Coupled canonical polyadic decompositions and multiple shift-invariance in array processing

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***Abstract*—The Canonical Polyadic Decomposition (CPD) plays an important role for signal separation in array processing. The CPD model requires arrays composed of several displaced but identical subarrays. Consequently, it is less appropriate for more complex array geometries. In this paper we explain that coupled CPD allows a much more flexible modeling that can handle multiple shift-invariance structures, i.e., arrays that can be decomposed into multiple but not identical displaced subarrays. Both deterministic and generic identifiability conditions are presented. We also point out that, under mild conditions, the signal separation problem can in the exact case be solved by means of an eigenvalue decomposition. This is similar to ESPRIT, although the working conditions are much more relaxed. Borrowing tools from algebraic geometry, we derive generic uniqueness bounds for L-shaped, frame-shaped and triangular-shaped arrays that come close to bounds that are necessary for uniqueness. Recognizing multiple shift-invariance can be a bit of an art by itself. We explain that any centro-symmetric array processing problem can be interpreted in terms of a coupled CPD. In addition, we demonstrate that the coupled CPD model allows us to significantly relax the far-field assumption commonly used in CPD-based array processing.**

***Index Terms*—array processing, identifiability, near-field, centro-symmetry, tensor, polyadic decomposition, coupled decomposition.**

I. Introduction

Sensor arrays have found important applications in radar, sonar, wireless communication and medical diagnostics. The connection between sensor array processing problems with regular array geometries composed of several displaced but identical subarrays and the Canonical Polyadic Decomposition (CPD) was made in [20]. In practice, array geometries may also be irregular or sparse. Hence, there is a need for a flexible algebraic framework for array processing.

To overcome some of the limitations of traditional tensor based array signal processing, we have already suggested a shift from the conventional CPD-based approach [20] to coupled tensor based array processing in [26], [27], [28], where links with multidimensional harmonic retrieval [11], [17] and multiple invariance ESPRIT [30] array processing problems were presented. In this paper we further broaden the approach. First, we argue that the coupled CPD framework presented in [24], [29] provides a unified algebraic framework for handling various types of one-, two-, three- or N-dimensional arrays with shift-invariance properties. More specifically, we present both deterministic and generic sufficient uniqueness conditions. By way of example, we discuss L-shaped, frame-shaped and triangular-shaped arrays [13]. However, the approach can also be used for other antenna array configurations, such as the hexagonal array [32]. Note that even though such antenna array configurations have been around for a long time, dedicated identifiability conditions have, to the knowledge of the authors, not yet been discussed in the literature. It is also interesting to note that in the noiseless case, the coupled CPD can in many cases be exactly computed via an EigenValue Decomposition (EVD). Our approach can therefore be understood as an extension of ESPRIT [19] to multiple shift-invariance structures. Part of this work was presented in the conference paper [22]. In addition, we explain that any centro-symmetric array (i.e., arrays that are symmetric around their "epicenter") can be interpreted as an array that enjoys multiple shift-invariance structures and hence fits within the coupled CPD framework. Finally, we demonstrate that the coupled CPD model allows us to relax the far-field assumption used in CPD-based array processing (e.g., [20]). As an illustration of "recognizing" coupled CPDs, we interpret near-field array processing using a linear array as far-field array processing using an array that enjoys multiple shift-invariance structures.

The paper is organized as follows. The rest of the introduction will present the notation. Section II reviews classical tensor based array processing. Section III reviews the coupled CPD model and introduces a new necessary identifiability condition. Section IV introduces the coupled CPD framework for array processing problems involving array geometries with multiple shift-invariance structures. Numerical experiments are reported in Section V. Section VI concludes the paper.

Notation: Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The rth column, con-

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jugate, transpose, conjugate-transpose, Moore-Penrose pseudoinverse, Frobenius norm, and rank of a matrix $\mathbf{A}$ are denoted by $\mathbf{a}_r$, $\mathbf{A}^*$, $\mathbf{A}^T$, $\mathbf{A}^H$, $\mathbf{A}^\dagger$, $\|\mathbf{A}\|_F$, rank($\mathbf{A}$), respectively.

The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao product, defined as

$$\mathbf{A}\otimes\mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$
$$\mathbf{A}\odot\mathbf{B} := [\mathbf{a}_1\otimes\mathbf{b}_1 \;\; \mathbf{a}_2\otimes\mathbf{b}_2 \;\dots],$$

in which $(\mathbf{A})_{mn} = a_{mn}$. The outer product of, say, three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is denoted by $\mathbf{a}\circ\mathbf{b}\circ\mathbf{c}$, such that $(\mathbf{a}\circ\mathbf{b}\circ\mathbf{c})_{ijk} = a_i b_j c_k$. The all-ones vector is denoted by $\mathbf{1}_R = [1,\dots,1]^T \in \mathbb{C}^R$. The I-element unit vector with unit entry at position i and zero elsewhere is denoted $\mathbf{e}_i^{(I)} \in \mathbb{C}^I$. $D_k(\mathbf{A}) \in \mathbb{C}^{J\times J}$ denotes the diagonal matrix holding row k of $\mathbf{A} \in \mathbb{C}^{I\times J}$ on its diagonal. Matlab index notation will be used for submatrices of a given matrix. For example, $\mathbf{A}(1{:}k,:)$ represents the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k of $\mathbf{A}$. Let $\mathbf{A} \in \mathbb{C}^{I\times R}$, then $\underline{\mathbf{A}} = \mathbf{A}(1:I-1,:) \in \mathbb{C}^{(I-1)\times R}$ and $\overline{\mathbf{A}} = \mathbf{A}(2:I,:) \in \mathbb{C}^{(I-1)\times R}$, i.e., $\underline{\mathbf{A}}$ and $\overline{\mathbf{A}}$ are obtained by deleting the bottom and upper row of $\mathbf{A}$, respectively. The number of nonzero elements of a vector $\mathbf{a}$ is denoted by $\omega(\mathbf{a})$. The binomial coefficient is denoted by $C_m^2 = \frac{m(m-1)}{2}$. The 2-nd compound matrix of $\mathbf{A} \in \mathbb{C}^{I\times R}$ is denoted by $C_2(\mathbf{A}) \in \mathbb{C}^{C_I^2\times C_R^2}$. It is the matrix containing the determinants of all 2×2 submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order, i.e., the $(C_I^2 + C_{I-i_1+1}^2 + i_2 - i_1, C_R^2 + C_{R-r_1+1}^2 + r_2 - r_1)$th entry of $C_2(\mathbf{A})$ is equal to $a_{i_1r_1}a_{i_2r_2} - a_{i_2r_1}a_{i_1r_2}$, where $1 \le i_1 < i_2 \le I$ and $1 \le r_1 < r_2 \le R$ (see [3] and references therein for a discussion).

II. Tensor based array processing

Consider R signals impinging on an array composed of I sensors such that the output of the ith sensor at the kth observation is

$$x_{ik} = \sum_{r=1}^{R} s_r(k - \tau_{ri}), \tag{1}$$

where τ_{ri} denotes the delay between the ith sensor and the rth source. The position of the ith sensor in cartesian coordinates is given by $\mathbf{p}_i = [x_i \; y_i \; z_i]^T$. Let us additionally assume that the sources are located in the far-field and that the narrowband assumption holds. Under these assumptions the array response vector $\mathbf{a}_r \in \mathbb{C}^I$ associated with the rth source can (approximately) be expressed as

$$\mathbf{a}_r = [e^{-\sqrt{-1}\omega_c\mathbf{b}_r^T\mathbf{p}_1/c} \;\dots\; e^{-\sqrt{-1}\omega_c\mathbf{b}_r^T\mathbf{p}_I/c}]^T, \tag{2}$$

where ω_c is the carrier frequency, $\mathbf{b}_r = [\sin(\phi_r)\cos(\theta_r),\; \sin(\phi_r)\sin(\theta_r),\; \cos(\phi_r)]^T$ is the bearing vector in which θ_r and ϕ_r denote the azimuth and elevation angle, respectively, c is the speed of propagation, and the product $\mathbf{b}_r^T\mathbf{p}_i/c$ corresponds to the propagation delay in (1) associated with the ith sensor and the rth source, that is $\tau_{ri} = \mathbf{b}_r^T\mathbf{p}_i/c$. Assume that R sources are impinging on the sensor array and that K snapshots are available such that $\mathbf{s}_r \in \mathbb{C}^K$ denotes the signal vector associated with the rth source. Then the observed data matrix admits the factorization

$$\mathbf{X} = \mathbf{A}\mathbf{S}^T \in \mathbb{C}^{I\times K}, \quad \mathbf{A} = [\mathbf{a}_1 \;\dots\; \mathbf{a}_R],\; \mathbf{S} = [\mathbf{s}_1 \;\dots\; \mathbf{s}_R]. \tag{3}$$

In Direction-Of-Arrival (DOA) applications the goal is to estimate $\{\theta_r, \phi_r\}$ via $\mathbf{A}$, observing $\mathbf{X}$. On the other hand, in source separation the goal is to estimate $\mathbf{S}$, observing $\mathbf{X}$. For both problems, uniqueness conditions and algorithms need to be developed. To this end, either arrays with simple geometries or signals with certain known properties (statistical independence, finite alphabet, sparsity, etc.) are used.

A. CPD model for array processing

Because of their simplicity, arrays composed of several displaced but identical subarrays are often considered in the literature (see [20] and references therein). More precisely, let x_{ijk} denote the output of the ith sensor of the jth subarray at the kth snapshot. Assume that J subarrays each composed of I sensors are used. Then the observed third-order data tensor $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ admits a so-called Canonical Polyadic Decomposition (CPD) [1], [12], [15], as pointed out in [20]:

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{d}_r \circ \mathbf{s}_r, \tag{4}$$

where $\mathbf{a}_r = [e^{-\sqrt{-1}\omega_c\mathbf{b}_r^T\mathbf{p}_1/c} \;\dots\; e^{-\sqrt{-1}\omega_c\mathbf{b}_r^T\mathbf{p}_I/c}]^T \in \mathbb{C}^I$, $\mathbf{d}_r = [e^{-\sqrt{-1}\omega_c\mathbf{b}_r^T\mathbf{t}_1/c} \;\dots\; e^{-\sqrt{-1}\omega_c\mathbf{b}_r^T\mathbf{t}_J/c}]^T \in \mathbb{C}^J$, and $\mathbf{s}_r \in \mathbb{C}^K$ denote the reference subarray, the displacement and signal vector associated with the rth source, respectively. In this case $\mathbf{p}_i$ denotes the position of the ith sensor in the reference subarray while $\mathbf{t}_j$ denotes the translation vector associated with the jth subarray. Note that the I element subarray vector $\mathbf{a}_r$ is repeated J times and scaled according to the entries of $\mathbf{d}_r$. This *polyadic structure* is the algebraic manifestation of the *shift-invariance* property of the array. Uniqueness conditions and algebraic algorithms for the CPD can be found in [14], [2], [3], [4], [5], [7], [21] and references therein.

Throughout the paper matrix representations of tensors will be used. Consider the horizontal matrix slice $\mathbf{X}^{(i\cdot\cdot)} \in \mathbb{C}^{J\times K}$ of $\mathcal{X}$, defined by $(\mathbf{X}^{(i\cdot\cdot)})_{jk} = x_{ijk} = \sum_{r=1}^R a_{ir}d_{jr}s_{kr}$. The tensor $\mathcal{X}$ can be interpreted as a collection of matrix slices $\mathbf{X}^{(1\cdot\cdot)},\dots,\mathbf{X}^{(I\cdot\cdot)}$, each admitting the factorization $\mathbf{X}^{(i\cdot\cdot)} = \sum_{r=1}^R a_{ir}\mathbf{d}_r\mathbf{s}_r^T = \mathbf{D}D_i(\mathbf{A})\mathbf{S}^T$. Stacking yields the matrix representation of (4):

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1\cdot\cdot)} \\ \vdots \\ \mathbf{X}^{(I\cdot\cdot)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}D_1(\mathbf{A}) \\ \vdots \\ \mathbf{D}D_I(\mathbf{A}) \end{bmatrix} \mathbf{S}^T = (\mathbf{A}\odot\mathbf{D})\mathbf{S}^T \in \mathbb{C}^{IJ\times K}. \tag{5}$$

B. ESPRIT method for array processing

One of the first proposed array processing methods that can be interpreted in terms of the CPD model (4) is ESPRIT [19]. Recall that ESPRIT considers a bilinear factorization of the form (3) in which $\mathbf{S}$ has full column rank and the columns of $\mathbf{A}$ are Vandermonde vectors of the form

$$\mathbf{a} = [a_0, a_1, \ldots, a_{I-1}]^T = [1, z, \ldots, z^{I-1}]^T \in \mathbb{C}^I. \quad (6)$$

Using the shift-property $z^{m+1} = z \cdot z^m$, the Vandermonde vector (6) can be related to the rank-1 Hankel matrix

$$\mathbf{H} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{I-2} \\ a_1 & a_2 & \cdots & a_{I-1} \end{bmatrix} = \begin{bmatrix} 1 \\ z \end{bmatrix} \underline{\mathbf{a}}^T = \begin{bmatrix} 1 \\ z \end{bmatrix} \circ \underline{\mathbf{a}}. \quad (7)$$

The shift-invariance property also implies that $\overline{\mathbf{X}} = \overline{\mathbf{A}}\mathbf{S}^T = \underline{\mathbf{A}} D_1([z_1, \ldots, z_R])\mathbf{S}^T = \underline{\mathbf{A}} D_2(\mathbf{A})\mathbf{S}^T$. Let us stack $\underline{\mathbf{X}}$ and $\overline{\mathbf{X}}$ in a two-slice tensor $\mathcal{X} \in \mathbb{C}^{2\times(I-1)\times K}$. According to (5), $\underline{\mathbf{X}} = \underline{\mathbf{A}}\mathbf{S}^T$ and $\overline{\mathbf{X}} = \underline{\mathbf{A}} D_2(\mathbf{A})\mathbf{S}^T$ can be seen as a CPD $\mathcal{X} = \sum_{r=1}^R \begin{bmatrix} 1 \\ z_r \end{bmatrix} \circ \underline{\mathbf{a}}_r \circ \mathbf{s}_r$, with matrix representation

$$\mathbf{X} = \begin{bmatrix} \underline{\mathbf{X}} \\ \overline{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{A}} \\ \underline{\mathbf{A}} D_2(\mathbf{A}) \end{bmatrix} = \left(\begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_R \end{bmatrix} \odot \underline{\mathbf{A}} \right) \mathbf{S}^T. \quad (8)$$

The first explicit mentioning of the link between CPD and ESPRIT in the context of array processing was probably in [20]. It is well-known that if $\underline{\mathbf{A}}$ and $\mathbf{S}$ have full column rank and the Vandermonde generators of $\mathbf{A}$ are distinct (i.e., $z_r \neq z_s$ for all $r \neq s$), then the CPD of $\mathcal{X}$ is unique and can be computed via an EVD [16]. In the next section we will see that the coupled CPD model provides multidimensional extensions of ESPRIT.

III. Coupled CPD model for array processing and identifiability conditions

Standard tensor-based methods for array processing are essentially limited to situations where the observed data admit a CPD interpretation of the form (4). In this section we explain that the coupled *CPD model* (9) below enables a more general modelling. It extends the CPD model to arrays enjoying *multiple shift-invariance* structures:

$$\mathbb{C}^{I_n \times J_n \times K} \ni \mathcal{X}^{(n)} = \sum_{r=1}^R \mathbf{a}_r^{(n)} \circ \mathbf{d}_r^{(n)} \circ \mathbf{s}_r, \quad n \in \{1, \ldots, N\}, \quad (9)$$

where N denotes the number of shift-invariance structures that have been taken into account, and where $\mathbf{a}_r^{(n)} \in \mathbb{C}^{I_n}$ and $\mathbf{d}_r^{(n)} \in \mathbb{C}^{J_n}$ denote the nth reference subarray response vector and its associated displacement vector, respectively. Obviously, if $N = 1$, then (9) reduces to (4). Similar to the CPD matrix representation (5), the coupled CPD of $\{\mathcal{X}^{(n)}\}$ has the following matrix representation:

$$\begin{bmatrix} \mathbf{X}^{(1)} \\ \vdots \\ \mathbf{X}^{(N)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} \odot \mathbf{D}^{(1)} \\ \vdots \\ \mathbf{A}^{(N)} \odot \mathbf{D}^{(N)} \end{bmatrix} \mathbf{S}^T \in \mathbb{C}^{(\sum_{n=1}^N I_n J_n) \times K}. \quad (10)$$

As for CPD, uniqueness conditions and algebraic algorithms for coupled CPD (9) have been developed [24], [29]. In this paper we limit the discussion to the multiple snapshot case, meaning that we assume that the factor matrix $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_R] \in \mathbb{C}^{K \times R}$ has full column rank. However, this does not mean that the factor matrices $\mathbf{A}^{(n)} = [\mathbf{a}_1^{(n)}, \ldots, \mathbf{a}_R^{(n)}] \in \mathbb{C}^{I_n \times R}$ and $\mathbf{D}^{(n)} = [\mathbf{d}_1^{(n)}, \ldots, \mathbf{d}_R^{(n)}] \in \mathbb{C}^{J_n \times R}$ are required to have full column rank. (In fact, in array processing it is not uncommon that some of matrices in the set $\{\mathbf{A}^{(n)}, \mathbf{D}^{(n)}\}_{n=1}^N$ do not have full column rank.) Before presenting uniqueness conditions for coupled CPD, a few concepts need to be explained.

1) Definitions of coupled rank, coupled CPD and coupled PD: We define the coupled rank of $\{\mathcal{X}^{(n)}\}$ as the minimal number of coupled rank-1 tensors $\mathbf{a}_r^{(n)} \circ \mathbf{d}_r^{(n)} \circ \mathbf{c}_r$ that yield $\{\mathcal{X}^{(n)}\}$ in a linear combination. Assume that the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R, then (9) will be called the coupled CPD of $\{\mathcal{X}^{(n)}\}$. On the other hand, if R in (9) is not necessarily minimal, then (9) will simply be called a coupled Polyadic Decomposition (PD) of $\{\mathcal{X}^{(n)}\}$.

2) Definition of uniqueness of coupled CPD: The coupled rank-1 tensors in (9) can be arbitrarily permuted and the vectors within the same coupled rank-1 tensor can be arbitrarily scaled provided the overall coupled rank-1 term remains the same. We say that the coupled CPD is unique when it is only subject to these trivial indeterminacies.

3) Uniqueness condition for coupled CPD: In this paper we consider the case where the signal matrix $\mathbf{S}$ has full column rank. A necessary and sufficient uniqueness condition for coupled CPD with full column rank $\mathbf{S}$ can be found in [24], and formulated in terms of compound matrices in Theorem III.1 below. It will make use of the matrix

$$\mathbf{G}^{(N)} = \begin{bmatrix} C_2\left(\mathbf{A}^{(1)}\right) \odot C_2\left(\mathbf{D}^{(1)}\right) \\ \vdots \\ C_2\left(\mathbf{A}^{(N)}\right) \odot C_2\left(\mathbf{D}^{(N)}\right) \end{bmatrix} \in \mathbb{C}^{\left(\sum_{n=1}^N C_{I_n}^2 C_{J_n}^2\right) \times C_R^2}. \quad (11)$$

Theorem III.1. *Consider the coupled PD of $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n \times J_n \times K}$, $n \in \{1, \ldots, N\}$ in (9). Assume that $\mathbf{S}$ has full column rank. The coupled rank of $\{\mathcal{X}^{(n)}\}$ is R and the coupled CPD of $\{\mathcal{X}^{(n)}\}$ is unique if and only if*

$$\boldsymbol{G}^{(N)} \boldsymbol{c}^{(2)} = \boldsymbol{0} \Rightarrow \omega(\boldsymbol{c}) \leq 1, \quad (12)$$

where $\boldsymbol{c}^{(2)} = [c_1 c_2, c_1 c_3, \ldots, c_{R-1} c_R]^T \in \mathbb{C}^{C_R^2}$ [24].

In practice, the necessary and sufficient uniqueness condition (12) can be hard to check. As an alternative, one can resort to the following more easy to check and yet very powerful uniqueness condition.

Theorem III.2. *Consider the coupled PD of $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n \times J_n \times K}$, $n \in \{1, \ldots, N\}$ in (9). If*

$$\begin{cases} \boldsymbol{S} \text{ in (9) has full column rank}, & (13a) \\ \boldsymbol{G}^{(N)} \text{ in (11) has full column rank}, & (13b) \end{cases}$$

then the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R, the coupled CPD of $\{\mathcal{X}^{(n)}\}$ is unique, and the coupled CPD can be computed via an EVD [24], [29].

From the definition of $\mathbf{G}^{(N)}$ it is clear that more relaxed uniqueness conditions are obtained by taking the coupling into account; indeed, every PD in the set contributes a submatrix to $\mathbf{G}^{(N)}$.

4) A necessary identifiability conditions for (constrained) coupled CPD: Subsections IV-A and IV-B demonstrate that the coupled CPD model (9) plays an important role in array processing problems involving shift-invariance structures. In all cases considered in this paper the matrix $\mathbf{A}^{(n)} \odot \mathbf{D}^{(n)}$ in (10) is subject to the following constraint:

$$\begin{aligned} &\text{the columns } \mathbf{a}_r^{(n)} \otimes \mathbf{d}_r^{(n)} \\ &\text{are values of a vector function } \mathbf{b}^{(n)} \in \mathbb{C}^{I_n J_n} \end{aligned} \tag{14}$$

such that each entry of $\mathbf{b}^{(n)}$ is an analytic function on $\mathbb{C}^l$. For example, the unconstrained coupled CPD corresponds to the case $\mathbf{b}^{(n)} = [z_1 \ \dots \ z_{I_n}]^T \otimes [z_{I_n+1} \ \dots \ z_{I_n+J_n}]^T$, i.e., $\mathbf{b}^{(n)}$ is a vector function of $l = I_n + J_n$ variables. In the case of constrained coupled CPD, $\mathbf{b}^{(n)}$ may depend on less than $I_n + J_n$ parameters. For instance, in (8), $\mathbf{b}^{(1)} = [1 \ z]^T \otimes [1 \ \dots \ z^{I-2}] \in \mathbb{C}^{2(I-1)}$ is a vector function of only one variable z.

The sufficient conditions for uniqueness in Theorems III.1 and III.2 can be used for the constrained coupled CPD as well by just ignoring the constraints. On the other hand, condition (12) in Theorem III.1 is not necessary for uniqueness of the constrained coupled CPD.

The following trivial bound on R is necessary for the uniqueness of (constrained) coupled CPD (10). We assume that constraint (14) holds and we put (10) in the form

$$\mathbf{X} = \mathbf{B}\mathbf{S}^T, \tag{15}$$

where $\mathbf{X}$ denotes the matrix in the left-hand side of (10), and $\mathbf{B}$ has columns $[(\mathbf{a}_r^{(1)} \otimes \mathbf{d}_r^{(1)})^T \ \dots \ (\mathbf{a}_r^{(N)} \otimes \mathbf{d}_r^{(N)})^T]^T$. Thus, uniqueness of the (constrained) coupled CPD (10) is equivalent to the uniqueness of the structured matrix factorization (15).

Theorem III.3. *Let the entries of the vector functions $\mathbf{a}^{(n)}(\mathbf{z}) : \mathbb{C}^l \to \mathbb{C}^{I_n}$ and $\mathbf{d}^{(n)}(\mathbf{z}) : \mathbb{C}^l \to \mathbb{C}^{J_n}$ be analytic functions on $\mathbb{C}^l$ and*

$$\mathbf{b}(\mathbf{z}) = [\mathbf{b}^{(1)}(\mathbf{z})^T \ \dots \ \mathbf{b}^{(N)}(\mathbf{z})^T]^T, \quad \mathbf{b}^{(n)}(\mathbf{z}) = \mathbf{a}^{(n)}(\mathbf{z}) \otimes \mathbf{d}^{(n)}(\mathbf{z}).$$

Let also δ denote the dimension of the subspace spanned by all possible vectors of $\mathbf{b}$. If $\mathbf{S}$ has full column rank, $\operatorname{rank}(\mathbf{B}) = \delta$, *and $R \geq \delta$, then the (constrained) coupled CPD* (10) *is not unique.*

Proof: See Appendix A. ∎

Note that in the CPD case (4) with $\mathbf{b}_r^{(n)} = \mathbf{a}_r^{(n)} \otimes \mathbf{d}_r^{(n)}$ and $N = 1$, Theorem III.3 yields the well-known necessary condition $R < IJ$. Similarly, for the coupled CPD case (9) with $\mathbf{b}_r^{(n)} = \mathbf{a}_r^{(n)} \otimes \mathbf{d}_r^{(n)}$ and $N \geq 2$, we obtain the analogous necessary condition $R < I_1 J_1 + \cdots + I_N J_N$. In both the CPD and coupled CPD cases, Theorem III.3 only yields rough identifiability bounds on R. However, in the constrained coupled CPD cases associated with L-shaped, frame-shaped and triangular-shaped arrays discussed in Section IV-A, we will see that the necessary bounds obtained via Theorem III.3 are close to the sufficient bounds obtained in Theorems IV.3, IV.5 and IV.7.

IV. Case studies and applications in array processing

A. Arrays composed of linear or rectangular arrays

Several practical array configurations are based on the combination of uniform linear or rectangular arrays (ULA, URA). Such configurations can be handled in the coupled CPD modeling framework. Despite the generality of configurations that can be described, coupled CPD yields explicit uniqueness conditions. Moreover, the EVD in Theorem III.2 can be interpreted as an extension of ESPRIT to such geometries. We discuss the increasingly complex cases of L-shaped, frame-shaped and triangular-shaped arrays.

1) L-shaped array: A simple first example is the L-shaped array depicted in Figure 1 (left), which is clearly composed of two ULAs. Assume that K snapshots are available. The outputs of the two ULAs can be stacked into the matrices $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ according to $(\mathbf{Y}^{(1)})_{ik} = y_{i1k}$ and $(\mathbf{Y}^{(2)})_{ik} = y_{1ik}$ where the index k denotes the snapshot number. This leads to the structured matrix factorizations

$$\mathbf{Y}^{(1)} = \mathbf{B}^{(1)}\mathbf{S}^T \ \text{ and } \ \mathbf{Y}^{(2)} = \mathbf{B}^{(2)}\mathbf{S}^T, \tag{16}$$

where $\mathbf{S} \in \mathbb{C}^{K \times R}$ is the collected signal matrix and

$$\begin{aligned} \mathbf{b}_r^{(1)} &= [1, z_{1,r}, \dots, z_{1,r}^{(I_1-1)}], \quad z_{1,r} = e^{-\sqrt{-1}\omega_c d_x \sin(\phi_r)\cos(\theta_r)/c}, \\ \mathbf{b}_r^{(2)} &= [1, z_{2,r}, \dots, z_{2,r}^{(I_2-1)}], \quad z_{2,r} = e^{-\sqrt{-1}\omega_c d_y \sin(\phi_r)\sin(\theta_r)/c}. \end{aligned} \tag{17}$$

Necessary identifiability condition: It is clear that (16) is equivalent to (15) with $\mathbf{X} = \begin{bmatrix} \mathbf{Y}^{(1)} \\ \mathbf{Y}^{(2)} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \end{bmatrix}$. Hence, according to Theorem III.3 with $\mathbf{b}(\mathbf{z}) = [1, z_1, \dots, z_1^{(I_1-1)}, 1, z_2, \dots, z_2^{(I_2-1)}]^T$, the factorization (16) cannot be unique when $R \geq \delta = I_1 + I_2 - 1$.

Link to Vandermonde constrained coupled CPD: As in ESPRIT, one can from each ULA subarray data matrix $\mathbf{Y}^{(n)}$ build a tensor $\mathcal{X}^{(n)} = \sum_{r=1}^{R} \begin{bmatrix} 1 \\ z_{n,r} \end{bmatrix} \circ \underline{\mathbf{b}}_r^{(n)} \circ \mathbf{s}_r \in \mathbb{C}^{2 \times (I_n-1) \times K}$ with matrix representation

$$\mathbf{X}^{(n)} = \begin{bmatrix} \mathbf{Y}^{(n)} \\ \overline{\mathbf{Y}}^{(n)} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{B}}^{(n)}\mathbf{S}^T \\ \overline{\mathbf{B}}^{(n)}\mathbf{S}^T \end{bmatrix} = (\mathbf{A}^{(n)} \odot \mathbf{D}^{(n)})\mathbf{S}^T, \tag{18}$$

where $\mathbf{A}^{(n)} = \begin{bmatrix} 1 & \cdots & 1 \\ z_{n,1} & \cdots & z_{n,R} \end{bmatrix}$ and $\mathbf{D}^{(n)} = \underline{\mathbf{B}}^{(n)}$. Considering the overall array, we obtain

$$\begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} \odot \mathbf{D}^{(1)} \\ \mathbf{A}^{(2)} \odot \mathbf{D}^{(2)} \end{bmatrix} \mathbf{S}^T. \tag{19}$$

From (10) it is clear that (19) corresponds to a matrix representation of a coupled CPD of the form (9) with $N = 2$. The CPD structure of the individual data tensors $\mathcal{X}^{(1)}, \mathcal{X}^{(2)}$ is thanks to the fact that each of them corresponds to a ULA. On the other hand, the coupling is due to the fact that $\mathbf{S}$ is the same for the two subarrays. Comparing (8) with (19) it is clear that this coupled CPD approach can be interpreted as an L-shaped array variant of ESPRIT.

Relaxation of Vandermonde constraints of $\boldsymbol{D}^{(1)}$ and $\boldsymbol{D}^{(2)}$: The following Theorem IV.1 states that we can relax the Vandermonde structure of the factor matrices $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ in (19) without affecting the identifiability of the decomposition.

Theorem IV.1. *Assume that the matrices $\left[\begin{smallmatrix}\boldsymbol{X}^{(1)}\\ \boldsymbol{X}^{(2)}\end{smallmatrix}\right]$ and $\boldsymbol{S}$ in* (19) *have full column rank. Let*

$$\begin{bmatrix}\boldsymbol{X}^{(1)}\\ \boldsymbol{X}^{(2)}\end{bmatrix} = \begin{bmatrix}\widehat{\boldsymbol{A}}^{(1)} \odot \widehat{\boldsymbol{D}}^{(1)}\\ \widehat{\boldsymbol{A}}^{(2)} \odot \widehat{\boldsymbol{D}}^{(2)}\end{bmatrix}\widehat{\boldsymbol{S}}^{T} \tag{20}$$

be an alternative factorization of $\left[\begin{smallmatrix}\boldsymbol{X}^{(1)}\\ \boldsymbol{X}^{(2)}\end{smallmatrix}\right]$, where $\widehat{\boldsymbol{A}}^{(n)} = \left[\begin{smallmatrix}1 & \cdots & 1\\ \widehat{z}_{n,1} & \cdots & \widehat{z}_{n,\widehat{R}}\end{smallmatrix}\right]$ with $\widehat{z}_{n,r} \neq 0$, $\widehat{\boldsymbol{D}}^{(n)} \in \mathbb{C}^{(I_n-1)\times\widehat{R}}$ (i.e., the factor matrices $\widehat{\boldsymbol{D}}^{(1)}$ and $\widehat{\boldsymbol{D}}^{(2)}$ are allowed to be unstructured), $\widehat{\boldsymbol{S}} \in \mathbb{C}^{K\times\widehat{R}}$ and $\widehat{R} \leq R$. Then $\widehat{R} = R$ and the columns of $\widehat{\boldsymbol{D}}^{(n)}$ are proportional to the columns of the Vandermonde matrix with generators $\widehat{z}_{n,1}, \ldots, \widehat{z}_{n,R}$.

Proof: see Appendix B. ∎

Theorem IV.1 further implies that we can compute the Vandermonde constrained bilinear factorization of $\{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}\}$ via a coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$ in which the structures of $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are ignored.

Deterministic sufficient identifiability condition: Even though the structure of $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ in (19) can be relaxed without affecting the identifiability of the decomposition, the structure of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$, i.e., the fact that the rows do not hold any zero entries, cannot be dropped. Consequently, (19) still corresponds to a (very mildly) constrained coupled CPD. However, if we also ignore the structure of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$, then (19) reduces to an unconstrained coupled CPD of the form (9). This leads to the following sufficient identifiability condition.

Theorem IV.2. *Consider the coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$ with matrix representation* (19)*. If condition* (12) *holds, then the bilinear factorization of $\{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}\}$ in* (16) *is unique. In particular, the uniqueness holds if $\boldsymbol{G}^{(2)}$ has full column rank.*

For a specific matrix $\mathbf{S}$ and values ϕ_r, θ_r the uniqueness of (19) can be checked with Theorem IV.2, despite the fact that the structures of $\{\mathbf{A}^{(n)}\}$ and $\{\mathbf{D}^{(n)}\}$ are ignored. As in the ordinary coupled CPD case, condition (12) can be hard to check in practice. On the other hand, if $\mathbf{G}^{(2)}$ in (11) has full column rank (i.e., if conditions (13a)–(13b) in Theorem III.2 are satisfied), then we know that $\omega(\mathbf{c}) = 0$ and condition (12) will automatically be satisfied. In the next paragraph we provide a condition that generically ensures that the constrained coupled CPD (19) with structured factors $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ (while factors $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are allowed to be unstructured) is unique.

Generic sufficient identifiability condition: The following bound on R holds if $\{\phi_r\}$ and $\{\theta_r\}$ can be assumed generic.

Theorem IV.3. *Consider the coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$ with matrix representation* (19)*. Assume that the matrix $\mathbf{S}$ has full column rank. If*

$$R \leq I_1 + I_2 - 4, \tag{21}$$

then the coupled CPD (19) *is generically unique, i.e., the Lebesgue measure of the set $\left\{(\phi_r, \theta_r) \ : \text{coupled CPD (19) is not unique}\right\}$ is zero.*

Proof: See Appendix C. ∎

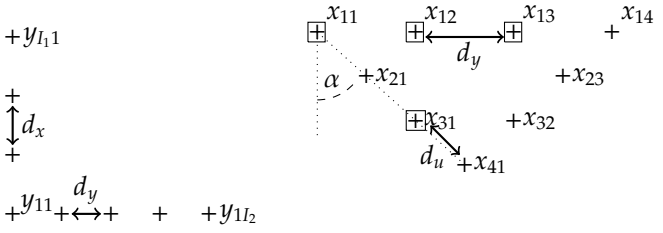

Fig. 1. L-shaped array (left) and triangular-shaped array (right) where '+' represents an antenna element.

Theorem IV.3 states that the Vandermonde constrained bilinear factorization of $\{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}\}$ in (16) is generically unique if $R \leq I_1 + I_2 - 4$. Note that we are quite close to what is necessary for uniqueness, which is $R \leq I_1 + I_2 - 2$. Note also that we have derived conditions that are significantly more relaxed than what can be obtained via ESPRIT/CPD, which is $R \leq \max(I_1, I_2) - 1$. As an example, if $I := I_1 = I_2$, then the gain in terms of identifiability is at least $\frac{2I-4}{I-1}$, i.e., for large dimensions I the bound on R is basically relaxed by a factor two. This is also intuitively clear since the coupled CPD approach works jointly with both ULAs in the L-shaped array while the ESPRIT/CPD approach only exploits the structure of one of the involved ULAs at a time.

2) Frame-shaped array: Consider the frame-shaped array depicted in Figure 2 (left). Compared to the L-shaped array, there is an additional shift-invariance structure between the two vertical subarrays and between the two horizontal subarrays. We explain that this additional shift-invariance structure can be integrated in the coupled CPD formulation. We build the observation matrices $\mathbf{Y}^{(1)} \in \mathbb{C}^{2I_1\times K}$ with columns $\mathbf{y}_k^{(1)} = [y_{11k}, \ldots, y_{I_1 1k}, y_{1I_2k}, \ldots, y_{I_1I_2k}]^T$ and $\mathbf{Y}^{(2)} \in \mathbb{C}^{2I_2\times K}$ with columns $\mathbf{y}_k^{(2)} = [y_{11k}, \ldots, y_{1I_2k}, y_{I_11k}, \ldots, y_{I_1I_2k}]^T$, where y_{ijk} denotes the output of sensor y_{ij} at the kth snapshot and with indexing as indicated in Figure 2 (left). The matrices $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ admit the factorizations

$$\mathbf{Y}^{(1)} = \left(\mathbf{B}^{(1)} \odot \mathbf{C}^{(1)}\right)\mathbf{S}^T, \quad \mathbf{Y}^{(2)} = \left(\mathbf{B}^{(2)} \odot \mathbf{C}^{(2)}\right)\mathbf{S}^T, \tag{22}$$

$$\mathbf{b}_r^{(1)} = [1 \ z_{1,r} \ \ldots \ z_{1,r}^{I_1-1}]^T, \quad \mathbf{c}_r^{(1)} = [1 \ z_{2,r}^{I_2-1}]^T,$$
$$\mathbf{b}_r^{(2)} = [1 \ z_{2,r} \ \ldots \ z_{2,r}^{I_2-1}]^T, \quad \mathbf{c}_r^{(2)} = [1 \ z_{1,r}^{I_1-1}]^T,$$

in which $z_{1,r}$ and $z_{2,r}$ are defined as in (17).

Necessary identifiability condition: It is clear that (22) is equivalent to (15) with $\mathbf{X} = \left[\begin{smallmatrix}\mathbf{Y}^{(1)}\\ \mathbf{Y}^{(2)}\end{smallmatrix}\right]$ and $\mathbf{B} = \left[\begin{smallmatrix}\mathbf{B}^{(1)}\odot\mathbf{C}^{(1)}\\ \mathbf{B}^{(2)}\odot\mathbf{C}^{(2)}\end{smallmatrix}\right]$. Theorem III.3 states that the factorization (22) cannot be unique when $R \geq \delta = 2I_1 + 2I_2 - 4$, where it has been

taken into account that four of the row vectors in $\mathbf{B}$ are redundant.

Link to Vandermonde constrained coupled CPD: Clearly (22) corresponds to a matrix representation of the decomposition of the two tensors $\mathcal{Y}^{(1)} \in \mathbb{C}^{2\times I_1\times K}$ and $\mathcal{Y}^{(2)} \in \mathbb{C}^{2\times I_2\times K}$ that take the shift-invariance between the two vertical subarrays and between the two horizontal subarrays into account, respectively. The next step is to build tensors that exploit the shift-invariance of the vertical and horizontal ULA. Similar to the L-shaped case, this structure can be taken into account by building a set of tensors $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$ whose matrix representation

$$\mathbf{X}^{(1)} = \begin{bmatrix} (\underline{\mathbf{I}}_{I_1} \otimes \mathbf{I}_2)\mathbf{Y}^{(1)} \\ (\bar{\mathbf{I}}_{I_1} \otimes \mathbf{I}_2)\mathbf{Y}^{(1)} \end{bmatrix}, \quad \mathbf{X}^{(2)} = \begin{bmatrix} (\underline{\mathbf{I}}_{I_2} \otimes \mathbf{I}_2)\mathbf{Y}^{(2)} \\ (\bar{\mathbf{I}}_{I_2} \otimes \mathbf{I}_2)\mathbf{Y}^{(2)} \end{bmatrix}$$

admits the coupled CPD interpretation:

$$\begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} \odot \mathbf{D}^{(1)} \\ \mathbf{A}^{(2)} \odot \mathbf{D}^{(2)} \end{bmatrix} \mathbf{S}^T, \tag{23}$$

where $\mathbf{A}^{(n)} = \begin{bmatrix} 1 & \cdots & 1 \\ z_{n,1} & \cdots & z_{n,R} \end{bmatrix}$ and $\mathbf{D}^{(n)} = \underline{\mathbf{B}}^{(n)} \odot \mathbf{C}^{(n)}$. Note that the latter is just an augmented version of the factor matrix ($\mathbf{D}^{(n)} = \underline{\mathbf{B}}^{(n)}$) in (18) associated with the L-shaped array. As in the L-shaped case, we expect that the structures of $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ can be relaxed without affecting the identifiability of the decomposition. However, we will defer from such a technical demonstration.

Deterministic sufficient identifiability condition: As in the L-shaped array case, if we also ignore the structures of $\{\mathbf{A}^{(n)}\}$ and $\{\mathbf{D}^{(n)}\}$, then (23) reduces to an unconstrained coupled CPD of the form (9). This leads to the following sufficient identifiability condition.

Theorem IV.4. *Consider the coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$ with matrix representation* (23)*. If condition* (12) *holds, then the factorization of $\{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}\}$ in* (22) *is unique. In particular, the uniqueness holds if $\boldsymbol{G}^{(2)}$ has full column rank.*

Generic sufficient identifiability condition: As in the L-shaped case, we know that condition (12) can be hard to verify. To alleviate this problem Theorem IV.5 below presents an easy-to-check condition that guarantees the generic uniqueness of the decomposition (23).

Theorem IV.5. *Consider the Vandermonde constrained coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$ with matrix representation* (23)*. Assume that the matrix $\mathbf{S}$ has full column rank. If*

$$R \leq 2I_1 + 2I_2 - 7, \tag{24}$$

then the coupled CPD (23) *is generically unique, i.e., the Lebesgue measure of the set $\left\{(\phi_r, \theta_r) \ : \text{coupled CPD (23) is not unique}\right\}$ is zero.*

Proof: See Appendix C. ∎

Theorem IV.5 states that the factorization of $\{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}\}$ in (22) is generically unique if $R \leq 2I_1 + 2I_2 - 7$, which is close to the necessary condition $R \leq 2I_1 + 2I_2 - 3$. It is clear that the coupling of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}\}$, as expressed by (22)–(23), leads to improved uniqueness conditions. By way of example, consider the case where $I_1 = I_2 = 5$ and $K \geq R$. If we only exploit the CPD structure of $\mathcal{X}^{(1)}$, which exploits the shift-invariance structures of the vertical subarrays, then from Theorem III.2 with $N = 1$ (i.e., ordinary CPD) we obtain the bound $R \leq 8$. Thanks to Theorem III.2 with $N = 2$ the coupled CPD structure of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}\}$ relaxes the bound to $R \leq 10$. Theorem IV.5 further relaxes the bound to $R \leq 14$.

3) Triangular-shaped array: Associating a coupled CPD with a given array configuration can be less straightforward than in the case of the L-shaped or frame-shaped array.

a) Special case: Let us first discuss the nine-element triangular array depicted in Figure 1 (right). Let x_{mnk} denote the output of sensor x_{mn} at snapshot k. Then the kth column of $\mathbf{X}$ and the rth column of $\mathbf{A}$ in (3) are given by

$$\mathbf{x}_k = [x_{11k}, x_{12k}, x_{13k}, x_{14k}, x_{21k}, x_{23k}, x_{31k}, x_{32k}, x_{41k}]^T, \tag{25}$$

$$\mathbf{a}_r = \left[1, y_r, y_r^2, y_r^3, z_r, z_r y_r^2, z_r^2, z_r^2 y_r, z_r^3\right]^T, \tag{26}$$

where $y_r = e^{-\sqrt{-1}\frac{\omega_c}{c} d_y \sin(\theta_r)\sin(\phi_r)}$ and $z_r = e^{-\sqrt{-1}\frac{\omega_c}{c} d_u \cos(\theta_r - \alpha)\sin(\phi_r)}$ in which α denotes the angle between the vertical axis and the diagonal axis of the triangular array, see Figure 1 (right). At first sight one may view the triangular array as three superimposed ULAs each composed of three sensors. However, we can also see the triangular array as a collection of three subarrays, each composed of *four* sensors that also enjoy a shift-invariance property (one of the four-element subarrays is highlighted in the figure by squared boxes). More precisely, the triangular array has the following three shift-invariance structures:

$$\mathbf{d}_r^{(1)} := \mathbf{a}_r([1,2,3,7]) = \mathbf{a}_r([2,3,4,8])\, y_r^{-1}, \tag{27}$$

$$\mathbf{d}_r^{(2)} := \mathbf{a}_r([1,5,7,3]) = \mathbf{a}_r([5,7,9,7])\, z_r^{-1}, \tag{28}$$

$$\mathbf{d}_r^{(3)} := \mathbf{a}_r([4,6,8,2]) = \mathbf{a}_r([6,8,9,5])\, (z_r/y_r)^{-1}. \tag{29}$$

This motivates the construction of the tensors $\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)} \in \mathbb{C}^{2\times 4\times K}$ with matrix representation

$$\begin{aligned} \mathbf{X}^{(1)} &= [\mathbf{X}([1,2,3,7],:)^T, \mathbf{X}([2,3,4,8],:)^T]^T, \\ \mathbf{X}^{(2)} &= [\mathbf{X}([1,5,7,3],:)^T, \mathbf{X}([5,7,9,7],:)^T]^T, \\ \mathbf{X}^{(3)} &= [\mathbf{X}([4,6,8,2],:)^T, \mathbf{X}([6,8,9,5],:)^T]^T. \end{aligned}$$

The matrix representation of the tensors $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)}\}$ admits the coupled decomposition

$$\begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \\ \mathbf{X}^{(3)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} \odot \mathbf{D}^{(1)} \\ \mathbf{A}^{(2)} \odot \mathbf{D}^{(2)} \\ \mathbf{A}^{(3)} \odot \mathbf{D}^{(3)} \end{bmatrix} \mathbf{S}^T, \tag{30}$$

where the columns of $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$, $\mathbf{D}^{(3)}$ are of the form (27)–(29) and the columns of $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ are given by $\mathbf{a}_r^{(1)} = \begin{bmatrix} 1 \\ y_r \end{bmatrix}$, $\mathbf{a}_r^{(2)} = \begin{bmatrix} 1 \\ z_r \end{bmatrix}$ and $\mathbf{a}_r^{(3)} = \begin{bmatrix} 1 \\ z_r/y_r \end{bmatrix}$, respectively. Note that if we only exploit the CPD structure of, say $\mathcal{X}^{(1)}$, then from Theorem III.2 with $N = 1$ we obtain the upper bound $R \leq 4$. By exploiting the coupled CPD structure of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)}\}$, Theorem III.2 with $N = 3$ relaxes the bound to $R \leq 6$.

b) General case: From Figure 1 (right) it is clear that the triangular-shaped array is composed of three ULAs so that

$$\mathbf{Z}^{(n)} = \mathbf{A}^{(n)}\mathbf{S}^T, \quad n \in \{1,2,3\}, \tag{31}$$

in which

$$\begin{aligned}
\mathbf{z}_k^{(1)} &= [x_{11k}, x_{12k}, x_{13k}, \ldots, x_{1Mk}]^T, \\
\mathbf{z}_k^{(2)} &= [x_{11k}, x_{21k}, x_{31k}, \ldots, x_{M1k}]^T, \\
\mathbf{z}_k^{(3)} &= [x_{1Mk}, x_{2,M-1,k}, x_{3,M-2,k}, \ldots, x_{M1k}]^T, \\
\mathbf{a}_r^{(1)} &= \left[1, y_r, y_r^2, \ldots, y_r^{M-1}\right]^T, \\
\mathbf{a}_r^{(2)} &= \left[1, z_r, z_r^2, \ldots, z_r^{M-1}\right]^T, \\
\mathbf{a}_r^{(3)} &= \left[y_r^{M-1}, y_r^{M-2}z_r, y_r^{M-3}z_r^2, \ldots, z_r^{M-1}\right]^T,
\end{aligned}$$

where y_r and z_r are defined as in (26) and the generators of the three ULAs are y_r, z_r and z_r/y_r.

Necessary identifiability condition: Clearly $\mathbf{X} = [\mathbf{Z}^{(1)T}, \mathbf{Z}^{(2)T}, \mathbf{Z}^{(3)T}]^T$ corresponds to (15) with $\mathbf{B} = [\mathbf{A}^{(1)T}, \mathbf{A}^{(2)T}, \mathbf{A}^{(3)T}]^T$. Theorem III.3 now tells us that the factorization (31) cannot be unique when $R \geq \delta = 3M-3$, where it has been taken into account that three of the row vectors in (31) are redundant.

Deterministic sufficient identifiability condition: Similar to the special triangular-shaped case (30), we can translate the factorization (31) into a Vandermonde constrained coupled CPD with matrix representation

$$\mathbf{X}^{(1)} = \begin{bmatrix} \underline{\mathbf{Z}}^{(1)} \\ \overline{\mathbf{Z}}^{(2)}(M-1,:) \\ \overline{\mathbf{Z}}^{(1)} \\ \mathbf{Z}^{(3)}(M-1,:) \end{bmatrix} = \left(\mathbf{A}^{(1)} \odot \mathbf{D}^{(1)}\right)\mathbf{S}^T \in \mathbb{C}^{2M\times K}, \tag{32}$$

$$\mathbf{X}^{(2)} = \begin{bmatrix} \underline{\mathbf{Z}}^{(2)} \\ \overline{\mathbf{Z}}^{(1)}(M-1,:) \\ \overline{\mathbf{Z}}^{(2)} \\ \mathbf{Z}^{(3)}(2,:) \end{bmatrix} = \left(\mathbf{A}^{(2)} \odot \mathbf{D}^{(2)}\right)\mathbf{S}^T \in \mathbb{C}^{2M\times K}, \tag{33}$$

$$\mathbf{X}^{(3)} = \begin{bmatrix} \underline{\mathbf{Z}}^{(3)} \\ \overline{\mathbf{Z}}^{(1)}(2,:) \\ \overline{\mathbf{Z}}^{(3)} \\ \mathbf{Z}^{(2)}(2,:) \end{bmatrix} = \left(\mathbf{A}^{(3)} \odot \mathbf{D}^{(3)}\right)\mathbf{S}^T \in \mathbb{C}^{2M\times K}, \tag{34}$$

where the columns of $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ are given by $\mathbf{a}_r^{(1)} = \begin{bmatrix} 1 \\ y_r \end{bmatrix}$, $\mathbf{a}_r^{(2)} = \begin{bmatrix} 1 \\ z_r \end{bmatrix}$ and $\mathbf{a}_r^{(3)} = \begin{bmatrix} 1 \\ z_r/y_r \end{bmatrix}$, respectively, and the columns of $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$, $\mathbf{D}^{(3)}$ are the extended versions of (27)–(29). Ignoring the structure of $\{\mathbf{A}^{(n)}\}$ and $\{\mathbf{D}^{(n)}\}$, leads to the following sufficient identifiability condition.

Theorem IV.6. *Consider the coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)}\}$ with matrix representation* (32)–(34)*. If condition* (12) *holds, then the factorization of $\{\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \mathbf{Z}^{(3)}\}$ in* (31) *is unique. In particular, the uniqueness holds if $\boldsymbol{G}^{(3)}$ has full column rank.*

Generic sufficient identifiability condition: As in the L-shaped and frame-shaped cases, we know that condition (12) can be hard to verify. Theorem IV.7 below presents an easy-to-check condition that guarantees the generic uniqueness of the decomposition (32)–(34).

Theorem IV.7. *Consider the Vandermonde constrained coupled CPD of $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)}\}$ with matrix representation* (32)–(34)*. Assume that the matrix $\mathbf{S}$ has full column rank. If*

$$R \leq 3M - 6, \tag{35}$$

then the coupled CPD (32)–(34) *is generically unique, i.e., the Lebesgue measure of the set $\left\{(\phi_r, \theta_r) \ : \text{coupled CPD (32)–(34) is not unique}\right\}$ is zero.*

The proof of Theorem IV.7 to is similar to the proofs of Theorems IV.3 and IV.5 and for that reason it is omitted. Note that for the special triangular case $M = 4$ discussed earlier, the bound obtained by Theorem III.2 with $N = 3$ coincides with the bound (35) in Theorem IV.7.

Other antenna arrays that consist of superimposed linear subarrays, possibly also with identical translated subarrays, include cross-shaped [8] and hexagonal arrays [32]. From the derivation in this section it follows that the coupled CPD based results in [26], [27], [28] can directly be used for such array configurations.

B. Centro-symmetric arrays

In this section we consider centro-symmetric arrays, i.e., arrays that are symmetric around their "epicenter". In particular, we will argue that the centro-symmetry of an array by itself can be sufficient to ensure identifiability. We will explain that the coupled CPD model, albeit not optimal, can actually be used for *any* centro-symmetric array. The key observation is that for a centro-symmetric array, we know that for any two sensors there exist two "mirrored" sensors so that the four sensors together form a 2-by-2 rectangular array. (For example, in Figure 2(right), sensor x_1 and x_2 together with the two corresponding "mirrored" sensors x_5 and x_6 form a 2-by-2 rectangular array. More details will be provided at the end of this subsection.) Formally speaking, let $\boldsymbol{\Xi}^{(n)} \in \mathbb{C}^{2\times R}$ denote a row selection matrix that selects two rows of $\mathbf{A}$ in (3), denoted by $\mathbf{D}^{(n)} = \boldsymbol{\Xi}^{(n)}\mathbf{A} \in \mathbb{C}^{2\times R}$. Similarly, let $\boldsymbol{\Xi}_{(n)} \in \mathbb{C}^{2\times R}$ denote a row selection matrix that selects the two corresponding "mirrored" rows of $\mathbf{A}$ so that $\mathbf{D}^{(n)}\boldsymbol{\Lambda}^{(n)} = \boldsymbol{\Xi}_{(n)}\mathbf{A}$, where $\boldsymbol{\Lambda}^{(n)} \in \mathbb{C}^{R\times R}$ is a diagonal matrix. We emphasize that due to the centro-symmetry, there exists a row-selection pair $(\boldsymbol{\Xi}^{(n)}, \boldsymbol{\Xi}_{(n)})$ for all choices of two sensors, except for the trivial case in which $\boldsymbol{\Xi}^{(n)}$ selects two sensors that already mirror each other (e.g., sensors x_1 and x_5 in Figure 2(right)). In other words, no matter which two rows the selection matrix $\boldsymbol{\Xi}^{(n)}$ has chosen, there always exists a selection matrix $\boldsymbol{\Xi}_{(n)}$ that picks two rows from $\mathbf{A}$ so that we can build a tensor $\mathcal{X}^{(n)} \in \mathbb{C}^{2\times 2\times R}$ whose matrix representation admits the decomposition

$$\mathbf{X}^{(n)} = \begin{bmatrix} \boldsymbol{\Xi}^{(n)}\mathbf{X} \\ \boldsymbol{\Xi}_{(n)}\mathbf{X} \end{bmatrix} = (\mathbf{A}^{(n)} \odot \mathbf{D}^{(n)})\mathbf{S}^T \in \mathbb{C}^{4\times K}, \tag{36}$$

where $\mathbf{A}^{(n)} = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_{11}^{(n)} & \cdots & \lambda_{RR}^{(n)} \end{bmatrix}$ in which $\lambda_{rr}^{(n)} \in \mathbb{C}$ corresponds to the rth diagonal element of $\boldsymbol{\Lambda}^{(n)}$. Assume that for N choices of sensor pairs, we construct tensors $\mathcal{X}^{(n)} \in$

$\mathbb{C}^{2\times 2\times R}$, $n \in \{1,\ldots,N\}$, each with a matrix factorization of the form (36). Theorem III.2 now tells us that if

$$\begin{cases} \mathbf{G} \text{ in (11) has full column rank,} & \text{(37a)} \\ \mathbf{S} \text{ in (3) has full column rank,} & \text{(37b)} \end{cases}$$

where $\mathbf{G}$ is built from $\{\mathbf{A}^{(n)}, \mathbf{D}^{(n)}\}_{n=1}^{N}$ in (36), then the coupled CPD of $\{\mathcal{X}^{(n)}\}$ is unique and can actually be computed via a matrix EVD. Consequently, if conditions (37a)–(37b) are satisfied, then $\mathbf{S}$ is essentially unique[1]. This in turn implies that $\mathbf{A} = \mathbf{X}(\mathbf{S}^T)^\dagger$ in the original matrix factorization (3) is also essentially unique.

By way of example, let us consider the specific Uniform Circular Array (UCA) depicted in Figure 2(right). For a UCA composed of I sensors the entries of $\mathbf{A}$ in (3) are given by $a_{i,r} = e^{\sqrt{-1}\cdot 2\pi\cdot d\cdot\cos(\theta_r-\gamma_i)\cdot\sin(\phi_r)/\lambda}$, in which d is the radius of the UCA, λ is the signal wavelength, $\gamma_i = 2\pi(i-1)/I$, θ_r is the azimuth angle of the rth source and ϕ_r is the elevation angle of the rth source. Note that any UCA with an even number of sensors is centro-symmetric. For the particular example in Figure 2(right) with $I = 8$, the two sensors indexed by the pair (x_1, x_2) together with the two "mirrored" sensors indexed by the pair (x_5, x_6) constitute a 2-by-2 rectangular subarray. Denote this correspondence by $(x_1, x_2) \leftrightarrow (x_5, x_6)$. We then have the following six couples $(x_1, x_2) \leftrightarrow (x_5, x_6)$, $(x_1, x_3) \leftrightarrow (x_5, x_7)$, $(x_1, x_4) \leftrightarrow (x_5, x_8)$ $(x_2, x_3) \leftrightarrow (x_6, x_7)$, $(x_2, x_4) \leftrightarrow (x_6, x_8)$, $(x_2, x_5) \leftrightarrow (x_7, x_8)$, each corresponding to a 2-by-2 rectangular subarray. More generally, we can extract $C_{I/2}^2$ 2-by-2 rectangular arrays from $\mathbf{X}$, each with a matrix representation:

$$\mathbf{X}^{(m,n)} = \begin{bmatrix} \mathbf{\Xi}^{(m,n)}\mathbf{X} \\ \mathbf{\Xi}^{(n+I/2,m+I/2)}\mathbf{X} \end{bmatrix} = (\mathbf{A}^{(m,n)} \odot \mathbf{D}^{(m,n)})\mathbf{S}^T \in \mathbb{C}^{4\times K}, \tag{38}$$

where $1 \le m \le I/2$ and $m < n \le I/2$, $\mathbf{\Xi}^{(p,q)} \in \mathbb{C}^{2\times I}$ is a row-selection matrix that selects rows p and q, $\mathbf{A}^{(m,n)} = \begin{bmatrix} 1 & \cdots & 1 \\ d_1^{(m,n)} & \cdots & d_R^{(m,n)} \end{bmatrix}$ in which $d_r^{(m,n)} \in \mathbb{C}$ has the property $\mathbf{a}_r^{(n+I/2,m+I/2)} = \mathbf{a}_r^{(m,n)} d_r^{(m,n)}$, and $\mathbf{D}^{(m,n)} = \mathbf{A}([m,n],:) \in \mathbb{C}^{2\times R}$. Clearly the set of matrices $\{\mathbf{X}^{(m,n)}\}$ with $1 \le m \le I/2$ and $m < n \le I/2$ corresponds to a matrix representation of a coupled CPD. Note that this coupled CPD approach does not fully exploit the *rotational* invariance of the UCA and, hence, does not provide the most relaxed bound on the number of signals that can be separated in a unique way. On the other hand, it provides a uniqueness condition that is easy-to-check and valid for *any* centro-symmetric array. Note that each row of $\mathbf{G}$ in condition (13b) is associated with a 2-by-2 rectangular subarray. Since we consider $C_4^2 = 6$ subarrays, $\mathbf{G}$ will be an $(6\times C_R^2)$ matrix. It can be verified that this matrix generically has full column rank if $R \le 3$. Consequently, Theorem III.2 generically guarantees uniqueness for this UCA if $R \le 3$.

Note that due to the centro-symmetry property, forward-backward averaging [18] can be used in a pre-processing step to relax condition (37b). To make things more tangible, let us focus again on the above UCA. Observe that any UCA with an even number of I sensors has the property $a_{(n+I/2),r} = a_{n,r}^*$. This property implies that

$$\mathbf{Y} = [\mathbf{X}, (\mathbf{J}_2 \otimes \mathbf{I}_{I/2})\mathbf{X}^*] = \mathbf{A}\widetilde{\mathbf{S}}^T, \quad \widetilde{\mathbf{S}} = \begin{bmatrix} \mathbf{S} \\ \mathbf{S}^* \end{bmatrix},$$

where $\mathbf{J}_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. We can now proceed with $\mathbf{S}$ in (3) and in (37b) replaced by the augmented matrix $\widetilde{\mathbf{S}} \in \mathbb{C}^{2K\times R}$, which generically has full column rank if $2K \ge R$ [25].

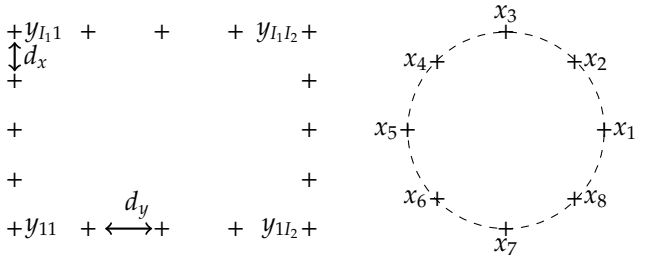

Fig. 2. Centro-symmetric frame-shaped array (left) and circular array (right) where '+' represents an antenna element.

C. Local low-rank modeling

In this section we investigate the use of coupled tensor decompositions for local low-rank modeling in the context of narrowband near-field DOA estimation. The goal of this section is to demonstrate that coupled decompositions can potentially be used to approximate complex models using more simple locally defined low-rank models.

1) Data model: We consider DOA estimation in the case of narrowband signals where the kth snapshot of the ith sensor can be expressed as

$$x_{ik} = \sum_{r=1}^{R} e^{-\sqrt{-1}\cdot\tau_{ir}} s_{kr}\,. \tag{39}$$

Assume that the array is equipped with I sensors and that K snapshots are available so that the data can be stored in an $(I \times K)$ observation matrix $\mathbf{X}$. Assume also that a ULA is used and that the sources can be located in the near-field so that [31]:

$$\tau_{ir} = 2\pi\frac{\Delta_r}{\lambda}\left(1 - \sqrt{1 + \frac{i^2\cdot d^2}{\Delta_r^2} - \frac{2\cdot i\cdot d\cdot\sin(\theta_r)}{\Delta_r}}\right), \tag{40}$$

where λ denotes the signal wavelength, Δ_r and θ_r denote the range and angle parameters associated with the rth source, and d denotes the inter-element spacing between the sensors in the ULA. We consider the standard ULA case where $d = \frac{\lambda}{2}$.

2) Local linear approximation: Recall that if the sources are located in the far-field ($\Delta_r >> (I-1)d$), then a linear approximation of (40) is commonly used, i.e., $\tau_{ir} \approx \frac{2\pi}{\lambda} i\cdot d\cdot\sin(\theta_r) = \pi\cdot i\cdot\sin(\theta_r)$ when $d = \frac{\lambda}{2}$. The core idea behind our coupled CPD approach for near-field array processing is to split the ULA into N possibly overlapping, say 3-element sub-ULAs. Since

[1]A matrix, say $\mathbf{S}$, is said to be essentially unique if it is unique up to column scaling and permutation ambiguities.

the array aperture of this sub-ULA is only $2d = \lambda$ meters when $d = \frac{\lambda}{2}$, it is reasonable to assume that if the sources are not located too close to the ULA (say $\Delta_r > 5\lambda$), then a linear approximation of this 3-element sub-ULA is acceptable. Overall, by ignoring the structural relations between the different sub-ULAs, (39) can be approximated by a coupled and Vandermonde constrained factorization:

$$\mathbf{X}^{(n)} \approx \beta_1^{(n)}\mathbf{a}_1^{(n)}\mathbf{s}_1^T + \cdots + \beta_R^{(n)}\mathbf{a}_R^{(n)}\mathbf{s}_R^T, \quad n \in \{1, \ldots, N\}, \quad (41)$$

where the scalars $\beta_1^{(n)}, \ldots, \beta_R^{(n)} \in \mathbb{C}$ take the individual time-delays between the sub-ULAs and the sources into account, and

$$\mathbf{a}_r^{(n)} = \left[x_{r,n}^{-1},\ 1,\ x_{r,n}\right]^T, \quad (42)$$

in which $x_{r,n} = e^{\sqrt{-1}\alpha_{r,n}}$ with $\alpha_{r,n} \in \mathbb{R}$. Let us drop the sub/super-scripts of $\mathbf{a}_r^{(n)}$, i.e., we now simply write $\mathbf{a} = [a_1\ a_2\ a_3]^T$. Using Hankelization, the Vandermonde vector (42) can be associated with the (2×2) rank-1 Hankel matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = \begin{pmatrix} x^{-1} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & x \end{pmatrix} = \mathbf{b} \cdot \mathbf{c}^T, \quad (43)$$

where $\mathbf{c} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{b}^*$. This in turn means that the coupled and Vandermonde constrained factorization (41) can be translated into a coupled CPD:

$$\mathcal{Y}^{(n)} \approx \beta_1^{(n)}\mathbf{b}_1^{(n)} \circ \mathbf{c}_1^{(n)} \circ \mathbf{s}_1 + \cdots + \beta_R^{(n)}\mathbf{b}_R^{(n)} \circ \mathbf{c}_R^{(n)} \circ \mathbf{s}_R, \quad n \in \{1, \ldots, N\}. \quad (44)$$

Note that if the sources are not too close to the ULA, larger sub-ULAs may be considered (e.g. 4- or 5-element sub-ULAs can be used, implying that $\mathbf{b}_r^{(n)}$ in (44) is now a 3- or 4-element Vandermonde vector.)

In practice, we first compress the $(I \times K)$ observation matrix $\mathbf{X}$ into an $(I \times R)$ matrix. Note that this is consistent with the ESPRIT method and can be done via an SVD. An estimate of the computational complexity of the coupled CPD of compressed observation matrices based on EVD was provided in [29]. In short, assume that I_n-element sub-ULAs are used, then the EVD approach for coupled CPD has a complexity of the order $4 \cdot N \cdot (I_n - 1)^2 R^2$ or $4 \cdot N \cdot (I_n - 1)R^4$ depending on the implementation. (If $I_n > R$, then the sub-ULA matrices can also be compressed so that I_n can be replaced by R.) For example, if $I_n = 3$, then the complexity of the algorithm is only of the order $8 \cdot N \cdot R^2$, which is cheap for practical problems with small R.

V. Numerical experiments

Let us consider the factorization $\mathbf{X} = \mathbf{A}\mathbf{S}^T$ in (3). The goal is to estimate the angles $\{\theta_r, \phi_r\}$ from $\mathbf{T} = \mathbf{X} + \beta\mathbf{N}$, where $\mathbf{N}$ is an unstructured perturbation matrix and $\beta \in \mathbb{R}$ controls the signal-to-noise level. We will consider Monte Carlo simulations, where for each trial the real and imaginary entries of $\mathbf{S}$ and $\mathbf{N}$ are randomly drawn from a Gaussian distribution with zero mean and unit variance. The following Signal-to-Noise Ratio (SNR) measure will be used: SNR [dB] $= 10\log(\|\mathbf{X}\|_F^2/\beta^2\|\mathbf{N}\|_F^2)$.

A. From separate to joint processing

Case 1: Consider the L-shaped array in subsection IV-A with $I = J = 5$, $K = 50$ and $R = 4$. We compare the algebraic coupled CPD method with a basic ESPRIT/CPD method [19] that first computes the azimuth angles $\{\theta_r\}$ and $\mathbf{S}$ via the horizontal ULA ($\mathbf{X}^{(1)} = \mathbf{B}^{(1)}\mathbf{S}^T$) and then finds the elevation angles $\{\phi_r\}$ via the vertical ULA ($\mathbf{X}^{(2)} = \mathbf{B}^{(2)}\mathbf{S}^T$) with $\mathbf{S}$ fixed. We set $\boldsymbol{\theta} = [\theta_1,\ \theta_2,\ \theta_3,\ \theta_4]^T = \frac{\pi}{180}[10,\ 30,\ 50,\ 70]^T$ and $\boldsymbol{\phi} = [\phi_1,\ \phi_2,\ \phi_3,\ \phi_4]^T = \frac{\pi}{180}[-70,\ -50,\ 50,\ 70]^T$. The $\mathrm{E_{RMS}} = \sqrt{\frac{1}{2RT_{\mathrm{MC}}}\sum_{t=1}^{T_{\mathrm{MC}}}\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_t\|_F^2 + \|\boldsymbol{\phi} - \hat{\boldsymbol{\phi}}_t\|_F^2}$ and Cramér-Rao Bound (CRB) values over $T_{\mathrm{MC}} = 100$ trials as a function of SNR can be seen in figure 3. We observe that the coupled CPD method performs better than the ESPRIT/CPD method. The reason is that the former method takes the whole L-shaped antenna array into account when estimating the DOAs while the latter method only exploits the horizontal (vertical) subarray when estimating the azimuth (elevation) angles.

Case 2: Consider the same experiment, but now we set $R = 3$ and, to illustrate that the DOAs are not required to be distinct, we also set $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T = \frac{\pi}{180}[10,\ 10,\ 70]^T$ and $\boldsymbol{\phi} = [\phi_1,\ \phi_2,\ \phi_3]^T = \frac{\pi}{180}[-70,\ 50,\ 50]^T$. The mean $\mathrm{E_{RMS}}$ and CRB values over 100 trials as a function of SNR can be seen in figure 4. For the same reason as before, coupled CPD performs better than the ESPRIT/CPD approach. The repeated azimuth and elevation angles make the estimation problem more difficult, which explains the degradation in performance despite R has been reduced to three.

B. Centro-symmetric array via coupled CPD

Case 3: Consider the centro-symmetric UCA in Subsection IV-B with $I = 8$ and $K = 50$. In order to illustrate that the coupled CPD can handle more sources than basic ESPRIT/CPD we set $R = 3$. (The basic ESPRIT/CPD method, which only exploits a single shift-invariance structure of this centro-symmetric array, can identify at most $R = 2$ sources.) By capitalizing on the centro-symmetric property of the array, the coupled CPD approach reduces the multi-source DOA recovery problem into a set of decoupled single-source DOA recovery problems. The latter problem is solved by means of a conventional beamformer with angular resolutions set to $\frac{\pi}{360}$ radians (i.e., 1 degrees). We set $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T = \frac{\pi}{180}[10,\ 40,\ 45]^T$ and $\boldsymbol{\phi} = [\phi_1,\ \phi_2,\ \phi_3]^T = \frac{\pi}{180}[-70,\ -60,\ 70]^T$. The mean $\mathrm{E_{RMS}}$ and CRB values over 100 trials as a function of SNR can be seen in figure 5. From the figure one can observe that it is indeed possible to recover the DOAs, despite only the centro-symmetric property of the array was exploited.

C. From far-field to near-field

Case 4: Consider now the near-field scenario discussed in Subsection IV-C in which $R = 2$ sources impinge on a ULA composed of I sensors. The SNR level

is fixed to 20 dB. The DOAs are fixed to $\boldsymbol{\theta} = [\theta_1,\ \theta_2]^T = [0.5545,\ 0.1917]^T$ while the distances $\{\Delta_1, \Delta_2\}$ in (40) are varying. We consider cases in which $\Delta := \Delta_1 = \Delta_2$. For the DOA extraction in the cases of coupled CPD and CPD/ESPRIT, a conventional beamformer is used. We compare the coupled CPD and CPD/ESPRIT methods with a dedicated near-field spectral MUSIC method. In both the MUSIC and the beamforming methods, the angular and range resolutions are set to $\frac{2\pi}{3600}$ radians (i.e., 0.1 degrees) and $\frac{\lambda}{10}$ meters. The median angular error $P(\theta) = \sqrt{\frac{1}{R}\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|_F^2}$ value over 10 runs as a function of the distance (measured in wavelengths λ) can be seen in figure 6. The performance is bounded by the SNR level and the angular and range resolutions of the MUSIC and beamforming methods. We observe that if the distance between the sources and the sensors is at least 6 wavelengths, then the approximate coupled CPD approach performs about the same as the MUSIC method. Note that in the MUSIC method, the spectral peak-picking is by itself a nontrivial task. As expected, the ESPRIT method did not perform well due to the incorrect far-field assumption.

Case 5: Let us consider the same experiment, but now the ULA is only composed of $I = 11$ sensors. We investigate when the far-field assumption is reasonable for this smaller array. The median angular distance $P(\theta)$ value over 10 runs as a function of the distance can be seen in figure 7. To reduce the computational complexity of the beamformer, the range resolution has been reduced to λ. We observe that only around $\Delta = 180\lambda$, the coupled CPD and the far-field ESPRIT/CPD methods perform about the same.

VI. Conclusion

Despite its importance, sensor array processing beyond the CPD model still presents important algebraic challenges. We first explained that the coupled CPD model provides multidimensional extensions of the CPD/ESPRIT framework to arrays with multiple shift-invariance structures. This includes algebraic and generic uniqueness conditions and exact factor recovery by means of a matrix EVD in the noiseless case. We have obtained mild bounds on the number of signals that can be handled by L-shaped, frame-shaped and triangular arrays, along with EVD-based algorithms. This approach is general. More examples can be found in [23],[13], [32], [9]. Note also that despite the fact that we only discussed two-dimensional arrays, the approach is also valid for N-dimensional arrays, e.g., the three-dimensional cross-shaped array in [8].

Second, we explained that centro-symmetry involves multiple shift-invariance, and hence array processing problems that involve a centro-symmetric array are related to coupled CPD. In particular, we have concluded that centro-symmetry by itself can be sufficient for identifiability. We also pointed out that the coupled CPD can be used to analyze data that locally enjoy CPD structure. Specifically, we have followed this approach to extend CPD to near-field scenarios.

Finally, numerical experiments demonstrated that by simultaneously exploiting multiple shift-invariance, improved performance can be achieved compared to the ordinary CPD model, which only exploits a single shift-invariance structure. The numerical experiments also supported the claim that the local low-rank property of coupled CPD model makes it more capable of handling near-field sources compared to the ordinary CPD model.

Appendix A
Proof of Theorem III.3

The proof of Theorem III.3 will be based on the following lemma.

Lemma A.1. *Let $M \geq 2$, $b_1, \ldots, b_M$ be analytic functions on $\mathbb{C}^l$, $\mathbf{b}(\cdot) = [b_1(\cdot)\ \ldots\ b_M(\cdot)]^T$, $\mathbf{z}_0 \in \mathbb{C}^l$ and $\mathbf{a} \in \mathbb{C}^M$. Let also Ω denote the set of perturbations $\Delta\mathbf{z} \in \mathbb{C}^l$ such that $\mathbf{b}(\mathbf{z}_0 + \Delta\mathbf{z})$ is proportional to $\mathbf{a}$, that is $\Omega = \{\Delta\mathbf{z} : \text{rank}([\mathbf{b}(\mathbf{z}_0 + \Delta\mathbf{z})\ \mathbf{a}]) \leq 1\}$. Then either the Lebesgue measure of Ω is zero or $\mathbf{b}(\cdot) = \phi(\cdot)\mathbf{a}$, where ϕ is an analytic function on $\mathbb{C}^l$.*

Proof: Let $d_1(\Delta\mathbf{z}), \ldots, d_{M(M-1)/2}(\Delta\mathbf{z})$ denote all 2×2 minors of $[\mathbf{b}(\mathbf{z}_0 + \Delta\mathbf{z})\ \mathbf{a}]$. Then $\text{rank}([\mathbf{b}(\mathbf{z}_0 + \Delta\mathbf{z})\ \mathbf{a}]) \leq 1$ if and only if $d_1(\Delta\mathbf{z}) = \cdots = d_{M(M-1)/2}(\Delta\mathbf{z}) = 0$. Thus, Ω is the zero set of the functions $d_1, \ldots, d_{M(M-1)/2}$. It is clear that $d_1, \ldots, d_{M(M-1)/2}$ are analytic functions on $\mathbb{C}^l$. Hence, if the Lebesgue measure of Ω is nonzero, then, by [10, Corollary 10], the functions $d_1, \ldots, d_{M(M-1)/2}$ are identically zero and consequently $\text{rank}([\mathbf{b}(\mathbf{z}_0 + \Delta\mathbf{z})\ \mathbf{a}]) \leq 1$ for all $\Delta\mathbf{z} \in \mathbb{C}^l$. One can show that this is possible if and only if $\mathbf{b}(\cdot) = \phi(\cdot)\mathbf{a}$, where ϕ is an analytic function on $\mathbb{C}^l$. ∎

Proof of Theorem III.3: Assume that (15) holds and that $R \geq \delta$. Let $\widehat{\mathbf{B}}$ be a small generic perturbation of $\mathbf{B}$ whose columns are structured like the columns of $\mathbf{B}$. Since the entries of $\mathbf{b}(\mathbf{z})$ are continuous functions on $\mathbb{C}^l$, $\text{rank}(\widehat{\mathbf{B}}) = \delta$. By definition of δ, $\text{rank}([\mathbf{B}\ \widehat{\mathbf{B}}]) \leq \delta$. Since $\text{rank}(\mathbf{B}) = \text{rank}(\widehat{\mathbf{B}}) = \delta$, it follows that the range of $\mathbf{B}$ coincides with the range of $\widehat{\mathbf{B}}$. Hence, $\mathbf{B} = \mathbf{U}\mathbf{B}_1$ and $\widehat{\mathbf{B}} = \mathbf{U}\widehat{\mathbf{B}}_1$, where the matrices $\mathbf{B}_1$ and $\widehat{\mathbf{B}}_1$ have full row rank. Thus, $\mathbf{X} = \mathbf{B}\mathbf{S}^T = \mathbf{U}\mathbf{B}_1\mathbf{S}^T = \mathbf{U}\widehat{\mathbf{B}}_1\widehat{\mathbf{B}}_1^H(\widehat{\mathbf{B}}_1\widehat{\mathbf{B}}_1^H)^{-1}\mathbf{B}_1\mathbf{S}^T = \mathbf{U}\widehat{\mathbf{B}}_1\left(\widehat{\mathbf{B}}_1^H(\widehat{\mathbf{B}}_1\widehat{\mathbf{B}}_1^H)^{-1}\mathbf{B}_1\mathbf{S}^T\right) = \widehat{\mathbf{B}}\left(\widehat{\mathbf{B}}_1^H(\widehat{\mathbf{B}}_1\widehat{\mathbf{B}}_1^H)^{-1}\mathbf{B}_1\mathbf{S}^T\right)$. To prove that the factorization (15) is not unique we show that $\widehat{\mathbf{B}}$ cannot be obtained from $\mathbf{B}$ by column permutation and scaling. Consider an arbitrary vector $\mathbf{z}_0 \in \mathbb{C}^l$. It is clear that the columns of $\widehat{\mathbf{B}}$ can be interpreted as generic perturbations of $\mathbf{b}(\mathbf{z}_0)$. By Lemma A.1, none of the columns of $\widehat{\mathbf{B}}$ is proportional to the first column of $\mathbf{B}$. In particular, $\widehat{\mathbf{B}}$ cannot be obtained from $\mathbf{B}$ by column permutation and scaling. ∎

Appendix B
Proof of Theorem IV.1

Proof: Observe that since $\left[\begin{smallmatrix}\mathbf{A}^{(1)}\odot\mathbf{D}^{(1)}\\ \mathbf{A}^{(2)}\odot\mathbf{D}^{(2)}\end{smallmatrix}\right]$ and $\mathbf{S}$ have full column rank and $\widehat{R}\le R$, $\left[\begin{smallmatrix}\widehat{\mathbf{A}}^{(1)}\odot\widehat{\mathbf{D}}^{(1)}\\ \widehat{\mathbf{A}}^{(2)}\odot\widehat{\mathbf{D}}^{(2)}\end{smallmatrix}\right]$ and $\widehat{\mathbf{S}}$ must also have full column rank and $\widehat{R}=R$. Denote $\mathbf{V}^{(n)}=\mathbf{Y}^{(n)}(\widehat{\mathbf{S}}^T)^\dagger$. Since $\widehat{\mathbf{S}}$ has full column rank, we obtain from (20):

$$\left[\begin{array}{c}\mathbf{V}^{(1)}\\ \mathbf{V}^{(2)}\end{array}\right]=\left[\begin{array}{c}\mathbf{X}^{(1)}(\widehat{\mathbf{S}}^T)^\dagger\\ \mathbf{X}^{(2)}(\widehat{\mathbf{S}}^T)^\dagger\end{array}\right]=\left[\begin{array}{c}\widehat{\mathbf{A}}^{(1)}\odot\widehat{\mathbf{D}}^{(1)}\\ \widehat{\mathbf{A}}^{(2)}\odot\widehat{\mathbf{D}}^{(2)}\end{array}\right],\qquad(45)$$

where $\mathbf{V}^{(n)}=\left[\begin{smallmatrix}\mathbf{V}^{(n,1)}\\ \mathbf{V}^{(n,2)}\end{smallmatrix}\right]$ in which $\mathbf{V}^{(n,1)}=\underline{\mathbf{I}}_{I_n}\mathbf{Y}^{(n)}(\widehat{\mathbf{S}}^T)^\dagger$ and $\mathbf{V}^{(n,2)}=\bar{\mathbf{I}}_{I_n}\mathbf{Y}^{(n)}(\widehat{\mathbf{S}}^T)^\dagger$. We have to verify that $\widehat{\mathbf{d}}_r^{(n)}$ is a nonzero scalar multiple of $[1\ \widehat{z}_{n,r}\ \dots\ \widehat{z}_{n,r}^{I_n-2}]^T$. Due to the relations $\mathbf{v}_r^{(n,1)}=\underline{\mathbf{I}}_{I_n}\mathbf{v}_r^{(n)}$ and $\mathbf{v}_r^{(n,2)}=\bar{\mathbf{I}}_{I_n}\mathbf{v}_r^{(n)}$, we know from (45) that

$$\widehat{z}_{n,r}\cdot[\widehat{d}_{1,r}^{(n)}\ \widehat{d}_{2,r}^{(n)}\ \dots\ \widehat{d}_{I_n-2,r}^{(n)}]^T=1\cdot[\widehat{d}_{2,r}^{(n)}\ \widehat{d}_{3,r}^{(n)}\ \dots\ \widehat{d}_{I_n-1,r}^{(n)}]^T,$$

implying that $[\widehat{d}_{1,r}^{(n)}\ \widehat{d}_{2,r}^{(n)}\ \dots\ \widehat{d}_{I_n-1,r}^{(n)}]^T=\widehat{d}_{1,r}^{(n)}[1\ \widehat{z}_{n,r}\ \dots\ \widehat{z}_{n,r}^{I_n-2}]^T$. Now we show that $\widehat{d}_{1,r}^{(1)}=\widehat{d}_{1,r}^{(2)}\neq 0$. From (16)–(18) it follows that $\mathbf{X}^{(1)}(1,:)=\mathbf{X}^{(2)}(1,:)=[1\dots1]\mathbf{S}^T$. Hence, by (45),

$$\widehat{\mathbf{D}}^{(1)}(1,:)=\mathbf{V}^{(1)}(1,:)=[1\dots1]\mathbf{S}^T(\widehat{\mathbf{S}}^T)^\dagger=\mathbf{V}^{(2)}(1,:)=\widehat{\mathbf{D}}^{(2)}(1,:),$$

that is, $\widehat{d}_{1,r}^{(1)}=\widehat{d}_{1,r}^{(2)}$. Since $\left[\begin{smallmatrix}\mathbf{A}^{(1)}\odot\mathbf{D}^{(1)}\\ \mathbf{A}^{(2)}\odot\mathbf{D}^{(2)}\end{smallmatrix}\right]$ has full column rank, its r-th column

$$\left[\widehat{d}_{1,r}^{(1)}[1\ \widehat{z}_{1,r}\ \dots\ \widehat{z}_{1,r}^{I_1-2}]\quad \widehat{d}_{1,r}^{(2)}[1\ \widehat{z}_{2,r}\ \dots\ \widehat{z}_{2,r}^{I_2-2}]\right]^T$$

is not zero. Hence $\widehat{d}_{1,r}^{(1)}=\widehat{d}_{1,r}^{(2)}\neq 0$. ∎

Appendix C
Proofs of Theorems IV.3 and IV.5

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$. The proofs of Theorems IV.3 and IV.5 rely on [6, Theorem 1], which guarantees the generic uniqueness of the following structured rank-1 decomposition of a $K\times N$ matrix $\mathbf{Y}$

$$\mathbf{Y}=\sum_{r=1}^{R}\mathbf{a}_r\mathbf{b}(\zeta_r)^T,\quad \mathbf{a}_r\in\mathbb{F}^K,\quad \zeta_r\in\mathbb{F}^l,\qquad(46)$$

where the vector function $\mathbf{b}:\mathbb{F}^l\to\mathbb{F}^N$ is known and constructed as explained below. It is clear that in (46) the rank-1 terms can be arbitrarily permuted. We say that decomposition (46) is unique when it is only subject to this trivial indeterminacy. We say that decomposition (46) is generically unique if it is unique for a generic choice of $\zeta_1,\dots,\zeta_R$. In this paper we will use a simplified version of [6, Theorem 1], namely, we will consider the case where the vector function $\mathbf{b}$ is structured as follows:

$$\begin{aligned}&\mathbf{b}(\cdot)=\mathbf{r}(\mathbf{f}(\cdot)),\\ &\mathbf{r}(\cdot)=(p_1(\cdot),\dots,p_N(\cdot)),\\ &p_1,\dots,p_N\ \text{ are polynomials in } l \text{ variables},\\ &\mathbf{f}(\cdot)=(f_1(\cdot),\dots,f_l(\cdot)),\\ &f_1,\dots,f_l \text{ are functions analytic on } \mathbb{C}^l.\end{aligned}\qquad(47)$$

The following result follows from [6, Theorem 1]. We use $\mathbf{J}_\mathbf{r}\in\mathbb{F}^{N\times l}$ and $\mathbf{J}_\mathbf{f}\in\mathbb{F}^{l\times l}$ to denote the Jacobian matrices of $\mathbf{r}$ and $\mathbf{f}$, respectively.

Theorem C.1. *Assume that*

i) *the matrix* $[\mathbf{a}_1\ \dots\ \mathbf{a}_R]$ *has full column rank;*
ii) *there exists* $\zeta^0\in\mathbb{C}^l$ *such that* $\det\mathbf{J}_\mathbf{f}(\zeta^0)\neq 0$*;*
iii) *the dimension of the subspace spanned by the vectors of the form* (47) *is at least* $\widehat{N}$*;*
iv) $\operatorname{rank}\mathbf{J}_\mathbf{r}(\mathbf{x})\le\widehat{l}$ *for a generic choice of* $\mathbf{x}\in\mathbb{C}^l$*;*
v) $R\le\widehat{N}-\widehat{l}-1$.

Then decomposition (46) *is generically unique.*

Proof of Theorem IV.3: It is clear that (19) is equivalent to (16), which can be put in the form (15) where $\mathbf{X}=\left[\begin{smallmatrix}\mathbf{Y}^{(1)}\\ \overline{\mathbf{Y}}^{(2)}\end{smallmatrix}\right]$ and $\mathbf{B}=\left[\begin{smallmatrix}\mathbf{B}^{(1)}\\ \overline{\mathbf{B}}^{(2)}\end{smallmatrix}\right]$. Thus uniqueness of the coupled CPD (19) is equivalent to the uniqueness of the structured matrix factorization (15). In the remaining part of the proof we interpret (15) as decomposition (46) and then check the assumptions in Theorem C.1. We set

$$t_x=-\sqrt{-1}\omega_c d_x/c,\quad t_y=-\sqrt{-1}\omega_c d_y/c,\quad \zeta=[\phi\ \theta]^T\qquad(48)$$

and define the vector functions

$$\begin{aligned}\mathbf{f}(\zeta)=(e^{t_x\sin(\phi)\cos(\theta)},e^{t_y\sin(\phi)\sin(\theta)}):\ \mathbb{R}^2\to\mathbb{C}^2,\\ \mathbf{r}^{(1)}(z_1,z_2)=(1,z_1,\dots,z_1^{I_1-1}):\ \mathbb{C}^2\to\mathbb{C}^{I_1},\\ \mathbf{r}^{(2)}(z_1,z_2)=(z_2,\dots,z_2^{I_2-1}):\ \mathbb{C}^2\to\mathbb{C}^{I_2-1},\\ \mathbf{r}(z_1,z_2)=(\mathbf{r}^{(1)}(z_1,z_2),\mathbf{r}^{(2)}(z_1,z_2)).\end{aligned}\qquad(49)$$

Then, by (17), the columns of $\mathbf{B}$ in (15) are sampled values of

$$\mathbf{b}(\zeta)=(\mathbf{r}^{(1)}(\mathbf{f}(\zeta)),\ \mathbf{r}^{(2)}(\mathbf{f}(\zeta)))\qquad(50)$$

at points $\zeta_1=[\phi_1\ \theta_1]^T,\dots,\zeta_R=[\phi_R\ \theta_R]^T$. Now taking the transpose of (15) and setting $\mathbf{a}_r=\mathbf{s}_r$ we obtain

$$\mathbf{Y}=\mathbf{X}^T=\mathbf{S}\mathbf{B}^T=\sum_{r=1}^{R}\mathbf{a}_r\mathbf{b}(\zeta_r)^T,\qquad(51)$$

which coincides with (46). Now we check the assumptions in Theorem C.1: i) by construction, $[\mathbf{a}_1\ \dots\ \mathbf{a}_R]$ coincides with the matrix $\mathbf{S}$, which has full column rank by assumption; ii) simple calculations yield that the determinant of the Jacobian of $\mathbf{f}$ is equal to $t_xt_yf_1f_2\frac{\sin(2\phi)}{2}$, i.e., it is not identically zero; iii) the vectors $[\mathbf{r}^{(1)}(z_1,z_2)\ \mathbf{r}^{(2)}(z_1,z_2)]^T\in\mathbb{C}^{I_1+I_2-1}$ obviously span the whole space, i.e., we can set $\widehat{N}=I_1+I_2-1$; iv) we set $\widehat{l}=2$. Finally, assumption v) holds since $R\le\widehat{N}-\widehat{l}-1=I_1+I_2-4$. ∎

Proof of Theorem IV.5: It is clear that the uniqueness of the coupled CPD (22) is equivalent to the uniqueness of the structured matrix factorization (15) where $\mathbf{X}=\left[\begin{smallmatrix}\mathbf{Y}^{(1)}\\ \mathbf{Y}^{(2)}\end{smallmatrix}\right]$ and $\mathbf{B}=\left[\begin{smallmatrix}\mathbf{B}^{(1)}\odot\mathbf{C}^{(1)}\\ \mathbf{B}^{(2)}\odot\mathbf{C}^{(2)}\end{smallmatrix}\right]$. We interpret (15) as decomposition (46) and then check the assumptions in Theorem C.1. We define t_x, t_y and $\mathbf{f}(\zeta)$ as in (48) and (49),

respectively, and set

$$\mathbf{r}^{(1)}(z_1,z_2) = [1\ z_1\ \dots\ z_1^{I_1-1}]^T \otimes [1\ z_2^{I_2-1}]^T :\ \mathbb{C}^2 \to \mathbb{C}^{2I_1},$$
$$\mathbf{r}^{(2)}(z_1,z_2) = [1\ z_2\ \dots\ z_2^{I_2-1}]^T \otimes [1\ z_1^{I_1-1}]^T :\ \mathbb{C}^2 \to \mathbb{C}^{2I_2},$$
$$\mathbf{r}(z_1,z_2) = (\mathbf{r}^{(1)}(z_1,z_2), \mathbf{r}^{(2)}(z_1,z_2)).$$

Taking the transpose of (15) we obtain (51), where $\mathbf{a}_r = \mathbf{s}_r$, $\mathbf{b}(\boldsymbol{\zeta})$ is defined as in (50), and $\boldsymbol{\zeta}_1 = [\phi_1\ \theta_1]^T, \dots, \boldsymbol{\zeta}_R = [\phi_R\ \theta_R]^T$. Now we check the assumptions in Theorem C.1: the derivation of i) and ii) is the same as in the proof of Theorem IV.3; iii) the vectors $[\mathbf{r}^{(1)}(z_1,z_2)\ \mathbf{r}^{(2)}(z_1,z_2)]^T \in \mathbb{C}^{I_1+I_2-1}$ obviously span a subspace of dimension $2I_1+2I_2-4$, i.e., we can set $\widehat{N} = 2I_1+2I_2-4$; iv) we set $\widehat{l} = 2$. Finally, assumption v) holds since $R \le \widehat{N} - \widehat{l} - 1 = 2I_1 + 2I_2 - 7$. ■

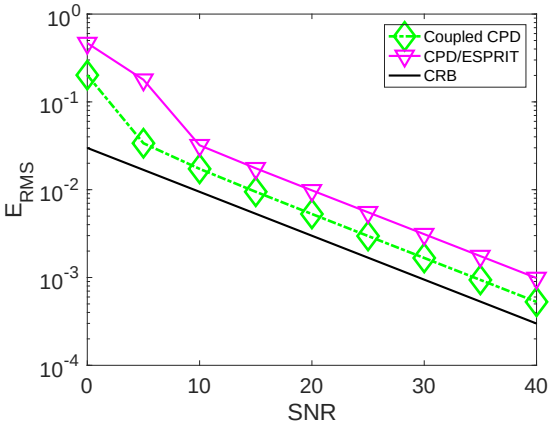

Fig. 3. Case 1: Coupled CPD for piecewise linear arrays, L-shaped array.

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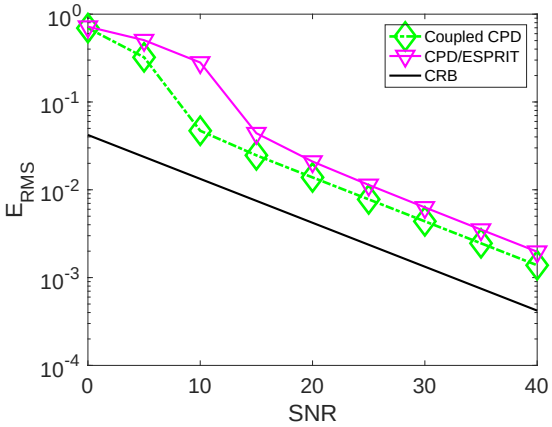

Fig. 4. Case 2: Coupled CPD for piecewise linear arrays, L-shaped array.

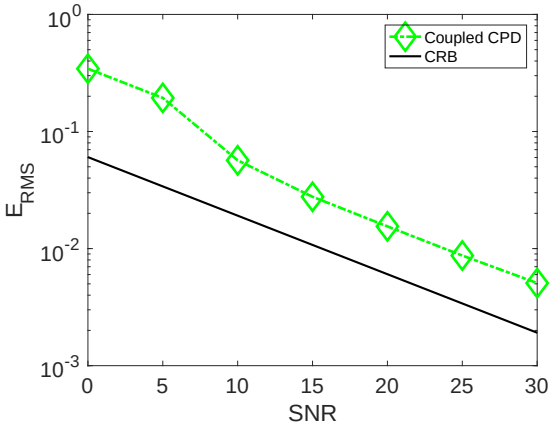

Fig. 5. Case 3: Coupled CPD for centro-symmeric arrays, UCA.

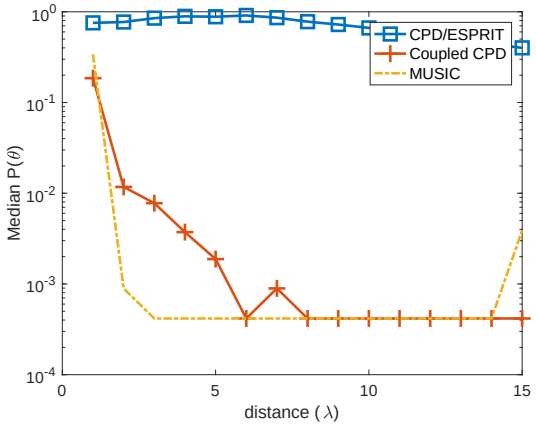

Fig. 6. Case 4: Coupled CPD for near-field sources, ULA.

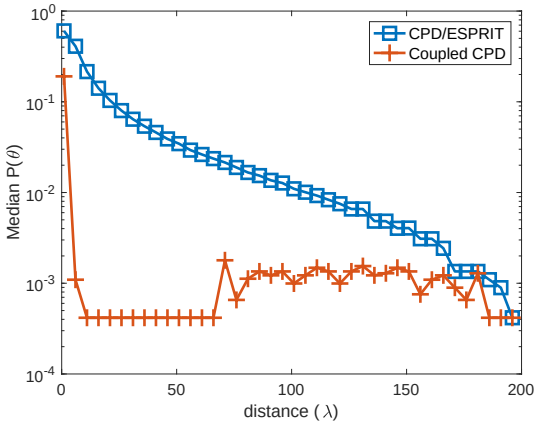

Fig. 7. Case 5: Coupled CPD for near-field sources, ULA.

Acknowledgment

Research supported by: (1) Research Council KU Leuven: C1 project C16/15/059-nD, (2) F.W.O.: project G.0830.14N, G.0881.14N, (3) EU: The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Advanced Grant: BIOTENSORS (no. 339804). This paper reflects only the authors' views and the Union is not liable for any use that may be made of the contained information.



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