcing canceller or linear precompensator performs near-optimally in an xDSL environment.

Even for these simple linear cancellers, the complexity grows with the square of the number of lines. For example, in a binder of 8 VDSL lines transmitting on 4096 tones at a block rate of 4000 blocks per second, the runtime complexity of crosstalk cancellation exceeds 1 billion multiplications per second.

However, crosstalk exhibits space and tone selectivity [8]. Measurements show that most of the crosstalk originates from a limited number of lines, e.g. those in close proximity. Moreover, crosstalk coupling is heavily dependent on the frequency.

Because most of the crosstalk originates from a limited number of lines on a limited number of tones, a fraction of the complexity of full crosstalk cancellation suffices to cancel most of the crosstalk. This is called partial crosstalk cancellation [8] [9].

The challenge is then to determine for every user which crosstalk to cancel on which tones. In [8], an algorithm based on resource allocation is presented to solve this single-user problem. This paper presents an alternative optimal algorithm, based on a dual decomposition. The complexity of the algorithm is found to be more favourable than the complexity of the resource allocation algorithm, where a multi-user extension is made to incorporate the rate constraints.

In section II, the partial crosstalk cancellation problem is presented and then solved following a dual decomposition approach. A number of observations is made to reduce the complexity without losing the optimality of the solution. In section III the

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complexity of the single-user version of the dual decomposition algorithm is compared to the complexity of the resource allocation algorithm for the single-user case, where each user has an individual complexity constraint. Section IV then extends these results to the multi-user case where all users share a complexity constraint. A search procedure is presented to dynamically distribute the available complexity for crosstalk cancellation according to the rate constraints. Section V provides some simulation results and finally section VI concludes the paper.

II. Dual Decomposition

A. System Model

Most current DSL systems use Discrete Multi-Tone (DMT) modulation. The available frequency band is divided in a number of parallel subchannels or tones. Each tone is capable of transmitting data independently from other tones, and so the transmit power and the number of bits can be assigned individually for each tone.

Transmission for a binder of N users can be modelled on each tone k by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k \qquad k = 1 \dots K.$$

The vector $\mathbf{x}_k = [x_k^1, x_k^2, \dots, x_k^N]^T$ contains the transmitted signals on tone k for all N users. $[\mathbf{H}_k]_{n,m} = h_k^{n,m}$ is an $N \times N$ matrix containing the channel transfer functions from transmitter m to receiver n. The diagonal elements are the direct channels, the off-diagonal elements are the crosstalk channels. $\mathbf{z}_k$ is the vector of additive noise on tone k, containing thermal noise, alien crosstalk, RFI,.... The vector $\mathbf{y}_k$ contains the received symbols.

To take crosstalk cancellation into account, an equivalent channel $\tilde{\mathbf{H}}$ is introduced. This is the same channel as the original channel $\mathbf{H}$, but with off-diagonal elements set to zero where the crosstalk is cancelled. If user n is cancelling crosstalk originating from user m on tone k, then $\tilde{h}_k^{n,m} = 0$.

We denote the transmit power as $s_k^n \triangleq \Delta_f E\{|x_k^n|^2\}$, the noise power as $\sigma_k^n \triangleq \Delta_f E\{|z_k^n|^2\}$. The DMT symbol rate is denoted as f_s, the tone spacing as Δ_f.

It is assumed that each modem treats interference from other modems as noise. When the number of interfering modems is large, the interference is well approximated by a Gaussian distribution. Under this assumption the achievable bit loading of user n on tone k, given the transmit spectra of all modems in the system and the crosstalk cancellation configuration, is

$$b_k^n \triangleq \log_2\left(1 + \frac{1}{\Gamma}\frac{|\tilde{h}_k^{n,n}|^2 s_k^n}{\sum_{m \neq n} |\tilde{h}_k^{n,m}|^2 s_k^m + \sigma_k^n}\right), \tag{1}$$

where Γ denotes the SNR-gap to capacity, which is function of the desired BER, the coding gain and noise margin. The data rate for user n is

$$R^n = f_s \sum_k b_k^n.$$

B. Partial Crosstalk Cancellation Problem

Because of the runtime complexity of full crosstalk cancellation, only a limited amount of crosstalk can be cancelled. The cancellation of the crosstalk from one user on some tone is done by a cancellation tap. The number of cancellation taps that can be used is constrained by the *cancellation tap constraint* C^{tot} [10]. The partial crosstalk cancellation problem amounts to finding an optimal selection of which crosstalk to cancel, thereby maximizing the capacity of the network.

Secondly, there is a *rate constraint* $R^{n,target}$ for each user. Typically, service providers offer a number of profiles to guarantee a certain Quality of Service. The rate constraint then indicates a minimum data rate required by the user.

The allocation of cancellation taps in partial crosstalk cancellation then results in the following maximization problem:

$$\begin{array}{lll} \text{maximize}_{\mathbf{c}} & \sum_{n=1}^{N} R^n & \\ \text{subject to} & C = \sum_k \sum_m \sum_n c_k^{n,m} \leq C^{tot} & n,m = 1 \dots N, k = 1 \dots K \\ & R^n \geq R^{n,target} & n = 1 \dots N \\ \text{with} & [\mathbf{c}_k]_{n,m} = c_k^{n,m} & c_k^{n,m} = \begin{cases} 0 & \Rightarrow \quad \tilde{h}_k^{n,m} = h_k^{n,m} \\ 1 & \Rightarrow \quad \tilde{h}_k^{n,m} = 0 \end{cases} \end{array} \tag{2}$$

where $\mathbf{c} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_K]$. $c_k^{n,m} = 1$ indicates that a cancellation tap is assigned on tone k for cancelling crosstalk on line n originating from line m.

To find the global optimum for this optimization problem, one has to exhaustively search through all possible cancellation tap configurations $\mathbf{c}$. Because the cancellation tap constraint and the rate constraints are coupled over the tones, this results in an exponential complexity in the number of tones. By using a dual decomposition this complexity can be made linear [11] [12]

[13] [14] [10]. This is done by using Lagrange multipliers to move the constraints coupled over tones into the unconstrained part of the optimization problem:

$$\begin{aligned} \mathbf{c}^{opt} = \text{argmax}_{\mathbf{c}} \sum_{n=1}^{N} \omega_n R^n + \lambda\big(C^{tot} - \sum_{k=1}^{K}\sum_{m=1}^{N}\sum_{n=1}^{N} c_k^{n,m}\big) \\ \text{subject to} \quad \lambda \geq 0 \\ \omega_n \geq 0 \quad n = 1 \ldots N \end{aligned} \tag{3}$$

where λ and ω_n are Lagrange multipliers. For a given set of λ and $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_N]^T$, the optimization problem can then be solved in a per-tone fashion:

$$\begin{aligned} &\text{for } k = 1 \ldots K, \\ &\mathbf{c}_k^{opt} = \text{argmax}_{\mathbf{c}_k} \sum_{n=1}^{N} \omega_n f_s b_k^n - \sum_{n=1}^{N}\sum_{m=1}^{N} \lambda c_k^{n,m} \\ &\qquad \text{subject to} \quad \lambda \geq 0 \\ &\qquad\qquad\quad \omega_n \geq 0 \quad n = 1 \ldots N \end{aligned} \tag{4}$$

Maximization of (4) for given Lagrange multipliers can be performed by an exhaustive search. For each tone, all possible combinations for the canceller taps of the users should be checked. The combination giving the largest value for this expression is the optimal allocation of canceller taps for this tone.

The constraints can be enforced by choosing appropriate values for the Lagrange multipliers. The λ can be viewed as a cost for crosstalk cancellation taps. Larger values for the Lagrange multiplier result in less cancellation taps being allocated. The data rates of the users are weighted by $\boldsymbol{\omega}$, thereby giving more importance to some users. In this way, all possible trade-offs can be made to enforce the data rate constraints.

To solve (2) by (4), $\boldsymbol{\omega}$ and λ should be tuned to enforce the constraints. In [11] [12], an efficient Lagrange multiplier search procedure is presented for a similar problem. This procedure can be easily adapted for this partial cancellation problem. The basis for this procedure is relation (5), which is proven in **Appendix I**:

$$\begin{bmatrix} -(\Delta\boldsymbol{\omega})^T & \Delta\lambda \end{bmatrix} \begin{bmatrix} \Delta\mathbf{R} \\ \Delta C \end{bmatrix} \leq 0 \tag{5}$$

$\mathbf{R} = [R^1, \ldots, R^N]^T$ is a vector with the data rates and C is the number of cancellation taps corresponding to the Lagrange multipliers at hand.

Following [11] [12], relation (5) leads to the following update formula for the Lagrange multipliers:

$$\begin{bmatrix} \Delta\boldsymbol{\omega} \\ \Delta\lambda \end{bmatrix} = -\mu \begin{bmatrix} \mathbf{R} - \mathbf{R}^{target} \\ C^{tot} - C \end{bmatrix} \Rightarrow \begin{bmatrix} \boldsymbol{\omega} \\ \lambda \end{bmatrix}^{t+1} = \left(\begin{bmatrix} \boldsymbol{\omega} \\ \lambda \end{bmatrix}^{t} - \mu \begin{bmatrix} \mathbf{R} - \mathbf{R}^{target} \\ C^{tot} - C \end{bmatrix} \right)^{+} \tag{6}$$

where $(x)^+$ means $max(0, x)$ and μ is a stepsize parameter. Note that all the Lagrange multipliers are updated in parallel. This update formula is used in **Algorithm 1**, adopted from [11], to converge to the Langrange multipliers that enforce the constraints.

Algorithm 1 Lagrange multiplier search algorithm

```
while distance > tolerance do
  Θ = [ω, λ]^T = best [ω, λ]^T so far
  μ = 1
  while distance ≤ previousDistance do
    previousDistance = distance
    μ = μ × 2
    ΔΘ = [Δω, Δλ]^T = update formula (6)
    [R^{Θ+ΔΘ}, C^{Θ+ΔΘ}, c] = exhaustiveSearch(Θ + ΔΘ)
    distance = ||[R^{Θ+ΔΘ} − R^{target}, C^{tot} − C^{Θ+ΔΘ}]^T||
  end while
end while
```

The partial crosstalk cancellation problem (2) is a non-convex constrained optimization problem. Without dual decomposition, finding the global optimum requires an exhaustive search over all possible solutions. On a certain tone, a user has to decide which crosstalk of $N-1$ other users has to be cancelled. There are 2^{N-1} possibilities to do this. For N users and K tones, this results in a total complexity of $\mathcal{O}((2^{N-1})^{NK})$.

In [10] it is shown that when using a dual decomposition in multi-carrier systems, the duality gap is zero. Therefore the solution for the dual problem is also the solution for the primal problem.

The dual decomposition decouples the problem over the tones, therefore reducing the exponential complexity in the number of tones K to linear complexity: $\mathcal{O}(K(2^{N-1})^N)$. This amounts to K exhaustive searches of complexity $\mathcal{O}((2^{N-1})^N)$. For an 8 user VDSL system, the complexity is reduced from $2^{7\times8\times4096}$ to $4096\times2^{7\times8}$. This is an enormous reduction in complexity. Moreover, as shown in the next subsection, the complexity can be even further reduced by observing that many cancellation tap configurations can be eliminated in advance.

C. Per-tone Search Complexity Reduction

To determine the optimal allocation of crosstalk cancellation taps on a certain tone, all of the $(2^{N-1})^N \approx 2^{N^2}$ possible allocations have to be evaluated. Even for a limited number of users this becomes complex. Fortunately, many of these possibilities can be eliminated based on two observations: user independence and line selection.

- *User Independence:* All users have to decide on a crosstalk cancellation configuration. This leads to an exponential complexity in the number of users N. However, from (1) it can be seen that if user n allocates a crosstalk cancellation tap to cancel crosstalk caused by user m (i.e. $\tilde{h}_k^{n,m}=0$) this only has an influence on the capacity of user n. This corresponds to a per-user decoupling of (4), leading to

$$\begin{aligned} &\text{for } k=1\dots K,\\ &\text{for } n=1\dots N,\\ &\mathbf{c}_k^{n,opt}=\text{argmax}_{\mathbf{c}_k^n}\,\omega_n f_s b_k^n-\textstyle\sum_{m=1}^{N}\lambda c_k^{n,m}\\ &\quad\text{subject to}\quad \lambda\geq 0\\ &\qquad\qquad\quad\ \omega_n\geq 0\quad n=1\dots N \end{aligned} \tag{7}$$

  As a consequence, the exponential complexity in N is reduced to linear complexity. Instead of one large search over all users, there are N independent searches for the users. This observation results in the following complexity reduction:

$$(2^{N-1})^N \rightarrow N(2^{N-1}) \tag{8}$$

- *Line Selection:* A user has to decide for $N-1$ other users whether or not to cancel the crosstalk originating from these other users. This leads to 2^{N-1} possible crosstalk cancellation configurations. However, from (1) it can be seen that to maximize the capacity, one should allocate crosstalk cancellation taps to cancel the users which are causing the largest crosstalk. Therefore, if n crosstalk cancellation taps are available, these should be used to cancel the n largest sources of crosstalk.
  As a consequence, the 2^{N-1} possibilities for crosstalk cancellation are reduced to N possibilities: cancel no crosstalker, cancel the strongest crosstalker, cancel the 2 strongest crosstalkers, ..., cancel all $N-1$ crosstalkers.

$$\begin{aligned} &\text{for } k=1\dots K,\\ &\text{for } n=1\dots N,\\ &\mathbf{c}_k^{n,opt}=\text{argmax}_{\mathbf{c}_k^n}\,\omega_n f_s b_k^n(r)-\lambda r\\ &\quad\text{subject to}\quad \lambda\geq 0\\ &\qquad\qquad\quad\ \omega_n\geq 0\quad n=1\dots N \end{aligned} \tag{9}$$

  where $b(r)$ is the capacity when the r largest crosstalkers are cancelled.

When both observations are combined, N users independently have to choose one of N possible crosstalk cancellation configurations. This results in the following total complexity reduction:

$$(2^{N-1})^N \rightarrow NN \tag{10}$$

In a 8-user case, these observations reduce the number of crosstalk cancellation configurations to be evaluated from 2^{56} to 2^6. Note that despite drastic complexity reductions, the solution is still optimal.

III. Single-User Algorithms and Complexity Comparison

In this section the complexity of the algorithm based on dual decomposition is analyzed and compared to the complexity of the optimal resource allocation algorithm of [8]. The resource allocation algorithm is a single-user algorithm. Therefore, a single-user formulation of the dual decomposition algorithm is used for the complexity comparison. The results will then be extended to the multi-user case in section IV.

A. Single-User Resource Allocation Algorithm

The resource allocation algorithm uses the average capacity increase per allocated crosstalk cancellation tap on a certain tone:

$$v_k(r) = \frac{b_k(r) - b_k(0)}{r}, \tag{11}$$

with $b_k(r)$ the capacity on tone k when the r largest crosstalkers are cancelled (cfr. section II-C, line selection). A greedy algorithm then selects the tone k and number of crosstalkers r to cancel by searching the largest value of $v_k(r)$. The average capacity increase per allocated crosstalk cancellation tap should then be recalculated on tone k_s, based on the selected value $v_{k_s}(r_s)$, as follows:

- the average capacity increase for allocating less or equal crosstalk cancellation taps than r_s is set to zero
- the average capacity increase for allocating more crosstalk cancellation taps than r_s is recalculated as $v_k(r) = \frac{(b_k(r)-b_k(r_s))}{(r-r_s)}$, where the increase is now referenced to $b_k(r_s)$.

This is repeated until all available crosstalk cancellation taps are allocated. Note that in each iteration of the algorithm a minimum of 1 and a maximum of $N-1$ crosstalk cancellation taps is allocated. Because of this varying granularity, the crosstalk cancellation tap constraint cannot always be enforced tightly. However, the granularity is small enough to get close to the constraint.

Algorithm 2 Single-User Resource allocation algorithm

	capacities	multiplications	additions	comparisons
init: $v_k(r) = \frac{(b_k(r)-b_k(0))}{r}$ $\begin{cases} k=1\ldots K \\ r=1\ldots N-1 \end{cases}$	KN	$K(N-1)$	$K(N-1)$	$K\sum_{i=1}^{N-1} log_2(i)$
sort $v_k(r)$	0	0	0	$\sum_{i=1}^{K(N-1)} log_2(i)$
repeat				
$(k_s, r_s) = \text{argmax}_{k,r}\, v_k(r)$	0	0	0	0
$v_{k_s}(r) = 0, \quad \forall r \leq r_s$	0	0	0	0
$v_{k_s}(r) = \frac{(b_k(r)-b_k(r_s))}{(r-r_s)}, \quad \forall r > r_s$	$\frac{N-1}{2}+1$	$\frac{N-1}{2}$	$N-1$	0
re-sort $v_k(r)$	0	0	0	$\sum_{i=K(N-1)-(\frac{N-1}{2}-1)}^{K(N-1)} log_2(i)$
while $\sum_k r_k < C^{tot}$	0	0	1	1

The procedure is presented in **Algorithm 2**. A $K \times (N-1)$ table is initialized containing the average capacity increases per allocated crosstalk cancellation tap. For each of K tones the capacity increase has to be calculated for all $N-1$ crosstalk cancellation configurations. To be able to calculate the capacity increase, the capacity without crosstalk cancellation $b_k(0)$ also has to be calculated for every tone. This results in KN capacity calculations. Another $K(N-1)$ multiplications and additions are required to calculate the average capacity increase per allocated crosstalk cancellation tap. The $N-1$ crosstalk cancellation configurations are based on the line selection observation of section II-C. This requires a sort over the crosstalkers for each tone. This sort can be accomplished by selecting the crosstalkers one by one and placing them in the correct position of a sorted list. Because the resulting list is sorted at all times, a binary search can be used to find the correct position to place the current crosstalker. This results in a complexity of $\sum_{i=1}^{N-1} log_2(i)$ comparisons to sort the list.

The table is then sorted to be able to efficiently find the maximum. This can be done analogous to the sorting of the crosstalkers and requires a complexity of $\sum_{i=1}^{K(N-1)} log_2(i)$ comparisons.

Crosstalk cancellation taps can now be allocated by selecting the element with the maximum average capacity increase of the table, located at the top of the sorted list. On average, $\frac{N-1}{2}$ crosstalk cancellation taps are thereby allocated. $\frac{N-1}{2}$ elements in the table then have to be recalculated to the new reference capacity $b_k(r_s)$. This requires $\frac{N-1}{2}+1$ capacity calculations, $\frac{N-1}{2}$ multiplications and $N-1$ additions.

To keep the list sorted, $\frac{N-1}{2}$ binary searches are performed to find the new positions for the $\frac{N-1}{2}$ updated elements. This requires $\sum_{i=K(N-1)-(\frac{N-1}{2}-1)}^{K(N-1)} log_2(i)$ comparisons. The number of currently allocated cancellation taps is updated and compared to the cancellation tap constraint C^{tot}.

This is repeated until all available crosstalk cancellation taps are allocated. In [8] it was shown that with a runtime complexity of 30% of full crosstalk cancellation, almost all crosstalk can be cancelled. This means that approximately $\frac{K(N-1)}{3}$ crosstalk cancellation taps have to be allocated. Taking into account that in each iteration of the algorithm $\frac{N-1}{2}$ taps are allocated, there are $\frac{K(N-1)}{3\frac{N-1}{2}}$ iterations required on average.

B. Single-User Dual Decomposition Algorithm

To be able to compare the algorithm based on dual decomposition to the resource allocation algorithm, a single-user formulation of the partial crosstalk cancellation problem (2) is used for user n:

$$\begin{array}{lll} \text{maximize}_{\mathbf{c}} & R^n & \\ \text{subject to} & C^n = \sum_k \sum_m c_k^{n,m} \leq C^{n,tot} & m = 1 \ldots N, k = 1 \ldots K \\ \text{with} & [\mathbf{c}_k]_{n,m} = c_k^{n,m} & c_k^{n,m} = \begin{cases} 0 & \Rightarrow \quad \tilde{h}_k^{n,m} = h_k^{n,m} \\ 1 & \Rightarrow \quad \tilde{h}_k^{n,m} = 0 \end{cases} \end{array} \tag{12}$$

This results in the following dual problem which is decoupled over the tones:

$$\begin{array}{c} \text{for } k = 1 \ldots K, \\ \mathbf{c}_k^{opt} = \text{argmax}_{\mathbf{c}_k} \, b_k^n - \sum_{m=1}^{N} \lambda c_k^{n,m} \\ \text{subject to} \quad \lambda \geq 0 \end{array} \tag{13}$$

This can be viewed as one optimization of the multi-user problem where all users are allocated a crosstalk cancellation tap budget in advance.

Algorithm 3 presents the single-user dual decomposition algorithm. It starts by initializing a $K \times N$ table of capacities for K tones and N possible crosstalk cancellation configurations. To obtain the N possible crosstalk cancellation configurations, the line selection observation of section II-C is used. This requires sorting the crosstalkers which uses $K \sum_{i=1}^{N-1} log_2(i)$ comparisons.

The algorithm then starts from some initial λ and performs K per-tone exhaustive searches. There are N possible values for λr, which can be calculated in advance. This requires N multiplications. These precalculated values are then subtracted from the corresponding elements of the $K \times N$ table. Finally, K exhaustive searches of N values are performed to obtain the maximum on each tone. This requires $K(N-1)$ comparisons.

The cancellation tap constraint is then checked by summing the number of taps allocated on each tone. If the constraint is not tightly satisfied, the Lagrange multiplier λ is updated and then the per-tone search is repeated. Because there is only one Lagrange multiplier, bisection can be used. This requires typically 10 iterations.

Table I summarizes the total complexity of the single-user resource allocation algorithm and the dual decomposition algorithm.

Algorithm 3 Single-User Dual decomposition algorithm

	capacities	multiplications	additions	comparisons
init: $b_k(r)$ $\begin{cases} k=1\ldots K \\ r=0\ldots N-1 \end{cases}$	KN	0	0	$K \sum_{i=1}^{N-1} log_2(i)$
repeat				
for $k = 1 \ldots K$				
$\mathbf{c}_k^{opt} = \text{argmax}_r \, b_k(r) - \lambda r$	0	N	KN	$K(N-1)$
endfor				
update λ based on (6)				
while $\sum_k \mathbf{c}_k^{opt} \neq C^{tot}$	0	0	$K-1$	1

TABLE I
COMPLEXITY COMPARISON SINGLE-USER ALGORITHMS

	resource allocation	dual decomposition
capacities	$KN + \frac{K(N-1)}{3\frac{N-1}{2}} \left(\frac{N-1}{2} + 1 \right)$	KN
multiplications	$K(N-1) + \frac{K(N-1)}{3\frac{N-1}{2}} \frac{N-1}{2}$	$10 \times N$
additions	$K(N-1) + \frac{K(N-1)}{3\frac{N-1}{2}} N$	$10 \times (KN + K - 1)$
comparisons	$K \sum_{i=1}^{N-1} log_2(i) + \sum_{i=1}^{K(N-1)} log_2(i) +$ $\frac{K(N-1)}{3\frac{N-1}{2}} \left(1 + \sum_{i=K(N-1)-(\frac{N-1}{2}-1)}^{K(N-1)} log_2(i) \right)$	$K \sum_{i=1}^{N-1} log_2(i) + 10 \times \left(K(N-1) + 1 \right)$

Figure 1 shows the complexities as a function of the number of users for the single-user resource allocation algorithm and the dual decomposition algorithm for $K = 1000$. Note the significant decrease in multiplications, capacity calculations and comparisons. This complexity decrease largely overpowers the increase in the number of additions.

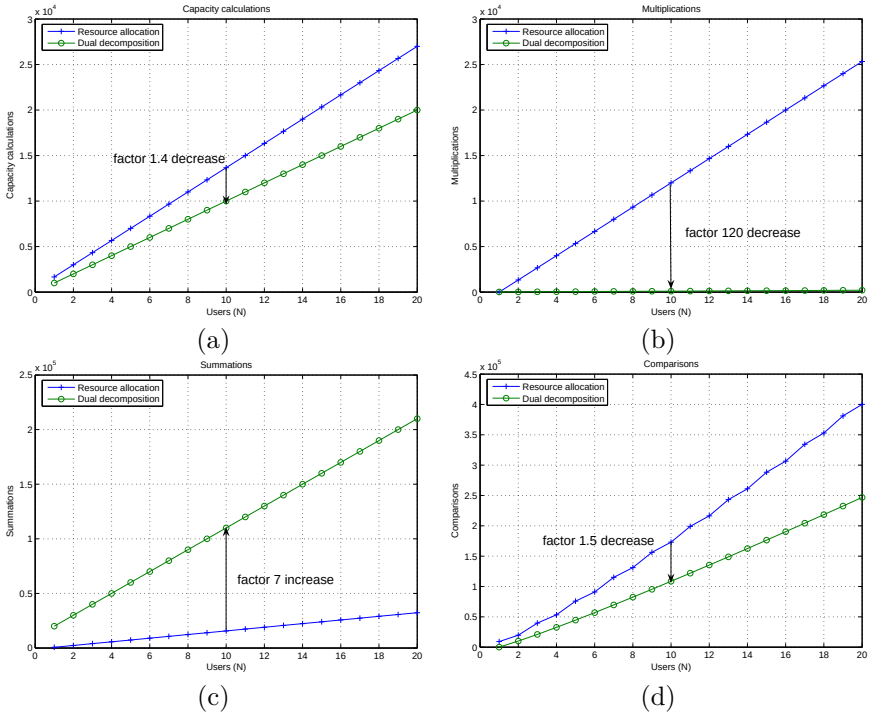


Fig. 1. Complexity comparison single-user algorithms: (a) capacity calculations (b) multiplications (c) additions (d) comparisons

IV. Multi-User Algorithms and Complexity Comparison

The extension to the multi-user case can be made by dividing the cancellation tap budget over the users in advance. By varying the cancellation tap budget allocated to each user, various trade-offs can be made in the data rates. This reduces the problem to multiple single-user problems. The core complexity of both the resource allocation algorithm and the dual decomposition algorithm is then increased by a factor N. Because of user independence and fixed individual cancellation tap budgets, optimization of the individual users also results in the optimization of the sum rate.

In this section, the single-user algorithms are extended to automatically determine the correct proportions of the cancellation tap budget to be allocated to the users such that the rate constraints are satisfied.

A. Multi-user resource allocation algorithm

For the resource allocation algorithm in [8], no procedure is available to automatically ditribute the cancellation tap budget over the users so that certain data rate constraints are satisfied. However, by introducing weights ω_n, some lines can be emphasized to meet the rate constraints. To achieve a higher data rate for a user, more crosstalk cancellation taps should be allocated to that user. In order to do this, the average benefit of adding a crosstalk cancellation tap for that user is increased by a factor ω_n. A larger weight leads to more crosstalk cancellation taps allocated and thus a higher data rate.

A given set of $\omega_n{}'$s implies a cancellation tap budget for each user (which is known after the optimization is done with these $\omega_n{}'$s). Because of the user independence, this again leads to an optimization of the sum rate. However, the rates are now weighted with $\omega_n{}'$s, thus a weighted rate sum is optimized.

Therefore, the following relation can be derived, analogous to the derivation in appendix I:

$$\Delta\boldsymbol{\omega}\Delta\mathbf{R} \geq 0 \tag{14}$$

This is a reduced form of (5), which leads to a simplified version of the update formula (6):

$$\Delta\boldsymbol{\omega} = -\mu(\mathbf{R} - \mathbf{R}^{target}) \Rightarrow \boldsymbol{\omega}^{t+1} = \left[\boldsymbol{\omega}^{t} - \mu(\mathbf{R} - \mathbf{R}^{target})\right]^{+} \tag{15}$$

During I iterations, this update formula can then be used to steer the $\omega_n{}'$s so that the rate constraints are satisfied.

Algorithm 4 presents the resulting multi-user resource allocation algorithm with its associated complexities. Note that the table of $KN(N-1)$ average capacity increases per crosstalk cancellation tap is now globally searched instead of individually per user.

Algorithm 4 Multi-user resource allocation algorithm

	capacities	multiplications	additions	comparisons
init: $v_k^n(r) = \frac{(b_k^n(r) - b_k^n(0))}{r}$ $\begin{cases} k{=}1\ldots K \\ r{=}1\ldots N{-}1 \\ n{=}1\ldots N \end{cases}$	KNN	$KN(N-1)$	$KN(N-1)$	$KN\sum_{i=1}^{N-1} log_2(i)$
repeat				
$v_k^{\boldsymbol{\omega},n}(r) = \omega_n v_k^n(r)$	0	$KN(N-1)$	0	0
sort $v_k^{\boldsymbol{\omega},n}(r)$	0	0	0	$\sum_{i=1}^{KN(N-1)} log_2(i)$
repeat				
$(k_s, r_s, n_s) = \text{argmax}_{k,r,n}\, v_k^{\boldsymbol{\omega},n}(r)$	0	0	0	0
$v_{k_s}^{\boldsymbol{\omega},n_s}(r) = 0, \quad \forall r \leq r_s$	0	0	0	0
$v_{k_s}^{\boldsymbol{\omega},n_s}(r) = \omega_{n_s} \frac{(b_k^{n_s}(r) - b_k^{n_s}(r_s))}{(r - r_s)}, \quad \forall r > r_s$	$\frac{N-1}{2} + 1$	$N-1$	$N-1$	0
re-sort $v_k^{\boldsymbol{\omega},n}(r)$	0	0	0	$\sum_{i=KN(N-1)-(\frac{N-1}{2}-1)}^{KN(N-1)} log_2(i)$
while $\sum_{n=1}^{N}\sum_{k=1}^{K} r_k^n < C^{tot}$	0	0	1	1
update $\boldsymbol{\omega}$ based on (6)				
while rate constraints not satisfied				

B. Multi-user dual decomposition algorithm

In the dual decomposition approach, Algorithm 1 can be used to find an appropriate distribution of the cancellation tap budget over the users, where the per-tone search is simplified based on the observations in section II-C. The resulting algorithm and complexities are shown in **Algorithm 5**. Because the updates of the Lagrange multipliers are based on the same update formula as in the resource allocation algorithm, roughly the same number of I iterations is required to enforce the constraints.

In **Table II** the total complexities of the multi-user resource allocation algorithm and the multi-user dual decomposition algorithm are compared.

Algorithm 5 Multi-user dual decomposition algorithm

	capacities	multiplications	additions	comparisons
init: $b_k^n(r)$ $\begin{cases} k{=}1\ldots K \\ r{=}0\ldots N{-}1 \\ n{=}1\ldots N \end{cases}$	KNN	0	0	$KN\sum_{i=1}^{N-1} log_2(i)$
repeat				
for $k = 1 \ldots K$				
for $n = 1 \ldots N$				
$\mathbf{c}_k^{n,opt} = \text{argmax}_r\, \omega_n b_k^n(r) - \lambda r$	0	$N + KNN$	KNN	$KN(N-1)$
endfor				
endfor				
update $\boldsymbol{\omega}, \lambda$ based on (6)				
while $\sum_{n=1}^{N}\sum_{k=1}^{K} \mathbf{c}_k^{n,opt} \neq C^{tot}$ and rate constraints not satisfied	0	0	$(N-1)(K-1)$	1

TABLE II

COMPLEXITY COMPARISON MULTI-USER ALGORITHMS

	resource allocation	dual decomposition
capacities	$KNN + I \times \frac{KN(N-1)}{3\frac{N-1}{2}}\left(\frac{N-1}{2} + 1\right)$	KNN
multiplications	$KN(N-1) + I \times \left(KN(N-1) + \frac{KN(N-1)}{3\frac{N-1}{2}}(N-1)\right)$	$I \times (N + KNN)$
additions	$KN(N-1) + I \times \frac{KN(N-1)}{3\frac{N-1}{2}} N$	$I \times \big(KNN + (N-1)(K-1)\big)$
comparisons	$KN\sum_{i=1}^{N-1} log_2(i) + I \times \sum_{i=1}^{KN(N-1)} log_2(i) +$ $\frac{KN(N-1)}{3\frac{N-1}{2}}\left(1 + \sum_{i=KN(N-1)-(\frac{N-1}{2}-1)}^{KN(N-1)} log_2(i)\right)$	$KN\sum_{i=1}^{N-1} log_2(i) + I \times \big(KN(N-1) + 1\big)$

Figure 2 shows the complexities as function of the number of users for the resource allocation algorithm and the dual decomposition algorithm for $K = 1000$, under the assumption that $I = 50$ iterations are required to enforce the constraints. Note the significant decrease in multiplications, capacity calculations and comparisons. This complexity decrease largely overpowers the increase in the number of additions.

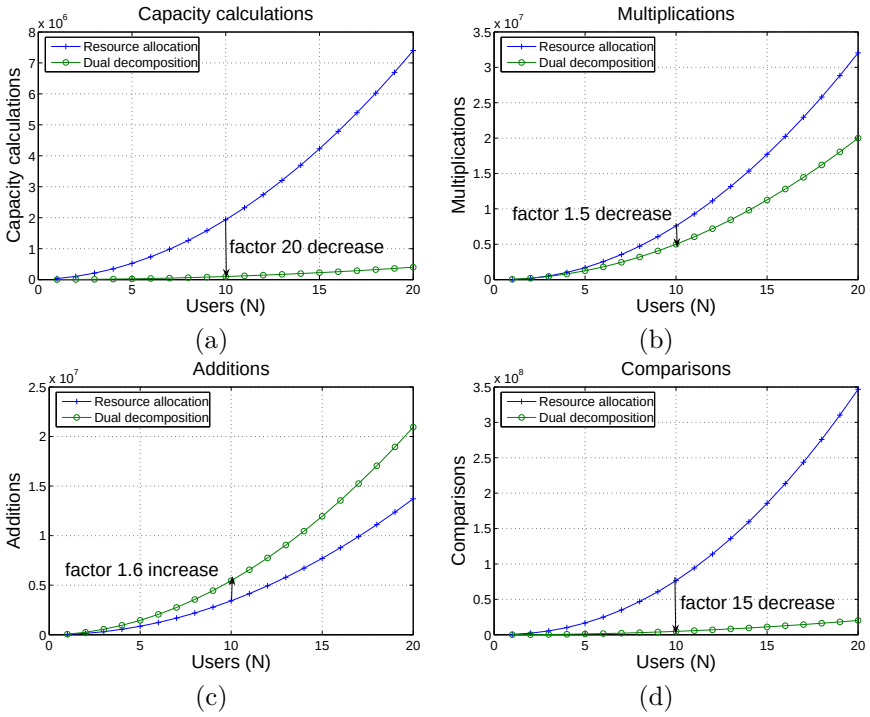


Fig. 2. Complexity comparison multi-user algorithms: (a) capacity calculations (b) multiplications (c) additions (d) comparisons

V. Simulation Results

In [8] a simplified joint line/tone selection algorithm is also presented. This algorithm has a much lower complexity than the algorithms discussed in this paper and is claimed to be near-optimal. This algorithm can also be extended to the multi-user case by introducing the weights ω. However, this near-optimality largely depends on the scenario. For simple scenarios with only two different line lengths, the simplified joint line/tone selection algorithm indeed performs near-optimal. However, for practical scenarios with lines of varying lengths, this simplified algorithm can be suboptimal depending on the runtime complexity that is allowed.

In **figure 3** the performance of both the optimal as well as the simplified line/tone selection is presented for different runtime complexities. This is done for an 8-user upstream VDSL scenario, with line lengths varying from 150 m to 1200 m in 150 m intervals. An empirical channel model [15] is used with line diameter of 0.5 mm (24 AWG). The transmit power is set to -60 dBm on all tones. The SNR gap Γ is set to 12.9 dB, corresponding to a target symbol error probability of 10^{-7}, coding gain of 3 dB and a noise margin of 6 dB. The tone spacing $\Delta_f = 4.3125$ kHz and the DMT symbol rate $f_s = 4$ kHz.

To allow for an easier comparison, cancellation taps are allocated to each line using a single-user algorithm, keeping all other lines at a fixed bitrate with no crosstalk cancellation. Note that for small runtime complexities, the optimal joint line/tone selection algorithm can gain up to 50% of the performance of the simplified joint line/tone selection algorithm. Especially for the far-end users, which should be protected most from crosstalk, this performance difference is large.

Secondly, note the difference in runtime complexity for different lines to approach the full crosstalk cancellation performance. For long lines, 30% of full crosstalk cancellation is sufficient because only few tones carry a significant amount of bits. As the lines get shorter, up to 50-60% of full crosstalk cancellation is necessary. Therefore, multi-user algorithms are more suitable to solve the partial crosstalk cancellation problem because they can automatically distribute the cancellation tap budget over the users, in contrast to single-user algorithms where the budget has to be distributed in advance, taking into account the different line lenghts.

The simplified joint line/tone selection algorithm requires a high runtime complexity before it starts performing optimal. For low runtime complexities however, the optimal algorithm reaches a much higher perfomance. Thus depending on the allowed runtime complexity, the optimal joint line/tone algorithm can be preferred over the simplified algorithm, trading of runtime complexity for initialization complexity when the required bitrate is fixed.

In **figure 4**, rate regions are shown for a symmetric upstream VDSL scenario with two 300m lines. Various crosstalk cancellation complexities are considered when allocating crosstalk cancellation taps optimally.

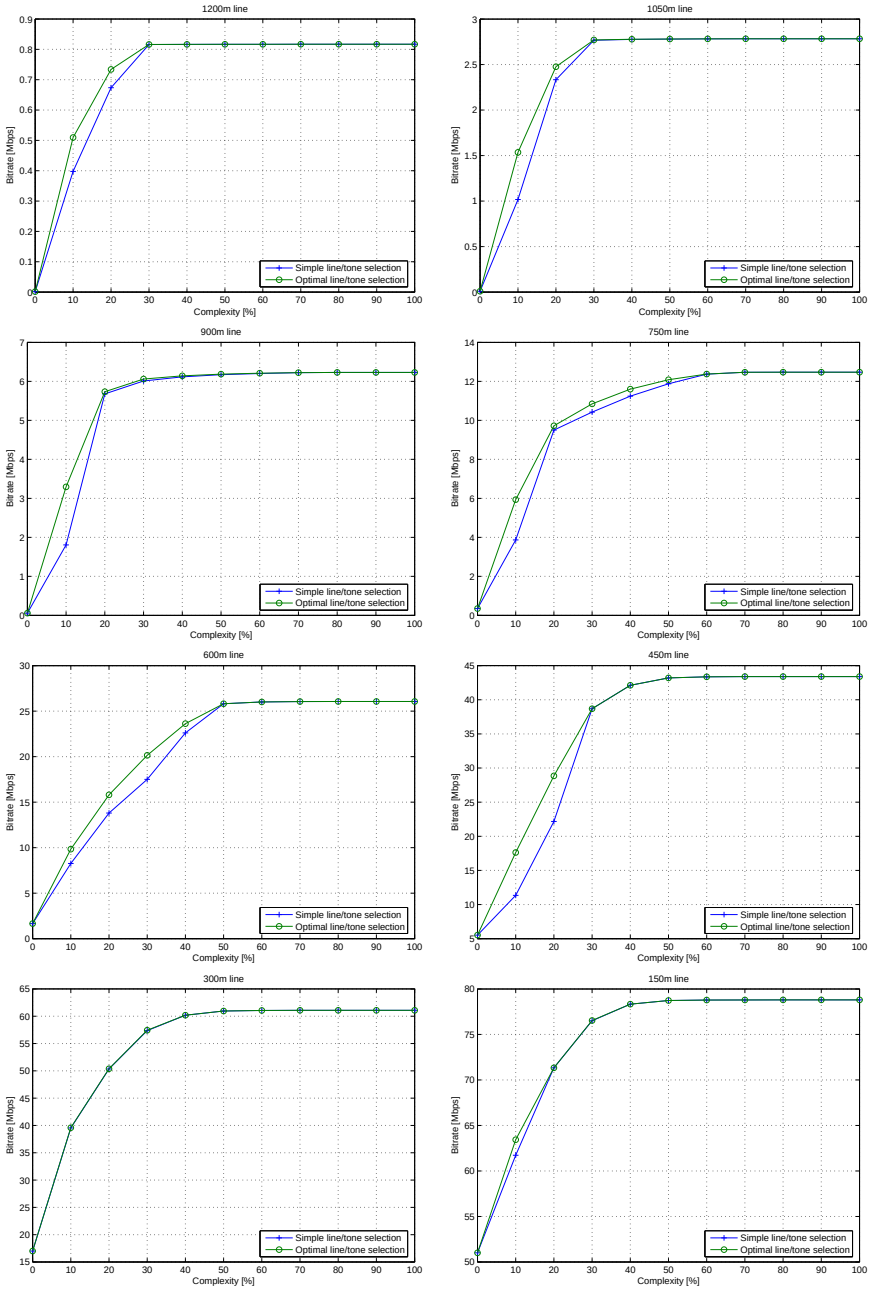


Fig. 3. Performance comparison Optimal and Simple line/tone selection algorithms

VI. Conclusion

In modern DSL systems, crosstalk is a major source of performance degradation. Crosstalk cancellation schemes have been proposed to mitigate the effect of crosstalk. However, the complexity of crosstalk cancellation grows with the square of the number of lines in the binder. Fortunately, most of the crosstalk originates from a limited number of lines on a limited number of tones. As a result, a fraction of the complexity of full crosstalk cancellation suffices to cancel most of the crosstalk, which is exploited by partial crosstalk cancellation. The challenge is then to determine which crosstalk to cancel on which tones, given a certain complexity constraint. In this paper we have presented an algorithm to optimally solve this problem, based on a dual decomposition.

Two cases were considered: single-user and multi-user. In the single-user case, each user has an individual cancellation tap budget to be allocated. It was shown that the dual decomposition algorithm has a favourable complexity compared to the optimal resource allocation algorithm.

In the multi-user case, all users have a common cancellation tap budget. This budget has to be distributed over the users in such a way that rate constraints are satisfied. The dual decomposition approach naturally incorporates these rate constraints. The resource allocation algorithms were extended to this multi-user case to also include these rate constraints. The extension

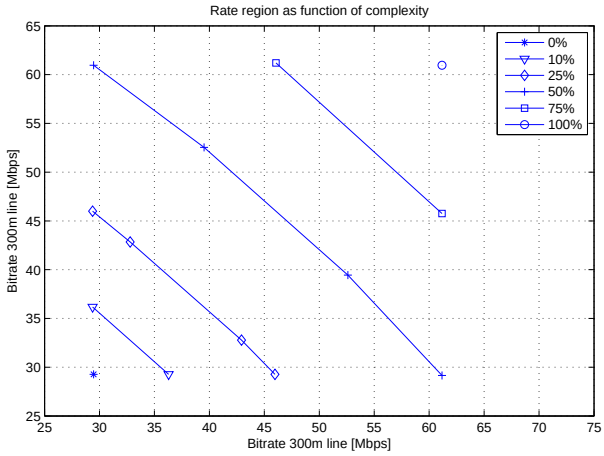


Fig. 4. Rate regions for various crosstalk cancellation complexities

allows for the same search procedure to be used to find the distribution of the cancellation tap budget over the users as used in the dual decomposition algorithm. Also in this multi-user case, the complexity of the dual decomposition algorithm was found to compare favourably with the complexity of the multi-user resource allocation algorithm.

APPENDIX I
SEARCH ALGORITHM FOR THE LAGRANGE MULTIPLIERS

The proof presented in [11] [12] can be easily adapted for partial crosstalk cancellation. Assume a two-user scenario with signal-level control. Starting from two optimal solutions $(R^{1,\boldsymbol{\omega}_A,\lambda_A}, R^{2,\boldsymbol{\omega}_A,\lambda_A}, C^{\boldsymbol{\omega}_A,\lambda_A})$ and $(R^{1,\boldsymbol{\omega}_B,\lambda_B}, R^{2,\boldsymbol{\omega}_B,\lambda_B}, C^{\boldsymbol{\omega}_B,\lambda_B})$ corresponding to $(\boldsymbol{\omega}_A, \lambda_A)$ and $(\boldsymbol{\omega}_B, \lambda_B)$ respectively, optimality for $(\boldsymbol{\omega}_A, \lambda_A)$ implies

$$\begin{aligned} & \omega_{1,A} R^{1,\boldsymbol{\omega}_B,\lambda_B} + \omega_{2,A} R^{2,\boldsymbol{\omega}_B,\lambda_B} - \lambda_A C^{\boldsymbol{\omega}_B,\lambda_B} \\ \leq\ & \omega_{1,A} R^{1,\boldsymbol{\omega}_A,\lambda_A} + \omega_{2,A} R^{2,\boldsymbol{\omega}_A,\lambda_A} - \lambda_A C^{\boldsymbol{\omega}_A,\lambda_A} \end{aligned} \tag{16}$$

Optimality for $(\boldsymbol{\omega}_B, \lambda_B)$ implies

$$\begin{aligned} & \omega_{1,B} R^{1,\boldsymbol{\omega}_A,\lambda_A} + \omega_{2,B} R^{2,\boldsymbol{\omega}_A,\lambda_A} - \lambda_B C^{\boldsymbol{\omega}_A,\lambda_A} \\ \leq\ & \omega_{1,B} R^{1,\boldsymbol{\omega}_B,\lambda_B} + \omega_{2,B} R^{2,\boldsymbol{\omega}_B,\lambda_B} - \lambda_B C^{\boldsymbol{\omega}_B,\lambda_B} \end{aligned} \tag{17}$$

Taking the sum of (16) and (17) results in

$$\begin{aligned} & -\underbrace{\left(\omega_{1,B} - \omega_{1,A}\right)}_{\Delta\omega_1} \underbrace{\left(R^{1,\boldsymbol{\omega}_B,\lambda_B} - R^{1,\boldsymbol{\omega}_A,\lambda_A}\right)}_{\Delta R^1} - \underbrace{\left(\omega_{2,B} - \omega_{2,A}\right)}_{\Delta\omega_2} \underbrace{\left(R^{2,\boldsymbol{\omega}_B,\lambda_B} - R^{2,\boldsymbol{\omega}_A,\lambda_A}\right)}_{\Delta R^2} \\ & + \underbrace{\left(\lambda_B - \lambda_A\right)}_{\Delta\lambda} \underbrace{\left(C^{\boldsymbol{\omega}_B,\lambda_B} - C^{\boldsymbol{\omega}_A,\lambda_A}\right)}_{\Delta C} \leq 0 \end{aligned} \tag{18}$$

Relation (18) is straightforwardly extended to a multi-user scenario:

$$\begin{bmatrix} -(\Delta\boldsymbol{\omega})^T & \Delta\lambda \end{bmatrix} \begin{bmatrix} \Delta\mathbf{R} \\ \Delta C \end{bmatrix} \leq 0 \tag{19}$$

$\boldsymbol{\omega} = [\omega_1, \ldots, \omega_N]$ is a vector containing the Lagrange multipliers for the weights for the users, λ is the Lagrange multiplier controlling the number of cancellation taps used. $\mathbf{R} = [R^1, \ldots, R^N]^T$ is a vector with the corresponding data rates and C is the corresponding number of cancellation taps.

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