NEW UNIQUENESS CONDITIONS FOR THE CANONICAL POLYADIC DECOMPOSITION OF HIGHER-ORDER TENSORS

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Abstract. The uniqueness properties of the Canonical Polyadic Decomposition (CPD) of higher-order tensors have made it an attractive tool for signal separation. However, CPD uniqueness is not yet fully understood. In this paper, we first present a new uniqueness condition for a CPD where one of the factor matrices is assumed to be known. Second, we show that this result can be used to obtain a new overall uniqueness condition for the CPD. We also provide a new uniqueness condition for a complex CPD with a columnwise orthonormal factor matrix. We also demonstrate that the result can be used to obtain a new uniqueness condition for a complex CPD with a partial Hermitian symmetry. Finally, we provide an algorithm for computing a CPD with a known factor matrix.

Key words. tensor, polyadic decomposition, parallel factor (PARAFAC), canonical decomposition (CANDECOMP), canonical polyadic decomposition.

1. Introduction. Tensor decompositions are finding more and more applications in signal processing, data analysis and machine learning. For instance, Canonical Polyadic Decomposition (CPD) is becoming a basic tool for signal separation. Essentially, this is due to the fact that CPD is unique under mild conditions, compared to decomposition of a matrix in rank-1 terms. Thanks to their uniqueness, rank-1 terms are easily associated with interpretable data components. Numerous applications have been reported in Independent Component Analysis, exploratory data analysis, multi-user access in telecommunication, radar, chemometrics, psychometrics, sensor array processing, and so on [3, 4, 19, 20, 16, 18, 12, 2, 13]. However, the understanding of uniqueness is lagging behind the use of CPD in practice.

In this paper we first present a new uniqueness condition for a CPD with a known factor matrix. Based on this result we propose a new overall uniqueness condition for the CPD. We provide a new uniqueness condition for a complex CPD with a columnwise orthonormal factor matrix. We also demonstrate that the result can be used to obtain a new uniqueness condition for a complex CPD with a partial Hermitian symmetry. Such constrained CPDs are very common in signal processing. Finally, we present an inexpensive algorithm for computing a CPD with a known factor matrix, which is also relevant for signal processing.

The paper is organized as follows. The rest of the introduction presents our notation followed by a quick review of the CPD. Section 2 reviews existing uniqueness results for the CPD. Section 3 presents a new uniqueness condition and numerical method for a CPD with a known factor matrix. Next, in sections 4 and 5 we present new uniqueness conditions for the overall CPD and for some variants. Numerical experiments are reported in section 6. We end the paper with a conclusion in section 7.

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1.1. Notation. Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The rth column vector of $\mathbf{A}$ is denoted by $\mathbf{a}_r$. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao products, defined as

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad \mathbf{A} \odot \mathbf{B} \triangleq \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \mathbf{a}_2 \otimes \mathbf{b}_2 & \dots \end{bmatrix}$$

in which $(\mathbf{A})_{mn} = a_{mn}$. Sometimes we denote the Khatri-Rao products of N matrices $\{\mathbf{A}^{(n)}\}_{n=1}^{N}$ by

$$\bigodot_{n=1}^{N} \mathbf{A}^{(n)} = \mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}.$$

The Hadamard product is denoted by $*$ and is given by

$$\mathbf{A} * \mathbf{B} \triangleq \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots \\ a_{21}b_{21} & a_{22}b_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $\circ$ denote the outer product of N vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ such that $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ satisfies

$$\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)}\right)_{i_1, i_2, \dots, i_N} = a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_N}^{(N)}.$$

The identity matrix, all-zero matrix and all-zero vector are denoted by $\mathbf{I}_M \in \mathbb{C}^{M \times M}$, $\mathbf{0}_{M,N} \in \mathbb{C}^{M \times N}$ and $\mathbf{0}_M \in \mathbb{C}^M$, respectively.

Let $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^{-1}$, $(\cdot)^\dagger$, $\|\cdot\|_F$, $|\cdot|$, $\mathrm{Col}(\cdot)$ and $\mathrm{Null}(\cdot)$ denote the transpose, conjugate, conjugate-transpose, inverse, Moore-Penrose pseudo-inverse, Frobenius norm, determinant, column space and null space of a matrix, respectively.

Given $\mathbf{A} \in \mathbb{C}^{I \times J}$, then $\mathrm{Vec}(\mathbf{A}) \in \mathbb{C}^{IJ}$ will denote the column vector defined by $(\mathrm{Vec}(\mathbf{A}))_{i+(j-1)I} = (\mathbf{A})_{ij}$. Given $\mathbf{a} \in \mathbb{C}^{IJ}$, then the reverse operation is $\mathrm{Unvec}(\mathbf{a}) = \mathbf{A} \in \mathbb{C}^{I \times J}$ such that $(\mathbf{a})_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $D_k(\mathbf{A}) \in \mathbb{C}^{J \times J}$ denote the diagonal matrix holding row k of $\mathbf{A} \in \mathbb{C}^{I \times J}$ on its diagonal.

Matlab index notation will be used to extract submatrices from a given matrix. For example $\mathbf{A}(1:k,:)$ denotes the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k of $\mathbf{A}$. Let $\mathbf{T} \in \mathbb{C}^{I \times R}$, then $\underline{\mathbf{T}} = \mathbf{T}(1:I-1,:) \in \mathbb{C}^{(I-1) \times R}$, i.e., $\underline{\mathbf{T}}$ is obtained by deleting the bottom row of $\mathbf{T}$.

The rank of a matrix $\mathbf{A}$ is denoted by $r(\mathbf{A})$. The k-rank of a matrix $\mathbf{A}$ is denoted by $k(\mathbf{A})$. It is equal to the largest integer $k(\mathbf{A})$ such that every subset of $k(\mathbf{A})$ columns of $\mathbf{A}$ is linearly independent.

Let $C_n^k = \frac{n!}{k!(n-k)!}$ denote the binomial coefficient. The k-th compound matrix of $\mathbf{A} \in \mathbb{C}^{m \times n}$ is denoted by $C_k(\mathbf{A}) \in \mathbb{C}^{C_m^k \times C_n^k}$ and its entries correspond to the k-by-k minors of $\mathbf{A}$ ordered lexicographically. As an example, let $\mathbf{A} \in \mathbb{C}^{4 \times 3}$, then

$$C_2(\mathbf{A}) = \begin{bmatrix} |\mathbf{A}([1,2],[1,2])| & |\mathbf{A}([1,2],[1,3])| & |\mathbf{A}([1,2],[2,3])| \\ |\mathbf{A}([1,3],[1,2])| & |\mathbf{A}([1,3],[1,3])| & |\mathbf{A}([1,3],[2,3])| \\ |\mathbf{A}([1,4],[1,2])| & |\mathbf{A}([1,4],[1,3])| & |\mathbf{A}([1,4],[2,3])| \\ |\mathbf{A}([2,3],[1,2])| & |\mathbf{A}([2,3],[1,3])| & |\mathbf{A}([2,3],[2,3])| \\ |\mathbf{A}([2,4],[1,2])| & |\mathbf{A}([2,4],[1,3])| & |\mathbf{A}([2,4],[2,3])| \\ |\mathbf{A}([3,4],[1,2])| & |\mathbf{A}([3,4],[1,3])| & |\mathbf{A}([3,4],[2,3])| \end{bmatrix}.$$

See [9, 5] for further details on compound matrices.

1.2. Canonical Polyadic Decomposition (CPD). Consider an Nth-order tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$. We say that $\mathcal{X}$ is a rank-1 tensor if it is equal to the outer product of some non-zero vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$, $n \in \{1, \ldots, N\}$, such that $x_{i_1 \ldots i_N} = \prod_{n=1}^{N} a_{i_n}^{(n)}$. More generally, the rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(N)}. \tag{1.1}$$

The decomposition (1.1) will in this paper be referred to as the Canonical Polyadic Decomposition (CPD) of $\mathcal{X}$. Let us stack the vectors $\{\mathbf{a}_r^{(n)}\}$ into the matrices

$$\mathbf{A}^{(n)} = \left[\begin{array}{ccc} \mathbf{a}_1^{(n)}, & \cdots & , \mathbf{a}_R^{(n)} \end{array} \right] \in \mathbb{C}^{I_n \times R}, \quad n \in \{1, \ldots N\}. \tag{1.2}$$

The matrices $\mathbf{A}^{(n)}$ in (1.2) will be referred to as the factor matrices of the CPD of the tensor $\mathcal{X}$ in (1.1). The following matrix representation of an Nth-order tensor will also be used throughout the paper. Consider $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R, then it has matrix representation $\mathbf{X}^{[P]} \in \mathbb{C}^{\prod_{m=1}^{P} I_m \times \prod_{n=P+1}^{N} I_n}$ given by

$$\begin{aligned} \mathbf{X}^{[P]} &= \left[\begin{array}{cccc} x_{1,\ldots,1,1,\ldots,1} & x_{1,\ldots,1,1,\ldots,2} & \cdots & x_{1,\ldots,1,I_{P+1},\ldots,I_N} \\ x_{1,\ldots,2,1,\ldots,1} & x_{1,\ldots,2,1,\ldots,2} & \cdots & x_{1,\ldots,2,I_{P+1},\ldots,I_N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{I_1,\ldots,I_P,1,\ldots,1} & x_{I_1,\ldots,I_P,1,\ldots,2} & \cdots & x_{I_1,\ldots,I_P,I_{P+1},\ldots,I_N} \end{array} \right] \\ &= \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)} \right) \left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)} \right)^T. \end{aligned} \tag{1.3}$$

In some cases it is convenient to map an Nth-order tensor to a third-order tensor. Let the integers M and P satisfy $1 \leq M < P < N$, then we can group the factor matrices of the CPD of $\mathcal{X}$ as follows

$$\begin{aligned} \mathbf{A} &= \mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(M)} \in \mathbb{C}^{I \times R}, & I &= \prod_{n=1}^{M} I_n \\ \mathbf{B} &= \mathbf{A}^{(M+1)} \odot \cdots \odot \mathbf{A}^{(P)} \in \mathbb{C}^{J \times R}, & J &= \prod_{n=M+1}^{P} I_n \\ \mathbf{C} &= \mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)} \in \mathbb{C}^{K \times R}, & K &= \prod_{n=P+1}^{N} I_n. \end{aligned}$$

By ignoring the structural relations among the column vectors of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ we obtain the third-order tensor

$$\mathbb{C}^{I \times J \times K} \ni \widetilde{\mathbf{X}} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r. \tag{1.4}$$

The tensor (1.4) has matrix representation

$$\mathbb{C}^{IJ \times K} \ni \mathbf{X} \triangleq \widetilde{\mathbf{X}}^{[2]} = \mathbf{X}^{[P]} = (\mathbf{A} \odot \mathbf{B}) \mathbf{C}^T. \tag{1.5}$$

2. Existing Uniqueness Results for CPD. Subsection 2.1 presents existing conditions that guarantee uniqueness of one factor matrix of a CPD while subsection 2.2 presents existing conditions that guarantee overall uniqueness of a CPD. We will in this section treat tensors of order $N > 3$ as third-order tensors by working with matrix representation (1.5).

2.1. Uniqueness Conditions for One Factor Matrix of a CPD. A factor matrix, say $\mathbf{A}^{(n)}$, of the CPD of $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is said to be unique if it can be determined up to the inherent column scaling and permutation ambiguities from $\mathcal{X}$. More formally, the factor matrix $\mathbf{A}^{(n)}$ is unique if alternative factor matrices $\widehat{\mathbf{A}}^{(n)}$ satisfy the condition

$$\widehat{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)} \mathbf{P} \Delta_{\mathbf{A}^{(n)}} ,$$

where $\mathbf{P}$ is a permutation matrix and $\Delta_{\mathbf{A}^{(n)}}$ is a nonsingular diagonal matrix. The conditions stated in this subsection guarantees the uniqueness of one factor matrix of a CPD. They will be used in the paper to obtain new overall uniqueness results for tensor decompositions.

THEOREM 2.1. *Let $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ be a tensor with rank R and matrix representation*

$$\boldsymbol{X} = (\boldsymbol{A} \odot \boldsymbol{B})\, \boldsymbol{C}^T ,$$

where $\boldsymbol{A} \in \mathbb{C}^{I \times R}$, $\boldsymbol{B} \in \mathbb{C}^{J \times R}$ and $\boldsymbol{C} \in \mathbb{C}^{K \times R}$. If

$$\begin{cases} k(\boldsymbol{C}) \geq 1 \\ \min\left(k(\boldsymbol{A}), k(\boldsymbol{B})\right) \geq R - r(\boldsymbol{C}) + 2 \\ r(\boldsymbol{C}) + k(\boldsymbol{B}) + k(\boldsymbol{C}) + \max\left(r(\boldsymbol{A}) + k(\boldsymbol{A}), r(\boldsymbol{B}) + k(\boldsymbol{B})\right) \geq 2\,(R+1) \end{cases} \tag{2.1}$$

then the factor matrix $\boldsymbol{C}$ is unique [8].

More recently the following more relaxed condition has been obtained.

THEOREM 2.2. *Let $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ be a tensor with rank R and matrix representation*

$$\boldsymbol{X} = (\boldsymbol{A} \odot \boldsymbol{B})\, \boldsymbol{C}^T ,$$

where $\boldsymbol{A} \in \mathbb{C}^{I \times R}$, $\boldsymbol{B} \in \mathbb{C}^{J \times R}$ and $\boldsymbol{C} \in \mathbb{C}^{K \times R}$. If

$$\begin{cases} k(\boldsymbol{C}) \geq 1 \\ \min\left(I, J\right) \geq R - r(\boldsymbol{C}) + 2 \\ C_{R-r(\boldsymbol{C})+2}\left(\boldsymbol{A}\right) \odot C_{R-r(\boldsymbol{C})+2}\left(\boldsymbol{B}\right) \;\; \textit{has full column rank} \end{cases} \tag{2.2}$$

then the factor matrix $\boldsymbol{C}$ is unique [5].

2.2. Uniqueness Conditions for CPD. A CPD of $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is said to be unique if all the N-tuplets $\left(\widehat{\mathbf{A}}^{(1)}, \ldots, \widehat{\mathbf{A}}^{(N)}\right)$ satisfying (1.1) are related via

$$\widehat{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)} \mathbf{P} \Delta_{\mathbf{A}^{(n)}}, \quad \forall n \in \{1, \ldots N\} ,$$

where $\{\Delta_{\mathbf{A}^{(n)}}\}$ are diagonal matrices satisfying $\prod_{n=1}^{N} \Delta_{\mathbf{A}^{(n)}} = \mathbf{I}_R$ and $\mathbf{P}$ is a permutation matrix. The following theorem states a well-known condition for overall CPD uniqueness.

THEOREM 2.3. *Let $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ be a tensor with rank R and matrix representation*

$$\boldsymbol{X} = (\boldsymbol{A} \odot \boldsymbol{B})\, \boldsymbol{C}^T ,$$

where $\boldsymbol{A} \in \mathbb{C}^{I\times R}$, $\boldsymbol{B} \in \mathbb{C}^{J\times R}$ and $\boldsymbol{C} \in \mathbb{C}^{K\times R}$. If

$$k(\boldsymbol{A}) + k(\boldsymbol{B}) + k(\boldsymbol{C}) \geq 2\,(R+1) , \tag{2.3}$$

then the CPD of $\mathcal{X}$ is unique.

Proof. The original proof can be found in [14] but a more accessible prove is given in [26]. An alternative and shortened proof has recently been presented in [17]. □

Recently the following more relaxed overall CPD uniqueness condition has been obtained.

THEOREM 2.4. *Let $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ be a tensor with rank R and matrix representation*

$$\boldsymbol{X} = (\boldsymbol{A} \odot \boldsymbol{B})\, \boldsymbol{C}^T ,$$

where $\boldsymbol{A} \in \mathbb{C}^{I\times R}$, $\boldsymbol{B} \in \mathbb{C}^{J\times R}$ and $\boldsymbol{C} \in \mathbb{C}^{K\times R}$. If

$$\begin{cases} \min\left(k\left(\boldsymbol{A}\right), k\left(\boldsymbol{B}\right)\right) + k\left(\boldsymbol{C}\right) \geq R+1 \\ \max\left(k\left(\boldsymbol{A}\right), k\left(\boldsymbol{B}\right)\right) + k\left(\boldsymbol{C}\right) \geq R+2 \\ C_{R-r(\boldsymbol{C})+2}\left(\boldsymbol{A}\right) \odot C_{R-r(\boldsymbol{C})+2}\left(\boldsymbol{B}\right) \;\; \textit{has full column rank} \end{cases} \tag{2.4}$$

then the CPD of $\mathcal{X}$ is unique [6].

3. New Uniqueness Result for CPD with a Known Factor Matrix. Consider the tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ with rank R and matrix representation $\mathbf{X}^{[P]} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)\left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)^T$. Assume that the factor matrix $\mathbf{A}^{(1)}$ is known and let the $(N-1)$-tuplet $\left(\widehat{\mathbf{A}}^{(2)}, \ldots, \widehat{\mathbf{A}}^{(N)}\right)$ yield an alternative decomposition of $\mathcal{X}$ with $\mathbf{A}^{(1)}$. The CPD of $\mathcal{X}$ with the known factor matrix $\mathbf{A}^{(1)}$ is said to be unique if all the N-tuplets $\left(\widehat{\mathbf{A}}^{(1)}, \ldots, \widehat{\mathbf{A}}^{(N)}\right)$ satisfying (1.1) are related via

$$\widehat{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)}\Delta_{\mathbf{A}^{(n)}} , \quad \forall\, n \in \{2, \ldots, N\} ,$$

where $\Delta_{\mathbf{A}^{(n)}} \in \mathbb{C}^{R\times R}$ are diagonal matrices with property $\prod_{n=2}^{N} \Delta_{\mathbf{A}^{(n)}} = \mathbf{I}_R$.

In this section we present a new uniqueness result for a CPD with a known factor matrix. If the known factor matrix has full column rank, then the overall decomposition is trivially unique. We state the result here for completeness.

PROPOSITION 3.1. *Consider $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ with rank R and matrix representation $\boldsymbol{X}^{[N-1]} = \left(\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(N-1)}\right)\boldsymbol{A}^{(N)T}$. If $\boldsymbol{A}^{(N)}$ is known and has full column rank, then the CPD with $\boldsymbol{A}^{(N)}$ is unique.*

Proof. The result is trivial, simply compute

$$\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_R] = \mathbf{X}^{[N-1]}\left(\mathbf{A}^{(N)T}\right)^{\dagger} ,$$

then the factor matrices follow from the best rank-1 approximation problems

$$\min_{\mathbf{a}_r^{(2)},\ldots,\mathbf{a}_r^{(N-1)}} \left\| \mathbf{y}_r - \mathbf{a}_r^{(2)} \otimes \cdots \otimes \mathbf{a}_r^{(N-1)} \right\|_F^2 , \quad r \in \{1,\ldots,R\} .$$

□

In this section we present a new result for the case where the known factor matrix does not necessarily have full column rank. The result is stated in Theorems 3.6 and 3.8. We first provide some auxiliary results.

LEMMA 3.2. *Given an analytic function $f : \mathbb{C}^n \to \mathbb{C}$. If there exists a vector $\boldsymbol{x} \in \mathbb{C}^n$ such that $f(\boldsymbol{x}) \neq 0$, then the set $\{ \boldsymbol{x} | f(\boldsymbol{x}) = 0 \}$ is of Lebesgue measure zero.*

Proof. For a proof, see for instance [10]. □

A matrix $\mathbf{A}^{(1)}$ is said to be Vandermonde if it takes the form

$$\mathbf{A}^{(1)} = \begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_R \\ \vdots & \ddots & \vdots \\ z_1^{I_1-1} & \cdots & z_R^{I_1-1} \end{bmatrix} \in \mathbb{C}^{I_1 \times R}, \tag{3.1}$$

where $\{z_r\}$ are called the generators of $\mathbf{A}^{(1)}$.

LEMMA 3.3. *Let $\boldsymbol{A}^{(1)} \in \mathbb{C}^{I_1 \times R}$ be a Vandermonde matrix and let $\boldsymbol{A}^{(2)} \in \mathbb{C}^{I_2 \times R}$, then the matrix $\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}$ generically has rank $\min(I_1 I_2, R)$.*

Proof. The result follows from Theorem 3 and Corollary 1 in [10]. □

PROPOSITION 3.4. *Given a Vandermonde matrix $\boldsymbol{A}^{(1)} = \left[\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_R^{(1)} \right] \in \mathbb{C}^{I_1 \times R}$ with distinct generators. Let the column vectors of $\boldsymbol{U} \in \mathbb{C}^{I_1 I_2 \times R}$ constitute a basis for $Col\left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)} \right)$, where $\boldsymbol{A}^{(2)} \in \mathbb{C}^{I_2 \times R}$. If the matrix $\underline{\boldsymbol{A}}^{(1)} \odot \boldsymbol{A}^{(2)}$ has full column rank, then the matrix*

$$\boldsymbol{G}^{(r)} = \left[\boldsymbol{U}, -\boldsymbol{a}_r^{(1)} \otimes \boldsymbol{I}_{I_2} \right] \in \mathbb{C}^{I_1 I_2 \times (R + I_2)} \tag{3.2}$$

has rank $R + I_2 - 1$, $\forall r \in \{1,\ldots,R\}$. Generically, $\boldsymbol{G}^{(r)}$ has rank $R + I_2 - 1$ if $(I_1 - 1) I_2 \geq R$.

Proof. Consider a Vandermonde matrix $\mathbf{A}^{(1)} \in \mathbb{C}^{I_1 \times R}$ as in (3.1). Let us first prove that if $\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}$ has full column rank, then $\mathbf{G}^{(r)}$ has a one-dimensional kernel. Let $\mathbf{U} = \mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}$, then

$$\begin{aligned} \mathbf{G}^{(r)} &= \left[\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}, -\mathbf{a}_r^{(1)} \otimes \mathbf{I}_{I_2} \right] \\ &= \begin{bmatrix} \mathbf{A}^{(2)} & \mathbf{I}_{I_2} \\ \mathbf{A}^{(2)}\mathbf{Z} & z_r \mathbf{I}_{I_2} \\ \vdots & \vdots \\ \mathbf{A}^{(2)}\mathbf{Z}^{I_1-1} & z_r^{I_1-1} \mathbf{I}_{I_2} \end{bmatrix} \in \mathbb{C}^{I_1 I_2 \times (R + I_2)}, \end{aligned}$$

where $\mathbf{Z} = D_1([z_1, z_2, \ldots, z_R])$. Consider the following nonsingular block-bidiagonal matrix

$$\mathbf{F}^{(r)} = \begin{bmatrix} \mathbf{I}_{I_2} & \mathbf{0}_{I_2,I_2} & \cdots & \cdots & \mathbf{0}_{I_2,I_2} & \mathbf{0}_{I_2,I_2} \\ -z_r\mathbf{I}_{I_2} & \mathbf{I}_{I_2} & \ddots & & \vdots & \vdots \\ \mathbf{0}_{I_2,I_2} & -z_r\mathbf{I}_{I_2} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0}_{I_2,I_2} & \mathbf{0}_{I_2,I_2} \\ \mathbf{0}_{I_2,I_2} & & \ddots & \ddots & \mathbf{I}_{I_2} & \mathbf{0}_{I_2} \\ \mathbf{0}_{I_2,I_2} & \cdots & \cdots & \mathbf{0}_{I_2,I_2} & -z_r\mathbf{I}_{I_2} & \mathbf{I}_{I_2} \end{bmatrix} \in \mathbb{C}^{I_1I_2 \times I_1I_2},$$

then

$$\mathbf{F}^{(r)}\mathbf{G}^{(r)} = \begin{bmatrix} \mathbf{A}^{(2)} & \mathbf{I}_{I_2} \\ \mathbf{A}^{(2)}\left(\mathbf{Z} - z_r\mathbf{I}_R\right) & \mathbf{0}_{I_2,I_2} \\ \vdots & \vdots \\ \mathbf{A}^{(2)}\left(\mathbf{Z}^{I_1-1} - z_r\mathbf{Z}^{I_1-2}\right) & \mathbf{0}_{I_2,I_2} \end{bmatrix}. \tag{3.3}$$

Hence, the problem of determining the rank of $\mathbf{G}^{(r)}$ reduces to finding the rank of

$$\begin{aligned} \mathbf{H}^{(r)} &= \begin{bmatrix} \mathbf{A}^{(2)}\left(\mathbf{Z} - z_r\mathbf{I}_R\right) \\ \vdots \\ \mathbf{A}^{(2)}\left(\mathbf{Z}^{I_1-1} - z_r\mathbf{Z}^{I_1-2}\right) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}^{(2)}\mathbf{I}_R\left(\mathbf{Z} - z_r\mathbf{I}_R\right) \\ \vdots \\ \mathbf{A}^{(2)}\mathbf{Z}^{I_1-2}\left(\mathbf{Z} - z_r\mathbf{I}_R\right) \end{bmatrix} \\ &= \left(\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}\right)\left(\mathbf{Z} - z_r\mathbf{I}_R\right). \end{aligned} \tag{3.4}$$

The deterministic condition with $\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}$ having full column rank follows from (3.4). Indeed, we note that if $\underline{\mathbf{A}}^{(1)} \odot \mathbf{A}^{(2)}$ has full column rank, then the matrix (3.4) has rank $R-1$ and consequently $\mathbf{G}^{(r)}$ in (3.2) has rank $R + I_2 - 1$, $\forall r \in \{1, \ldots, R\}$. More generally, any $\mathbf{U}$ of which the columns form a basis for Col $\left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)$ yields a matrix $\mathbf{G}^{(r)}$ of rank $R + I_2 - 1$, $\forall r \in \{1, \ldots, R\}$.

Let us now prove that if $(I_1 - 1)I_2 \geq R$, then generically $\mathbf{G}^{(r)}$ has a one-dimensional kernel. Due to Lemma 3.2 we just need to find one example where the statement made in this proposition holds. Note that any minor of $\mathbf{G}^{(r)}$ is an analytic function of the elements $\mathbf{G}^{(r)}$. Due to Lemma 3.3 we know that when $(I_1 - 1)I_2 \geq R$, the matrix $\mathbf{H}^{(r)}$ given by (3.4) generically has rank $R - 1$. Hence, due to Lemma 3.2 we can now conclude that when $(I_1 - 1)I_2 \geq R$, the matrix $\mathbf{G}^{(r)}$ in (3.2) generically has rank $R + I_2 - 1$, $\forall r \in \{1, \ldots, R\}$. □

A direct consequence of Proposition 3.4 is the following result, which generalizes the generic rank condition to the case where $\mathbf{A}^{(1)}$ is not Vandermonde. The result will be instrumental in the proof of Theorem 3.6.

COROLLARY 3.5. *Given* $\boldsymbol{A}^{(1)} = \left[\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_R^{(1)}\right] \in \mathbb{C}^{I_1 \times R}$ *and let the column vectors of* $\boldsymbol{U} \in \mathbb{C}^{I_1I_2 \times R}$ *constitute a basis for Col* $\left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right)$ *in which* $\boldsymbol{A}^{(2)} \in$

$\mathbb{C}^{I_2 \times R}$. *If* $(I_1 - 1)I_2 \geq R$, *then the matrix*

$$\boldsymbol{G}^{(r)} = \left[\boldsymbol{U}, -\boldsymbol{a}_r^{(1)} \otimes \boldsymbol{I}_{I_2} \right] \in \mathbb{C}^{I_1 I_2 \times (R + I_2)}$$

generically has rank $R + I_2 - 1$, $\forall r \in \{1, \ldots, R\}$.

Proof. The result follows directly from Lemma 3.2 and Proposition 3.4. ☐

THEOREM 3.6. *Consider* $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ *with rank* R *and matrix representation* $\boldsymbol{X}^{[P]} = \left(\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)} \right) \left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(N)} \right)^T$. *Assume that the factor matrix* $\boldsymbol{A}^{(1)}$ *is known. If*

$$\begin{cases} r \left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(N)} \right) = R \\ r \left(\left[\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)}, -\boldsymbol{a}_r^{(1)} \otimes \boldsymbol{I}_{\prod_{n=2}^{P} I_n} \right] \right) = \prod_{n=2}^{P} I_n + R - 1 \,, \; \forall r \in \{1, \ldots, R\} \end{cases}, \tag{3.5}$$

then the CPD of $\mathcal{X}$ *is unique. Generically, condition* (3.5) *is satisfied when*

$$\min \left((I_1 - 1) \prod_{n=2}^{P} I_n, \prod_{m=P+1}^{N} I_m \right) \geq R \,. \tag{3.6}$$

Proof. Let us first prove that condition (3.5) guarantees the uniqueness of the CPD of $\mathcal{X}$ with $\mathbf{A}^{(1)}$. Note that, trivially, $\mathbf{a}_r^{(1)} \otimes \cdots \otimes \mathbf{a}_r^{(P)} \in \text{Col} \left(-\mathbf{a}_r^{(1)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n} \right)$. Condition (3.5) implies that this is the only dependency between the column vectors of $\left[\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}, -\mathbf{a}_r^{(1)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n} \right]$. Hence, we know that $r \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)} \right) = R$. Given are $\mathbf{X}^{[P]} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)} \right) \left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)} \right)^T$ and $\mathbf{A}^{(1)}$. We now know that, under condition (3.5), the matrices $\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}$ and $\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}$ have full column rank. Thus, let $\mathbf{X}^{[P]} = \mathbf{U} \Sigma \mathbf{V}^H$ denote the compact SVD of $\mathbf{X}^{[P]}$, where $\mathbf{U} \in \mathbb{C}^{(\prod_{n=1}^{P} I_n) \times R}$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R \times R}$ such that

$$\mathbf{U}\mathbf{M} = \mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)} \Leftrightarrow \mathbf{U}\mathbf{m}_r = \mathbf{a}_r^{(1)} \otimes \cdots \otimes \mathbf{a}_r^{(P)}, \quad r \in \{1, \ldots, R\} \,. \tag{3.7}$$

Relation (3.7) can also be written as

$$\mathbf{G}^{(r)} \begin{bmatrix} \mathbf{m}_r \\ \mathbf{a}_r^{(2)} \otimes \cdots \otimes \mathbf{a}_r^{(P)} \end{bmatrix} = \mathbf{0}_{\prod_{p=1}^{P} I_p} \,, \quad r \in \{1, \ldots, R\} \,, \tag{3.8}$$

where

$$\mathbf{G}^{(r)} = \left[\mathbf{U}, -\mathbf{a}_r^{(1)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n} \right] \in \mathbb{C}^{\prod_{n=1}^{P} I_n \times \left(\prod_{n=2}^{P} I_n + R \right)} . \tag{3.9}$$

Since we assume that $\mathbf{G}^{(r)}$ has a one-dimensional kernel the factor matrices $\mathbf{A}^{(1)}$ up to $\mathbf{A}^{(P)}$ follow from (3.8), as explained next. Let $\mathbf{x}_r \in \mathbb{C}^{\prod_{n=2}^{P} I_n + R}$ denote the right singular vector associated with the zero singular value of the matrix in (3.9). Set $\mathbb{C}^{\prod_{n=2}^{P} I_n} \ni \mathbf{y}_r = \mathbf{x}_r \left(R + 1 : R + \prod_{n=2}^{P} I_n \right)$, then due to relation (3.8) the factor

matrices $\left\{\mathbf{A}^{(n)}\right\}_{n=2}^{P}$ follow from the decoupled best rank-1 tensor approximation problems

$$\min_{\mathbf{a}_r^{(2)},\ldots,\mathbf{a}_r^{(P)}} \left\|\mathbf{y}_r - \mathbf{a}_r^{(2)} \otimes \cdots \otimes \mathbf{a}_r^{(P)}\right\|_F^2, \quad \forall r \in \{1,\ldots,R\}. \tag{3.10}$$

Recall that the matrix $\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}$ has full column rank. Let

$$\mathbf{F}^T = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)^{\dagger} \mathbf{X}^{[P]} = \left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)^T,$$

then the remaining factor matrices follow from the decoupled best rank-1 tensor approximation problems

$$\min_{\mathbf{a}_r^{(P+1)},\ldots,\mathbf{a}_r^{(N)}} \left\|\mathbf{f}_r - \mathbf{a}_r^{(P+1)} \otimes \cdots \otimes \mathbf{a}_r^{(N)}\right\|_F^2, \quad \forall r \in \{1,\ldots,R\}. \tag{3.11}$$

Let us now prove that condition (3.6) generically guarantees the uniqueness of the CPD of $\mathcal{X}$ with $\mathbf{A}^{(1)}$. Given are $\mathbf{X}^{[P]} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)\left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)^T$ and $\mathbf{A}^{(1)}$. Due to Lemma 3.3 we know that $\min\left((I_1-1)\prod_{n=2}^{P} I_n, \prod_{m=P+1}^{N} I_m\right) \geq R$ implies that generically the matrices $\left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)$ and $\left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)$ have full column rank. Due to Corollary 3.5 we know that if $(I_1-1)\prod_{n=2}^{P} I_n \geq R$, then the rank of $\mathbf{G}^{(r)}$ given by (3.9) is generically equal to $\prod_{n=2}^{P} I_n + R - 1$. Hence, generically we can work as in the proof of the deterministic case. □

An important remark concerning the proof of Theorem 3.6 is that it is constructive, i.e., it provides us with an algorithm to compute a CPD with a known factor. In other words, if one of the factor matrices of the CPD is known, say $\mathbf{A}^{(1)}$, and the conditions stated in Theorem 3.6 are satisfied, then even if the known factor matrix does not have full column rank (as opposed to Proposition 3.1), we can find the remaining factor matrices from decoupled best rank-1 tensor approximation problems. The algorithm is useful for applications in wireless communication and array processing, see [23, 22].

Note that if $P = 2$, then $\mathbf{A}^{(2)}$ follows directly from (3.10). Furthermore, if $P = 3$, then $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ follow from (3.10) via best rank-1 matrix approximation problems, i.e., via standard numerical linear algebra. Similarly, if $N = P+1$, then $\mathbf{A}^{(N)}$ follows directly from (3.11). Again, if $N = P+2$, then $\mathbf{A}^{(N-1)}$ and $\mathbf{A}^{(N)}$ follow from (3.11) via best rank-1 matrix approximation problems.

An outline of the proposed method for computing a CPD with a known factor matrix is given in Algorithm 1. We exploited the relation [11]

$$\left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)^{\dagger} = \left((\mathbf{A}^{(1)H}\mathbf{A}^{(1)}) * \cdots * (\mathbf{A}^{(P)H}\mathbf{A}^{(P)})\right)^{-1} (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)})^H.$$

From the first part of the proof of Theorem 3.6 we obtain the following result, which may be useful in problems where only some of the columns of the factor matrix $\mathbf{A}^{(1)}$ are known, see [23] for concrete examples in signal processing.

COROLLARY 3.7. *Consider $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R and matrix representation $\boldsymbol{X}^{[P]} = \left(\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)}\right)\left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(N)}\right)^T$. Assume that the rth column*

Algorithm 1 Outline of the procedure for computing a CPD with a known factor matrix based on Theorem 3.6.

Input: $\mathbf{X}^{[P]} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)\left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)^T$ and $\mathbf{A}^{(1)}$.

Compute SVD $\mathbf{X}^{[P]} = \mathbf{U}\Sigma\mathbf{V}^H$.

Solve $\left[\mathbf{U}, -\mathbf{a}_r^{(1)} \otimes \mathbf{I}_{\prod_{n=2}^P I_n}\right]\mathbf{x}_r = \mathbf{0}_{\prod_{p=1}^P I_p}$, $\forall r \in \{1,\ldots,R\}$.

Set $\mathbf{y}_r = \mathbf{x}_r\left(R+1 : R+\prod_{n=2}^P I_n\right), \quad \forall r \in \{1,\ldots,R\}$.

Solve $\min_{\mathbf{a}_r^{(2)},\ldots,\mathbf{a}_r^{(P)}} \left\|\mathbf{y}_r - \mathbf{a}_r^{(2)} \otimes \cdots \otimes \mathbf{a}_r^{(P)}\right\|_F^2, \quad \forall r \in \{1,\ldots,R\}$.

Compute $\mathbf{G} = \left(\left(\mathbf{A}^{(1)H}\mathbf{A}^{(1)}\right) * \cdots * \left(\mathbf{A}^{(P)H}\mathbf{A}^{(P)}\right)\right)^{-1}$.

Compute $\mathbf{F}^T = \mathbf{G}\left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)^H \mathbf{X}^{[P]}$.

Solve $\min_{\mathbf{a}_r^{(P+1)},\ldots,\mathbf{a}_r^{(N)}} \left\|\mathbf{f}_r - \mathbf{a}_r^{(P+1)} \otimes \cdots \otimes \mathbf{a}_r^{(N)}\right\|_F^2, \quad \forall r \in \{1,\ldots,R\}$.

Output: $\{\mathbf{A}^{(n)}\}_{n=2}^N$.

vector $\boldsymbol{a}_r^{(1)}$ *of the factor matrix* $\boldsymbol{A}^{(1)}$ *is known. If*

$$\begin{cases} r\left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(N)}\right) = R \\ r\left(\left[\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)}, -\boldsymbol{a}_r^{(1)} \otimes \boldsymbol{I}_{\prod_{n=2}^P I_n}\right]\right) = \prod_{n=2}^P I_n + R - 1\,, \end{cases} \tag{3.12}$$

then the column vectors $\{\boldsymbol{a}_r^{(p)}\}_{p=2}^P$ *are unique. Generically, this is true if*

$$\min\left((I_1 - 1)\prod_{n=2}^P I_n, \prod_{m=P+1}^N I_m\right) \geq R\,. \tag{3.13}$$

Let $\mathbf{X}^{[P]} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{X}^{[P]}$ and assume that the column vector $\mathbf{a}_r^{(1)}$ is known. Then numerically we first solve the homogeneous linear system

$$\left[\mathbf{U}, -\mathbf{a}_r^{(1)} \otimes \mathbf{I}_{\prod_{n=2}^P I_n}\right]\mathbf{x}_r = \mathbf{0}_{\prod_{p=1}^P I_p}\,,$$

and thereafter we obtain the column vectors $\{\mathbf{a}_r^{(p)}\}_{p=2}^P$ via the rank-1 approximation problem

$$\min_{\mathbf{a}_r^{(2)},\ldots,\mathbf{a}_r^{(P)}} \left\|\mathbf{y}_r - \mathbf{a}_r^{(2)} \otimes \cdots \otimes \mathbf{a}_r^{(P)}\right\|_F^2\,,$$

in which $\mathbf{y}_r = \mathbf{x}_r\left(R+1 : R+\prod_{n=2}^P I_n\right)$.

We now derive a relaxed version of Theorem 3.6 in which the matrices $\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}$ and $\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}$ are not required to have full column rank. We will assume that the known factor $\mathbf{A}^{(1)}$ has rank S. In that case we can w.l.o.g. assume that $r\left(\mathbf{A}^{(1)}(1:S, 1:S)\right) = S$. Indeed, if this is not the case, then there exist permutation matrices $\mathbf{P}_{I_1} \in \mathbb{C}^{I_1 \times I_1}$ and $\mathbf{P}_R \in \mathbb{C}^{R \times R}$ such that $r\left(\left(\mathbf{P}_{I_1}\mathbf{A}^{(1)}\mathbf{P}_R\right)(1:S, 1:S)\right) = S$.

THEOREM 3.8. *Consider* $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ *with rank* R *and matrix representation* $\boldsymbol{X}^{[P]} = \left(\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)}\right)\left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(N)}\right)^T$. *Denote* $S = r\left(\boldsymbol{A}^{(1)}\right)$. *Assume that* $\boldsymbol{A}^{(1)}$ *is known and* $S \geq 2$. *W.l.o.g. we also assume that* $r\left(\boldsymbol{A}^{(1)}(1:S, 1:S)\right) = S$. *Consider an integer* $m \in \{2, \ldots, S\}$ *and define*

$$\boldsymbol{A}^{(1,m,S)} = \left[\boldsymbol{a}_1^{(1)}(1:m), \ldots, \boldsymbol{a}_m^{(1)}(1:m), \boldsymbol{a}_{S+1}^{(1)}(1:m), \ldots, \boldsymbol{a}_R^{(1)}(1:m)\right] \in \mathbb{C}^{m \times (R-S+m)}$$
$$\boldsymbol{A}^{(n,m,S)} = \left[\boldsymbol{a}_1^{(n)}, \ldots, \boldsymbol{a}_m^{(n)}, \boldsymbol{a}_{S+1}^{(n)}, \ldots, \boldsymbol{a}_R^{(n)}\right] \in \mathbb{C}^{I_n \times (R-S+m)}, \quad n \in \{2, \ldots, N\}.$$

If

$$\begin{cases} r\left(\boldsymbol{A}^{(P+1,m,S)} \odot \cdots \odot \boldsymbol{A}^{(N,m,S)}\right) = R - S + m \\ r\left(\left[\boldsymbol{A}^{(1,m,S)} \odot \cdots \odot \boldsymbol{A}^{(P,m,S)}, -\boldsymbol{a}_r^{(1,m,S)} \otimes \boldsymbol{I}_{\prod_{n=2}^P I_n}\right]\right) = J + R - S + m - 1, \\ \forall r \in \{1, \ldots, R - S + m\}, \quad \textit{for some} \quad m \in \{2, \ldots, S\}, \end{cases} \tag{3.14}$$

in which $J = \prod_{n=2}^P I_n$, *then the CPD of* $\mathcal{X}$ *is unique. Generically, condition* (3.14) *is satisfied if*

$$\begin{cases} R \leq \frac{(K+S)(J-1)+S-J}{J} & \textit{when} \quad K < R, \\ R \leq (S-1)J & \textit{when} \quad K \geq R, \end{cases} \tag{3.15}$$

where $J = \prod_{n=2}^P I_n$, $K = \prod_{n=P+1}^N I_n$ *and* $S = \min(I_1, R)$.

Proof. Let us first prove that condition (3.14) guarantees the uniqueness of the CPD of $\mathcal{X}$ with $\mathbf{A}^{(1)}$. W.l.o.g. we assume that $\mathbf{A}^{(1)}(1:S, 1:S)$ is nonsingular. Observe that

$$\begin{aligned} \mathbf{B}^{(1)} &= \left(\mathbf{A}^{(1)}(1:S, 1:S)\right)^{-1} \mathbf{A}^{(1)}(1:S, :) \\ &= \left[\mathbf{I}_S, \left(\mathbf{A}^{(1)}(1:S, 1:S)\right)^{-1} \mathbf{A}(1:S, S+1:R)\right]. \end{aligned} \tag{3.16}$$

Compute

$$\mathbf{Y}^{[1]} = \left(\mathbf{A}^{(1)}(1:S, 1:S)\right)^{-1} \mathbf{X}^{[1]}(1:S, :).$$

Due to the structure of the matrix given by (3.16) we can pick $m \leq S$ rows of $\mathbf{Y}^{[1]}$ so that $(S-m)$ columns of $\mathbf{B}^{(1)}$ do not contribute. Hence, the tensor $\mathcal{Z}$ with matrix representation

$$\mathbf{Z}^{[1]} = \mathbf{Y}^{[1]}(1:m, :)$$

is composed of $(R - S + m)$ rank-1 terms. Its matrix representation $\mathbf{Z}^{[P]}$ is given by

$$\mathbf{Z}^{[P]} = \left(\bigodot_{n=1}^{P} \mathbf{A}^{(n,m,S)}\right)\left(\bigodot_{p=P+1}^{N} \mathbf{A}^{(p,m,S)}\right)^T. \tag{3.17}$$

We assume that $r\left(\left[\mathbf{A}^{(1,m,S)} \odot \cdots \odot \mathbf{A}^{(P,m,S)}, -\mathbf{a}_r^{(1,m,S)} \otimes \mathbf{I}_{\prod_{n=2}^P I_n}\right]\right) = \prod_{n=2}^P I_n + R - S + m - 1$. As explained in Theorem 3.6 this implies that $r\left(\mathbf{A}^{(1,m,S)} \odot \cdots \odot \mathbf{A}^{(P,m,S)}\right) =$

$R-S+m$, i.e., $\bigodot_{n=1}^{P} \mathbf{A}^{(n,m,S)}$ has full column rank. We also assume that $\bigodot_{p=P+1}^{N} \mathbf{A}^{(p,m,S)}$ has full column rank. Let $\mathbf{Z}^{[P]} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{Z}^{[P]}$ in which $\mathbf{U} \in \mathbb{C}^{(m\prod_{n=2}^{P} I_n)\times(R-S+m)}$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{(R-S+m)\times(R-S+m)}$ such that

$$\mathbf{UM} = \bigodot_{n=1}^{P} \mathbf{A}^{(n,m,S)}$$

or equivalently

$$\mathbf{G}^{(r,m,S)} \begin{bmatrix} \mathbf{m}_r \\ \mathbf{a}_r^{(2,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(P,m,S)} \end{bmatrix} = \mathbf{0}_{m\prod_{n=2}^{P} I_n}, \tag{3.18}$$

in which

$$\mathbf{G}^{(r,m,S)} = \left[\mathbf{U}, -\mathbf{a}_r^{(1,m,s)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n}\right] \in \mathbb{C}^{(m\prod_{n=2}^{P} I_n)\times(\prod_{n=2}^{P} I_n+R-S+m)}. \tag{3.19}$$

We assume that $\mathbf{G}^{(r,m,S)}$ has rank $\prod_{n=2}^{P} I_n + R - S + m - 1$, $\forall r \in \{1,\ldots,R-S+m\}$ which implies that we can obtain the matrices $\{\mathbf{A}^{(n,m,S)}\}_{n=2}^{P}$ from (3.18) as follows. Let $\mathbf{x}_r \in \mathbb{C}^{(\prod_{n=2}^{P} I_n+R-S+m)}$ denote the right singular vector associated with the zero singular value of the matrix given by (3.19). Set

$$\mathbb{C}^{\prod_{n=2}^{P} I_n} \ni \mathbf{y}_r = \mathbf{x}_r\left(R-S+m+1 : R-S+m+\prod_{n=2}^{P} I_n\right),$$

then due to relation (3.18) the factor matrices $\left\{\mathbf{A}^{(n,m,S)}\right\}_{n=2}^{P}$ follow from the decoupled best rank-1 tensor approximation problems

$$\min_{\mathbf{a}_r^{(2,m,S)},\ldots,\mathbf{a}_r^{(P,m,S)}} \left\|\mathbf{y}_r - \mathbf{a}_r^{(2,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(P,m,S)}\right\|_F^2, \quad r \in \{1,\ldots,R-S+m\}.$$

The next step is to determine the matrices $\{\mathbf{A}^{(n,m,S)}\}_{n=P+1}^{N}$. Let

$$\mathbf{F}^T = \left(\mathbf{A}^{(1,m,S)} \odot \cdots \odot \mathbf{A}^{(P,m,S)}\right)^{\dagger} \mathbf{Z}^{[P]} = \left(\mathbf{A}^{(P+1,m,S)} \odot \cdots \odot \mathbf{A}^{(N,m,S)}\right)^T,$$

then the factor matrices $\{\mathbf{A}^{(n,m,S)}\}_{n=P+1}^{N}$ follow from the decoupled best rank-1 tensor approximation problems

$$\min_{\mathbf{a}_r^{(P+1,m,S)},\ldots,\mathbf{a}_r^{(N,m,S)}} \left\|\mathbf{f}_r - \mathbf{a}_r^{(P+1,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(N,m,S)}\right\|_F^2, \quad r \in \{1,\ldots,R-S+m\}.$$

Finally, given $\left\{\mathbf{A}^{(n,m,S)}\right\}_{n=1}^{N}$ we determine the matrices

$$\widetilde{\mathbf{A}}^{(n,m,S)} = \left[\mathbf{a}_{m+1}^{(n)},\ldots,\mathbf{a}_S^{(n)}\right] \in \mathbb{C}^{I_n\times(S-m)}, \quad n \in \{2,\ldots,N\}.$$

Let $\widehat{\mathbf{A}}^{(1,m,S)} = \left[\mathbf{a}_1^{(1)},\ldots,\mathbf{a}_m^{(1)},\mathbf{a}_{S+1}^{(1)},\ldots,\mathbf{a}_R^{(1)}\right]$ and compute

$$\mathbf{Q}^{[1]} = \mathbf{X}^{[1]} - \widehat{\mathbf{A}}^{(1,m,S)}\left(\bigodot_{p=2}^{N} \mathbf{A}^{(p,m,S)}\right)^T = \widetilde{\mathbf{A}}^{(1,m,S)}\left(\bigodot_{p=2}^{N} \widetilde{\mathbf{A}}^{(p,m,S)}\right)^T,$$

where $\widetilde{\mathbf{A}}^{(1,m,S)} = \left[\mathbf{a}_{m+1}^{(1)}, \ldots, \mathbf{a}_{S}^{(1)}\right] \in \mathbb{C}^{I_1 \times (S-m)}$ is known. Recall that w.l.o.g. we assumed that the matrix $\mathbf{A}^{(1)}(1:S, 1:S)$ is nonsingular which in turn means that the matrix $\widetilde{\mathbf{A}}^{(1,m,S)}$ has full column rank. Hence, we obtain

$$\mathbf{H} = \mathbf{Q}^{[1]T} \left(\widetilde{\mathbf{A}}^{(1,m,S)T}\right)^{\dagger}$$

and the remaining unknowns $\{\widetilde{\mathbf{A}}^{(n,m,S)}\}_{n=2}^{N}$ now follow from the best rank-1 tensor approximation problems

$$\min_{\mathbf{a}_r^{(2,m,S)}, \ldots, \mathbf{a}_r^{(N,m,S)}} \left\| \mathbf{h}_r - \mathbf{a}_r^{(2,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(N,m,S)} \right\|_F^2 , \quad r \in \{m+1, \ldots, S\} .$$

Let us now prove that condition (3.15) generically guarantees the uniqueness of the CPD of $\mathcal{X}$ with $\mathbf{A}^{(1)}$. Let $m \in \{2, \ldots, S\}$. Due to Lemma 3.3 we know that $\mathbf{A}^{(P+1,m,S)} \odot \cdots \odot \mathbf{A}^{(N,m,S)}$ generically has full column rank if

$$K \geq R - S + m \Leftrightarrow K - R + S \geq m .$$

Since $m \in \{2, \ldots, S\}$ we obtain the following upper bound on m:

$$m \leq \min\left(K - R + S, S\right) . \tag{3.20}$$

Due to Corollary 3.5 we know that $\left[\mathbf{A}^{(1,m,S)} \odot \cdots \odot \mathbf{A}^{(P,m,S)}, -\mathbf{a}_r^{(1,m,S)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n}\right]$ generically has rank $J + R - S + m - 1$ if

$$(m-1)J \geq R - S + m \Leftrightarrow m \geq \frac{R - S + J}{J - 1} , \tag{3.21}$$

which necessitates $m > 1$. Consider first the case when $K - R + S < S$. This in turn means that $K < R$. From (3.20) and (3.21) we conclude that if the inequality

$$\frac{R - S + J}{J - 1} \leq K - R + S \Leftrightarrow R \leq \frac{(K+S)(J-1) + S - J}{J}$$

is satisfied, then generically the CPD of $\mathcal{X}$ with $\mathbf{A}^{(1)}$ is unique. Consider now the case when $K - R + S \geq S$ or equivalently $K \geq R$. In that case we know from (3.20) that $m \leq S$. Combining this condition with (3.21) we conclude that if the inequality

$$\frac{R - S + J}{J - 1} \leq S \Leftrightarrow R \leq (S-1)J$$

is satisfied, then generically the CPD of $\mathcal{X}$ with $\mathbf{A}^{(1)}$ is unique. □

We observe that the proof is constructive. An algorithm based on Theorem 3.8 is given in Algorithm 2.

Analogous to the above, the following result may be useful in signal processing applications where only some of the columns of the factor matrix $\mathbf{A}^{(1)}$ are known, see [23] for concrete examples.

COROLLARY 3.9. *Consider $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R and matrix representation $\boldsymbol{X}^{[P]} = \left(\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)}\right)\left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(N)}\right)^T$. Assume that the submatrix*

Algorithm 2 Outline of the procedure for computing a CPD with a known factor matrix $\mathbf{A}^{(1)}$ based on Theorem 3.8.

Input: $\mathbf{X}^{[P]} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}\right)\left(\mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)^T$ and

$\mathbf{A}^{(1)}$ with $S = r\left(\mathbf{A}^{(1)}\right)$ and $S = r\left(\mathbf{A}^{(1)}(1:S, 1:S)\right)$.

Choose integer $m \in \{2, \ldots, S\}$.

Compute $\mathbf{Y}^{[1]} = \left(\mathbf{A}^{(1)}(1:S, 1:S)\right)^{-1} \mathbf{X}^{[1]}$.

Set $\mathbf{Z}^{[1]} = \mathbf{Y}^{[1]}(1:m, :)$.

Build $\mathbf{Z}^{[P]}$.

Compute SVD $\mathbf{Z}^{[P]} = \mathbf{U}\Sigma\mathbf{V}^H$.

Solve $\left[\mathbf{U}, -\mathbf{a}_r^{(1,m,S)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n}\right] \mathbf{x}_r = \mathbf{0}_{m \prod_{p=2}^{P} I_p}, \quad r \in \{1, \ldots, R-S+m\}$.

Set $\mathbf{y}_r = \mathbf{x}_r\left(R-S+m+1 : R-S+m+\prod_{n=2}^{P} I_n\right)$.

Solve $\min_{\{\mathbf{a}_r^{(n,m,S)}\}_{n=2}^{P}} \left\|\mathbf{y}_r - \mathbf{a}_r^{(2,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(P,m,S)}\right\|_F^2, \quad r \in \{1, \ldots, R-S+m\}$.

Compute $\mathbf{G} = \left(\left(\mathbf{A}^{(1,m,S)H}\mathbf{A}^{(1,m,S)}\right) * \cdots * \left(\mathbf{A}^{(P,m,S)H}\mathbf{A}^{(P,m,S)}\right)\right)^{-1}$.

Compute $\mathbf{F}^T = \mathbf{G}\left(\mathbf{A}^{(1,m,S)} \odot \cdots \odot \mathbf{A}^{(P,m,S)}\right)^H \mathbf{Z}^{[P]}$.

Solve $\min_{\{\mathbf{a}_r^{(n,m,S)}\}_{n=P+1}^{N}} \left\|\mathbf{f}_r - \mathbf{a}_r^{(P+1,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(N,m,S)}\right\|_F^2, \; r \in \{1, \ldots, R-S+m\}$.

Build $\widehat{\mathbf{A}}^{(1,m,S)} = \left[\mathbf{a}_1^{(1)}, \ldots, \mathbf{a}_m^{(1)}, \mathbf{a}_{S+1}^{(1)}, \ldots, \mathbf{a}_R^{(1)}\right]$

Compute $\mathbf{Q}^{[1]} = \mathbf{X}^{[1]} - \mathbf{A}^{(1,m,S)}\left(\bigodot_{p=1}^{N} \mathbf{A}^{(p,m,S)}\right)^T$.

Build $\widetilde{\mathbf{A}}^{(1,m,S)} = \left[\mathbf{a}_{m+1}^{(1)}, \ldots, \mathbf{a}_S^{(1)}\right]$

Compute $\mathbf{H} = \mathbf{Q}^{[1]T}\left(\widetilde{\mathbf{A}}^{(1,m,S)T}\right)^{\dagger}$.

Solve $\min_{\{\mathbf{a}_r^{(n,m,S)}\}_{n=2}^{N}} \left\|\mathbf{h}_r - \mathbf{a}_r^{(2,m,S)} \otimes \cdots \otimes \mathbf{a}_r^{(N,m,S)}\right\|_F^2, \; r \in \{m+1, \ldots, S\}$.

Output: $\{\mathbf{A}^{(n)}\}_{n=2}^{N}$.

$\left[\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_Q^{(1)}\right]$ *of* $\boldsymbol{A}^{(1)}$ *is known and denote* $r\left(\left[\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_Q^{(1)}\right]\right) = Q$. *W.l.o.g. we assume that* $r\left(\boldsymbol{A}^{(1)}(1:Q, 1:Q)\right) = Q$. *Consider an integer* $m \in \{2, \ldots, Q\}$ *and define*

$$\boldsymbol{A}^{(n,m,Q)} = \left[\boldsymbol{a}_1^{(n)}, \ldots, \boldsymbol{a}_m^{(n)}, \boldsymbol{a}_{Q+1}^{(n)}, \ldots, \boldsymbol{a}_R^{(n)}\right] \in \mathbb{C}^{I_n \times (R-Q+m)}, \quad n \in \{2, \ldots, N\}$$

$$\boldsymbol{A}^{(1,m,Q)} = \left[\boldsymbol{a}_1^{(1)}(1:m), \ldots, \boldsymbol{a}_m^{(1)}(1:m), \boldsymbol{a}_{Q+1}^{(1)}(1:m), \ldots, \boldsymbol{a}_R^{(1)}(1:m)\right] \in \mathbb{C}^{m \times (R-Q+m)}.$$

If

$$\begin{cases} r\left(\boldsymbol{A}^{(P+1,m,Q)} \odot \cdots \odot \boldsymbol{A}^{(N,m,Q)}\right) = R - Q + m \\ r\left(\left[\boldsymbol{A}^{(1,m,Q)} \odot \cdots \odot \boldsymbol{A}^{(P,m,Q)}, -\boldsymbol{a}_r^{(1,m,Q)} \otimes \boldsymbol{I}_{\prod_{n=2}^{P} I_n}\right]\right) = J + R - Q + m - 1, \\ \forall r \in \{1, \ldots, R-Q+m\}, \quad \textit{for some} \quad m \in \{2, \ldots, Q\}, \end{cases} \tag{3.22}$$

in which $J = \prod_{n=2}^{P} I_n$*, then the column vectors* $\{\boldsymbol{a}_q^{(p,m,Q)}\}_{p=2}^{P}$ *are unique. Generically, condition* (3.22) *is satisfied if*

$$\begin{cases} R \leq \frac{(K+Q)(J-1)+Q-J}{J} & \text{when} \quad K < R\,, \\ R \leq (S-1)J & \text{when} \quad K \geq R\,, \end{cases} \tag{3.23}$$

where $J = \prod_{n=2}^{P} I_n$*,* $K = \prod_{n=P+1}^{N} I_n$ *and* $S = \min(I_1, R)$*.*

We compute the column vectors $\{\mathbf{a}_q^{(p,m,Q)}\}_{p=2}^{P}$ as follows. Compute $\mathbf{Y}^{[1]} = \left(\mathbf{A}^{(1)}(1:Q, 1:Q)\right)^{-1} \mathbf{X}^{[1]}$ and $\mathbf{Z}^{[1]} = \mathbf{Y}^{[1]}(1:m, :)$. Let $\mathbf{Z}^{[P]} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD $\mathbf{Z}^{[P]}$ and solve the homogenous linear system

$$\left[\mathbf{U}, -\mathbf{a}_r^{(1,m,Q)} \otimes \mathbf{I}_{\prod_{n=2}^{P} I_n}\right] \mathbf{x}_r = \mathbf{0}_{m \prod_{p=2}^{P} I_p}\,.$$

Finally, we obtain the vectors $\{\mathbf{a}_q^{(p,m,Q)}\}_{p=2}^{P}$ via the rank-1 approximation

$$\min_{\{\mathbf{a}_r^{(p,m,Q)}\}_{p=2}^{P}} \left\| \mathbf{y}_r - \mathbf{a}_r^{(2,m,Q)} \otimes \cdots \otimes \mathbf{a}_r^{(P,m,Q)} \right\|_F^2 .$$

4. New Uniqueness Results for Overall CPD. By combining the existing results presented in subsection 2.1 with the new results presented in section 3 we will in this section present new uniqueness conditions for the overall CPD and some variants that are important in signal processing.

4.1. New Uniqueness Condition for CPD. We are now ready to present a deterministic uniqueness condition for CPD that does not involve k-ranks.

THEOREM 4.1. *Consider* $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ *with rank* R *and matrix representation* $\boldsymbol{X}^{[P]} = \left(\boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)}\right)\left(\boldsymbol{A}^{(P+1)} \odot \cdots \odot \boldsymbol{A}^{(Q)} \odot \boldsymbol{A}^{(Q+1)} \odot \cdots \odot \boldsymbol{A}^{(N)}\right)^T$*, where* $P \in \{1, \ldots, N-2\}$*,* $Q \in \{2, \ldots, N-1\}$ *and* $P < Q$*. Denote* $\boldsymbol{D} = \boldsymbol{A}^{(1)} \odot \cdots \odot \boldsymbol{A}^{(P)} \in \mathbb{C}^{(\prod_{p=1}^{P} I_p) \times R}$ *and* $S = r(\boldsymbol{D})$*. Assume that* $S \geq 2$ *and w.l.o.g. assume also that* $r(\boldsymbol{D}(1:S, 1:S)) = S$*. Consider an integer* $m \in \{2, \ldots, S\}$ *and define*

$$\begin{aligned} \boldsymbol{D}^{(m,S)} &= [\boldsymbol{d}_1(1:m), \ldots, \boldsymbol{d}_m(1:m), \boldsymbol{d}_{S+1}(1:m), \ldots, \boldsymbol{d}_R(1:m)] \in \mathbb{C}^{m \times (R-S+m)} \\ \boldsymbol{A}^{(n,m,S)} &= \left[\boldsymbol{a}_1^{(n)}, \ldots, \boldsymbol{a}_m^{(n)}, \boldsymbol{a}_{S+1}^{(n)}, \ldots, \boldsymbol{a}_R^{(n)}\right] \in \mathbb{C}^{I_n \times (R-S+m)}\,, \quad n \in \{P+1, \ldots, N\}\,. \end{aligned}$$

Define also $J = \prod_{n=P+1}^{Q} I_n$*. If*

$$\begin{cases} r\left(C_{R-S+2}\left(\bigodot_{p=P+1}^{Q} \boldsymbol{A}^{(p)}\right) \odot C_{R-S+2}\left(\bigodot_{q=Q+1}^{N} \boldsymbol{A}^{(q)}\right)\right) = C_R^{R-S+2} \\ r\left(\boldsymbol{A}^{(Q+1,m,S)} \odot \cdots \odot \boldsymbol{A}^{(N,m,S)}\right) = R - S + m \\ r\left(\left[\boldsymbol{D}^{(m,S)} \odot \boldsymbol{A}^{(P+1,m,S)} \odot \cdots \odot \boldsymbol{A}^{(Q,m,S)}, -\boldsymbol{d}_r^{(m,S)} \otimes \boldsymbol{I}_J\right]\right) = J + R - S + m - 1 \\ \forall r \in \{1, \ldots, R-S+m\}\,, \quad \textit{for some} \quad m \in \{2, \ldots, S\}\,, \\ \textit{for some}\, P \in \{1, \ldots, N-2\} \;\; \textit{and} \;\; Q \in \{2, \ldots, N-1\} \;\; \textit{with} \;\; P < Q\,, \end{cases} \tag{4.1}$$

then the CPD of $\mathcal{X}$ *is unique.*

Proof. Let us treat $\mathbf{X}$ as a third-order tensor such that $\mathbf{X}^{[P]} = \mathbf{C}(\mathbf{A} \odot \mathbf{B})^T$ in which $\mathbf{C} = \mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)}$, $\mathbf{A} = \mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(Q)}$ and $\mathbf{B} = \mathbf{A}^{(Q+1)} \odot \cdots \odot \mathbf{A}^{(N)}$.

Due to Theorem 2.2, the conditions

$$\begin{cases} r\left(C_{R-S+2}\left(\bigodot_{p=P+1}^{Q}\mathbf{A}^{(p)}\right)\odot C_{R-S+2}\left(\bigodot_{q=Q+1}^{N}\mathbf{A}^{(q)}\right)\right)=C_R^{R-S+2} \\ \text{for some}\, P\in\{1,\ldots,N-2\}\ \text{and}\ Q\in\{1,\ldots,N-1\}\ \text{with}\ P<Q\,, \end{cases}$$

stated in (4.1) guarantee the uniqueness of $\mathbf{C}=\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(P)}$ which due to the rank-1 structure of the column vectors of $\mathbf{C}$ also implies uniqueness of the factor matrices $\{\mathbf{A}^{(n)}\}_{n=1}^{P}$. We now treat $\mathbf{X}^{[P]}=\mathbf{C}\left(\mathbf{A}\odot\mathbf{B}\right)^T$ as a third-order CPD with known factor $\mathbf{C}$. Due to Theorem 3.8, the conditions

$$\begin{cases} r\left(\mathbf{A}^{(Q+1,m,S)}\odot\cdots\odot\mathbf{A}^{(N,m,S)}\right)=R-S+m \\ r\left(\left[\mathbf{D}^{(m,S)}\odot\mathbf{A}^{(P+1,m,S)}\odot\cdots\odot\mathbf{A}^{(Q,m,S)},-\mathbf{d}_r^{(m,S)}\otimes\mathbf{I}_J\right]\right)=J+R-S+m-1 \\ \forall r\in\{1,\ldots,R-S+m\}\,,\quad \text{for some}\quad m\in\{2,\ldots,S\}\,, \\ \text{for some}\, P\in\{1,\ldots,N-2\}\ \text{and}\ Q\in\{2,\ldots,N-1\}\ \text{with}\ P<Q\,, \end{cases}$$

stated in (4.1) now guarantee the uniqueness of $\mathbf{A}$ and $\mathbf{B}$. Finally, due the rank-1 structure of the column vectors $\mathbf{A}$ and $\mathbf{B}$ we obtain the overall uniqueness of the CPD. □

Remark that no k-ranks are involved in the deterministic condition (4.1). No efficient method for computing the k-rank has been proposed so far. The computation of the k-rank of a matrix is in fact expected to be NP-hard, see for instance [15] and references therein for a discussion.

For a discussion of Theorem 4.1, let us first consider the generic case, where factor matrices have maximal rank and k-rank. We notice that, generically, if the one factor matrix uniqueness condition stated in Theorem 2.2 is satisfied, then the overall uniqueness condition stated in Theorem 2.4 is automatically satisfied. This means that in the generic case, the one factor matrix uniqueness condition stated in Theorem 2.2, which corresponds to the first condition in (4.1), makes that Theorem 4.1 is not more relaxed than Theorem 2.2.

We will now give an example to show that Theorem 4.1 can lead to relaxed overall uniqueness results in the non-generic case. Consider the third-order tensor $\mathcal{X}\in\mathbb{C}^{4\times4\times4}$ with matrix representation $\mathbf{X}^{[1]}=\left(\mathbf{A}\odot\mathbf{B}\right)\mathbf{C}^T$ in which $\mathbf{A}\in\mathbb{C}^{4\times5}$, $\mathbf{B}\in\mathbb{C}^{4\times5}$ and $\mathbf{C}\in\mathbb{C}^{4\times5}$, where

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 & 0 & a_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & b_{25} \\ 0 & 0 & 1 & 0 & b_{35} \\ 0 & 0 & 0 & 1 & b_{45} \end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix} 1 & 0 & 0 & 0 & c_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c_{35} \\ 0 & 0 & 0 & 1 & c_{45} \end{bmatrix}.$$

For a random choice of parameter entries we have $k\left(\mathbf{A}\right)=3$, $k\left(\mathbf{B}\right)=3$ and $k\left(\mathbf{C}\right)=3$. On the other hand, we also have $r\left(\mathbf{A}\right)=4$, $r\left(\mathbf{B}\right)=4$ and $r\left(\mathbf{C}\right)=4$. For this problem

the conditions stated in Theorem 2.4 are not satisfied, which means that Theorem 2.4 fails. However, the conditions stated in Theorem 4.3 are satisfied with $m = 3$, which implies overall uniqueness of the CPD of $\mathcal{X}$.

4.2. New Uniqueness Condition for CPD with Partial Hermitian Symmetry. We say that CPD of a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_1 \times I_3}$ is partially Hermitian symmetric if $\mathbf{X}^{[2]} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(1)*}\right) \mathbf{A}^{(3)T}$. This structure is very common in signal separation applications. For instance, tensors with partial Hermitian symmetry are obtained by stacking sets of covariance matrices of complex data, see [24, 1, 3, 4, 27, 28] for references to concrete applications. It has recently been shown in [24] that a complex CPD with a partial Hermitian symmetry is unique under milder conditions than an unconstrained CPD. The result presented in [24] was limited to the case where the matrix $\mathbf{D} = \left[\mathbf{A}^{(3)T}, \mathbf{A}^{(3)H}\right]^T$ has full column rank. We will now provide a new uniqueness condition for a complex CPD with a partial Hermitian symmetry that is even more relaxed than the one obtained in [24], in the sense that $\mathbf{D}$ is not required to have full column rank.

THEOREM 4.2. *Consider a third-order tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_1 \times I_3}$ with rank R and matrix representation $\boldsymbol{X}^{[2]} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(1)*}\right) \boldsymbol{A}^{(3)T}$, i.e., $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$. Construct*

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{A}^{(3)} \\ \boldsymbol{A}^{(3)*} \end{bmatrix}.$$

Denote $S = r(\boldsymbol{D}) \leq \min(2I_3, R)$ and assume that $S \geq 2$. W.l.o.g. we assume that $r(\boldsymbol{D}(1:S, 1:S)) = S$. Consider an integer $m \in \{2, \ldots, S\}$ and define

$$\begin{aligned} \boldsymbol{A}^{(m,S)} &= \left[\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_m^{(1)}, \boldsymbol{a}_{S+1}^{(1)}, \ldots, \boldsymbol{a}_R^{(1)}\right] \in \mathbb{C}^{I_1 \times (R-S+m)}, \\ \boldsymbol{D}^{(m,S)} &= [\boldsymbol{d}_1(1:m), \ldots, \boldsymbol{d}_m(1:m), \boldsymbol{d}_{S+1}(1:m), \ldots, \boldsymbol{d}_R(1:m)] \in \mathbb{C}^{m \times (R-S+m)}. \end{aligned}$$

If

$$\begin{cases} r\left(C_{R-S+2}\left(\boldsymbol{A}^{(1)}\right) \odot C_{R-S+2}\left(\boldsymbol{A}^{(1)*}\right)\right) = C_R^{R-S+2}, \\ r\left(\boldsymbol{A}^{(m,S)}\right) = R - S + m, \\ r\left(\left[\boldsymbol{D}^{(m,S)} \odot \boldsymbol{A}^{(m,S)}, -\boldsymbol{d}_r^{(m,S)} \otimes \boldsymbol{I}_{I_1}\right]\right) = I_1 + R - S + m - 1, \\ \forall r \in \{1, \ldots, R - S + m\}, \quad \textit{for some} \quad m \in \{2, \ldots, S\}, \end{cases} \tag{4.2}$$

then the CPD of $\mathcal{X}$ with the partial Hermitian symmetry $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is unique.

Proof. Let us define the tensor $\mathcal{Y} \in \mathbb{C}^{I_1 \times I_1 \times I_3}$ by $y_{i_1 i_2 i_3} = x^*_{i_2 i_1 i_3}$ for all indices. We have

$$\mathbf{Z}^{[2]} = \left[\mathbf{X}^{[2]}, \mathbf{Y}^{[2]}\right] = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(1)*}\right) \mathbf{D}^T \in \mathbb{C}^{I_1^2 \times 2I_3}, \tag{4.3}$$

where $\mathbf{D}^T = \left[\mathbf{A}^{(3)}, \mathbf{A}^{(3)*}\right]^T$. We may now study the uniqueness of the CPD of $\mathcal{Z}$ in order to obtain uniqueness results for $\mathcal{X}$.

Since $r(\mathbf{D}) = S \leq \min(2I_3, R)$ and $r\left(C_{R-S+2}\left(\mathbf{A}^{(1)}\right) \odot C_{R-S+2}\left(\mathbf{A}^{(1)*}\right)\right) = C_R^{R-S+2}$ we know from Theorem 2.2 that $\mathbf{D}$ is unique.

Due to Theorem 3.8, the conditions

$$\begin{cases} r\left(\mathbf{D}\right) = S \leq \min\left(2I_3, R\right) \\ r\left(\mathbf{A}^{(m,S)}\right) = R - S + m \\ r\left(\left[\mathbf{D}^{(m,S)} \odot \mathbf{A}^{(m,S)}, -\mathbf{d}_r^{(m,S)} \otimes \mathbf{I}_{I_1}\right]\right) = I_1 + R - S + m - 1\,, \\ \forall r \in \{1, \ldots, R - S + m\}\,, \quad \text{for some} \quad m \in \{2, \ldots, S\}\,, \end{cases}$$

guarantee overall CPD uniqueness.□

4.3. New Uniqueness Conditions for CPD with Columnwise Orthonormal Factor Matrix. We say that the CPD of a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ has a columnwise orthonormal factor if $\mathbf{X}^{[2]} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T}$ in which $\mathbf{A}^{(3)H}\mathbf{A}^{(3)} = \mathbf{I}_R$. It has recently been demonstrated in [25] that a complex CPD with a columnwise orthonormal factor matrix is unique under milder conditions than an unconstrained CPD. Orthogonality-constrained CPD is very common in statistical signal processing. Typically, the orthonormal columns of $\mathbf{A}^{(3)}$ model signals that are uncorrelated. See [25] for an overview and references. We will now provide a new uniqueness condition for a complex CPD with a columnwise orthonormal factor matrix.

THEOREM 4.3. *Consider a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R and matrix representation $\boldsymbol{X}^{[2]} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right)\boldsymbol{A}^{(3)T}$ in which $\boldsymbol{A}^{(3)}$ is columnwise orthonormal. Let*

$$N_{\max} = \begin{cases} 2 & \text{if} \quad I_2 \geq I_1 \\ 1 & \text{if} \quad I_1 > I_2 \end{cases}, \quad N_{\min} = \begin{cases} 2 & \text{if} \quad I_2 < I_1 \\ 1 & \text{if} \quad I_1 \leq I_2 \end{cases} \tag{4.4}$$

$$I_{\max} = \max\left(I_1, I_2\right), \quad I_{\min} = \min\left(I_1, I_2\right)\,. \tag{4.5}$$

Construct $\boldsymbol{D} = \boldsymbol{A}^{(N_{\min})*} \odot \boldsymbol{A}^{(N_{\min})} \in \mathbb{C}^{I_{\min}^2 \times R}$. Denote $S = r\left(\boldsymbol{D}\right)$ and assume that $S \geq 2$. W.l.o.g. we assume that $r\left(\boldsymbol{D}(1:S, 1:S)\right) = S$. Consider an integer $m \in \{2, \ldots, S\}$ and define

$$\boldsymbol{D}^{(m,S)} = [\boldsymbol{d}_1(1:m), \ldots, \boldsymbol{d}_m(1:m), \boldsymbol{d}_{S+1}(1:m), \ldots, \boldsymbol{d}_R(1:m)] \in \mathbb{C}^{m \times (R-S+m)},$$
$$\boldsymbol{A}^{(N_{\max},m,S)} = \left[\boldsymbol{a}_1^{(I_{N_{\max}})}, \ldots, \boldsymbol{a}_m^{(I_{N_{\max}})}, \boldsymbol{a}_{S+1}^{(I_{N_{\max}})}, \ldots, \boldsymbol{a}_R^{(I_{N_{\max}})}\right] \in \mathbb{C}^{I_{N_{\max}} \times (R-S+m)}.$$

If

$$\begin{cases} r\left(C_{R-S+2}\left(\boldsymbol{A}^{(N_{\max})*}\right) \odot C_{R-S+2}\left(\boldsymbol{A}^{(N_{\max})}\right)\right) = C_R^{R-S+2} \\ r\left(\boldsymbol{A}^{(N_{\max},m,S)}\right) = R - S + m \\ r\left(\left[\boldsymbol{D}^{(m,S)} \odot \boldsymbol{A}^{(N_{\max},m,S)}, -\boldsymbol{d}_r^{(m,S)} \otimes \boldsymbol{I}_{I_{\max}}\right]\right) = I_{\max} + R - S + m - 1 \\ \forall r \in \{1, \ldots, R - S + m\}\,, \quad \textit{for some} \quad m \in \{2, \ldots, S\}\,, \end{cases} \tag{4.6}$$

then the CPD of $\mathcal{X}$ with a columnwise orthonormal factor matrix $\boldsymbol{A}^{(3)}$ is unique.

Proof. Denote $\mathbf{X}^{(i_1 \cdot \cdot)} = \mathcal{X}(i_1, :, :)$ and construct the fourth-order tensor $\mathcal{Y} \in \mathbb{C}^{I_2 \times I_2 \times I_1 \times I_1}$ with matrix slices

$$\mathbb{C}^{I_2 \times I_2} \ni \mathbf{Y}^{(\cdot\cdot i_3, i_4)} \triangleq \mathcal{Y}\left(:, :, i_3, i_4\right) = \mathbf{X}^{(i_3 \cdot \cdot)}\mathbf{X}^{(i_4 \cdot \cdot)H}.$$

Since $\mathbf{X}^{(i_1 \cdot \cdot)} = \mathbf{A}^{(2)} D_{i_1}\left(\mathbf{A}^{(1)}\right) \mathbf{A}^{(3)T}$ and since $\mathbf{A}^{(3)}$ is columnwise orthonormal, the matrix slice $\mathbf{Y}^{(\cdot\cdot i_3, i_4)}$ admits the factorization

$$\mathbf{Y}^{(\cdot\cdot i_3, i_4)} = \mathbf{A}^{(2)} D_{i_3}\left(\mathbf{A}^{(1)}\right) D_{i_4}\left(\mathbf{A}^{(1)*}\right) \mathbf{A}^{(2)H}, \quad i_3, i_4 \in \{1, \ldots, I_1\}.$$

Thus, the fourth-order tensor $\mathcal{Y}$ has the following matrix representation

$$\begin{aligned} \mathbf{Y} &= \left[\mathrm{Vec}\left(\mathbf{Y}^{(\cdot\cdot 1,1)}\right), \mathrm{Vec}\left(\mathbf{Y}^{(\cdot\cdot 1,2)}\right), \cdots, \mathrm{Vec}\left(\mathbf{Y}^{(\cdot\cdot I_1, I_1)}\right)\right] \\ &= \left(\mathbf{A}^{(2)*} \odot \mathbf{A}^{(2)}\right)\left(\mathbf{A}^{(1)*} \odot \mathbf{A}^{(1)}\right)^T. \end{aligned} \tag{4.7}$$

If the fourth-order CPD with matrix representation $\mathbf{Y}$ in (4.7) is unique, then the CPD of $\mathcal{X}$ with columnwise orthonormal factor matrix is also unique. Let us interpret $\mathbf{Y}$ in (4.7) as a matrix representation of a CPD with factor matrices $\mathbf{A}^{(N_{\max})*}$, $\mathbf{A}^{(N_{\max})}$ and $\mathbf{A}^{(N_{\min})*} \odot \mathbf{A}^{(N_{\min})}$, in which the Khatri-Rao structure of the latter is in first instance ignored. Assuming that condition (4.6) is satisfied, we know that Theorem 2.2 guarantees the uniqueness of the factor matrix $\mathbf{A}^{(N_{\min})*} \odot \mathbf{A}^{(N_{\min})}$. Since $\mathbf{A}^{(N_{\min})*} \odot \mathbf{A}^{(N_{\min})}$ is unique, $\mathbf{A}^{(N_{\min})}$ is unique. Furthermore, since $\mathbf{A}^{(N_{\min})}$ is unique and condition (4.6) holds, Theorem 3.8 guarantees the uniqueness of the factor matrix $\mathbf{A}^{(N_{\max})}$. Finally, the columnwise orthonormal matrix $\mathbf{A}^{(3)}$ follows from $\mathbf{X}^{[2]} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right) \mathbf{A}^{(3)T}$ with $\mathbf{X}^{[2]}$, $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ known, via an orthogonal Procrustes problem [7]. □

The uniqueness condition stated in Theorem 4.3 is more relaxed than those presented in [25]. For instance, consider a third-order tensor $\mathcal{X} \in \mathbb{C}^{8\times 4\times 17}$ with matrix representation $\mathbf{X}^{[1]} = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T$ in which $\mathbf{A} \in \mathbb{C}^{8\times 17}$, $\mathbf{B} \in \mathbb{C}^{4\times 17}$ and columnwise orthonormal $\mathbf{C} \in \mathbb{C}^{17\times 17}$. We randomly generate $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. For this setting the uniqueness results presented in [25] fail. On the hand, the conditions stated in Theorem 4.3 with $m = 7$ are generically satisfied.

Uniqueness may be established beyond Theorem 4.3 and the results presented in [25]. As an example, consider the third-order tensor $\mathcal{X} \in \mathbb{C}^{4\times 2\times 5}$ with matrix representation $\mathbf{X}^{[2]} = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T$ in which $\mathbf{A} \in \mathbb{C}^{4\times 5}$, $\mathbf{B} \in \mathbb{C}^{2\times 5}$ and columnwise orthonormal $\mathbf{C} \in \mathbb{C}^{5\times 5}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & a_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \end{bmatrix}.$$

For this case Theorem 4.3 fails. However, uniqueness can be established as follows. As in Theorem 4.3 we construct a matrix $\mathbf{Y}$, which can be decomposed as $\mathbf{Y} = \mathbf{D}\left(\mathbf{A}^* \odot \mathbf{A}\right)^T$ in which $\mathbf{D} = \mathbf{B}^* \odot \mathbf{B}$. This decomposition is now interpreted as a matrix representation of a CPD with factor matrices $\mathbf{A}^*$, $\mathbf{D}$ and $\mathbf{A}$. Note that $r(\mathbf{A}) = 4$ and that generically $C_3\left(\mathbf{A}^*\right) \odot C_3\left(\mathbf{D}\right)$ has full column rank. Theorem 2.2 now generically guarantees the uniqueness of $\mathbf{A}$. Further, $\mathbf{A}^* \odot \mathbf{A}$ has generically full column rank and therefore we also obtain $\mathbf{B}$ via $\left(\mathbf{A}^* \odot \mathbf{A}\right)^\dagger \mathbf{Y}^T = \left(\mathbf{B}^* \odot \mathbf{B}\right)^T$. Finally, the columnwise orthonormal matrix $\mathbf{C}$ follows from $\mathbf{X}^{[2]} = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T$ with $\mathbf{X}^{[2]}$, $\mathbf{A}$ and $\mathbf{B}$ known, via an orthogonal Procrustes problem.

5. A New Partial Uniqueness Condition for the CPD. In [8] a partial uniqueness condition for the CPD was obtained (given below as Lemma 5.1). Partial

uniqueness means that when the overall CPD is not unique, some submatrices of the factor matrices may still be unique. In Theorem 5.2 we provide a new partial uniqueness condition for the CPD. The derivation will make use of the following result.

LEMMA 5.1. *Consider a tensor* $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ *with rank* R *and matrix representation* $\boldsymbol{X}^{[2]} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right) \boldsymbol{A}^{(3)T}$ *in which* $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n \times R}$. *Assume that* $\boldsymbol{A}^{(1)}$ *is given and let* $\boldsymbol{P} \in \mathbb{C}^{R \times R}$ *be a permutation matrix such that the partitioning*

$$\boldsymbol{A}^{(n)}\boldsymbol{P} = \left[\boldsymbol{A}^{(n,1)}, \ldots, \boldsymbol{A}^{(n,Q)}\right], \quad \boldsymbol{A}^{(n,q)} \in \mathbb{C}^{I_n \times R_q}, \quad R = \sum_{q=1}^{Q} R_q, \quad n \in \{1,2,3\} \tag{5.1}$$

has the following property

$$Col\left(\boldsymbol{A}^{(1,q)}\right) \bigcap \left(\bigcup_{\substack{p=1 \\ p \neq q}}^{Q} Col\left(\boldsymbol{A}^{(1,p)}\right) \right) = \{\boldsymbol{0}_{I_1}\}. \tag{5.2}$$

Then the pair $\left(\boldsymbol{A}^{(2,q)}, \boldsymbol{A}^{(3,q)}\right)$ *is unique up to the intrinsic ambiguities of the partial CPD with matrix representation* $\boldsymbol{X}_q = \left(\boldsymbol{A}^{(1,q)} \odot \boldsymbol{A}^{(2,q)}\right) \boldsymbol{A}^{(3,q)T}$.

Proof. A proof can be found in [8] where the result is stated as Theorem 3.1. □

Note that in the case $R_q = 1$, due to the rank-1 structure of

$$\mathbf{X}_q = \left(\mathbf{a}^{(1,q)} \otimes \mathbf{a}^{(2,q)}\right) \mathbf{a}^{(3,q)T}$$

the pair $\left(\mathbf{a}^{(2,q)}, \mathbf{a}^{(3,q)}\right)$ is obviously unique when $\mathbf{a}^{(1,q)}$ is given. Consequently, the following Theorem 5.2 will only consider the case where $R_q \geq 2$.

THEOREM 5.2. *Consider* $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ *with rank* R *and matrix representation* $\boldsymbol{X}^{[2]} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right) \boldsymbol{A}^{(3)T}$. *Let* $\boldsymbol{P} \in \mathbb{C}^{R \times R}$ *be a permutation matrix such that* $\boldsymbol{A}^{(n)}$, $n \in \{1,2,3\}$, *has the partitioning*

$$\boldsymbol{A}^{(n)}\boldsymbol{P} = \left[\boldsymbol{A}^{(n,1)}, \ldots, \boldsymbol{A}^{(n,Q)}\right], \quad \boldsymbol{A}^{(n,q)} \in \mathbb{C}^{I_n \times R_q}, \quad R = \sum_{q=1}^{Q} R_q, \tag{5.3}$$

with the following property

$$Col\left(\boldsymbol{A}^{(1,q)}\right) \bigcap \left(\bigcup_{\substack{p=1 \\ p \neq q}}^{Q} Col\left(\boldsymbol{A}^{(1,p)}\right) \right) = \{\boldsymbol{0}_{I_1}\}. \tag{5.4}$$

Denote $S = r\left(\boldsymbol{A}^{(1)}\right)$ *and* $S_q = r\left(\boldsymbol{A}^{(1,q)}\right)$. *Assume that* $S \geq 2$ *and* $S_q \geq 2$ *(and therefore also* $R_q \geq 2$*). W.l.o.g. we also assume that* $S_q = r\left(\boldsymbol{A}^{(1,q)}\right)(1 : S_q, 1 : S_q)$. *Consider an integer* $m_q \in \{2, \ldots, S_q\}$. *Define* $\boldsymbol{A}^{(1,q,m_q,S_q)} \in \mathbb{C}^{m_q \times (R_q - S_q + m_q)}$ *and*

$\boldsymbol{A}^{(n,q,m_q,S_q)} \in \mathbb{C}^{I_n \times (R_q - S_q + m_q)}$, $n \in \{2,3\}$, *as follows*

$$\begin{aligned}
\boldsymbol{A}^{(1,q,m_q,S_q)} &= \left[\boldsymbol{a}_1^{(1,q)}(1:m_q), \ldots, \boldsymbol{a}_{m_q}^{(1,q)}(1:m_q), \boldsymbol{a}_{S_q+1}^{(1,q)}(1:m_q), \ldots, \boldsymbol{a}_{R_q}^{(1,q)}(1:m_q) \right], \\
\boldsymbol{A}^{(n,q,m_q,S_q)} &= \left[\boldsymbol{a}_1^{(n,q)}, \ldots, \boldsymbol{a}_{m_q}^{(n,q)}, \boldsymbol{a}_{S_q+1}^{(n,q)}, \ldots, \boldsymbol{a}_{R_q}^{(n,q)} \right], \quad n \in \{2,3\}.
\end{aligned}$$

If

$$\begin{cases}
r\left(C_{R-S+2}\left(\boldsymbol{A}^{(2)}\right) \odot C_{R-S+2}\left(\boldsymbol{A}^{(3)}\right)\right) = C_R^{R-S+2} \\
r\left(\boldsymbol{A}^{(3,q,m_q,S_q)}\right) = R_q - S_q + m_q \\
r\left(\left[\boldsymbol{A}^{(1,q,m_q,S_q)} \odot \boldsymbol{A}^{(2,q,m_q,S_q)}, -\boldsymbol{a}_{r_q}^{(1,q,m_q,S_q)} \otimes \boldsymbol{I}_{I_2}\right]\right) = I_2 + R_q - S_q + m_q - 1,
\end{cases} \tag{5.5}$$

then $\boldsymbol{A}^{(1)}$ *and the pair* $\left(\boldsymbol{A}^{(2,q)}, \boldsymbol{A}^{(3,q)}\right)$ *are unique.*

Proof. Due to Theorem 2.2, the conditions

$$\begin{cases}
r\left(C_{R-S+2}\left(\mathbf{A}^{(2)}\right) \odot C_{R-S+2}\left(\mathbf{A}^{(3)}\right)\right) = C_R^{R-S+2} \\
S \geq 2
\end{cases}$$

stated in (5.5) guarantee the uniqueness of $\mathbf{A}^{(1)}$. Lemma 5.1 together with assumption (5.4) tells us that the study of the overall uniqueness of $\mathcal{X}$ decouples into the study of the CPDs

$$\mathbf{X}_q = \left(\mathbf{A}^{(1,q)} \odot \mathbf{A}^{(2,q)}\right) \mathbf{A}^{(3,q)T}, \quad q \in \{1, \ldots, Q\},$$

where $\mathbf{A}^{(n,q)} \in \mathbb{C}^{I_n \times R_q}$ is given by (5.3) and $\mathbf{A}^{(1,q)}$ is known. Theorem 3.8 together with the conditions

$$\begin{cases}
r\left(\mathbf{A}^{(3,q,m_q,S_q)}\right) = R_q - S_q + m_q \\
r\left(\left[\mathbf{A}^{(1,q,m_q,S_q)} \odot \mathbf{A}^{(2,q,m_q,S_q)}, -\mathbf{a}_{r_q}^{(1,q,m_q,S_q)} \otimes \mathbf{I}_{I_2}\right]\right) = I_2 + R_q - S_q + m_q - 1,
\end{cases}$$

now guarantee the uniqueness of the pair $\left(\mathbf{A}^{(2,q)}, \mathbf{A}^{(3,q)}\right)$. ∎

6. Numerical Experiments. Let $\mathcal{T} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ denote a rank-R tensor, the CPD of which we attempt to estimate from the observed tensor $\mathcal{X} = \mathcal{T} + \beta\mathcal{N}$, where $\mathcal{N}$ is an unstructured perturbation tensor and $\beta \in \mathbb{R}$ controls the noise level. The entries of all the involved factor matrices and perturbation tensors are randomly drawn from a Gaussian distribution with zero mean and unit variance. The following Signal-to-Noise Ratio (SNR) measure will be used

$$\text{SNR [dB]} = 10 \log\left(\left\| \mathbf{T}^{[1]} \right\|_F^2 / \left\| \beta \mathbf{N}^{[1]} \right\|_F^2 \right).$$

The distance between the factor matrix $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$ with $I_n \geq R$ and its estimate $\widehat{\mathbf{A}}^{(n)}$, is measured according to the following criterion:

$$\mathrm{P}\left(\mathbf{A}^{(n)}\right) = \min_{\Lambda} \left\| \mathbf{A}^{(n)} - \widehat{\mathbf{A}}^{(n)} \Lambda \right\|_F / \left\| \mathbf{A}^{(n)} \right\|_F,$$

where Λ denotes a diagonal matrix. In the case $I_n < R \leq I_n^2$ we resort to the following criterion:

$$\mathrm{Q}\left(\mathbf{A}^{(n)}\right) = \min_{\Lambda} \left\| \mathbf{A}^{(n)} \odot \mathbf{A}^{(n)} - \left(\widehat{\mathbf{A}}^{(n)} \odot \widehat{\mathbf{A}}^{(n)}\right) \Lambda \right\|_F / \left\| \mathbf{A}^{(n)} \odot \mathbf{A}^{(n)} \right\|_F ,$$

where Λ denotes a diagonal matrix.

We consider the problem of computing a CPD with a known factor matrix. We compare Algorithms 1 and 2 with the classical Alternating Least Squares (ALS) method which alternates between the update of the two unknown factor matrices. For other optimization-based algorithms [21] the conclusions are similar. Let $f\left(\widehat{\mathbf{X}}_{(k)}^{[1]}\right) = \left\| \mathbf{X}^{[1]} - \widehat{\mathbf{X}}_{(k)}^{[1]} \right\|_F$, where $\widehat{\mathbf{X}}_{(k)}^{[1]}$ denotes the estimated tensor at iteration k, then we decide that ALS method has converged when $\left| f\left(\widehat{\mathbf{X}}_{(k)}^{[1]}\right) - f\left(\widehat{\mathbf{X}}_{(k+1)}^{[1]}\right) \right| < \epsilon_{\mathrm{ALS}} = 1e{-}8$ or when the number of iterations exceeds 5000. The ALS method is randomly initialized. Since the best out of ten random initializations is used, the ALS method will be referred to as ALS-CP-10. Algorithms 1 and 2 will be referred to as Alg1 and Alg2, respectively. When Alg1 and Alg2 are followed by at most 500 ALS refinement iterations they will be referred to as Alg1-ALS and Alg2-ALS, respectively.

Note that the computational cost of the Alg1 and Alg2 methods is not much more than one ALS iteration, so they are computationally very cheap.

Case 1: $I_1, I_3 < R$ and $I_2 > R$. Let the model parameters be $I_1 = 3$, $I_2 = 10$, $I_3 = 4$ and $R = 9$. The factor matrix $\mathbf{A}^{(3)}$ is known. The mean $\mathrm{Q}\left(\mathbf{A}^{(1)}\right)$ and $\mathrm{P}\left(\mathbf{A}^{(2)}\right)$ values over 100 trials while SNR is varying from 10 to 40 dB SNR can be seen in figure 7.1. We notice that the Alg1 and Alg1-ALS methods perform better than the ALS-CP-10 method. Hence, the proposed Alg1 method, possibly followed by few ALS refinement steps, seems to be a good method for computing a CPD with a known factor matrix in which one unknown factor matrix has full column rank.

Case 2: $I_1, I_2, I_3 < R$. Let the model parameters be $I_1 = 4$, $I_2 = 6$, $I_3 = 5$ and $R = 8$. The factor matrix $\mathbf{A}^{(3)}$ is known. The mean $\mathrm{Q}\left(\mathbf{A}^{(1)}\right)$ and $\mathrm{Q}\left(\mathbf{A}^{(2)}\right)$ values over 100 trials while SNR is varying from 10 to 40 dB SNR can be seen in figure 7.2. We first notice that in the presence of noise Alg2 performs worse than ALS-CP-10. However, we also observe that Alg2-10 and ALS-CP-10 perform about the same. Since the Alg2-ALS method is also computationally cheaper than the ALS-CP-10 method, the Alg2 method seems to be a good method for initializing the ALS method. Thus, the Alg2-ALS method seems to be the method of choice for computing a CPD with a known factor in which none of the unknown factor matrices have full column rank.

7. Conclusion. Many problems in signal processing can be formulated as tensor decomposition problems with a known or partly known factor matrix. In the first part of the paper we provided a new uniqueness condition and an algorithm for computing a CPD with a known or partly known factor matrix.

Based on the results obtained in the first part of the paper, we provided in the second part of the paper a new deterministic overall uniqueness condition for the CPD that only considers the ranks of the involved factor matrices. Since no k-ranks are involved the condition should be easy to check in practice. Next, we presented a new uniqueness condition for a complex CPD with an columnwise orthonormal factor

matrix. Furthermore, we provided a new uniqueness condition for a complex CPD with a partial Hermitian symmetry. We also showed that the results presented in this paper also leads to a new partial uniqueness condition for the CPD.

Finally, numerical experiments confirmed the practical use of the proposed Algorithms 1 and 2 for computing a CPD with a known factor matrix. More precisely, in the case where one of the unknown factor matrices had full column rank, Algorithm 1 performed better than the standard ALS method. In the case where none of the unknown factor matrices had full column rank, Algorithm 2 provided at a low computational cost a good initial value for an optimization-based algorithm.

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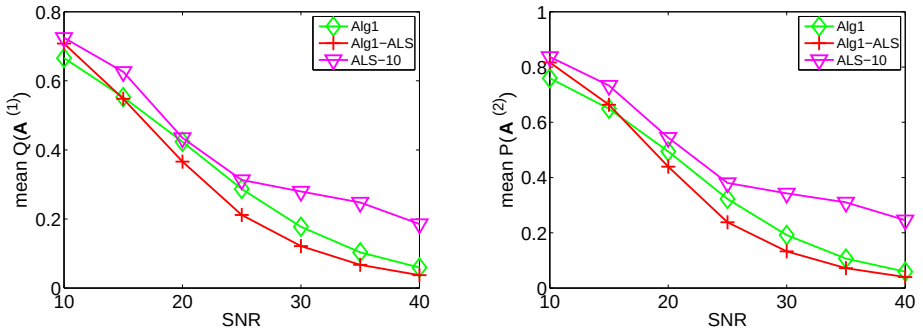
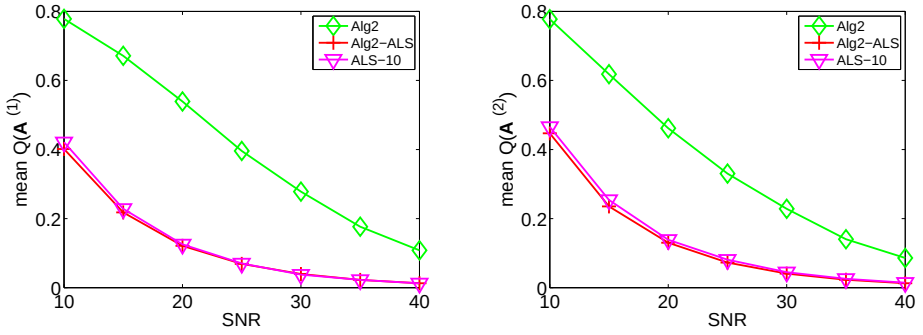


(a) Mean Q $\left(\mathbf{A}^{(1)}\right)$. (b) Mean P $\left(\mathbf{A}^{(2)}\right)$.

FIG. 7.1. *Mean* $Q\left(\boldsymbol{A}^{(1)}\right)$ *and* $P\left(\boldsymbol{A}^{(2)}\right)$ *values over 100 trials while SNR is varying from* 10 *to* 40 *dB, case 1.*



(a) Mean Q $\left(\mathbf{A}^{(1)}\right)$. (b) Mean Q $\left(\mathbf{A}^{(2)}\right)$.

FIG. 7.2. *Mean* $Q\left(\boldsymbol{A}^{(1)}\right)$ *and* $Q\left(\boldsymbol{A}^{(2)}\right)$ *values over 100 trials while SNR is varying from* 10 *to* 40 *dB, case 2.*