A Bézoutian-Based Method for Solving Overdetermined Systems of Polynomial Equations

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***Abstract*—The solution of overdetermined systems of polynomial equations shows promise for solving complicated problems in signal processing and related fields. Polynomials can represent more complicated interactions than their linear counterparts, while overdeterminedness enables the mitigation of noise. In the overdetermined case, the number of common roots can correspond to the number of physically meaningful solutions (which can be less than the Bézout number from the non-overdetermined case). This article extends an algebraic method based on Bézoutian tensors to the non-square case to find all (approximate) roots of an overdetermined system of polynomial equations. The problem is shown to be a partially coupled canonical polyadic decomposition, computable via a rank-1 map. The practical utility of solving overdetermined systems of polynomial equations is shown in an underdetermined blind signal separation problem with constant modulus.**

***Index Terms*—Bézoutian, polynomial, root solving, tensor**

I. Introduction

Overdetermined systems of linear equations form a cornerstone of many engineering problems, allowing the robust determination of solutions in the presence of noise or uncertainty. Extending linear equations to multivariate polynomial equations enables the representation of higher-order interactions, essential for solving more complicated problems, with potentially multiple solutions as shown in Fig. 1. Polynomial equations find applications in diverse areas such as computer graphics, robotics and cryptography; see, e.g., [1]–[6].

Square systems of polynomial equations—where the number of equations equals the number of variables—have been studied extensively [7], [8]. Two common solution strategies are continuous deformation-based approaches, such as homotopy continuation methods (see, e.g., [9], [10]), and algebra-based methods. The latter encompasses two subcategories: symbolic methods, for instance, Gröbner-based solvers, which manipulate exact algebraic expressions, and numerical methods, which use, e.g., Macaulay matrices or Bézoutians; see, e.g., [11]–[16]. In these square systems, an issue is the rapid growth of the number of solutions. This can be mitigated by adding equations, which could also reduce the effect of noise.

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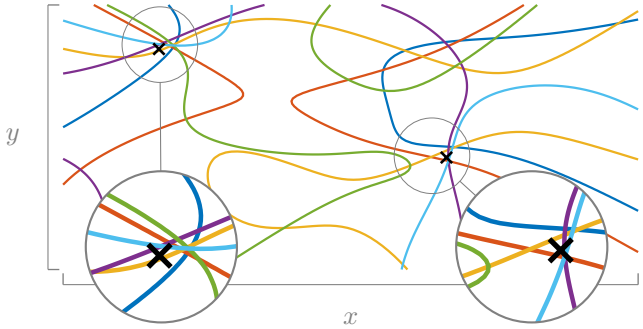


Fig. 1. The two true solutions (crosses) of this overdetermined system of polynomial equations in two variables can only be found approximately, which can be challenging as not all lines cross because of perturbations; see in particular the green line in the bottom right zoom plot. The coefficients of the polynomials are perturbed by 10% due to noise (coefficient-to-noise ratio of 20 dB). The six polynomials have at most total degree four.

While solving overdetermined systems of linear equations in a least-squares sense is well-established in numerical algebra [17], solution methods for its polynomial counterpart are still underdeveloped. Some methods (e.g., [13], [14], [18]) work for overdetermined systems, but were not designed to handle noise. Specifically for noisy settings, solution methods were published that rely on a simultaneous diagonalization problem starting from the nullspace of the Macaulay matrix [19], [20]. This paper studies a promising alternative based on Bézoutian tensors, where, for square systems, simultaneous diagonalization can directly be used without first computing a nullspace [21].

The roots of a square system can be obtained from a canonical polyadic decomposition (CPD) of the (tensorized) Macaulay nullspace or of the Bézoutian tensor, though the latter is cheaper to compute. However, extending Bézoutians to overdetermined systems is nontrivial. Instead, we propose a joint decomposition of several Bézoutians associated with different square systems, which noteworthily are in part coupled through the inverse of their factors. This can be resolved by using an extension of the rank-1 mapping technique [22], [23] to the case where only a few instead of all columns of a certain

matrix are vectorized rank-1 terms [24]. We numerically show the benefits of overdetermined polynomial systems and present how they can be used to recover constant modulus signals with more sources than measured signals.

Section II briefly introduces systems of polynomial equations. Section III introduces notation and concepts from multilinear algebra, including various tensor decompositions and the rank-1 mapping technique. The Bézoutian tensor, of which the low-rank properties will be exploited to determine the roots, is then introduced in Section IV. Section V then describes the method, which is tested in Section VI.

II. Systems of Polynomial Equations

The goal is to find the R solutions $\mathbf{x}^{(r)} \in \mathbb{F}^N$, $r = 1, \ldots, R$, to the system of S polynomial equations $\mathbf{\Sigma} = \{p_1(\mathbf{x}) = 0, \ldots, p_S(\mathbf{x}) = 0\}$ of degree D in N variables, with $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$. Polynomials are written as $p_s(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^N} \mathbf{c}_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with monomials $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$. The (total) degree D is the maximum sum of the exponents $(\alpha_1 + \ldots + \alpha_N)$ of any monomial with a nonzero coefficient.

The *number of roots* of a square (i.e., $S = N$) system of (independent) polynomial equations equals the Bézout number, $L = \prod_{n=1}^{N} D_n$, with D_n the degree of the nth polynomial, counting roots at infinity and multiplicity; see, e.g., [25]. By adding equations to form an overdetermined system, the number of roots can vary between zero and L, enabling the solution of systems with one or a few roots, an interesting feature for applications; see, e.g., Section VI-B or [20].

Algebraic root-finding methods introduce additional (typically undesired) roots, called *roots at infinity*, see, e.g., [8]. This is due to homogenization which changes polynomial $p(\mathbf{x})$ to $t^D p(\frac{x_1}{t}, \ldots, \frac{x_N}{t})$ with t an additional variable such that every monomial with a nonzero coefficient has the same degree D. Any root $(t, x_1, \ldots, x_N)$ of the homogenized polynomial corresponds with a root $(x_1/t, \ldots, x_N/t)$ of the original polynomial $p(\mathbf{x})$, except if $t = 0$, indicating a root at infinity. For ease of exposition, we assume no roots at infinity are present and replace the homogenization variable t by 1. Roots at infinity do not pose an issue for the proposed method as long as their multiplicity equals one and, in any case, they can be removed from the Bézoutian tensors [15].

III. Tensor Algebra

Scalars, vectors, matrices, and tensors are denoted as lowercase (e.g., a), bold lowercase (e.g., $\mathbf{a}$), bold capital (e.g., $\mathbf{A}$), and calligraphic letters (e.g., $\mathcal{A}$), respectively. The frontal slice of tensor $\mathcal{A}$ with index i_3 is the matrix $\mathbf{A}_{:,:,i_3}$. The outer product $\mathbf{C} = \mathbf{a} \otimes \mathbf{b}$ satisfies $c_{i,j} = a_i b_j$. The Kronecker product $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$ for $\mathbf{a} \in \mathbb{F}^{I_1}$, $\mathbf{b} \in \mathbb{F}^{J_1}$, $\mathbf{c} \in \mathbb{F}^{I_1 J_1}$ stacks all $a_{i_1}\mathbf{b}$ into $\mathbf{c}$ in the natural order. The mode-n product $\mathcal{C} = \mathcal{A} \cdot_n \mathbf{B}$ satisfies $c_{i_1,\ldots,i_{n-1},j_n,i_{n+1},\ldots,i_N} = \sum_{i_n} b_{j_n,i_n} a_{i_1,\ldots,i_N}$. The diagonal matrix with $\mathbf{b}$ on its diagonal is denoted $\mathrm{diag}(\mathbf{b})$. The determinant is denoted $\det(\cdot)$.

The *multilinear singular value decomposition* (MLSVD) decomposes a tensor $\mathcal{T} \in \mathbb{F}^{I_1 \times I_2 \times I_3}$ as $\mathcal{T} = \mathcal{S} \cdot_1 \mathbf{U}_1 \cdot_2 \mathbf{U}_1 \cdot_3 \mathbf{U}_3$ with core tensor $\mathcal{S} \in \mathbb{F}^{R_1 \times R_2 \times R_3}$ and column-wise orthonormal factor matrices $\{\mathbf{U}^{(n)} \in \mathbb{F}^{I_n \times R_n}\}_{n=1}^{N}$ with $I_n \geq R_n$ for $n = 1, 2, 3$, [26], [27]. The core tensor $\mathcal{S}$ is all-orthogonal and its order-$(N-1)$ subtensors are ordered with nonincreasing norm along each mode [26].

The *canonical polyadic decomposition* [27] decomposes a tensor in a minimal sum of rank-1 terms (outer products of nonzero vectors) and is denoted by $[\![\mathbf{w}; \mathbf{A}, \mathbf{B}, \mathbf{C}]\!] := \sum_{r=1}^{R} w_r \mathbf{a}_{:,r} \otimes \mathbf{b}_{:,r} \otimes \mathbf{c}_{:,r}$ or $[\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!]$ if $\mathbf{w} = \mathbf{1}$.

The *rank-1 map* [22], [23] can be used to find a linear combination of matrices such that the result has rank 1, i.e., to find vectors $\mathbf{w}$ such that $\mathrm{rank}(\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}) = 1$. This problem can be seen as a linear system with a CPD-constrained solution (LSCPD) of the form $\mathbf{Q}(\mathbf{w} \otimes \mathbf{w}) = \mathbf{0}$ with $\mathbf{Q} \in \mathbb{F}^{I^2 J^2 \times K^2}$ the compound matrix [22], [24]. This matrix, multiplied by $\mathbf{w} \otimes \mathbf{w}$, yields every 2×2-determinant of $\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}$, all of which are zero if $\mathcal{T} \cdot_3 \mathbf{w}^{\mathrm{T}}$ has rank one. To find the R solutions to this LSCPD, a basis of the symmetric nullspace $\mathbf{K} \in \mathbb{F}^{K^2 \times R}$ of the compound matrix $\mathbf{Q}$ is computed. $\mathbf{K}$ is then reshaped into a tensor $\mathcal{K} \in \mathbb{F}^{K \times K \times R}$, from which all R solutions $\mathbf{w}_r$ can be obtained through its rank-R CPD, i.e. $\mathcal{K} = [\![\mathbf{W}, \mathbf{W}, \mathbf{M}]\!]$ [24]. The symmetric nullspace is defined as the intersection of the nullspace and all vectorized symmetric matrices.

IV. Bézoutian Tensor

Multivariate Bézoutians [15], [16], [21] are polynomials associated with a square system of polynomial equations $\mathbf{\Sigma}$ and an additional polynomial $q(\mathbf{x})$. Through a CPD of a Bézoutian tensor, which is obtained by stacking the coefficient matrix of Bézoutians associated with different $q_n(\mathbf{x})$, the value of $q_n(\mathbf{x})$ evaluated in the roots $\{\mathbf{r}^{(l)}\}_{l=1}^{L}$ of $\mathbf{\Sigma}$ is obtained, which can be used to determine the roots by choosing, e.g., $q_n(\mathbf{x}) = x_n$; see Section IV-C.

A. Definitions

Let $\mathbf{\Sigma}$ be a square system in N variables, and $q(\mathbf{x})$ a particular polynomial. The Bézoutian polynomial $\mathfrak{B}_{\mathbf{\Sigma},q}(\mathbf{x}, \mathbf{y})$ in $\mathbf{x} \in \mathbb{F}^N$ and auxiliary variables $\mathbf{y} \in \mathbb{F}^N$ is defined as

$$\mathfrak{B}_{\mathbf{\Sigma},q}(\mathbf{x}, \mathbf{y}) = \frac{\det\left(\begin{bmatrix} q(\mathbf{z}_0) & q(\mathbf{z}_1) & \ldots & q(\mathbf{z}_N) \\ p_1(\mathbf{z}_0) & p_1(\mathbf{z}_1) & \ldots & p_1(\mathbf{z}_N) \\ \vdots & \vdots & \ddots & \vdots \\ p_N(\mathbf{z}_0) & p_N(\mathbf{z}_1) & \ldots & p_N(\mathbf{z}_N) \end{bmatrix}\right)}{\prod_{n=1}^{N}(y_n - x_n)}$$

$$\text{with } \mathbf{z}_i = \begin{bmatrix} y_1 & \ldots & y_i & x_{i+1} & \ldots & x_N \end{bmatrix}^{\mathrm{T}}.$$

Note that the numerator is a determinant of polynomials.

The corresponding Bézoutian matrix $\mathbf{B}_{\mathbf{\Sigma},q} \in \mathbb{F}^{L \times L}$ (with L the Bézout number, see Section II) is its coefficient matrix, i.e., $\mathfrak{B}_{\mathbf{\Sigma},q}(\mathbf{x}, \mathbf{y}) = \mathbf{v}_x(\mathbf{x})^{\mathrm{T}} \mathbf{B}_{\mathbf{\Sigma},q} \mathbf{v}_y(\mathbf{y})$, where $\mathbf{v}_x(\mathbf{x})$ and $\mathbf{v}_y(\mathbf{y})$ are multivariate Vandermonde vectors containing the required monomials to express $\mathfrak{B}_{\mathbf{\Sigma},q}(\mathbf{x}, \mathbf{y})$, e.g., for $N = 2$:

$$\mathbf{v}_x(\mathbf{x}) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 & x_1^3 & x_1^2 x_2 & \ldots \end{bmatrix}^{\mathrm{T}}.$$

We define the Bézoutian tensor $\mathcal{B}_{\mathbf{\Sigma},\mathbf{q}} \in \mathbb{F}^{L \times L \times (N+1)}$ as the tensor formed by stacking Bézoutian matrices associated with

different $q_n(\mathbf{x})$ along the third mode. Unless stated otherwise, $\mathbf{q}(\mathbf{x}) = \{q_i(\mathbf{x})\}_{i=0}^{N}$ with $q_0(\mathbf{x}) = 1$ and $q_n(\mathbf{x}) = x_n$ for $n = 1, \ldots, N$.

B. Computation

The Bézoutian tensor can be computed cheaply using the fast Fourier transformation [15]. Instead of symbolic manipulation, the Bézoutian polynomial is evaluated in roots of unity. Its coefficients are then determined through interpolation, see [15] for more details.

C. Low-Rank Structure of Bézoutians

The following property holds for Bézoutian matrices, assuming $\mathbf{\Sigma}$ has only roots with multiplicity one [21]:

$$\begin{aligned}
\mathbf{B}_{\mathbf{\Sigma},q} &= \mathbf{V}_x^{-\mathrm{T}} \mathrm{diag}(\delta_1 q(\mathbf{r}^{(1)}), \ldots, \delta_L q(\mathbf{r}^{(L)})) \mathbf{V}_y^{-1} \\
\text{with } \delta_l &= \det \left(\begin{bmatrix} \frac{\partial p_1}{\partial x_1}(\mathbf{r}^{(l)}) & \ldots & \frac{\partial p_1}{\partial x_N}(\mathbf{r}^{(l)}) \\ \vdots & \ddots & \vdots \\ \frac{\partial p_N}{\partial x_1}(\mathbf{r}^{(l)}) & \ldots & \frac{\partial p_N}{\partial x_N}(\mathbf{r}^{(l)}) \end{bmatrix} \right) \forall l \\
\mathbf{V}_x &= \begin{bmatrix} \mathbf{v}_x(\mathbf{r}^{(1)}) & \ldots & \mathbf{v}_x(\mathbf{r}^{(L)}) \end{bmatrix} \in \mathbb{F}^{L \times L}, \\
\mathbf{V}_y &= \begin{bmatrix} \mathbf{v}_y(\mathbf{r}^{(1)}) & \ldots & \mathbf{v}_y(\mathbf{r}^{(L)}) \end{bmatrix} \in \mathbb{F}^{L \times L}.
\end{aligned}$$

where $\mathbf{r}^{(l)}$ holds the coordinates of the lth root of system $\mathbf{\Sigma}$. A subtlety to note is that both $\mathbf{v}_x(\mathbf{r}^{(l)})$ and $\mathbf{v}_y(\mathbf{r}^{(l)})$ represent multivariate Vandermonde vectors evaluated in root $\mathbf{r}^{(l)}$, but these do not contain the same monomials. In a Bézoutian tensor—in which Bézoutian matrices associated with the same system $\mathbf{\Sigma}$ but different $q_i(\mathbf{x})$ are stacked—$\mathbf{V}_x, \mathbf{V}_y$ and $\boldsymbol{\delta}$ are in common, resulting in the Bézoutian tensor having the following CPD:

$$\begin{aligned}
\mathcal{B}_{\mathbf{\Sigma},\mathbf{q}} &= \left[\!\left[\boldsymbol{\delta}; \mathbf{V}_x^{-\mathrm{T}}, \mathbf{V}_y^{-\mathrm{T}}, \mathbf{X} \right]\!\right] \\
\text{with } \boldsymbol{\delta} &= \begin{bmatrix} \delta_1 & \ldots & \delta_L \end{bmatrix}^{\mathrm{T}} \\
\mathbf{X} &= \begin{bmatrix} \mathbf{q}(\mathbf{r}^{(1)}) & \ldots & \mathbf{q}(\mathbf{r}^{(L)}) \end{bmatrix} \in \mathbb{F}^{N+1 \times L}.
\end{aligned}$$

By choosing $\mathbf{q} = \{q_i(\mathbf{x})\}_{i=0}^{N}$ with $q_0(\mathbf{x}) = 1$ to fix the scaling indeterminacy of a CPD and $q_n(\mathbf{x}) = x_n$ for $n = 1, \ldots, N$, $\mathbf{X}$ reveals the roots:

$$\mathbf{X} = \begin{bmatrix} 1 & \ldots & 1 \\ r_1^{(1)} & \ldots & r_1^{(L)} \\ \vdots & \ddots & \vdots \\ r_N^{(1)} & \ldots & r_N^{(L)} \end{bmatrix}.$$

V. Method

If $\mathbf{r}$ is a root of an overdetermined system $\mathbf{\Sigma}$, it is also a root of any subsystem $\mathbf{\Sigma}_i$ of that overdetermined system $\mathbf{\Sigma}$. We show that finding common roots of square subsystems amounts to finding partially coupled rank-1 terms in their Bézoutians. This can be done with the rank-1 mapping technique as shown in Section V-A and proven in in Section V-B. The resulting algorithm is presented in Section V-C. Finally, the complexity is presented in Section V-D.

A. Recovery of Corresponding Rank-1 Terms

The rank-1 map is used to find corresponding rank-1 terms between multiple low-rank tensors. This differs from other uses, e.g., [22]–[24], where all rank-1 terms of a single tensor are sought. Assume M tensors $\mathcal{T}^{(m)} \in \mathbb{F}^{I \times J \times K}$ that have a rank-K CPD of the form $\mathcal{T}^{(m)} = \left[\!\left[\mathbf{A}^{(m)}, \mathbf{B}^{(m)}, (\mathbf{C}^{(m)})^{-\mathrm{T}} \right]\!\right]$ with full column rank factors are given. The factors for each $\mathcal{T}^{(m)}$ are partitioned according to the $R < K$ shared rank-1 terms which are coupled through $\mathbf{B}_1 \in \mathbb{F}^{J \times R}$ and $\mathbf{C}_1 \in \mathbb{F}^{K \times R}$, and the unshared terms: $\mathbf{A}^{(m)} = \begin{bmatrix} \mathbf{A}_1^{(m)} & \mathbf{A}_2^{(m)} \end{bmatrix}$, $\mathbf{B}^{(m)} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2^{(m)} \end{bmatrix}$ and $\mathbf{C}^{(m)} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2^{(m)} \end{bmatrix}$. It holds that

$$\begin{aligned}
\mathcal{T}^{(m)} \cdot_3 \mathbf{C}_1^{\mathrm{T}} &= \left[\!\left[\mathbf{A}^{(m)}, \mathbf{B}^{(m)}, \mathbf{C}_1^{\mathrm{T}} (\mathbf{C}^{(m)})^{-\mathrm{T}} \right]\!\right] \\
&= \left[\!\left[\mathbf{A}^{(m)}, \mathbf{B}^{(m)}, \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \right]\!\right] \\
&= \left[\!\left[\mathbf{A}_1^{(m)}, \mathbf{B}_1, \mathbf{I} \right]\!\right], \quad m = 1, \ldots, M.
\end{aligned}$$

By stacking the tensors along the first mode, we obtain

$$\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{C}_1^{\mathrm{T}} = \begin{bmatrix} \mathcal{T}^{(1)} \\ \vdots \\ \mathcal{T}^{(M)} \end{bmatrix} \cdot_3 \mathbf{C}_1^{\mathrm{T}} = \left[\!\left[\begin{bmatrix} \mathbf{A}_1^{(1)} \\ \vdots \\ \mathbf{A}_1^{(M)} \end{bmatrix}, \mathbf{B}_1, \mathbf{I} \right]\!\right].$$

This implies that only the R columns $\mathbf{c}_r$ of $\mathbf{C}_1$ yield a rank-1 matrix, i.e., $\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{c}_r^{\mathrm{T}} = \mathbf{a}_r^{(\mathrm{tot})} \mathbf{b}_r^{\mathrm{T}}$, with $\mathbf{b}_r$ the rth column of $\mathbf{B}_1$ and $\mathbf{a}_r^{(\mathrm{tot})}$ the concatenated rth columns of $\mathbf{A}_1^{(m)}$ for all m, which is proven in the next subsection.

B. Proof of Recovery of Corresponding Rank-1 Terms

We prove that if and only if $\mathbf{w} = \mathbf{c}_r$ is a column of $\mathbf{C}_1$, and $\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{w}^{\mathrm{T}} = \mathbf{a}_r^{(\mathrm{tot})} \mathbf{b}_r^{\mathrm{T}}$, $\mathbf{w}$ satisfies $\mathrm{rank}(\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{w}^{\mathrm{T}}) = 1$. Exploiting the CPD structure of $\mathcal{T}^{(m)}$, we find

$$\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{w}^{\mathrm{T}} = \begin{bmatrix} \mathbf{A}^{(1)} \mathrm{diag}\left(\left(\mathbf{C}^{(1)} \right)^{-1} \mathbf{w} \right) \left(\mathbf{B}^{(1)} \right)^{\mathrm{T}} \\ \vdots \\ \mathbf{A}^{(M)} \mathrm{diag}\left(\left(\mathbf{C}^{(M)} \right)^{-1} \mathbf{w} \right) \left(\mathbf{B}^{(M)} \right)^{\mathrm{T}} \end{bmatrix}.$$

For block row m to have rank 1, it is required that $\left(\mathbf{C}^{(m)} \right)^{-1} \mathbf{w} = \mathbf{e}_{k_m}$ for some k_m, with $\mathbf{e}_{k_m}$ a column of the $K \times K$ identity matrix, as $\mathbf{A}^{(m)}$ and $\mathbf{B}^{(m)}$ have full column rank. This implies that $\mathbf{w} = \mathbf{C}^{(m)} \mathbf{e}_{k_m}$, i.e., $\mathbf{w}$ is a column of $\mathbf{C}^{(m)}$ for any m, meaning that all $\mathbf{C}^{(m)}$ share a column. At the same time, for the complete matrix $\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{w}^{\mathrm{T}}$ to have rank one, $\mathbf{B}^{(m)} \left(\mathbf{C}^{(m)} \right)^{-1} \mathbf{w} = \mathbf{B}^{(m)} \mathbf{e}_{k_m}$ should be equal for all m, meaning all $\mathbf{B}^{(m)}$ share a column. Thus, only if column k_m of $\mathbf{B}^{(m)}$ is shared for all m and column k_m of $\mathbf{C}^{(m)}$ is shared for all m, $\mathbf{w} = \mathbf{C}^{(m)} \mathbf{e}_{k_m}$ will satisfy $\mathrm{rank}(\mathcal{T}^{(\mathrm{tot})} \cdot_3 \mathbf{w}^{\mathrm{T}}) = 1$. If R columns satisfy this, R vectors $\mathbf{w}^{(r)}$ will satisfy $\mathrm{rank}\left(\mathcal{T}^{(\mathrm{tot})} \cdot_3 \left(\mathbf{w}^{(r)} \right)^{\mathrm{T}} \right) = 1$.

C. Proposed Method and Improvements

The proposed method is summarized in Algorithm 1. It first computes the Bézoutian tensor $\mathcal{B}_{\mathbf{\Sigma}_m,\mathbf{q}}$ for every square subsystem $\mathbf{\Sigma}_m$ of $\mathbf{\Sigma}$. After permuting the last two modes of all $\mathcal{B}_{\mathbf{\Sigma}_m,\mathbf{q}}$, the latter become partially coupled for each root of $\mathbf{\Sigma}$ (and thus also of any $\mathbf{\Sigma}_m$) in the same way as presented in Section V-A thanks to their CPD, see Section IV-C, allowing for these roots to be found through a rank-1 map.

Algorithm 1 Bézoutian-based solution for overdetermined polynomial systems.

Input a system $\mathbf{\Sigma}$.
Output matrix $\mathbf{X}$ which contains the R roots of $\mathbf{\Sigma}$.

1: Compute the Bézoutian tensors associated with each square subsystem $\mathbf{\Sigma}_m$ of $\mathbf{\Sigma}$ (see Section IV-B).
2: Stack these tensors along the first mode to form $\mathcal{Y}$.
3: Permute the last two modes of $\mathcal{Y}$ to obtain $\mathcal{E}$.
4: Compute the compound matrix $\mathbf{Q}$ associated with the rank-1 map applied to $\mathcal{E}$ (see Section III and [24]).
5: Determine the dimension R of the symmetric nullspace of $\mathbf{Q}$ and compute $\mathbf{K}$ which holds a basis for that nullspace.
6: Tensorize $\mathbf{K}$ and compute its CPD: $\mathcal{K} = [\![\mathbf{W}, \mathbf{W}, \mathbf{M}]\!]$.
7: **for** $r = 1, \ldots, R$ **do**
8: Compute $\mathbf{x}^{(r)}$, the vector that generates the row space of rank-1 matrix $\mathcal{E} \cdot_3 (\mathbf{w}^{(r)})^{\mathrm{T}}$
9: **end for**

The following alterations are needed to handle noise.

- Step 3: $\mathcal{Y}$ can be compressed using the MLSVD to improve numerical stability and reduce computations. Moreover, orthonormalizing the frontal slices of $\mathcal{E}$ is numerically beneficial in the presence of noise.
- Steps 5-9: When noise is present, right singular vectors associated with small singular values should be used for the approximate nullspace. The number of small singular values indicates the number of roots of the system R. Similarly in Step 6, rather than an exact CPD, a rank-R CP approximation should be computed. In Step 8, the dominant singular vector can be used for the row space.

Some further alterations can be used to reduce the number of computations but were not implemented for the experiments.

- Step 1: Computing Bézoutians for every square subsystem, of which there are $\binom{S}{N}$, is not strictly necessary, although only using some might affect the accuracy of the solution under noise.
- Step 2: Roots at infinity can be removed in this step [15].
- Step 5: The vectors $\mathbf{w}^{(r)}$, for which $\mathbf{K}$ holds a basis, are multivariate Vandermonde vectors. This means that $\mathbf{w}^{(r)} \otimes \mathbf{w}^{(r)}$ will have repeated entries. For efficiency, the columns of $\mathbf{Q}$ associated with repeated entries can thus be combined before computing the nullspace.

D. Computational Complexity

Assume all S polynomials are of degree D with N variables. The three potential bottlenecks are:

1) computing $\binom{S}{N}$ Bézoutian tensors which take $O((N!D^N)^2(\log_2(N!D^N) + N^3))$ operations each,
2) compressing $\mathcal{Y}$ requiring $O(\binom{S}{N} D^{3N} N^2)$ operations,
3) finding a basis for the nullspace of the Compound matrix $\mathbf{Q}$, requiring $O(N^4 D^{6N})$ operations (or $O(N^4 D^{4N})$ by exploiting the Vandermonde structure in Step 5).

VI. Experiments

Two experiments are included to showcase the robustness and applicability of the proposed method in solving overdetermined polynomial systems: an example to analyze the effect of the number of equations and signal-to-noise ratio (SNR), and a blind source separation problem with constant modulus.

Both experiments compare the proposed Bézoutian-based method with a Macaulay-based method [20] (abbreviated Béz and Mac, respectively). All experiments were done in MATLAB, using Tensorlab [28]. Every data point is the median of 1000 randomly generated instances.

Note that these randomly generated polynomial systems can become ill-conditioned far more easily than linear systems. By adding a tiny bit of noise to the coefficients, polynomials can go from having two close-by roots to one (with multiplicity two resulting in numerical issues) or the system can even become inconsistent (no common roots, see Fig. 1).

A. Effect of Overdeterminedness and Noise

Similar to systems of linear equations, the solution of a polynomial system improves as more equations (with a similar precision) are added. This can be seen in Fig. 2, which shows the accuracy of an experiment with polynomials of degree $D = 4$ in $N = 2$ variables with $R = 2$ common roots.

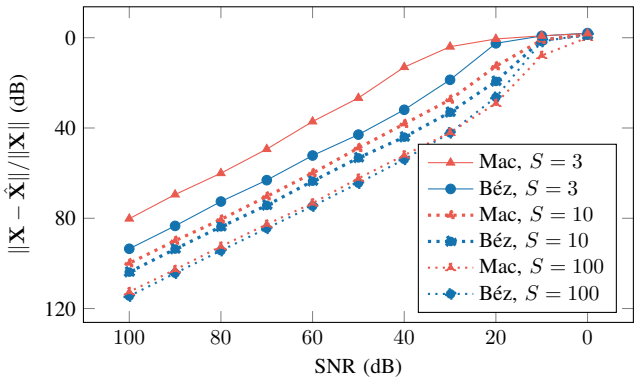


Fig. 2. Using more equations improves the accuracy of the solution, even in the low SNR regime, where a random guess would be as good as the estimate obtained from only $S = 3$ equations with either method. At high SNR, Béz outperforms Mac, especially with fewer equations.

B. Underdetermined Blind Source Separation

Consider a channel matrix $\mathbf{A}$ mixing I source signals x_i into J observed signals y_j, according to $\mathbf{y} = \mathbf{A}\mathbf{x}$ [29], [30]. Additionally, it is known that each source signal $x_i[k]$ has constant modulus, i.e., $|x_i[k]| = b_i, \forall k$. We consider the (challenging) underdetermined case, where the number of sources exceeds the number of observation channels, $I > J$.

We assume that the channel matrix was already estimated (with, e.g., [23]) and the constant moduli estimated, using, e.g., $\text{cov}(\mathbf{Y}) = \mathbf{A}\mathbf{\Sigma}^2\mathbf{A}^{\text{H}}$, such that $|x_i| = 1$ holds.

This problem is transformed into the following system of $2J + I$ polynomial equations by splitting each source signal into its real and imaginary parts $x_i = c_i + js_i$:

$$\mathbf{\Sigma} = \begin{cases} \text{real}(\mathbf{y}) = \text{real}(\mathbf{A})\mathbf{c} - \text{imag}(\mathbf{A})\mathbf{s} \\ \text{imag}(\mathbf{y}) = \text{real}(\mathbf{A})\mathbf{s} + \text{imag}(\mathbf{A})\mathbf{c} \\ c_i^2 + s_i^2 = 1 \qquad\qquad \text{for } i = 1, \ldots, I \end{cases}$$

with $2I$ unknowns, namely the entries of $\mathbf{c}$ and $\mathbf{s}$. To reduce the number of variables, the $2J$ first linear equations are substituted in the other equations. Figure 3 shows the results.

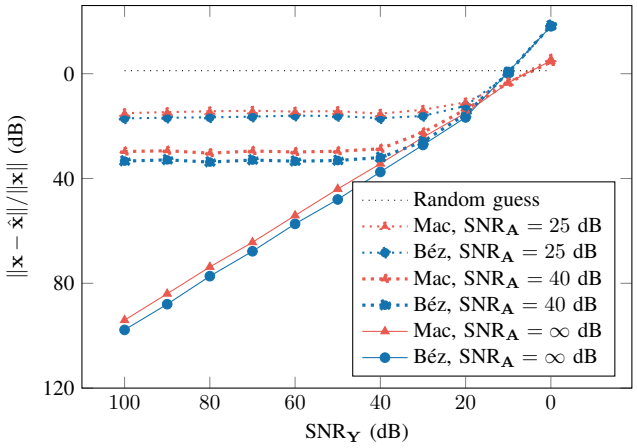


Fig. 3. Both the Bézoutian-based and Macaulay-based method can approximate the solution to the source separation problem, even with noise present. The experiment uses $I = 3$ sources and $J = 2$ observed signals. $\text{SNR}_{\mathbf{Y}}$ and $\text{SNR}_{\mathbf{A}}$ indicate the accuracy to which $\mathbf{Y}$ and $\mathbf{A}$ are estimated, respectively.

VII. Conclusion

We proposed a new method to solve overdetermined systems of noisy polynomial equations, by finding all partially coupled rank-1 terms in multiple Bézoutian tensors, using an extension of the rank-1 mapping technique. The method is effective at mitigating noise using the additional equations provided. The potential benefit of overdetermined polynomial systems in signal processing is demonstrated in a blind signal separation problem with more sources than measured signals.

Further research could be geared towards improving the computational complexity of the proposed method. Due to the abundant structure in the Bézoutian tensor, there is potential to reduce the number of operations significantly. This includes finding a more effective way of combining the Bézoutian tensors, although a generalization of Bézoutian tensors for overdetermined systems might be even better. Also of interest could be a more in-depth analysis of each step to improve noise mitigation through orthogonalization, normalization or other techniques.

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