Multidimensional Harmonic Retrieval via Coupled Canonical Polyadic Decomposition — Part I: Model and Identifiability

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***Abstract*—Multidimensional Harmonic Retrieval (MHR) is a common problem in signal processing. Uniqueness conditions for MHR have been developed based on the Canonical Polyadic Decomposition (CPD). However, existing results do not fully exploit the rich structure of the MHR problem. As our contribution, we present a link between MHR and the recently proposed coupled CPD model. Compared to existing single CPD methods, coupled CPD allows a better exploitation of the MHR structure. This leads to an improved uniqueness condition for MHR.**

***Index Terms*—coupled canonical polyadic decomposition, tensor, Vandermonde matrix, multidimensional harmonic retrieval, multirate sampling.**

I. Introduction

During the past two decades Multidimensional Harmonic Retrieval (MHR) has become an important problem in signal processing. A link between MHR and the Canonical Polyadic Decomposition (CPD) was established in [30]. Roughly speaking, it was observed that MHR problems can be addressed as Vandermonde constrained CPD (VDM-CPD) problems. Based on the VDM-CPD, uniqueness conditions and algebraic methods for MHR have been developed in [30], [14], [20], [18], [19]. In fact, such VDM-CPD approaches do not fully exploit the structure of the MHR problem.

The authors have recently extended the CPD modeling framework to coupled models in [36], [39]. Coupled matrix/tensor decompositions are basic tools for data fusion, i.e., for the joint analysis of multiple related data sets. Data fusion has important applications in telecommunication, biomedical signal processing, chemometrics, bioinformatics, social network analysis, artificial intelligence, etc., see [36], [39], [32], [31] and references therein.

In this paper, we present a link between MHR and the coupled CPD modeling framework [36], [39]. Coupled CPD allows one to take more of the structure into account than single CPD. This leads to an improved uniqueness condition for the MHR problem. Interestingly, the coupled CPD approach can be interpreted as a generalization of the classical ESPRIT method [25] for one-dimensional (1D) Harmonic Retrieval (HR) to MHR. Part of this work appeared in the conference paper [33].

The paper is organized as follows. The rest of the introduction will present the notation used throughout the paper. Section II reviews the necessary algebraic prerequisites. In Section III we briefly discuss the connections between 1D HR, CPD and the ESPRIT method. Section IV presents a link between MHR and coupled CPD, which in turn leads to a new uniqueness condition for MHR. Section V briefly presents a link between 1D HR based on multirate sampling and coupled CPD. Section VI concludes the paper.

Notation: Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The modulus of $a \in \mathbb{C}$ is denoted by $|a|$. The rth column vector of $\mathbf{A}$ is denoted by $\mathbf{a}_r$. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao product, defined as

$$\mathbf{A} \otimes \mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathbf{A} \odot \mathbf{B} := [\mathbf{a}_1 \otimes \mathbf{b}_1 \ \mathbf{a}_2 \otimes \mathbf{b}_2 \ \dots],$$

in which $(\mathbf{A})_{mn} = a_{mn}$. We denote the Kronecker and Khatri-Rao products of N matrices $\{\mathbf{A}^{(n)}\}_{n=1}^N$ by $\bigotimes_{n=1}^N \mathbf{A}^{(n)} = \mathbf{A}^{(1)} \otimes \cdots \otimes \mathbf{A}^{(N)}$ and $\bigodot_{n=1}^N \mathbf{A}^{(n)} = \mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}$, respectively. The symbol $\mathbf{A} * \mathbf{B}$ denotes the Hadamard product, i.e., $(\mathbf{A} * \mathbf{B})_{ij} = a_{ij}b_{ij}$ and $\ast_{n=1}^N \mathbf{A}^{(n)} = \mathbf{A}^{(1)} * \mathbf{A}^{(2)} * \cdots * \mathbf{A}^{(N)}$. The Cartesian product of N sets $\{\mathbb{C}^{I_n}\}_{n=1}^N$ is denoted by $\mathbb{C}^{\overset{N}{\underset{n=1}{\times}} I_n} = \mathbb{C}^{I_1 \times \cdots \times I_N}$. The outer product of N vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ is denoted by $\mathbf{a}^{(1)} \circ \cdots \circ \mathbf{a}^{(N)} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, such that $(\mathbf{a}^{(1)} \circ \cdots \circ \mathbf{a}^{(N)})_{i_1,i_2,\dots,i_N} = a^{(1)}_{i_1} a^{(2)}_{i_2} \cdots a^{(N)}_{i_N}$.

The conjugate, transpose, conjugate-transpose and Moore-Penrose pseudo-inverse of a matrix are denoted

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by $(\cdot)^*$, $(\cdot)^T$, $(\cdot)^H$ and $(\cdot)^\dagger$, respectively. From the context it should be clear when i denotes the imaginary unit number, i.e., $i = \sqrt{-1}$.

The identity matrix and all-zero vector are denoted by $\mathbf{I}_R \in \mathbb{C}^{R\times R}$ and $\mathbf{0}_R \in \mathbb{C}^R$, respectively. The exchange matrix with unit entries on the antidiagonal and zeros elsewhere is denoted by $\mathbf{J}_I \in \mathbb{C}^{I\times I}$. Note that $\bigotimes_{n=1}^N \mathbf{J}_{I_n} = \mathbf{J}_{\prod_{n=1}^N I_n}$, in which the order of the factors does not matter.

Matlab index notation will be used for submatrices of a given matrix. For example, $\mathbf{A}(1{:}k,:)$ represents the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k of $\mathbf{A}$. $D_k(\mathbf{A}) \in \mathbb{C}^{I\times J}$ denotes the diagonal matrix holding row k of $\mathbf{A} \in \mathbb{C}^{I\times J}$ on its diagonal. Similarly, $\mathrm{Vecd}(\mathbf{A})$ denotes the column vector defined by $(\mathrm{Vecd}(\mathbf{A}))_i = (\mathbf{A})_{ii}$.

The k-th compound matrix of $\mathbf{A} \in \mathbb{C}^{I\times R}$ is denoted by $C_k(\mathbf{A}) \in \mathbb{C}^{C_I^k \times C_R^k}$, where $C_m^k = \frac{m!}{k!(m-k)!}$. It is the matrix containing the determinants of all $k\times k$ submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order. See [7] and references therein for a discussion of compound matrices.

The k-rank of a matrix $\mathbf{A}$ is denoted by $k_\mathbf{A}$. It is equal to the largest integer $k_\mathbf{A}$ such that every subset of $k_\mathbf{A}$ columns of $\mathbf{A}$ is linearly independent.

II. Algebraic prerequisites

A. Vandermonde matrices

The matrix $\mathbf{A} \in \mathbb{C}^{I\times R}$ is called Vandermonde if

$$\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_R], \quad \mathbf{a}_r = \left[1, z_r, z_r^2, \ldots, z_r^{I-1}\right]^T, \tag{1}$$

where the scalars $\{z_r\}$ are called the generators of $\mathbf{A}$.

1) Spatial smoothing: When working with Vandermonde matrices it is common to do spatial smoothing in a preprocessing step in order to make sure that all factor matrices have full column rank [28]. Consider the factorization $\mathbf{X} = \mathbf{A}\mathbf{C}^T \in \mathbb{C}^{I\times K}$ in which $\mathbf{A} \in \mathbb{C}^{I\times R}$ is of the form (1) and $\mathbf{C} \in \mathbb{C}^{K\times R}$ does not have full column rank (e.g. $K < R$). Using $z_r^{l_1+k_1-2} = z_r^{l_1-1} z_r^{k_1-1}$, spatial smoothing maps $\mathbf{X}$ to $\mathbf{Y} = \mathbf{A}^{(K_1)}(\mathbf{A}^{(L_1)} \odot \mathbf{C})^T$ as follows

$$y_{k_1,(l_1-1)L_1+k} = x_{k_1+l_1-1,k}, \quad 1 \leqslant k \leqslant K_1\,,\ 1 \leqslant l \leqslant L_1\,,$$

in which $\mathbf{A}^{(K_1)} \in \mathbb{C}^{K_1\times R}$ and $\mathbf{A}^{(L_1)} \in \mathbb{C}^{L_1\times R}$ are given by

$$\mathbf{A}^{(L_1)} = [\mathbf{a}_1^{(L_1)} \ \ldots \ \mathbf{a}_R^{(L_1)}],\ \mathbf{a}_r^{(L_1)} = [1\ z_r\ z_r^2\ \ldots, z_r^{L_1-1}]^T, \tag{2}$$

$$\mathbf{A}^{(K_1)} = [\mathbf{a}_1^{(K_1)} \ \ldots \ \mathbf{a}_R^{(K_1)}],\ \mathbf{a}_r^{(K_1)} = [1\ z_r\ z_r^2\ \ldots\ z_r^{K_1-1}]^T, \tag{3}$$

subject to $K_1 + L_1 = I + 1$. In this way one may obtain factor matrices $\mathbf{A}^{(K_1)}$ and $\mathbf{A}^{(L_1)} \odot \mathbf{C}$ that have full column rank.

2) Forward-Backward Averaging (FBA): If the generators of the Vandermonde matrix $\mathbf{A}$ are located on the unit circle ($|z_r| = 1$), then we can also make use of the FBA procedure (e.g. [26]) to deal with rank deficiency. More precisely, consider the factorization $\mathbf{X} = \mathbf{A}\mathbf{C}^T$ in which $\mathbf{A}$ is Vandermonde with generators $z_r = e^{i\alpha_r}$ where $\alpha_r \in \mathbb{R}$, $\forall r \in \{1, \ldots, R\}$. Then $\mathbf{J}_I\mathbf{A}^* = \mathbf{A}\mathbf{D}_\mathbf{A}$ in which $\mathbf{D}_\mathbf{A} = D_1\left(\left[z_1^{-(I-1)}, z_2^{-(I-1)}, \ldots, z_R^{-(I-1)}\right]\right)$. FBA now provides the augmented factorization $[\mathbf{X}, \mathbf{J}_I\mathbf{X}^*] = \mathbf{A}[\mathbf{C}^T, \mathbf{D}_\mathbf{A}\mathbf{C}^H]$. In short, FBA virtually doubles the amount of data samples.

B. Canonical Polyadic Decomposition (CPD)

Consider the third-order tensor $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$. We say that $\mathcal{X}$ is a rank-1 tensor if it is equal to the outer product of some non-zero vectors $\mathbf{a} \in \mathbb{C}^I$, $\mathbf{b} \in \mathbb{C}^J$ and $\mathbf{c} \in \mathbb{C}^K$ such that $x_{ijk} = a_i b_j c_k$. A Polyadic Decomposition (PD) is a decomposition of $\mathcal{X}$ into rank-1 terms:

$$\mathbb{C}^{I\times J\times K} \ni \mathcal{X} = \sum_{r=1}^R \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r\,. \tag{4}$$

The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then (4) is called the CPD of $\mathcal{X}$, i.e., a PD of $\mathcal{X}$ with a minimal number of terms is a CPD. Let us stack the vectors $\{\mathbf{a}_r\}$, $\{\mathbf{b}_r\}$ and $\{\mathbf{c}_r\}$ into the matrices $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_R]$, $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R]$ and $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R]$. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ will be referred to as the factor matrices of the PD or CPD of $\mathcal{X}$ in (4).

1) Matrix Representation: Let $\mathbf{X}^{(i\cdot\cdot)} \in \mathbb{C}^{J\times K}$ denote the matrix such that $\left(\mathbf{X}^{(i\cdot\cdot)}\right)_{jk} = x_{ijk}$, then $\mathbf{X}^{(i\cdot\cdot)} = \mathbf{B}D_i(\mathbf{A})\mathbf{C}^T$ and we obtain the following matrix representation of (4):

$$\mathbb{C}^{IJ\times K} \ni \mathbf{X} := \left[\mathbf{X}^{(1\cdot\cdot)T}, \ldots, \mathbf{X}^{(I\cdot\cdot)T}\right]^T = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T. \tag{5}$$

2) Uniqueness: The rank-1 tensors in (4) can be arbitrarily permuted without changing the decomposition. The vectors within the same rank-1 tensor can also be arbitrarily scaled provided that the overall rank-1 term remains the same. The CPD is said to be unique when it is only subject to the mentioned indeterminacies. The development of uniqueness conditions for the CPD has been the subject of intensive investigation, see [7], [8] and references therein. In this paper we assume that $\mathbf{C}$ in (5) has full column rank, possibly after spatial smoothing and/or FBA. For the case where one factor matrix has full column rank the following sufficient condition has been obtained.

Theorem II.1. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ in* (4)*. If*

$$\begin{cases} \boldsymbol{C} \text{ has full column rank,} & \text{(6a)} \\ C_2(\boldsymbol{A}) \odot C_2(\boldsymbol{B}) \text{ has full column rank,} & \text{(6b)} \end{cases}$$

then the rank of $\mathcal{X}$ is R and the CPD of $\mathcal{X}$ is unique [13], [6], [7]. Generically[1]*, conditions* (6a) *and* (6b) *are satisfied if $2R(R-1) \leqslant I(I-1)J(J-1)$ and $R \leqslant K$ [6], [41].*

Under (6a), condition (6b) guarantees that the only Kronecker-structured vectors in range$(\mathbf{X})$ are scaled versions of $\mathbf{a}_r \otimes \mathbf{b}_r$, $r \in \{1, \ldots, R\}$. Furthermore, under conditions (6a) and (6b), scaled versions of $\{\mathbf{a}_r \otimes \mathbf{b}_r\}$ can be computed from range$(\mathbf{X})$ in (5) via standard linear algebra (low-rank matrix approximation and EigenValue Decomposition (EVD)); the robustness of the computation can be increased via Simultaneous matrix Diagonalization (SD) techniques [6], [9].

[1] A tensor decomposition property is called generic if it holds with probability one when the entries of the factor matrices are drawn from absolutely continuous probability measures.

In the case where two factor matrices, say $\mathbf{A}$ and $\mathbf{C}$, have full column rank, Theorem II.1 simplifies to the following.

Corollary II.2. *Consider the PD of $\mathcal{X}$ in (4). Assume that $\boldsymbol{A}$ and $\boldsymbol{C}$ have full column rank. The rank of $\mathcal{X}$ is R and the CPD of $\mathcal{X}$ is unique if and only if $k_{\boldsymbol{B}} \geqslant 2$ (e.g. [16]). Generically, this is satisfied if $R \leqslant \min(I, K)$ and $2 \leqslant J$.*

Furthermore, if the conditions stated in Corollary II.2 are satisfied, then the CPD of $\mathcal{X}$ can be computed by means of an EVD (e.g. [16]).

We refer to [3], [4] and references therein for further background information about tensor decompositions.

C. Coupled CPD

We say that a collection of tensors $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n \times J_n \times K}$, $n \in \{1, \ldots, N\}$, admits an R-term coupled PD if each tensor $\mathcal{X}^{(n)}$ can be written as [36]:

$$\mathcal{X}^{(n)} = \sum_{r=1}^{R} \mathbf{a}_r^{(n)} \circ \mathbf{b}_r^{(n)} \circ \mathbf{c}_r\,, \quad n \in \{1, \ldots, N\}, \tag{7}$$

with factor matrices $\mathbf{A}^{(n)} = \left[\mathbf{a}_1^{(n)}, \ldots, \mathbf{a}_R^{(n)}\right]$, $\mathbf{B}^{(n)} = \left[\mathbf{b}_1^{(n)}, \ldots, \mathbf{b}_R^{(n)}\right]$ and $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R]$. We define the coupled rank of $\{\mathcal{X}^{(n)}\}$ as the minimal number of coupled rank-1 tensors $\mathbf{a}_r^{(n)} \circ \mathbf{b}_r^{(n)} \circ \mathbf{c}_r$ that yield $\{\mathcal{X}^{(n)}\}$ in a linear combination. Assume that the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R, then (7) will be called the coupled CPD of $\{\mathcal{X}^{(n)}\}$.

1) Matrix representation: The coupled (C)PD of $\{\mathcal{X}^{(n)}\}$ given by (7) has the following matrix representation

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \vdots \\ \mathbf{X}^{(N)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} \odot \mathbf{B}^{(1)} \\ \vdots \\ \mathbf{A}^{(N)} \odot \mathbf{B}^{(N)} \end{bmatrix} \mathbf{C}^T \in \mathbb{C}^{(\sum_{n=1}^{N} I_n J_n) \times K}. \tag{8}$$

2) Uniqueness: The coupled rank-1 tensors in (7) can be arbitrarily permuted and the vectors within the same coupled rank-1 tensor can be arbitrarily scaled provided the overall coupled rank-1 term remains the same. We say that the coupled CPD is unique when it is only subject to these trivial indeterminacies. Uniqueness conditions for the coupled CPD were derived in [36]. Theorem II.3 extends Theorem II.1 to coupled CPD.

Theorem II.3. *Consider the coupled PD of $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n \times J_n \times K}$, $n \in \{1, \ldots, N\}$ in (7). Define*

$$\boldsymbol{G} = \begin{bmatrix} C_2\left(\boldsymbol{A}^{(1)}\right) \odot C_2\left(\boldsymbol{B}^{(1)}\right) \\ \vdots \\ C_2\left(\boldsymbol{A}^{(N)}\right) \odot C_2\left(\boldsymbol{B}^{(N)}\right) \end{bmatrix} \in \mathbb{C}^{\left(\sum_{n=1}^{N} C_{I_n}^2 C_{J_n}^2\right) \times C_R^2}.$$

If

$$\begin{cases} \boldsymbol{C} \text{ has full column rank,} & \text{(9a)} \\ \boldsymbol{G} \text{ has full column rank,} & \text{(9b)} \end{cases}$$

then the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R and the coupled CPD of $\{\mathcal{X}^{(n)}\}$ is unique [36].

Under (9a), condition (9b) guarantees that only scaled versions of $[(\mathbf{a}_r^{(1)} \otimes \mathbf{b}_r^{(1)})^T, \ldots, (\mathbf{a}_r^{(N)} \otimes \mathbf{b}_r^{(N)})^T]^T$, $r \in \{1, \ldots, R\}$, are contained in range$(\mathbf{X})$. Comparing Theorem II.1 with Theorem II.3, we observe that the coupling in (7) leads to a more relaxed uniqueness condition. As in ordinary CPD, conditions (9a) and (9b) also ensure that scaled versions of $\{[(\mathbf{a}_r^{(1)} \otimes \mathbf{b}_r^{(1)})^T, \ldots, (\mathbf{a}_r^{(N)} \otimes \mathbf{b}_r^{(N)})^T]^T\}$ can be computed from range$(\mathbf{X})$ in (8) via SD techniques [33], [39].

In case at least one of the factor matrices in $\{\mathbf{A}^{(n)}\}$ also has full column rank, we may use the following Corollary II.4, which can be understood as an extension of Corollary II.2 to coupled CPD.

Corollary II.4. *Consider the coupled PD of $\{\mathcal{X}^{(n)}\}$ in (7). Consider also a subset S of $\{1, \ldots, N\}$. W.l.o.g. we assume that $S = \{1, \ldots, Q\}$. If for some $Q \leqslant N$, we have*

$$\begin{cases} \boldsymbol{C} \text{ has full column rank,} \\ \boldsymbol{A}^{(n)} \text{ has full column rank,} \quad \forall n \in S, \\ \forall r \in \{1, \ldots, R\},\ \forall s \in \{1, \ldots, R\} \backslash r,\ \exists\, n \in S:\ k_{\left[\boldsymbol{b}_r^{(n)}, \boldsymbol{b}_s^{(n)}\right]} = 2\,, \end{cases}$$

then the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R and the coupled CPD of $\{\mathcal{X}^{(n)}\}$ is unique [36].

D. CPD with a Vandermonde factor matrix

Consider the PD of $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ in (4). If $\mathbf{A}$ is Vandermonde, then this may increase the minimal R in (4). This minimal value will be denoted by $r_{\text{VDM}}(\mathcal{X})$. If some of the factor matrices are Vandermonde and $R = r_{\text{VDM}}(\mathcal{X})$, then (4) will be called a VDM-CPD of $\mathcal{X}$. In a VDM-CPD the scaling/counterscaling indeterminacies do not involve the Vandermonde factors. Besides preventing rank-deficient factor matrices, spatial smoothing can also be used to increase the diversity of a system. Let us revisit the factorization $\mathbf{X} = \mathbf{A}\mathbf{C}^T \in \mathbb{C}^{I \times K}$ in which $\mathbf{A} \in \mathbb{C}^{I \times R}$ is of the form (1). Using $z_r^{l_1 + k_1 - 2} = z_r^{l_1 - 1} z_r^{k_1 - 1}$, spatial smoothing maps the matrix $\mathbf{X}$ to the tensor $\mathcal{Y} \in \mathbb{C}^{K_1 \times L_1 \times K}$ subject to $K_1 + L_1 = I + 1$ via $y_{k_1, l_1, k} = x_{k_1 + l_1 - 1, k}$, $1 \leqslant k \leqslant K_1$, $1 \leqslant l \leqslant L_1$. Note that $\mathcal{Y}$ has matrix representation $\mathbf{Y} = (\mathbf{A}^{(K_1)} \odot \mathbf{A}^{(L_1)})\mathbf{C}^T$ in which $\mathbf{A}^{(K_1)} \in \mathbb{C}^{K_1 \times R}$ and $\mathbf{A}^{(L_1)} \in \mathbb{C}^{L_1 \times R}$ are given by (3) and (2), respectively. By going from the matrix decomposition $\mathbf{X} = \mathbf{A}\mathbf{C}^T$ to the tensor decomposition $\mathcal{Y} = \sum_{r=1}^{R} \mathbf{a}_r^{(K_1)} \circ \mathbf{a}_r^{(L_1)} \circ \mathbf{c}_r$ we have increased the order of the system by one. This property leads to the following VDM-CPD uniqueness condition [34].

Theorem II.5. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ in (4). Assume that $\boldsymbol{A}$ is a Vandermonde with generators $\{z_r\}$. If there exists a pair (K_1, L_1) subject to $K_1 + L_1 = I + 1$ such that*

$$\begin{cases} \boldsymbol{A}^{(L_1)} \odot \boldsymbol{C} \text{ has full column rank,} \\ \boldsymbol{A}^{(K_1)}(1 : K_1 - 1, :) \odot \boldsymbol{B} \text{ has full column rank,} \\ z_r \neq z_s\,, \forall r \neq s\,, \end{cases} \tag{10}$$

then $R = r_{VDM}(\mathcal{X})$ *and the VDM-CPD of* $\mathcal{X}$ *is unique. Generically*[2]*, condition* (10) *is satisfied if and only if* $\left\lceil \frac{R}{J} \right\rceil + \left\lceil \frac{R}{K} \right\rceil \leqslant I$.

Note that if $K \geqslant R$, then the generic condition in Theorem II.5 simplifies to $(I-1)J \geqslant R$.

An algebraic algorithm for computing the CPD with a Vandermonde factor matrix was provided in [34].

III. Connections between 1D HR, ESPRIT and CPD

Let us briefly review the link between the ESPRIT method for 1D HR and CPD. This will provide us with an understanding of why the coupled CPD approach introduced in section IV is a natural framework for MHR. The ESPRIT method [25], [27] was developed to solve the 1D HR problem

$$\mathbf{X} = \mathbf{A}\mathbf{C}^T \in \mathbb{C}^{I \times K}, \tag{11}$$

where $\mathbf{A} \in \mathbb{C}^{I \times R}$ is a tall ($I > R$) Vandermonde matrix with full column rank and $\mathbf{C} \in \mathbb{C}^{K \times R}$ is an unstructured matrix with full column rank. Define the row-selection matrices

$$\underline{\mathbf{I}}_P = \mathbf{I}_P(1:P-1, 1:P) \in \mathbb{C}^{(P-1)\times P}, \tag{12}$$

$$\overline{\mathbf{I}}_P = \mathbf{I}_P(2:P, 1:P) \in \mathbb{C}^{(P-1)\times P}. \tag{13}$$

Then ESPRIT works with the submatrices $\underline{\mathbf{I}}_I\mathbf{X} = \mathbf{A}(1:I-1,:)\mathbf{C}^T = \mathbf{A}^{(I-1)}\mathbf{C}^T$ and $\overline{\mathbf{I}}_I\mathbf{X} = \mathbf{A}(2:I,:)\mathbf{C}^T = \mathbf{A}^{(I-1)}D_2(\mathbf{A})\mathbf{C}^T$, where $\mathbf{A}^{(I-1)} = \mathbf{A}(1{:}I-1,:) \in \mathbb{C}^{(I-1)\times R}$ and where the shift-invariance property $\mathbf{A}(2:I,:) = \mathbf{A}^{(I-1)}D_2(\mathbf{A})$ of the Vandermonde matrix $\mathbf{A}$ was exploited.

Using that the factorizations $\underline{\mathbf{I}}_I\mathbf{X} = \mathbf{A}^{(I-1)}\mathbf{C}^T$ and $\overline{\mathbf{I}}_I\mathbf{X} = \mathbf{A}^{(I-1)}D_2(\mathbf{A})\mathbf{C}^T$ are coupled via $\mathbf{C}$, ESPRIT turns the Vandermonde constrained bilinear decomposition of $\mathbf{X} \in \mathbb{C}^{I\times K}$ in (11) into an unconstrained CPD of $\mathcal{Y} \in \mathbb{C}^{2\times(I-1)\times K}$ with matrix representation

$$\mathbf{Y} = \begin{bmatrix} \underline{\mathbf{I}}_I \cdot \mathbf{X} \\ \overline{\mathbf{I}}_I \cdot \mathbf{X} \end{bmatrix} = \left(\mathbf{A}^{(2)} \odot \mathbf{A}^{(I-1)}\right)\mathbf{C}^T \in \mathbb{C}^{2(I-1)\times K}, \tag{14}$$

where $\mathbf{A}^{(2)} = \mathbf{A}(1{:}2,:) \in \mathbb{C}^{2\times R}$. Corollary II.2 now tells us that the CPD of $\mathcal{Y}$ is unique and that in the exact case it can be computed by means of an EVD. The latter EVD is in fact the same as in the common matrix formulation of ESPRIT.

IV. New uniqueness condition for multidimensional harmonic retrieval based on the coupled CPD

It was recognized in [30] that N-dimensional HR problems can be formulated in terms of the constrained PD of a tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times M}$,

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(N)} \circ \mathbf{c}_r\,, \tag{15}$$

[2]A Vandermonde constrained (C)PD property is called generic if it holds with probability one when the entries of the unstructured factor matrices and the generators of the Vandermonde factor matrices are drawn from absolutely continuous probability measures.

with Vandermonde factor matrices $\mathbf{A}^{(n)} = \left[\mathbf{a}_1^{(n)}, \ldots, \mathbf{a}_R^{(n)}\right] \in \mathbb{C}^{I_n\times R}$ with $\mathbf{a}_r^{(n)} = \left[1, z_{r,n}, z_{r,n}^2, \ldots, z_{r,n}^{I_n-1}\right]^T$ and unstructured $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R] \in \mathbb{C}^{M\times R}$, where M is the number of snapshots and R is the number of exponentials. MHR has found a wide range of applications in traditional signal processing, such as radar, sonar, wireless communication and channel sounding, see [30], [14], [15], [20], [18], [19], [11], [34] and references therein. The Vandermonde structure can be due to Doppler effects, structured receive and/or transmit antenna arrays, small delay spread, sinusoidal carriers, frequency hopping, carrier frequency off-sets, and so on. Another classical signal processing application of MHR is multidimensional NMR spectroscopy (e.g. [17]). More recent MHR applications in signal processing include sampling of parametric nonbandlimited 2D signals [21], phase retrieval of parametric 2D signals [29], phase retrieval of multidimensional spike models [2] and antenna array design [23]. The PD of $\mathcal{X}$ in (15) has the following matrix representation

$$\mathbf{X} = (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)})\mathbf{C}^T \in \mathbb{C}^{(\prod_{n=1}^{N} I_n)\times M}. \tag{16}$$

Note that (16) is a MHR generalization of (11). The cases where $M = 1$ are referred to as single-snapshot problems while the cases where $M > 1$ are referred to as multiple snapshot problems. The goal of MHR is to recover the generators $\{z_{r,n}\}$ from the observed data tensor $\mathcal{X}$.

As our contribution, we will present a link between the MHR problem and the *coupled* CPD. Subsection IV-A presents a new sufficient uniqueness condition for MHR based on the coupled CPD. In several signal processing applications, such as Direction-Of-Arrival (DOA) estimation, the generators are located on the unit circle ($|z_{r,n}| = 1$). Subsection IV-B briefly extends the result to this special but important case. We explain that if $\mathbf{C}$ in (16) does not have full column rank, then the constraint $|z_{r,n}| = 1$ may lead to a relaxation of the presented MHR uniqueness condition. Subsection IV-C considers the special case of strictly Non-Circular (NC) signals which is relevant for communication systems.

A. Sufficient uniqueness condition for MHR

1) MHR uniqueness in cases where $\mathbf{C}$ *has full column rank:* Let us first consider multiple snapshot MHR cases where $\mathbf{C}$ has full column rank (implying $M \geqslant R$). Recall that by capitalizing on the Vandermonde structure of $\mathbf{A}$ in (11), spatial smoothing turns a 1D HR problem into a CPD. We will do this for all N dimensions of the MHR problem, overall obtaining a coupled CPD. More precisely, using $z_{n,r}^{I_n+k_n-2} = z_{n,r}^{I_n-1} z_{n,r}^{k_n-1}$, spatial smoothing in the nth dimension produces the tensor $\mathcal{X}^{(n)} \in$

$\mathbb{C}^{2\times(\times_{p=1}^{n-1}I_p)\times(I_n-1)\times(\times_{q=n+1}^{N}I_q)\times M}$ as follows

$$\begin{aligned} x^{(n)}_{k_n,i_1,\ldots,i_{n-1},l_n,i_{n+1},\ldots,i_N,m} &= x_{i_1,\ldots,i_{n-1},k_n+l_n-1,i_{n+1},\ldots,i_N,m} \\ &= \sum_{r=1}^{R}\prod_{p=1,p\neq n}^{N} a^{(p)}_{i_p,r} z_{n,r}^{k_n-1} z_{n,r}^{l_n-1} c_{m,r}\ , \end{aligned}$$

where $k_n \in \{1,2\}$ and $l_n \in \{1,\ldots,I_n-1\}$. The PD of $\mathcal{X}^{(n)}$ has the following matrix representation

$$\mathbf{X}^{(n)} = \left(\mathbf{A}^{(2,n)} \odot \mathbf{B}^{(n)}\right)\mathbf{C}^T, \tag{17}$$

where $\mathbf{A}^{(2,n)} = \mathbf{A}^{(n)}(1:2,:)$, and

$$\mathbf{B}^{(n)} = (\bigodot_{p=1}^{n-1}\mathbf{A}^{(p)}) \odot \mathbf{A}^{(I_n-1,n)} \odot (\bigodot_{p=1}^{n-1}\mathbf{A}^{(p)}), \tag{18}$$

in which $\mathbf{A}^{(I_n-1,n)} = \mathbf{A}^{(n)}(1:I_n-1,:)$.

Define the row-selection matrices

$$\underline{\mathbf{S}}^{(I_1,\ldots,I_N)}_{(n)} = \mathbf{I}_{\prod_{p=1}^{n-1}I_p} \otimes \underline{\mathbf{I}}_{I_n} \otimes \mathbf{I}_{\prod_{q=n+1}^{N}I_q}\ , \tag{19}$$

$$\overline{\mathbf{S}}^{(I_1,\ldots,I_N)}_{(n)} = \mathbf{I}_{\prod_{p=1}^{n-1}I_p} \otimes \overline{\mathbf{I}}_{I_n} \otimes \mathbf{I}_{\prod_{q=n+1}^{N}I_q}\ , \tag{20}$$

where $\underline{\mathbf{I}}_{I_n}$ and $\overline{\mathbf{I}}_{I_n}$ are of the form (12) and (13), respectively. In other words, $\underline{\mathbf{S}}^{(I_1,\ldots,I_N)}_{(n)}$ and $\overline{\mathbf{S}}^{(I_1,\ldots,I_N)}_{(n)}$ delete the rows of $\mathbf{X}$ associated with the bottom and upper row of $\mathbf{A}^{(n)}$, respectively. In the notation of (14) we now have

$$\mathbf{X}^{(n)} = \begin{bmatrix} \underline{\mathbf{S}}^{(I_1,\ldots,I_N)}_{(n)}\mathbf{X} \\ \overline{\mathbf{S}}^{(I_1,\ldots,I_N)}_{(n)}\mathbf{X} \end{bmatrix} = \left(\mathbf{A}^{(2,n)} \odot \mathbf{B}^{(n)}\right)\mathbf{C}^T, \tag{21}$$

where $\mathbf{X}$ is given by (16). Note that the matrix $\mathbf{C}$ does not depend on n, i.e., (21) represents a decomposition of the form (7). By working with the tensors $\{\mathcal{X}^{(1)},\ldots,\mathcal{X}^{(N)}\}$, coupled CPD provides a natural framework for MHR that allows us to jointly exploit the shift-invariance structure contained in all Vandermonde matrices $\{\mathbf{A}^{(n)}\}$. Theorem IV.1 below states a uniqueness condition for MHR that follows from the uniqueness of the coupled PD of the tensors $\{\mathcal{X}^{(1)},\ldots,\mathcal{X}^{(N)}\}$. It makes use of a matrix that we define for further use as

$$\mathbf{G}^{(N)} = \begin{bmatrix} C_2\left(\mathbf{A}^{(2,1)}\right) \odot C_2\left(\mathbf{B}^{(1)}\right) \\ \vdots \\ C_2\left(\mathbf{A}^{(2,N)}\right) \odot C_2\left(\mathbf{B}^{(N)}\right) \end{bmatrix}. \tag{22}$$

The theorem follows directly from application of Theorem II.3 to $\{\mathcal{X}^{(1)},\ldots,\mathcal{X}^{(N)}\}$, ignoring the Khatri-Rao structure of $\{\mathbf{B}^{(1)},\ldots,\mathbf{B}^{(N)}\}$.

Theorem IV.1. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times M}$ in* (15) *where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde. If*

$$\begin{cases} \boldsymbol{C} \text{ in } (16) \text{ has full column rank,} \\ \boldsymbol{G}^{(N)} \text{ in } (22) \text{ has full column rank,} \end{cases} \tag{23}$$

then $r_{VDM}(\mathcal{X}) = R$ and the VDM-CPD of $\mathcal{X}$ is unique.

2) MHR uniqueness in cases where $\mathbf{C}$ does not have full column rank: Using spatial smoothing in a preprocessing step, Theorem IV.1 can be generalized to cases where $\mathbf{C}$ does not have full column rank. This is for instance relevant for single-snapshot MHR where $M = 1$. Spatial smoothing can be applied on $\mathcal{X}$ in (15) to obtain a $(2N+1)$th-order tensor $\mathcal{X}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{K_1\times\cdots\times K_N\times L_1\times\cdots\times L_N\times M}$ as follows

$$\begin{aligned} x^{(K_1,\ldots,K_N)}_{k_1,\ldots,k_N,l_1,\ldots,l_N,m} &= x_{l_1+k_1-1,\ldots,l_N+k_N-1,m} \\ &= \sum_{r=1}^{R}\prod_{p=1}^{N} z_{p,r}^{k_p-1}\prod_{q=1}^{N} z_{q,r}^{l_q-1} c_{m,r}, \end{aligned}$$

where $K_n + L_n = I_n + 1$. Combining the first N modes and the last $N+1$ modes in a matrix representation $\mathbf{X}^{(K_1,\ldots,K_N)}_{\mathrm{SS}} \in \mathbb{C}^{(\prod_{n=1}^{N}K_n)\times(\prod_{n=1}^{N}L_n)M}$, we obtain

$$\mathbf{X}^{(K_1,\ldots,K_N)}_{\mathrm{SS}} = \left(\mathbf{A}^{(K_1,1)} \odot \cdots \odot \mathbf{A}^{(K_N,N)}\right)\mathbf{C}^T_{\mathrm{SS}}, \tag{24}$$

where 'SS' stands for spatial smoothing, $\mathbf{A}^{(K_n,n)} = \mathbf{A}^{(n)}(1:K_n,:) \in \mathbb{C}^{K_n\times R}$, and

$$\mathbf{C}_{\mathrm{SS}} = \mathbf{A}^{(L_1,1)} \odot \cdots \odot \mathbf{A}^{(L_N,N)} \odot \mathbf{C}, \tag{25}$$

in which $\mathbf{A}^{(L_n,n)} = \mathbf{A}^{(n)}(1:L_n,:) \in \mathbb{C}^{L_n\times R}$. With the spatial smoothing we want to make sure that $\mathbf{C}_{\mathrm{SS}}$, replacing $\mathbf{C}$, has full column rank. With the $(N+1)$th-order tensor $\mathcal{X}^{(K_1,\ldots,K_N)}_{\mathrm{SS}} \in \mathbb{C}^{K_1\times\cdots\times K_N\times(M\prod_{n=1}^{N}L_n)}$, obtained from the $(2N+1)$th-order tensor $\mathcal{X}^{(K_1,\ldots,K_N)}$ by combining the last $N+1$ modes, and represented by $\mathbf{X}^{(K_1,\ldots,K_N)}_{\mathrm{SS}}$, we can now proceed as in Subsection IV-A1. Briefly, spatial smoothing in the nth dimension yields the tensor $\mathcal{X}^{(n)}_{\mathrm{SS}} \in \mathbb{C}^{2\times(\times_{p=1}^{n-1}K_p)\times(K_n-1)\times(\times_{q=n+1}^{N}K_q)\times(\times_{s=1}^{N}L_s)\times M}$ with matrix representation $\mathbf{X}^{(n)}_{\mathrm{SS}} \in \mathbb{C}^{2(\prod_{p=1,p\neq n}^{N}K_p)(K_n-1)\times(\prod_{q=1}^{N}L_q)M}$ admitting the factorization

$$\mathbf{X}^{(n)}_{\mathrm{SS}} = \begin{bmatrix} \underline{\mathbf{S}}^{(K_1,\ldots,K_N)}_{(n)}\mathbf{X}^{(K_1,\ldots,K_N)}_{\mathrm{SS}} \\ \overline{\mathbf{S}}^{(K_1,\ldots,K_N)}_{(n)}\mathbf{X}^{(K_1,\ldots,K_N)}_{\mathrm{SS}} \end{bmatrix} = \left(\mathbf{A}^{(2,n)} \odot \mathbf{B}^{(n)}_{\mathrm{SS}}\right)\mathbf{C}^T_{\mathrm{SS}}\ , \tag{26}$$

where $\mathbf{B}^{(n)}_{\mathrm{SS}} \in \mathbb{C}^{J_n\times R}$ with $J_n = (\prod_{p=1,p\neq n}^{N}K_p)(K_n-1)$ is given by

$$\mathbf{B}^{(n)}_{\mathrm{SS}} = \bigodot_{p=1}^{n-1}\mathbf{A}^{(K_p,p)} \odot \mathbf{A}^{(K_n-1,n)} \odot \bigodot_{p=n+1}^{N}\mathbf{A}^{(K_p,p)}. \tag{27}$$

Overall, we obtain

$$\begin{bmatrix} \mathbf{X}^{(1)}_{\mathrm{SS}} \\ \vdots \\ \mathbf{X}^{(N)}_{\mathrm{SS}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(2,1)} \odot \mathbf{B}^{(1)}_{\mathrm{SS}} \\ \vdots \\ \mathbf{A}^{(2,N)} \odot \mathbf{B}^{(N)}_{\mathrm{SS}} \end{bmatrix}\mathbf{C}^T_{\mathrm{SS}} \in \mathbb{C}^{(2\sum_{n=1}^{N}J_n)\times R}. \tag{28}$$

Theorem IV.2 below extends Theorem IV.1 to cases where $\mathbf{C}$ does not have full column rank. It makes use of the following matrix

$$\mathbf{G}^{(N)}_{\mathrm{SS}} = \begin{bmatrix} C_2\left(\mathbf{A}^{(2,1)}\right) \odot C_2\left(\mathbf{B}^{(1)}_{\mathrm{SS}}\right) \\ \vdots \\ C_2\left(\mathbf{A}^{(2,N)}\right) \odot C_2\left(\mathbf{B}^{(N)}_{\mathrm{SS}}\right) \end{bmatrix} \in \mathbb{C}^{(\sum_{n=1}^{N}C^2_{J_n})\times C^2_R}. \tag{29}$$

The theorem follows directly from application of Theorem IV.1 to $\{\mathcal{X}_{\text{SS}}^{(1)},\ldots,\mathcal{X}_{\text{SS}}^{(N)}\}$.

Theorem IV.2. *Consider the PD of $\mathcal{X}\in\mathbb{C}^{I_1\times\cdots\times I_N\times M}$ in (15) where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde. If there exist pairs $\{(K_n,L_n)\}$ subject to $K_n+L_n=I_n+1$ such that*

$$\begin{cases}\boldsymbol{C}_{SS} \text{ in (25) has full column rank,}\\ \boldsymbol{G}_{SS}^{(N)} \text{ in (29) has full column rank,}\end{cases} \tag{30}$$

then $r_{VDM}(\mathcal{X})=R$ and the VDM-CPD of $\mathcal{X}$ is unique.

Note that Theorem IV.2 does not prevent that $k_{\mathbf{A}^{(1)}}=k_{\mathbf{A}^{(2)}}=\cdots=k_{\mathbf{A}^{(N)}}=k_{\mathbf{C}}=1$, i.e., it can handle problems in which the factor matrices contain collinear columns. However, since $C_2\left(\mathbf{A}^{(2,n)}\right)=[z_{2,n}-z_{1,n},z_{3,n}-z_{1,n},\ldots,z_{R,n}-z_{R-1,n}]$, it is also clear that condition (30) requires that for every $r\in\{1,\ldots,R\}$ there exists an $n\in\{1,\ldots,N\}$ and an $s\in\{1,\ldots,R\}$ such that $z_{r,n}\neq z_{s,n}$. This is not a limitation of Theorem IV.2 but an obvious necessary requirement for MHR uniqueness.

3) Coupled CPD leads to relaxed MHR uniqueness condition: We will demonstrate that Theorem IV.2 yields a more relaxed uniqueness condition than Theorem II.5 does in the case of MHR. Let us first explain how MHR uniqueness can be established via Theorem II.5. Note that the factorization of each $\mathbf{X}_{\text{SS}}^{(n)}$ in (28), and given by (26), corresponds to a matrix representation of a Vandermonde constrained CPD. Assume that condition (10) in Theorem II.5 is satisfied for $\mathbf{X}_{\text{SS}}^{(n)}$ in (28) for some $n\in\{1,\ldots,N\}$. Then $\mathbf{A}^{(2,n)}$, $\mathbf{B}_{\text{SS}}^{(n)}$ and $\mathbf{C}_{\text{SS}}$ are unique and $\mathbf{C}_{\text{SS}}$ has full column rank. Since $\mathbf{C}_{\text{SS}}$ has full column rank, the matrices $\{\mathbf{A}^{(2,n)},\mathbf{B}_{\text{SS}}^{(n)}\}$ can be uniquely determined via $\mathbf{X}_{\text{SS}}^{(n)}(\mathbf{C}_{\text{SS}}^{T})^{\dagger}=\mathbf{A}^{(2,n)}\odot\mathbf{B}_{\text{SS}}^{(n)}$, $\forall n\in\{1,\ldots,N\}$, implying MHR uniqueness. We now explain that condition (10) implies condition (30). Condition (10) also implies that $z_{r,n}\neq z_{s,n}$, $\forall r\neq s$, and that $\mathbf{B}_{\text{SS}}^{(n)}$ has full column rank. This is equivalent to $C_2(\mathbf{A}^{(2,n)})\odot C_2(\mathbf{B}_{\text{SS}}^{(n)})$ having full column rank. Indeed, since the generators of $\mathbf{A}^{(2,n)}$ are assumed distinct, we know that $C_2(\mathbf{A}^{(2,n)})$ is a row-vector with only non-zero entries and since $\mathbf{B}_{\text{SS}}^{(n)}$ is assumed to have full column rank we also know that $C_2(\mathbf{B}_{\text{SS}}^{(n)})$ has full column rank [40]. This means that the submatrix $C_2(\mathbf{A}^{(2,n)})\odot C_2(\mathbf{B}_{\text{SS}}^{(n)})$ of $\mathbf{G}_{\text{SS}}^{(N)}$ has full column rank, which in turn implies that $\mathbf{G}_{\text{SS}}^{(N)}$ has full column rank. We conclude that if condition (10) in Theorem II.5 is satisfied for some $n\in\{1,\ldots,N\}$, then the condition (30) in Theorem IV.2 is automatically satisfied. However, the converse is not true.

4) Efficiently checking the condition: Let us now explain how to efficiently check the conditions in Theorem IV.2 (and in Theorem IV.1) when the dimensions $\{I_1,\ldots,I_N,M\}$ are large. Recall that $\mathbf{G}_{\text{SS}}^{(N)}\in\mathbb{C}^{(\sum_{n=1}^{N}C_{J_n}^2)\times C_R^2}$ in (29) has full column rank if and only if $\mathbf{G}_{\text{SS}}^{(N)H}\mathbf{G}_{\text{SS}}^{(N)}\in\mathbb{C}^{C_R^2\times C_R^2}$ is nonsingular. Due to the block-wise Khatri-Rao structure of $\mathbf{G}_{\text{SS}}^{(N)}$, the latter condition can be checked efficiently, as explained next.

Using the properties $C_2(\mathbf{A})^H=C_2(\mathbf{A}^H)$, $C_2(\mathbf{A})C_2(\mathbf{B})=C_2(\mathbf{AB})$ and $(\mathbf{A}\odot\mathbf{B})^H(\mathbf{A}\odot\mathbf{B})=(\mathbf{A}^H\mathbf{A})*(\mathbf{B}^H\mathbf{B})$ we observe that

$$\begin{aligned}&\mathbf{G}_{\text{SS}}^{(N)H}\mathbf{G}_{\text{SS}}^{(N)}\\&=\sum_{n=1}^{N}\left(C_2(\mathbf{A}^{(2,n)})^HC_2(\mathbf{A}^{(2,n)})\right)*\left(C_2(\mathbf{B}_{\text{SS}}^{(n)})^HC_2(\mathbf{B}_{\text{SS}}^{(n)})\right)\\&=\sum_{n=1}^{N}D_1(C_2(\mathbf{A}^{(2,n)}))C_2(\mathbf{B}_{\text{SS}}^{(n)H}\mathbf{B}_{\text{SS}}^{(n)})D_1(C_2(\mathbf{A}^{(2,n)*})),\end{aligned} \tag{31}$$

where we also exploited that $C_2(\mathbf{A}^{(2,n)})$ is a row-vector. Using the identity $(\mathbf{A}\odot\mathbf{B})^H(\mathbf{A}\odot\mathbf{B})=(\mathbf{A}^H\mathbf{A})*(\mathbf{B}^H\mathbf{B})$ we also note that $\mathbf{B}_{\text{SS}}^{(n)H}\mathbf{B}_{\text{SS}}^{(n)}=(\circledast_{m=1,m\neq n}^{N}\widetilde{\mathbf{A}}^{(K_m,m)})*\widetilde{\mathbf{A}}^{(K_n-1,n)}$, in which $\widetilde{\mathbf{A}}^{(m,n)}=\mathbf{A}^{(m,n)H}\mathbf{A}^{(m,n)}$. Due to the Vandermonde structure of $\mathbf{A}^{(m,n)}$, the entries of the matrix $\mathbf{B}_{\text{SS}}^{(n)H}\mathbf{B}_{\text{SS}}^{(n)}$ can easily be determined as $(\mathbf{B}_{\text{SS}}^{(n)H}\mathbf{B}_{\text{SS}}^{(n)})_{ij}=\frac{1-(x_{in}^*x_{jn})^{K_n}}{1-x_{in}^*x_{jn}}\prod_{m=1,m\neq n}^{N}(\frac{1-(x_{im}^*x_{jm})^{(K_m+1)}}{1-x_{im}^*x_{jm}})$. It is now clear that checking if $\mathbf{G}_{\text{SS}}^{(N)}$ has full column rank can efficiently be done by simply checking if $\mathbf{G}_{\text{SS}}^{(N)H}\mathbf{G}_{\text{SS}}^{(N)}$ is nonsingular. This is possible even for large dimensions $\{I_1,\ldots,I_N,M\}$.

In cases where R is very large, the construction of the $(C_R^2\times C_R^2)$ matrix $\mathbf{G}_{\text{SS}}^{(N)H}\mathbf{G}_{\text{SS}}^{(N)}$ nevertheless becomes difficult. In such cases we may instead resort to the following MHR variant of Corollary II.4, which can easily be checked for large R.

Corollary IV.3. *Consider the PD of $\mathcal{X}\in\mathbb{C}^{I_1\times\cdots\times I_N\times M}$ in (15) where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde. Consider also a subset S of $\{1,\ldots,N\}$. W.l.o.g. we assume that $S=\{1,\ldots,Q\}$. If there exist an integer $Q\leqslant N$ and pairs $\{(K_n,L_n)\}$ subject to $K_n+L_n=I_n+1$ such that*

$$\begin{cases}\boldsymbol{C}_{SS} \text{ in (25) has full column rank,} & \text{(32a)}\\ \boldsymbol{B}_{SS}^{(n)} \text{ in (27) has full column rank,}\quad \forall n\in S, & \text{(32b)}\\ \forall r\in\{1,\ldots,R\},\forall s\in\{1,\ldots,R\}\backslash r,\exists n\in S\colon\ z_{r,n}\neq z_{s,n}, & \text{(32c)}\end{cases}$$

then $r_{VDM}(\mathcal{X})=R$ and the VDM-CPD of $\mathcal{X}$ is unique.

The conditions (32a) and (32b) can easily be verified by simply checking the nonsingularity of the $(R\times R)$ matrices $\mathbf{C}_{\text{SS}}^H\mathbf{C}_{\text{SS}}=(\circledast_{n=1}^{N}\widetilde{\mathbf{A}}^{(L_n,n)})*(\mathbf{C}^H\mathbf{C})$ and $\mathbf{B}_{\text{SS}}^{(n)H}\mathbf{B}_{\text{SS}}^{(n)}=(\circledast_{m=1,m\neq n}^{N}\widetilde{\mathbf{A}}^{(K_m,m)})*\widetilde{\mathbf{A}}^{(K_n-1,n)}$, respectively.

Obviously Corollary IV.3 is more restrictive than Theorem IV.2 since it requires the matrices in condition (32b) to have full column rank. However, Corollary IV.3 is more relaxed than Theorem II.5 since it does not require a mode in which all generators are distinct, as expressed by condition (32c).

5) Generic MHR uniqueness results: Generically $z_{r,n}\neq z_{s,n},\forall r\neq s,\forall n\in\{1,\ldots,N\}$. Consequently, Corollary IV.3 yields the same generic bound on R as Theorem II.5. However, Theorem IV.2 yields improved generic bounds on R. Let us compare Theorem IV.2 with a result obtained in [14] for 2D HR ($N=2$) in which $M\geqslant R$. It was explained in [14] that 2D HR uniqueness is generically guaranteed if $I_1I_2-\min(I_1,I_2)\geqslant R$ and $M\geqslant R$. Note

that this condition coincides with the generic condition in Theorem II.5. This can easily be seen by replacing $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ in Theorem II.5 with $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{C})$ in which the Vandermonde structure of $\mathbf{A}^{(2)}$ is ignored. For this reason the condition is in fact not better than conditions that can be obtained via CPD with only one Vandermonde factor matrix (e.g. Theorem II.5). However, by exploiting that both $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are Vandermonde, improved generic bounds on R can be obtained. We use the following tool for checking generic conditions to demonstrate this claim.

Lemma IV.4. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be an analytic function. If there exists an element $\boldsymbol{x} \in \mathbb{C}^n$ such that $f(\boldsymbol{x}) \neq 0$, then the set $\{\, \boldsymbol{x} \,|\, f(\boldsymbol{x}) = 0 \,\}$ is of Lebesgue measure zero (e.g. [10]).*

Recall that an $I \times R$ matrix has full column rank R if it has a non-vanishing $R \times R$ minor. A minor is an analytic function (namely, it is a polynomial in several variables). Lemma IV.4 implies that it suffices to numerically check condition (30) for a matrix $\mathbf{C}_{\text{SS}}$ and a matrix $\mathbf{G}_{\text{SS}}^{(N)}$ built from random Vandermonde matrices $\{\mathbf{A}^{(n)}\}$ and a random matrix $\mathbf{C}$, leading to a generic bound on R.

Table I (left) indicates the maximum value for R for which Theorems II.5 and IV.2 generically guarantee uniqueness for 2D HR with $4 \leqslant I_1, I_2 \leqslant 8$ and $R \leqslant M$. Theorem II.5 tells us that if we only exploit that either $\mathbf{A}^{(1)}$ or $\mathbf{A}^{(2)}$ is Vandermonde, then R in (15) is bounded by $I_1 I_2 - \min(I_1, I_2) \geqslant R$. This encompasses the bounds obtained in [14], [18], [19]. In the cases listed in Table I, Theorem IV.2 relaxes the bound on R in (15) to $I_1 I_2 - 3 \geqslant R$.

Table I (right) considers single snapshot 2D HR cases ($M = 1$) in which $9 \leqslant I_1 = I_2 \leqslant 11$. Here the choice of spatial smoothing parameters (K_1, K_2) affects the bound. Consider the example $(N, I_1, I_2, M) = (2, 9, 9, 1)$. If we choose $K_1 = 4$ and $K_2 = 6$, then Theorem IV.2 yields the upper bound $R \leqslant 21$. On the other hand, if we choose $K_1 = K_2 = 5$, then Theorem IV.2 relaxes the upper bound to $R \leqslant 22$, as indicated in the table.

To summarize, the generic bounds on R for MHR in [14], [18], [19] are not better than the one for VDM-CPD involving a single Vandermonde factor (e.g. Theorem II.5). We have demonstrated that this bound is not necessary for MHR. In fact, by better exploiting the MHR structure, the bound can be slightly relaxed in the 2D HR case. In the following subsection we show that the relaxation can be more pronounced for HR problems of order greater than two.

B. Vandermonde matrices with unit norm generators

Let us now consider the special case where $|z_{r,n}| = 1$, $\forall r \in \{1, \ldots, R\}$, $\forall n \in \{1, \ldots, N\}$. As explained in Subsection II-A2, FBA can be used to virtually increase the number of snapshots. More precisely, FBA yields

$$\mathbf{J}_{\prod_{n=1}^N I_n} \mathbf{X}^* = (\bigodot_{n=1}^{N} \mathbf{A}^{(n)}) \left(\mathbf{D}_{\mathbf{A}^{(1)}} \mathbf{D}_{\mathbf{A}^{(2)}} \cdots \mathbf{D}_{\mathbf{A}^{(N)}} \right) \mathbf{C}^H, \quad (33)$$

where $\mathbf{D}_{\mathbf{A}^{(n)}} = D_1 \left(\left[z_{1,n}^{-(I_n-1)}, z_{2,n}^{-(I_n-1)}, \ldots, z_{R,n}^{-(I_n-1)} \right] \right)$. Observe that the factor matrices $\{\mathbf{A}^{(n)}\}_{n=1}^N$ appear in both the given data matrix $\mathbf{X}$ and the virtual data matrix $\mathbf{J}_{\prod_{n=1}^N I_n} \mathbf{X}^*$. Concatenating $\mathbf{X}$ and $\mathbf{J}_{\prod_{n=1}^N I_n} \mathbf{X}^*$ yields

$$\widehat{\mathbf{X}} = \left[\mathbf{X}, \mathbf{J}_{\prod_{n=1}^N I_n} \mathbf{X}^* \right] = (\bigodot_{n=1}^{N} \mathbf{A}^{(n)}) \widehat{\mathbf{C}}^T \in \mathbb{C}^{(\prod_{n=1}^N I_n) \times 2M}, \quad (34)$$

where $\widehat{\mathbf{C}} = \left[\mathbf{C}^T, \left(\mathbf{D}_{\mathbf{A}^{(1)}} \mathbf{D}_{\mathbf{A}^{(2)}} \cdots \mathbf{D}_{\mathbf{A}^{(N)}} \right) \mathbf{C}^H \right]^T \in \mathbb{C}^{2M \times R}$. Compared to the general case (16), the unit-norm variant involves the global matrix $\widehat{\mathbf{C}}$. Obviously, $\widehat{\mathbf{C}}$ has more easily full column rank than its submatrix $\mathbf{C}$. Generically, the rank of $\widehat{\mathbf{C}}$ is $\min(2M, R)$ while the rank of $\mathbf{C}$ is only $\min(M, R)$, see for instance [38]. In this way FBA can lead to improved MHR uniqueness in cases where $\mathbf{C}$ does not have full column rank. Substituting $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{C}}$ in (26), we obtain:

$$\widehat{\mathbf{X}}_{\text{SS}}^{(n)} = \left(\mathbf{A}^{(2,n)} \odot \mathbf{B}_{\text{SS}}^{(n)} \right) \widehat{\mathbf{C}}_{\text{SS}}^T, \quad (35)$$

where $\mathbf{B}_{\text{SS}}^{(n)}$ is of the form (27) and

$$\widehat{\mathbf{C}}_{\text{SS}} = \mathbf{A}^{(L_1,1)} \odot \cdots \odot \mathbf{A}^{(L_N,N)} \odot \widehat{\mathbf{C}}, \quad (36)$$

and the following variant of Theorem IV.2.

Theorem IV.5. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times M}$ in (15) where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde. If $|z_{r,n}| = 1$, $\forall r \in \{1, \ldots, R\}$, $\forall n \in \{1, \ldots, N\}$, and there exist pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$ such that*

$$\begin{cases} \widehat{\boldsymbol{C}}_{SS} \text{ in (36) has full column rank,} \\ \boldsymbol{G}_{SS}^{(N)} \text{ in (29) has full column rank,} \end{cases} \quad (37)$$

then $r_{VDM}(\mathcal{X}) = R$ and the Vandermonde constrained CPD of $\mathcal{X}$ is unique.

Let us compare Theorem IV.5 with results obtained in the literature for MHR with unit norm generators [18], [19]. It was explained in [18], [19] that uniqueness is generically guaranteed if $R \leqslant \min\left((K_1 - 1) \prod_{n=2}^N K_n, 2M \prod_{n=1}^N L_n \right)$ in which the roles of $(K_1, L_1), \ldots, (K_N, L_N)$ can be interchanged. This condition means that for some $n \in \{1, \ldots, N\}$ the matrices $\mathbf{B}_{\text{SS}}^{(n)}$ and $\widehat{\mathbf{C}}_{\text{SS}}$ in (35) generically have full column rank and $\mathbf{A}^{(2,n)}$ in (35) generically does not contain any collinear columns. Consequently, this condition is in fact not better than conditions that can be obtained for CPD with only one Vandermonde factor matrix (e.g. Theorem II.5) applied on $\widehat{\mathbf{X}}_{\text{SS}}^{(n)}$ in (35). However, by simultaneously exploiting that all $\{\mathbf{A}^{(n)}\}$ are Vandermonde, improved uniqueness conditions are obtained in the coupled CPD framework. Table II indicates the maximum value of R for which Theorems II.5 and IV.5 generically guarantee uniqueness for different 3D and 4D HR configurations, including both multiple and single snapshot cases.

M	$\geqslant R$					$\geqslant R$				$\geqslant R$			$\geqslant R$		$\geqslant R$	1	1	1
$I_{\min}$	4					5				6			7		8	9	10	11
$I_{\max}$	4	5	6	7	8	5	6	7	8	6	7	8	7	8	8	9	10	11
Theorem II.5	12	16	20	24	28	20	25	30	35	30	36	42	42	49	56	20	25	30
Theorem IV.2	13	17	21	25	29	22	27	32	37	33	39	45	46	53	61	22	27	33

TABLE I

MAXIMUM VALUE FOR R IN (15) FOR WHICH THEOREMS II.5 AND IV.2 GENERICALLY GUARANTEE UNIQUENESS. (LEFT) MULTIPLE SNAPSHOT 2D HR ($M \geqslant R$). (RIGHT) SINGLE SNAPSHOT 2D HR ($M = 1$). THEOREM II.5 EXPLOITS HARMONIC STRUCTURE IN ONE MODE; THEOREM IV.2 EXPLOITS HARMONIC STRUCTURE IN BOTH MODES. WE DENOTE $I_{\min} = \min(I_1, I_2)$ AND $I_{\max} = \max(I_1, I_2)$.

M	$\geqslant \lceil R/2 \rceil$						1	1	$\geqslant \lceil R/2 \rceil$	$\geqslant \lceil R/2 \rceil$	$\geqslant \lceil R/2 \rceil$	$\geqslant \lceil R/2 \rceil$	$\geqslant \lceil R/2 \rceil$
I_4	1						1	1	2	3	3	3	3
(I_1, I_2, I_3)	(2,3,3)	(3,3,3)	(3,3,4)	(3,4,4)	(4,4,4)	(4,4,5)	(5,5,5)	(6,6,6)	(2,2,2)	(2,2,3)	(2,3,3)	(3,3,3)	(3,3,3)
Theorem II.5	12	18	27	36	48	64	27	48	8	16	24	36	54
Theorem IV.5	14	23	31	43	59	75	31	54	11	18	30	47	73

TABLE II

MAXIMUM VALUE FOR R IN (15) FOR WHICH THEOREMS II.5 AND IV.5 GENERICALLY GUARANTEE UNIQUENESS. (LEFT) 3D UNIT NORM HR ($M \geqslant \lceil R/2 \rceil$ OR $M = 1$). (RIGHT) 4D UNIT NORM HR ($M \geqslant \lceil R/2 \rceil$). THEOREM II.5 EXPLOITS HARMONIC STRUCTURE IN ONE MODE; THEOREM IV.2 EXPLOITS HARMONIC STRUCTURE IN ALL THREE OR FOUR MODES.

C. Strictly non-circular signals and unit norm generators

As in [42] we consider the case of strictly NC signals. Strictly NC signal matrices $\mathbf{S} \in \mathbb{C}^{K \times R}$ admit the factorization $\mathbf{S} = \mathbf{S}_0 \mathbf{D}_\mathbf{S}$, where $\mathbf{S}_0 \in \mathbb{R}^{K \times R}$ and $\mathbf{D}_\mathbf{S} = D_1\left([e^{i\varphi_1}, \ldots, e^{i\varphi_R}]\right)$ contains the phase shifts $\varphi_r \in \mathbb{R}$. The problem where additionally the generators of the Vandermonde matrices $\{\mathbf{A}^{(n)}\}$ in (15) are located on the unit circle appears in communication systems (e.g. [42]).

Recall that the result in subsection IV-B was obtained by a *horizontal* stacking of the observed data $\mathbf{X}$ and $\mathbf{J}_{\prod_{n=1}^N I_n}\mathbf{X}^*$, yielding the global $\widehat{\mathbf{X}}$ in (34). Here, as an alternative to Theorem IV.5, we explain that by capitalizing on the NC property $\mathbf{S} = \mathbf{S}_0\mathbf{D}_\mathbf{S}$ it is also possible to obtain a result by a *vertical* stacking of the observed data. Due to relation $\mathbf{S} = \mathbf{S}_0\mathbf{D}_\mathbf{S}$ we obtain from (26) the additional tensors $\mathcal{Y}_{\text{SS}}^{(n)} \in \mathbb{C}^{2\times(\times_{p=1}^{n-1}K_p)\times(K_n-1)\times(\times_{q=n+1}^{N}K_q)\times(\times_{s=1}^{N}L_s)\times M}$, $n \in \{1, \ldots, N\}$, subject to $K_n + L_n = I_n + 1$ and with matrix representation $\mathbf{Y}_{\text{SS}}^{(n)} \in \mathbb{C}^{2(\prod_{p=1,p\neq n}^{N}K_p)(K_n-1)\times(\prod_{q=1}^{N}L_q)M}$:

$$\begin{aligned}\mathbf{Y}_{\text{SS}}^{(n)} &= \mathbf{J}_{2\prod_{p=1,p\neq n}^{N}K_p(K_n-1)}\mathbf{X}_{\text{SS}}^{(n)*}(\mathbf{J}_{\prod_{n=1}^{N}L_n}\otimes\mathbf{I}_M)^T \\ &= \left(\mathbf{A}^{(2,n)}\odot\mathbf{B}_{\text{SS}}^{(n)}\right)\mathbf{D}\mathbf{D}_\mathbf{S}^*\widetilde{\mathbf{S}}_{\text{SS}}^T, \end{aligned} \tag{38}$$

where $\mathbf{B}_{\text{SS}}^{(n)}$ is given by (27) and

$$\begin{aligned}\widetilde{\mathbf{S}}_{\text{SS}} &= \mathbf{A}^{(L_1,1)}\odot\cdots\odot\mathbf{A}^{(L_N,N)}\odot\mathbf{S}_0\,, \\ \mathbf{D} &= \prod_{n=1}^{N} D_1\left(\left[z_{1,n}^{-(I_n-1)}, z_{2,n}^{-(I_n-1)}, \ldots, z_{R,n}^{-(I_n-1)}\right]\right).\end{aligned} \tag{39}$$

Stacking $\mathbf{X}_{\text{SS}}^{(n)}$ in (26) and $\mathbf{Y}_{\text{SS}}^{(n)}$ in (38) results in the tensor $\mathcal{Z}_{\text{SS}}^{(n)} \in \mathbb{C}^{4\times(\times_{p=1}^{n-1}K_p)\times(K_n-1)\times(\times_{q=n+1}^{N}K_q)\times(\times_{s=1}^{N}L_s)\times M}$, $n \in \{1, \ldots, N\}$ with matrix representation

$$\begin{aligned}\mathbf{Z}_{\text{SS}}^{(n)} &= \begin{bmatrix}\mathbf{X}_{\text{SS}}^{(n)} \\ \mathbf{Y}_{\text{SS}}^{(n)}\end{bmatrix} = \begin{bmatrix}(\mathbf{A}^{(2,n)}\odot\mathbf{B}_{\text{SS}}^{(n)})\mathbf{D}_\mathbf{S}\widetilde{\mathbf{S}}_{\text{SS}}^T \\ (\mathbf{A}^{(2,n)}\odot\mathbf{B}_{\text{SS}}^{(n)})\mathbf{D}\mathbf{D}_\mathbf{S}^*\widetilde{\mathbf{S}}_{\text{SS}}^T\end{bmatrix} \\ &= \left(\widetilde{\mathbf{D}}\odot\mathbf{A}^{(2,n)}\odot\mathbf{B}_{\text{SS}}^{(n)}\right)\widetilde{\mathbf{S}}_{\text{SS}}^T,\end{aligned} \tag{40}$$

where $\widetilde{\mathbf{D}} = \begin{bmatrix}\text{Vecd}(\mathbf{D}_\mathbf{S})^T \\ \text{Vecd}\left(\mathbf{D}\mathbf{D}_\mathbf{S}^*\right)^T\end{bmatrix}$. We exploited that $\mathbf{S}_{\text{SS}} = \widetilde{\mathbf{S}}_{\text{SS}}\mathbf{D}_\mathbf{S}$ in (26) when $\mathbf{S} = \mathbf{S}_0\mathbf{D}_\mathbf{S}$. Using the coupled PD of $\{\mathcal{Z}_{\text{SS}}^{(1)}, \ldots, \mathcal{Z}_{\text{SS}}^{(N)}\}$, Theorem IV.6 below extends Theorem IV.2 to the strictly NC case where the generators are located on the unit circle. Theorem IV.6 involves the following matrix

$$\widetilde{\mathbf{G}}^{(N)} = \begin{bmatrix} C_2\left(\mathbf{A}^{(2,1)}\right)\odot C_2\left(\widetilde{\mathbf{D}}\odot\mathbf{B}_{\text{SS}}^{(n)}\right) \\ \vdots \\ C_2\left(\mathbf{A}^{(2,N)}\right)\odot C_2\left(\widetilde{\mathbf{D}}\odot\mathbf{B}_{\text{SS}}^{(n)}\right)\end{bmatrix}. \tag{41}$$

Theorem IV.6. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K}$ in (15) where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde and $\boldsymbol{S} = \boldsymbol{S}_0\boldsymbol{D}_\boldsymbol{S}$ in which $\boldsymbol{D}_\boldsymbol{S} \in \mathbb{C}^{R\times R}$ is a unitary diagonal matrix and $\boldsymbol{S}_0 \in \mathbb{R}^{K\times R}$. If $|z_{r,n}| = 1$, $\forall r$, $\forall n$, and if there exist pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$ such that*

$$\begin{cases}\widetilde{\boldsymbol{S}}_{SS} \text{ in (39) has full column rank,} \\ \widetilde{\boldsymbol{G}}^{(N)} \text{ in (41) has full column rank,}\end{cases} \tag{42}$$

then $r_{VDM}(\mathcal{X}) = R$ and the Vandermonde constrained CPD of $\mathcal{X}$ is unique.

Theorem IV.6 leads to improved uniqueness conditions. As an example, consider the 2D HR problem in which $I_1 = I_2 = 3$ and $K \geqslant R$. Applying Theorem II.5 to $\mathcal{Z}_{\text{SS}}^{(1)}$ and ignoring the Vandermonde structure of $\mathbf{A}^{(2)}$, we obtain the upper bound $R \leqslant 12$. Theorem IV.6 relaxes the bound to $R \leqslant 14$.

V. LINK BETWEEN MULTIRATE 1D HR AND COUPLED CPD

Consider a recorded time series of the form

$$x(n) = \sum_{r=1}^{R}\alpha_r e^{(\gamma_r + i\omega_r)\cdot\beta\cdot n}, \quad n \in \mathbb{N}\,, \tag{43}$$

in which $\alpha_r \in \mathbb{C}$ is the complex amplitude, $\gamma_r \in \mathbb{R}$ is the damping factor (i.e., it is usually assumed that $\gamma_r < 0$),

$0 \leqslant \omega_r < \pi$ is the normalized radial frequency, and $\beta \in \mathbb{R}$ is the sampling distance between two consecutive samples. As pointed out in [12] the choice of sampling rate affects the spectral resolution. This motivated the development of a decimation-based ESPRIT/CPD method for 1D HR of the form (14). Related ESPRIT/CPD-based decimation methods for 1D HR have been proposed in [24]. However, since the methods in [12], [24] make use of uniform sampling, the Nyquist sampling condition $\omega_r\beta < 2\pi$, $\forall r \in \{1,\ldots,R\}$, must be satisfied. A way to overcome this restriction is to employ nonuniform sampling (e.g., [44], [22]), as will be discussed next.

We will now present a multirate version of the result in [12] that can support sub-Nyquist sampling ($\omega_r\beta > 2\pi$). Let us work with N different sampling rates with associated interleaving factors $\beta_1,\ldots,\beta_N \in \mathbb{N}$. Assume that for sampling rate n we can construct matrices $\mathbf{X}^{(n)} \in \mathbb{C}^{M_n \times P_n}$ with factorizations

$$\mathbf{X}^{(n)} = \begin{bmatrix} x(k_{1,n}) & \cdots & x(k_{P_n,n}) \\ x(k_{1,n}+\beta_n) & \cdots & x(k_{P_n,n}+\beta_n) \\ \vdots & \ddots & \vdots \\ x(k_{1,n}+(M_n-1)\beta_n) & \cdots & x(k_{P_n,n}+(M_n-1)\beta_n) \end{bmatrix} = \mathbf{A}^{(n)}\mathbf{S}^{(n)T}, \quad (44)$$

where $\{k_{m,n}\}$ are sample indices and

$$\mathbf{A}^{(n)} = \begin{bmatrix} 1 & \cdots & 1 \\ e^{(\gamma_1+i\omega_1)\beta_n} & \ldots & e^{(\gamma_R+i\omega_R)\beta_n} \\ \vdots & \ddots & \vdots \\ e^{(\gamma_1+i\omega_1)(M_n-1)\beta_n} & \ldots & e^{(\gamma_R+i\omega_R)(M_n-1)\beta_n} \end{bmatrix},$$

$$\mathbf{S}^{(n)T} = \begin{bmatrix} \alpha_1 e^{(\gamma_1+i\omega_1)k_{1,n}} & \ldots & \alpha_1 e^{(\gamma_1+i\omega_1)k_{P_n,n}} \\ \alpha_2 e^{(\gamma_2+i\omega_2)k_{1,n}} & \ldots & \alpha_2 e^{(\gamma_2+i\omega_2)k_{P_n,n}} \\ \vdots & \ddots & \vdots \\ \alpha_R e^{(\gamma_R+i\omega_R)k_{1,n}} & \ldots & \alpha_R e^{(\gamma_R+i\omega_R)k_{P_n,n}} \end{bmatrix}.$$

Spatial smoothing applied on $\mathbf{X}^{(m,n)}$ yields the tensor $\mathcal{Y}^{(n)} \in \mathbb{C}^{(M_n-1)\times 2\times P_n}$ for which the matrix representation $\mathbf{Y}^{(m,n)} \in \mathbb{C}^{(M_n-1)2\times P_n}$ admits the factorization

$$\mathbf{Y}^{(n)} = \begin{bmatrix} \underline{\mathbf{I}}_{I_{M_n}} \cdot \mathbf{X}^{(n)} \\ \overline{\mathbf{I}}_{I_{M_n}} \cdot \mathbf{X}^{(n)} \end{bmatrix} = \left(\mathbf{A}^{(2,n)} \odot \mathbf{B}^{(n)}\right)\mathbf{S}^{(n)T},$$

where $\mathbf{A}^{(2,n)} = \begin{bmatrix} 1 & \cdots & 1 \\ e^{(\gamma_1+i\omega_1)\beta_n} & \cdots & e^{(\gamma_R+i\omega_R)\beta_n} \end{bmatrix}$ and $\mathbf{B}^{(n)} = \underline{\mathbf{A}}^{(n)}$.

If the uniqueness condition stated in Corollary II.2 is satisfied for the CPD of each tensor in the set $\{\mathbf{Y}^{(m,n)}\}$, then the factor matrices $\{\mathbf{A}^{(2,n)}\}$ are unique up to individual column scaling and permutation ambiguities. If one of the interleaving factors, say β_1, is chosen such that $\omega_r\beta_1 < 2\pi$, $\forall r \in \{1,\ldots,R\}$, then unique recovery of $\{\omega_r\}$ is guaranteed, even if for all other interleaving factors we have $\omega_r\beta_n \geqslant 2\pi$. However, if $\omega_r\beta_n \geqslant 2\pi$, $\forall n \in \{1,\ldots,N\}$, then in general $\{\omega_r\}$ cannot be recovered in a unique manner. On the other hand, if β_p and β_q are co-prime for some $p,q \in \{1,\ldots,N\}$, then by capitalizing on both the CPD structures of $\{\mathbf{Y}^{(m,n)}\}$ and the power constraints of $\{\mathbf{A}^{(2,n)}\}$, a unique recovery of $\{\omega_r\}$ from $\{\mathbf{Y}^{(m,n)}\}$ can be guaranteed, see [35] for an explanation.

Furthermore, if additionally the matrices $\mathbf{X}^{(n)}$ can be chosen of the same size (i.e., $P_1 = \cdots = P_N$, $k_{1,1} = \cdots = k_{1,N},\ldots,k_{P_1,1} = \cdots = k_{P_N,N}$), then we obtain a common matrix factor $\mathbf{S}^{(1)} = \cdots = \mathbf{S}^{(N)}$. In other words, the multirate sampled 1D HR problem has been reduced to a coupled CPD problem.

VI. Conclusion

MHR is an important problem in signal processing that is however not fully understood. We have first provided a link between the MHR problem and the coupled CPD. This has led to a new uniqueness condition for MHR. To the authors' knowledge, this is the most relaxed uniqueness condition for MHR that has been reported so far. We also explained that the coupled CPD modeling framework can be understood as a natural generalization of classical ESPRIT for 1D HR to MHR. Finally, a link between multirate sampled 1D HR and coupled CPD was provided that can support sub-Nyquist sampling.

Prior knowledge about the MHR decomposition can be incorporated into the coupled CPD framework. We considered the use of unit-norm generators, also in combination with strictly non-circular signals. Other structures that can be incorporated into the coupled CPD model include beamspace processing [46], [45] and frequency-selective methods (e.g., [43]).

The exposition was limited to PD with Vandermonde factor matrices. However, the results can be extended to PD with generalized Vandermonde factor matrices representing exponential polynomials [1], [5].

In this paper we explained that the link between MHR and coupled CPD leads to relaxed uniqueness conditions. In the companion paper [37] we discuss the actual computation and an application.

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