Multidimensional Harmonic Retrieval via Coupled Canonical Polyadic Decomposition

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***Abstract*—Multidimensional Harmonic Retrieval (MHR) is a common problem in signal processing. Despite their importance, uniqueness conditions and algebraic methods for MHR have not yet been fully developed. We present a link between MHR and the coupled Canonical Polyadic Decomposition (CPD), leading to an improved uniqueness condition and an algorithm. The coupled CPD approach can be interpreted as an MHR generalization of the classical ESPRIT method. We also explain that variants of classical ESPRIT, such as unitary and non-circular ESPRIT, can be extended to MHR via the coupled CPD approach.**

***Index Terms*—coupled canonical polyadic decomposition, Vandermonde matrix, multidimensional harmonic retrieval.**

I. Introduction

During the past two decades MHR has become an important problem in signal processing. A link between MHR and the CPD was established in [18]. Roughly speaking, it was observed that MHR problems can be addressed as Vandermonde constrained CPD problems. Based on the Vandermonde constrained CPD, uniqueness conditions and algebraic methods for MHR have been developed in [18], [11], [16], [14], [15]. In fact, such Vandermonde constrained CPD approaches do not fully exploit the structure of the MHR problem, leading to suboptimal results.

The authors have recently extended the CPD modeling framework to coupled models in [23], [26]. Coupled matrix/tensor decompositions are basic tools for data fusion, i.e., for the joint analysis of multiple related data sets. Data fusion has important applications in telecommunication, biomedical signal processing, chemometrics, bioinformatics, social network analysis, artificial intelligence, etc., see [23], [26], [20], [19] and references therein. In this paper, we present a link between MHR and the coupled CPD modeling framework [23], [26]. This leads to a new improved uniqueness condition for the MHR problem. Moreover, it leads to a versatile algebraic method for MHR. Interestingly, the coupled CPD approach can be interpreted as a generalization of the classical ESPRIT method [17] for one-dimensional (1D) Harmonic Retrieval (HR) to MHR. Part of this work appeared in the conference paper [24].

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The paper is organized as follows. The rest of the introduction will present the notation. Section II reviews the necessary algebraic prerequisites. Section III proposes a new uniqueness condition for MHR. Section IV develops a new algorithm for MHR. Numerical experiments are reported in section V. Section VI concludes the paper.

A. Notation

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The rth column vector of $\mathbf{A}$ is denoted by $\mathbf{a}_r$. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao product, defined as

$$\mathbf{A}\otimes\mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}, \ \mathbf{A}\odot\mathbf{B} \triangleq [\mathbf{a}_1\otimes\mathbf{b}_1 \ \ \mathbf{a}_2\otimes\mathbf{b}_2 \ \ \dots],$$

in which $(\mathbf{A})_{mn} = a_{mn}$. We denote the Kronecker and Khatri-Rao products of N matrices $\{\mathbf{A}^{(n)}\}_{n=1}^N$ by $\bigotimes_{n=1}^N \mathbf{A}^{(n)} = \mathbf{A}^{(1)}\otimes\cdots\otimes\mathbf{A}^{(N)}$ and $\bigodot_{n=1}^N \mathbf{A}^{(n)} = \mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(N)}$, respectively. The Cartesian product of N sets $\{\mathbb{C}^{I_n}\}_{n=1}^N$ is denoted by $\mathbb{C}^{\overset{N}{\underset{n=1}{\times}} I_n} = \mathbb{C}^{I_1\times\cdots\times I_N}$. The outer product of N vectors $\mathbf{a}^{(n)}\in\mathbb{C}^{I_n}$ is denoted by $\mathbf{a}^{(1)}\circ\cdots\circ\mathbf{a}^{(N)}\in\mathbb{C}^{I_1\times I_2\times\cdots\times I_N}$, such that $(\mathbf{a}^{(1)}\circ\cdots\circ\mathbf{a}^{(N)})_{i_1,i_2,\dots,i_N} = a^{(1)}_{i_1}a^{(2)}_{i_2}\cdots a^{(N)}_{i_N}$.

The conjugate, transpose, conjugate-transpose, Frobenius norm, range, real and imaginary part of a matrix are denoted by $(\cdot)^*$, $(\cdot)^T$, $(\cdot)^H$, $\|\cdot\|_F$, range$(\cdot)$, Re$\{\cdot\}$ and Im$\{\cdot\}$, respectively. From the context it should be clear when i denotes the imaginary unit number, i.e., $i=\sqrt{-1}$.

The identity matrix and all-zero vector are denoted by $\mathbf{I}_R\in\mathbb{C}^{R\times R}$ and $\mathbf{0}_R\in\mathbb{C}^R$, respectively. The exchange matrix with unit entries on the counterdiagonal and zeros elsewhere is denoted by $\mathbf{J}_I\in\mathbb{C}^{I\times I}$. Note that

$$\bigotimes_{n=1}^N \mathbf{J}_{I_n} = \mathbf{J}_{\prod_{n=1}^N I_n}, \tag{1}$$

in which the order of the factors does not matter.

Matlab index notation will be used for submatrices of a given matrix. For example, $\mathbf{A}(1{:}k,:)$ represents the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k of $\mathbf{A}$. $D_k(\mathbf{A})\in\mathbb{C}^{J\times J}$ denotes the diagonal matrix holding row k of $\mathbf{A}\in\mathbb{C}^{I\times J}$ on its diagonal. Similarly, Vecd$(\mathbf{A})$ denotes the column vector defined by $(\text{Vecd}(\mathbf{A}))_i = (\mathbf{A})_{ii}$.

The k-th compound matrix of $\mathbf{A}\in\mathbb{C}^{I\times R}$ is denoted by $C_k(\mathbf{A})\in\mathbb{C}^{C_I^k\times C_R^k}$, where $C_m^k = \frac{m!}{k!(m-k)!}$. It is the matrix

containing the determinants of all $k\times k$ submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order. See [5] and references therein for a discussion of compound matrices.

The k-rank of a matrix $\mathbf{A}$ is denoted by $k_{\mathbf{A}}$. It is equal to the largest integer $k_{\mathbf{A}}$ such that every subset of $k_{\mathbf{A}}$ columns of $\mathbf{A}$ is linearly independent.

II. Algebraic prerequisites

A. Vandermonde matrices

The matrix $\mathbf{A}\in\mathbb{C}^{I\times R}$ is called Vandermonde if

$$\mathbf{A}=[\mathbf{a}_1,\ldots,\mathbf{a}_R],\quad \mathbf{a}_r=\left[1,z_r,z_r^2,\ldots,z_r^{I-1}\right]^T, \tag{2}$$

where the scalars $\{z_r\}$ are called the generators of $\mathbf{A}$.

1) Spatial smoothing: When working with Vandermonde matrices it is common to do spatial smoothing in a preprocessing step in order to overcome the difficulties connected with factor matrices with collinear columns. Using $z_r^{l_1+k_1-2}=z_r^{l_1-1}z_r^{k_1-1}$, spatial smoothing maps $\mathbf{A}\in\mathbb{C}^{I\times R}$ to $\mathbf{A}^{(K_1)}\mathbf{A}^{(L_1)T}\in\mathbb{C}^{K_1\times L_1}$, in which $\mathbf{A}^{(K_1)}\in\mathbb{C}^{K_1\times R}$ and $\mathbf{A}^{(L_1)}\in\mathbb{C}^{L_1\times R}$ are given by

$$\mathbf{A}^{(L_1)}=\left[\mathbf{a}_1^{(L_1)}\ \ldots\ \mathbf{a}_R^{(L_1)}\right],\ \mathbf{a}_r^{(L_1)}=\left[1\ z_r\ z_r^2\ \ldots,z_r^{L_1-1}\right]^T, \tag{3}$$

$$\mathbf{A}^{(K_1)}=\left[\mathbf{a}_1^{(K_1)}\ \ldots\ \mathbf{a}_R^{(K_1)}\right],\ \mathbf{a}_r^{(K_1)}=\left[1\ z_r\ z_r^2\ \ldots\ z_r^{K_1-1}\right]^T, \tag{4}$$

subject to $K_1+L_1=I+1$.

2) Forward-Backward Averaging (FBA): If the generators of the Vandermonde matrix $\mathbf{A}$ are located on the unit circle ($|z_r|=1$), then we can make use of the well-known FBA procedure. More precisely, if $z_r=e^{i\alpha_r}$ where $\alpha_r\in\mathbb{R}$, $\forall r\in\{1,\ldots,R\}$, then $\mathbf{J}_I\mathbf{A}^*=\mathbf{A}\mathbf{D}_{\mathbf{A}}$ in which $\mathbf{D}_{\mathbf{A}}=D_1\left(\left[z_1^{-(I-1)},z_2^{-(I-1)},\ldots,z_R^{-(I-1)}\right]\right)$.

B. Canonical Polyadic Decomposition (CPD)

Consider the third-order tensor $\mathcal{X}\in\mathbb{C}^{I\times J\times K}$. We say that $\mathcal{X}$ is a rank-1 tensor if it is equal to the outer product of some non-zero vectors $\mathbf{a}\in\mathbb{C}^I$, $\mathbf{b}\in\mathbb{C}^J$ and $\mathbf{c}\in\mathbb{C}^K$ such that $x_{ijk}=a_ib_jc_k$. A Polyadic Decomposition (PD) is a decomposition of $\mathcal{X}$ into rank-1 terms:

$$\mathbb{C}^{I\times J\times K}\ni\mathcal{X}=\sum_{r=1}^{R}\mathbf{a}_r\circ\mathbf{b}_r\circ\mathbf{c}_r\,. \tag{5}$$

The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then (5) is called the CPD of $\mathcal{X}$. Let us stack the vectors $\{\mathbf{a}_r\}$, $\{\mathbf{b}_r\}$ and $\{\mathbf{c}_r\}$ into the matrices $\mathbf{A}=[\mathbf{a}_1,\ldots,\mathbf{a}_R]$, $\mathbf{B}=[\mathbf{b}_1,\ldots,\mathbf{b}_R]$ and $\mathbf{C}=[\mathbf{c}_1,\ldots,\mathbf{c}_R]$. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ will be referred to as the factor matrices of the PD or CPD of $\mathcal{X}$ in (5).

1) Matrix Representation: Let $\mathbf{X}^{(i\cdot\cdot)}\in\mathbb{C}^{J\times K}$ denote the matrix such that $\left(\mathbf{X}^{(i\cdot\cdot)}\right)_{jk}=x_{ijk}$, then $\mathbf{X}^{(i\cdot\cdot)}=\mathbf{B}D_i(\mathbf{A})\mathbf{C}^T$ and we obtain the following matrix representation of (5):

$$\mathbb{C}^{IJ\times K}\ni\mathbf{X}_{(1)}\triangleq\left[\mathbf{X}^{(1\cdot\cdot)T},\ldots,\mathbf{X}^{(I\cdot\cdot)T}\right]^T=(\mathbf{A}\odot\mathbf{B})\mathbf{C}^T. \tag{6}$$

2) Uniqueness: The development of uniqueness conditions for the CPD has been the subject of intense investigation, see [5], [6] and references therein. For the case where one factor matrix has full column rank the following sufficient condition has been obtained.

Theorem II.1. *Consider the PD of $\mathcal{X}\in\mathbb{C}^{I\times J\times K}$ in (5). If*

$$\begin{cases}\boldsymbol{C}\ \textit{has full column rank,}\\ C_2(\boldsymbol{A})\odot C_2(\boldsymbol{B})\ \textit{has full column rank,}\end{cases} \tag{7}$$

then the rank of $\mathcal{X}$ is R and the CPD of $\mathcal{X}$ is unique [10], [3], [5]. Generically[1], condition (7) is satisfied if $2R(R-1)\leq I(I-1)J(J-1)$ and $R\leq K$ [3], [28].

An algorithm for computing the CPD of $\mathcal{X}$ under the condition (7) was developed in [3] and further elaborated on in [4]. More precisely, it reduces the CPD problem to a Simultaneous matrix Diagonalization (SD) problem.

C. Coupled CPD

We say that a collection of tensors $\mathcal{X}^{(n)}\in\mathbb{C}^{I_n\times J_n\times K}$, $n\in\{1,\ldots,N\}$, admits an R-term coupled PD if each tensor $\mathcal{X}^{(n)}$ can be written as [23]:

$$\mathcal{X}^{(n)}=\sum_{r=1}^{R}\mathbf{a}_r^{(n)}\circ\mathbf{b}_r^{(n)}\circ\mathbf{c}_r\,,\quad n\in\{1,\ldots,N\}, \tag{8}$$

with factor matrices $\mathbf{A}^{(n)}=\left[\mathbf{a}_1^{(n)},\ldots,\mathbf{a}_R^{(n)}\right]$, $\mathbf{B}^{(n)}=\left[\mathbf{b}_1^{(n)},\ldots,\mathbf{b}_R^{(n)}\right]$ and $\mathbf{C}=[\mathbf{c}_1,\ldots,\mathbf{c}_R]$. We define the coupled rank of $\{\mathcal{X}^{(n)}\}$ as the minimal number of coupled rank-1 tensors $\mathbf{a}_r^{(n)}\circ\mathbf{b}_r^{(n)}\circ\mathbf{c}_r$ that yield $\{\mathcal{X}^{(n)}\}$ in a linear combination. Assume that the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R, then (8) will be called the coupled CPD of $\{\mathcal{X}^{(n)}\}$.

1) Matrix representation: The coupled PD or CPD of $\{\mathcal{X}^{(n)}\}$ given by (8) has the following matrix representation

$$\mathbf{X}=\begin{bmatrix}\mathbf{X}_{(1)}^{(1)}\\ \vdots\\ \mathbf{X}_{(1)}^{(N)}\end{bmatrix}=\begin{bmatrix}\mathbf{A}^{(1)}\odot\mathbf{B}^{(1)}\\ \vdots\\ \mathbf{A}^{(N)}\odot\mathbf{B}^{(N)}\end{bmatrix}\mathbf{C}^T\in\mathbb{C}^{(\sum_{n=1}^{N}I_nJ_n)\times K}. \tag{9}$$

2) Uniqueness: The coupled rank-1 tensors in (8) can be arbitrarily permuted and the vectors within the same coupled rank-1 tensor can be arbitrarily scaled provided the overall coupled rank-1 term remains the same. We say that the coupled CPD is unique when it is only subject to these trivial indeterminacies. Uniqueness conditions for the coupled CPD were derived in [23]. Theorem II.2 extends Theorem II.1 to coupled CPD.

Theorem II.2. *Consider the coupled PD of $\mathcal{X}^{(n)}\in\mathbb{C}^{I_n\times J_n\times K}$, $n\in\{1,\ldots,N\}$ in (8). Define*

$$\boldsymbol{G}=\begin{bmatrix}C_2\left(\boldsymbol{A}^{(1)}\right)\odot C_2\left(\boldsymbol{B}^{(1)}\right)\\ \vdots\\ C_2\left(\boldsymbol{A}^{(N)}\right)\odot C_2\left(\boldsymbol{B}^{(N)}\right)\end{bmatrix}\in\mathbb{C}^{\left(\sum_{n=1}^{N}C_{I_n}^2C_{J_n}^2\right)\times C_R^2}. \tag{10}$$

[1] A tensor decomposition property is called generic if it holds with probability one when the entries of the factor matrices are drawn from absolutely continuous probability density functions.

If

$$\begin{cases} \boldsymbol{C} \text{ has full column rank}, \\ \boldsymbol{G} \text{ has full column rank}, \end{cases} \tag{11}$$

then the coupled rank of $\{\mathcal{X}^{(n)}\}$ *is* R *and the coupled CPD of* $\{\mathcal{X}^{(n)}\}$ *is unique [23].*

Furthermore, an extension of the SD method in [3] to the coupled CPD has been developed in [24], [26].

D. CPD with a Vandermonde factor matrix

Consider the PD of $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ in (5). If $\mathbf{A}$ is Vandermonde, then this may increase the minimal R in (5). This minimal value will be denoted by $r_{\text{VDM}}(\mathcal{X})$. If some of the factor matrices are Vandermonde and $R = r_{\text{VDM}}(\mathcal{X})$, then (5) will be called a Vandermonde constrained CPD of $\mathcal{X}$. In a Vandermonde constrained CPD the scaling/counterscaling indeterminacies do not involve the Vandermonde factors.

Spatial smoothing yields $\mathcal{Y} \in \mathbb{C}^{K_1\times J\times L_1\times K}$ with

$$y_{(k_1-1)J+j,(l_1-1)K+k} = x_{l_1+k_1-1,j,k} = \sum_{r=1}^{R} z_r^{l_1-1} b_{j,r} z_r^{k_1-1} c_{k,r}\,,$$

and matrix factorization

$$\mathbb{C}^{K_1J\times L_1K} \ni \mathbf{Y} = \left(\mathbf{A}^{(K_1)} \odot \mathbf{B}\right)\left(\mathbf{A}^{(L_1)} \odot \mathbf{C}\right)^T\,, \tag{12}$$

where $\mathbf{A}^{(L_1)} \in \mathbb{C}^{L_1\times R}$ and $\mathbf{A}^{(K_1)} \in \mathbb{C}^{K_1\times R}$ are given by (3) and (4) subject to $K_1 + L_1 = I + 1$. The following uniqueness condition for Vandermonde constrained CPD was obtained in [22].

Theorem II.3. *Let* $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ *be given by* (5)*. Assume that* $\boldsymbol{A}$ *is a Vandermonde matrix with distinct generators. Consider the matrix representation* (12)*. If there exists a pair* (K_1, L_1) *subject to* $K_1 + L_1 = I + 1$ *such that*

$$\begin{cases} \boldsymbol{A}^{(K_1)}(1 : K_1 - 1, :) \odot \boldsymbol{B} \text{ has full column rank}, \\ \boldsymbol{A}^{(L_1)} \odot \boldsymbol{C} \text{ has full column rank}, \end{cases} \tag{13}$$

then $R = r_{VDM}(\mathcal{X})$ *and the Vandermonde constrained CPD of* $\mathcal{X}$ *is unique. Generically, condition* (13) *is satisfied if and only if* $\left\lceil \frac{R}{J} \right\rceil + \left\lceil \frac{R}{K} \right\rceil \leq I$.

An algorithm for computing the CPD with a Vandermonde factor matrix was also provided in [22].

III. New uniqueness condition for multidimensional harmonic retrieval based on the coupled CPD

It was recognized in [18] that N-dimensional HR problems can be formulated in terms of the constrained PD of a tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K}$,

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(N)} \circ \mathbf{s}_r\,, \tag{14}$$

with Vandermonde factor matrices $\mathbf{A}^{(n)} = \left[\mathbf{a}_1^{(n)}, \ldots, \mathbf{a}_R^{(n)}\right] \in \mathbb{C}^{I_n\times R}$ with $\mathbf{a}_r^{(n)} = \left[1, z_{r,n}, z_{r,n}^2, \ldots, z_{r,n}^{I_n-1}\right]^T$ and unstructured $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_R] \in \mathbb{C}^{K\times R}$. The PD of $\mathcal{X}$ in (14) has the following matrix representation

$$\mathbf{X} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\mathbf{S}^T \in \mathbb{C}^{(\prod_{n=1}^{N} I_n)\times K}. \tag{15}$$

Note that $N = 1$ corresponds to 1D HR, $N = 2$ corresponds to two-dimensional (2D) HR, and so on.

The goal of MHR is to recover the generators $\{z_{r,n}\}$ from the observed data tensor $\mathcal{X}$. As our contribution, we will present a link between the MHR problem and the *coupled* CPD. This will lead to an improved uniqueness condition and an algorithm for the MHR problem. Subsection III-A discusses a necessary uniqueness condition for MHR. Subsection III-B presents a new sufficient uniqueness condition for MHR based on the coupled CPD. Subsection III-C briefly extends the result to the special but important case of undamped exponentials ($|z_r| = 1$). Subsection III-D considers the special case of strictly non-circular signals which is relevant for communication systems.

A. Necessary uniqueness condition for MHR

An obvious necessary uniqueness condition for MHR is that the involved N-dimensional harmonics are distinct, as stated in the following proposition.

Proposition III.1. *Consider the PD of* $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K}$ *in* (14) *where the factor matrices* $\{\boldsymbol{A}^{(n)}\}$ *are Vandermonde with generators* $\{z_{r,n}\}$*. Define*

$$\boldsymbol{P} = \left[\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_R\right] \in \mathbb{C}^{N\times R}, \quad \boldsymbol{p}_r = \left[z_{r,1}\ ,\ z_{r,2}\ ,\ldots,\ z_{r,N}\right]^T.$$

If the Vandermonde constrained CPD of $\mathcal{X}$ *is unique, then*

$$\boldsymbol{p}_r \neq \boldsymbol{p}_s, \quad \forall r \neq s\,. \tag{16}$$

Condition (16) does not prevent $k_{\mathbf{A}^{(1)}} = \cdots = k_{\mathbf{A}^{(N)}} = k_{\mathbf{S}} = 1$ even in the case where $N = 2$. This is in contrast to ordinary CPD, where $k_{\mathbf{A}^{(1)}} \geq 2$, $k_{\mathbf{A}^{(2)}} \geq 2$ and $k_{\mathbf{S}} \geq 2$ are necessary if $N = 2$ (e.g. [5]). This observation already tells us that it is important to take the rich structure of the MHR decomposition in (14) into account.

B. Sufficient uniqueness condition for MHR

Consider the PD of $\mathcal{X}$ in (14) with matrix representation (15) and where $\{\mathbf{A}^{(n)}\}$ are Vandermonde. By spatial smoothing (see [22] for details) we obtain $\mathcal{X}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{K_1\times\cdots\times K_N\times L_1\times\cdots\times L_N\times K}$ subject to $K_n + L_n = I_n + 1$ and with matrix representation $\mathbf{X}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\prod_{n=1}^{N} K_n)\times(\prod_{n=1}^{N} L_n)K}$:

$$\mathbf{X}^{(K_1,\ldots,K_N)} = \left(\bigodot_{n=1}^{N} \mathbf{A}^{(K_n,n)}\right)\left(\bigodot_{n=1}^{N} \mathbf{A}^{(L_n,n)} \odot \mathbf{S}\right)^T, \tag{17}$$

where $\mathbf{A}^{(K_n,n)} = \mathbf{A}(1 : K_n, :) \in \mathbb{C}^{K_n\times R}$ and $\mathbf{A}^{(L_n,n)} = \mathbf{A}(1 : L_n, :) \in \mathbb{C}^{L_n\times R}$. By an additional spatial smoothing we obtain the tensors $\mathcal{X}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\times_{p=1}^{n-1} K_p)\times(K_n-1)\times 2\times(\times_{q=n+1}^{N} K_q)\times(\times_{s=1}^{N} L_s)\times K}$ subject

to $K_n + L_n = I_n + 1$ and with matrix representation $\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\prod_{p=1,p\neq n}^{N} K_p)(K_n-1)\times 2\times(\prod_{q=1}^{N} L_q)K}$ with factorization

$$\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} = \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\mathbf{C}^{[L_1,\ldots,L_N]T}, \quad (18)$$

where

$$\mathbf{B}^{[n,K_1,\ldots,K_N]} = \bigodot_{p=1}^{n-1} \mathbf{A}^{(K_p,p)} \odot \mathbf{A}^{(K_n-1,n)} \odot \bigodot_{p=n+1}^{N} \mathbf{A}^{(K_p,p)}, \quad (19)$$

$$\mathbf{C}^{[L_1,\ldots,L_N]} = \bigodot_{n=1}^{N} \mathbf{A}^{(L_n,n)} \odot \mathbf{S}. \quad (20)$$

The second spatial smoothing expressed by (18) is the crucial step that connects the coupled CPD framework to the MHR problem. Proposition III.2 below states a uniqueness condition for MHR that will make use of the coupled PD of the tensors $\{\mathcal{X}_{(1)}^{(K_1,\ldots,K_N)},\ldots,\mathcal{X}_{(N)}^{(K_1,\ldots,K_N)}\}$ with matrix decomposition $\mathbf{Y}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\sum_{n=1}^{N}(\prod_{p=1,p\neq n}^{N} K_p)(K_n-1))\times 2\times(\prod_{q=1}^{N} L_q)K}$ given by

$$\mathbf{Y}^{(K_1,\ldots,K_N)} = \begin{bmatrix} \mathbf{X}_{(1)}^{(K_1,\ldots,K_N)} \\ \vdots \\ \mathbf{X}_{(N)}^{(K_1,\ldots,K_N)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{[1,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,1)} \\ \vdots \\ \mathbf{B}^{[N,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,N)} \end{bmatrix} \mathbf{C}^{[L_1,\ldots,L_N]T}. \quad (21)$$

Proposition III.2 makes use of the following matrix

$$\mathbf{G}^{(K_1,\ldots,K_N)} = \begin{bmatrix} C_2\left(\mathbf{B}^{[1,K_1,\ldots,K_N]}\right) \odot C_2\left(\mathbf{A}^{(2,1)}\right) \\ \vdots \\ C_2\left(\mathbf{B}^{[N,K_1,\ldots,K_N]}\right) \odot C_2\left(\mathbf{A}^{(2,N)}\right) \end{bmatrix}. \quad (22)$$

The proposition follows directly from application of Theorem II.2 to $\{\mathcal{X}_{(1)}^{(K_1,\ldots,K_N)},\ldots,\mathcal{X}_{(N)}^{(K_1,\ldots,K_N)}\}$.

Proposition III.2. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K}$ in* (14) *where the factor matrices $\{A^{(n)}\}$ are Vandermonde. If there exist pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$ such that*

$$\begin{cases} \boldsymbol{C}^{[L_1,\ldots,L_N]} \text{ in (20) has full column rank,} \\ \boldsymbol{G}^{(K_1,\ldots,K_N)} \text{ in (22) has full column rank,} \end{cases} \quad (23)$$

then $r_{VDM}(\mathcal{X}) = R$ and the Vandermonde constrained CPD of $\mathcal{X}$ is unique.

Note that Proposition III.2 does not prevent that $k_{\mathbf{A}^{(1)}} = k_{\mathbf{A}^{(2)}} = \cdots = k_{\mathbf{A}^{(N)}} = k_{\mathbf{S}} = 1$, i.e., it can handle problems in which the factor matrices contain collinear columns. However, since

$$C_2\left(\mathbf{A}^{(2,n)}\right) = \left[z_{2,n} - z_{1,n}, z_{3,n} - z_{1,n}, \ldots, z_{R,n} - z_{R-1,n}\right],$$

it is also clear that condition (23) requires that for every $r \in \{1,\ldots,R\}$ there exists an $n \in \{1,\ldots,N\}$ and an $s \in \{1,\ldots,R\}$ such that $z_{r,n} \neq z_{s,n}$. From Proposition III.1 we know that this is not a limitation of Proposition III.2 but a necessary requirement for MHR uniqueness.

Let us now explain that Proposition III.2 yields a more relaxed uniqueness condition than Theorem II.3 does for MHR. Theorem II.3 establishes MHR uniqueness using $\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} = \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\mathbf{C}^{[L_1,\ldots,L_N]T}$. More precisely, if $\mathbf{B}^{[n,K_1,\ldots,K_N]}$ and $\mathbf{C}^{[L_1,\ldots,L_N]}$ have full column rank and $k_{\mathbf{A}^{(2,n)}} \geq 2$, then Theorem II.3 guarantees MHR uniqueness. This is equivalent to $C_2\left(\mathbf{B}^{[n,K_1,\ldots,K_N]}\right) \odot C_2\left(\mathbf{A}^{(2,n)}\right)$ and $\mathbf{C}^{[L_1,\ldots,L_N]}$ having full column rank. Indeed, since the generators of $\mathbf{A}^{(2,n)}$ are assumed distinct we know that $C_2\left(\mathbf{A}^{(2,n)}\right)$ is a row-vector with only non-zero entries and since $\mathbf{B}^{[n,K_1,\ldots,K_N]}$ is assumed to have full column rank we also know that $C_2\left(\mathbf{B}^{[n,K_1,\ldots,K_N]}\right)$ has full column rank [27]. This in turn implies that $\mathbf{G}^{(K_1,\ldots,K_N)}$ given by (22) has full column rank. We conclude that if the conditions stated in Theorem II.3 are satisfied for some $n \in \{1,\ldots,N\}$, then the condition (23) in Proposition III.2 is automatically satisfied. However, the converse is not true.

Proposition III.2 also yields improved generic bounds on R. Let us compare Proposition III.2 with a result obtained in [11] for 2D HR ($N = 2$) with $K \geq R$. It was explained in [11] that if $I_1I_2 - \min(I_1, I_2) \geq R$ and $K \geq R$, then 2D HR uniqueness is generically guaranteed. This condition is in fact not better than conditions that can be obtained for CPD with only one Vandermonde factor matrix (e.g. Theorem II.3). However, by exploiting that both $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are Vandermonde, improved generic bounds on R can be obtained. The following Lemma III.3 implies that it suffices to numerically check condition (23) for a random example, leading to a generic bound on R.

Lemma III.3. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be an analytic function. If there exists an element $\boldsymbol{x} \in \mathbb{C}^n$ such that $f(\boldsymbol{x}) \neq 0$, then the set $\{\boldsymbol{x} \mid f(\boldsymbol{x}) = 0\}$ is of Lebesgue measure zero (e.g. [11]).*

Table I indicates the maximum value for R for which Theorem II.3 and Proposition III.2 generically guarantee uniqueness for 2D HR with $4 \leq I_1, I_2 \leq 10$ and $R \leq K$. We already know from [11] that if $N = 2$ and $K \geq R$ and if we only exploit that either $\mathbf{A}^{(1)}$ or $\mathbf{A}^{(2)}$ is Vandermonde, then R in (14) is bounded by $I_1I_2 - \min(I_1, I_2) \geq R$ (see also Theorem II.3). In the cases listed in Table I, Proposition III.2 relaxes the bound on R in (14) to $I_1I_2 - 3 \geq R$.

C. Vandermonde matrices with unit norm generators

Let us now consider the special case where $|z_{r,n}| = 1$, $\forall r \in \{1,\ldots,R\}$, $\forall n \in \{1,\ldots,N\}$. The FBA procedure yields

$$\mathbf{J}_{\prod_{n=1}^{N} I_n}\mathbf{X}^* = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\left(\mathbf{D}_{\mathbf{A}^{(1)}}\mathbf{D}_{\mathbf{A}^{(2)}}\cdots\mathbf{D}_{\mathbf{A}^{(N)}}\right)\mathbf{S}^H, \quad (24)$$

where $\mathbf{D}_{\mathbf{A}^{(n)}} = D_1\left(\left[z_{1,n}^{-(I_n-1)}, z_{2,n}^{-(I_n-1)}, \ldots, z_{R,n}^{-(I_n-1)}\right]\right)$. Observe that the factor matrices $\{\mathbf{A}^{(n)}\}_{n=1}^{N}$ appear in both $\mathbf{X}$ and $\mathbf{J}_{\prod_{n=1}^{N} I_n}\mathbf{X}^*$. Concatenating $\mathbf{X}$ and $\mathbf{J}_{\prod_{n=1}^{N} I_n}\mathbf{X}^*$ produces

$$\widehat{\mathbf{X}} = \left[\mathbf{X}, \mathbf{J}_{\prod_{n=1}^{N} I_n}\mathbf{X}^*\right] = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\widehat{\mathbf{S}}^T \in \mathbb{C}^{(\prod_{n=1}^{N} I_n)\times 2K}, \quad (25)$$

where $\widehat{\mathbf{S}} = \left[\mathbf{S}^T, \left(\mathbf{D}_{\mathbf{A}^{(1)}}\mathbf{D}_{\mathbf{A}^{(2)}}\cdots\mathbf{D}_{\mathbf{A}^{(N)}}\right)\mathbf{S}^H\right]^T \in \mathbb{C}^{2K\times R}$. Comparison of (15) and (25) yields the following counterpart of (18):

$$\widehat{\mathbf{X}}_{(n)}^{(K_1,\ldots,K_N)} = \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\widehat{\mathbf{C}}^{[L_1,\ldots,L_N]T}, \quad (26)$$

$I_{\min}$	4							5						6					7				8			9	10
$I_{\max}$	4	5	6	7	8	9	10	5	6	7	8	9	10	6	7	8	9	10	7	8	9	10	8	9	10	9	10	10
Thm. II.3	12	16	20	24	28	32	36	20	25	30	35	40	45	30	36	42	48	54	42	49	56	63	56	64	72	72	81	90
Prop. III.2	13	17	21	25	29	33	37	22	27	32	37	42	47	33	39	45	51	57	46	53	60	67	61	69	77	78	87	97

TABLE I
MAXIMUM VALUE FOR R IN (14) FOR WHICH THEOREM II.3 AND PROPOSITION III.2 GENERICALLY GUARANTEE UNIQUENESS FOR 2D HR WITH $K \geq R$. WE DENOTE $I_{\min} = \min(I_1, I_2)$ AND $I_{\max} = \max(I_1, I_2)$.

where $\mathbf{B}^{[n,K_1,\ldots,K_N]}$ is of the form (19) and

$$\widehat{\mathbf{C}}^{[L_1,\ldots,L_N]} = \mathbf{A}^{(L_1,1)} \odot \cdots \odot \mathbf{A}^{(L_N,N)} \odot \widehat{\mathbf{S}}. \quad (27)$$

Using the coupled PD of $\{\widehat{\mathcal{X}}_{(1)}^{(K_1,\ldots,K_N)}, \ldots, \widehat{\mathcal{X}}_{(N)}^{(K_1,\ldots,K_N)}\}$, Proposition III.4 below extends Proposition III.2 to the case where the generators are located on the unit circle.

Proposition III.4. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K}$ in (14) where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde. If $|z_{r,n}| = 1$, $\forall r \in \{1,\ldots,R\}$, $\forall n \in \{1,\ldots,N\}$, and there exist pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$ such that*

$$\begin{cases} \widehat{\boldsymbol{C}}^{[L_1,\ldots,L_N]} \text{ in (27) has full column rank,} \\ \boldsymbol{G}^{(K_1,\ldots,K_N)} \text{ in (22) has full column rank,} \end{cases} \quad (28)$$

then $r_{VDM}(\mathcal{X}) = R$ and the Vandermonde constrained CPD of $\mathcal{X}$ is unique.

Proposition III.4 yields an improved uniqueness condition for MHR. Let us compare Proposition III.4 with the results obtained in [14], [15] for 2D HR with unit norm generators. It was explained in [14], [15] that uniqueness is generically guaranteed if $R \leq \min((K_1 - 1)K_2, L_1 L_2 2K)$. This condition is in fact not better than conditions that can be obtained for CPD with only one Vandermonde factor matrix (e.g. Theorem II.3). However, by exploiting that both $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are Vandermonde, improved uniqueness conditions can be obtained. As an example, consider the 2D HR problem in which $I_1 = I_2 = 6$ and $K = 1$. For this problem the results in [14], [15] and Theorem II.3 require that $R \leq 12$ while Proposition III.4 relaxes the bound to $R \leq 13$.

D. Strictly non-circular signals and unit norm generators

As in [8] we consider the case of strictly Non-Circular (NC) signals. Strictly NC signal matrices $\mathbf{S} \in \mathbb{C}^{K \times R}$ admit the factorization $\mathbf{S} = \mathbf{S}_0 \mathbf{D}_{\mathbf{S}}$, where $\mathbf{S}_0 \in \mathbb{R}^{K \times R}$ and $\mathbf{D}_{\mathbf{S}} = D_1\left([e^{i\varphi_1}, \ldots, e^{i\varphi_R}]\right)$ contains the phase shifts in which $\varphi_r \in \mathbb{R}$. The problem where additionally the generators of the Vandermonde matrices $\{\mathbf{A}^{(n)}\}$ in (14) are located on the unit circle appears in several communication systems (e.g. [8]).

Recall that the result in subsection III-C was obtained by a *horizontal* stacking of the observed data $\mathbf{X}$ and $\mathbf{J}_{\prod_{n=1}^N I_n} \mathbf{X}^*$, yielding $\widehat{\mathbf{X}}$ in (25). Here as an alternative to Proposition III.4, we explain that by capitalizing on the NC property $\mathbf{S} = \mathbf{S}_0 \mathbf{D}_{\mathbf{S}}$ it is also possible to obtain a result by a *vertical* stacking of the observed data. Due to relation $\mathbf{S} = \mathbf{S}_0 \mathbf{D}_{\mathbf{S}}$ we obtain from (18) the additional tensors $\mathcal{Y}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\times_{p=1}^{n-1} K_p) \times (K_n - 1) \times 2 \times (\times_{q=n+1}^{N} K_q) \times (\times_{s=1}^{N} L_s) \times K}$, $n \in \{1,\ldots,N\}$, subject to $K_n + L_n = I_n + 1$ and with matrix representation $\mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\prod_{p=1,p\neq n}^N K_p)(K_n-1) \times 2 \times (\prod_{q=1}^N L_q)K}$:

$$\begin{aligned} \mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)} &= (\mathbf{J}_{\prod_{p=1,p\neq n}^N K_p (K_n - 1)} \otimes \mathbf{J}_2) \mathbf{X}_{(n)}^{(K_1,\ldots,K_N)*} (\mathbf{J}_{\prod_{n=1}^N L_n} \otimes \mathbf{I}_K)^T \\ &= \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right) \mathbf{D} \mathbf{D}_{\mathbf{S}}^* \mathbf{C}_0^{[L_1,\ldots,L_N]T}, \end{aligned} \quad (29)$$

where $\mathbf{B}^{[n,K_1,\ldots,K_N]}$ is given by (19) and

$$\mathbf{C}_0^{[L_1,\ldots,L_N]} = \mathbf{A}^{(L_1,1)} \odot \cdots \odot \mathbf{A}^{(L_N,N)} \odot \mathbf{S}_0, \quad (30)$$

$$\mathbf{D} = \prod_{n=1}^{N} \mathbf{D}_{\mathbf{A}^{(n)}}, \quad \mathbf{D}_{\mathbf{A}^{(n)}} = D_1\left(\left[z_{1,n}^{-(I_n-1)}, z_{2,n}^{-(I_n-1)}, \ldots, z_{R,n}^{-(I_n-1)}\right]\right).$$

Combining $\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)}$ in (18) and $\mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)}$ in (29) results in the tensor $\mathcal{Z}_{(n)}^{(K_1,\ldots,K_N)}$ with matrix representation

$$\begin{aligned} \mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)} &= \begin{bmatrix} \mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} \\ \mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}) \mathbf{D}_{\mathbf{S}} \mathbf{C}_0^{[L_1,\ldots,L_N]T} \\ (\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}) \mathbf{D} \mathbf{D}_{\mathbf{S}}^* \mathbf{C}_0^{[L_1,\ldots,L_N]T} \end{bmatrix} \\ &= \left(\widetilde{\mathbf{D}} \odot \mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right) \mathbf{C}_0^{[L_1,\ldots,L_N]T}, \end{aligned} \quad (31)$$

where $\widetilde{\mathbf{D}} = \begin{bmatrix} \mathrm{Vecd}(\mathbf{D}_{\mathbf{S}})^T \\ \mathrm{Vecd}\left(\mathbf{D}\mathbf{D}_{\mathbf{S}}^*\right)^T \end{bmatrix}$. We exploited that $\mathbf{C}^{[L_1,\ldots,L_N]} = \mathbf{C}_0^{[L_1,\ldots,L_N]} \mathbf{D}_{\mathbf{S}}$ in (18) when $\mathbf{S} = \mathbf{S}_0 \mathbf{D}_{\mathbf{S}}$. Using the coupled PD of $\{\mathcal{Z}_{(1)}^{(K_1,\ldots,K_N)}, \ldots, \mathcal{Z}_{(N)}^{(K_1,\ldots,K_N)}\}$, Proposition III.5 below extends Proposition III.2 to the strictly NC case where the generators are located on the unit circle. Proposition III.5 involves the following matrix

$$\widetilde{\mathbf{G}}^{(K_1,\ldots,K_N)} = \begin{bmatrix} C_2\left(\widetilde{\mathbf{D}} \odot \mathbf{B}^{[1,K_1,\ldots,K_N]}\right) \odot C_2\left(\mathbf{A}^{(2,1)}\right) \\ \vdots \\ C_2\left(\widetilde{\mathbf{D}} \odot \mathbf{B}^{[N,K_1,\ldots,K_N]}\right) \odot C_2\left(\mathbf{A}^{(2,N)}\right) \end{bmatrix}. \quad (32)$$

Proposition III.5. *Consider the PD of $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K}$ in (14) where the factor matrices $\{\boldsymbol{A}^{(n)}\}$ are Vandermonde and $\boldsymbol{S} = \boldsymbol{S}_0 \boldsymbol{D}_{\boldsymbol{S}}$ in which $\boldsymbol{D}_{\boldsymbol{S}} \in \mathbb{C}^{R \times R}$ is a unitary diagonal matrix and $\boldsymbol{S}_0 \in \mathbb{R}^{K \times R}$. If $|z_{r,n}| = 1$, $\forall r$, $\forall n$, and if there exist pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$ such that*

$$\begin{cases} \boldsymbol{C}_0^{[L_1,\ldots,L_N]} \text{ in (30) has full column rank,} \\ \widetilde{\boldsymbol{G}}^{(K_1,\ldots,K_N)} \text{ in (32) has full column rank,} \end{cases} \quad (33)$$

then $r_{VDM}(\mathcal{X}) = R$ and the Vandermonde constrained CPD of $\mathcal{X}$ is unique.

Proposition III.5 leads to improved uniqueness conditions. As an example, consider the 2D HR problem in which $I_1 = I_2 = 3$ and $K \geq R$. Applying Theorem II.3 to $\mathcal{Z}_{(1)}^{(K_1,K_2)}$ and ignoring the Vandermonde structure of

$\mathbf{A}^{(2)}$, we obtain the upper bound $R \leq 12$. Proposition III.5 relaxes the bound to $R \leq 14$.

IV. New ESPRIT method for multidimensional harmonic retrieval via coupled CPD

In subsection IV-A we generalize the ESPRIT method [17] for 1D HR to MHR using the coupled CPD framework. Subsections IV-B and IV-C explain that variants of the standard ESPRIT, such as unitary ESPRIT and unitary NC-ESPRIT, can also be generalized to MHR via the coupled CPD approach.

A. ESPRIT, FBA-ESPRIT and NC-ESPRIT for MHR

We will now explain that the MHR problem, like classical ESPRIT, reduces to an eigenvalue problem. Assume that the conditions stated in Proposition III.2 are satisfied. Let $\mathbf{Y}^{(K_1,\ldots,K_N)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{Y}^{(K_1,\ldots,K_N)}$ in (21), where $\mathbf{U} \in \mathbb{C}^{2(\sum_{n=1}^{N}(\prod_{p=1,p\neq n}^{N} K_p)(K_n-1))\times R}$ and $\mathbf{V} \in \mathbb{C}^{(\prod_{q=1}^{N} L_q)K)\times R}$ are columnwise orthonormal and $\Sigma \in \mathbb{C}^{R\times R}$ is diagonal. Then there exists a nonsingular matrix $\mathbf{F} \in \mathbb{C}^{R\times R}$ such that from relation (18) we obtain

$$\mathbf{U}\Sigma = \begin{bmatrix} \mathbf{B}^{[1,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,1)} \\ \vdots \\ \mathbf{B}^{[N,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,N)} \end{bmatrix} \mathbf{F}, \quad (34)$$

$$\mathbf{V}^*\mathbf{G}^T = \mathbf{C}^{[L_1,\ldots,L_N]} = \mathbf{A}^{(L_1,1)} \odot \cdots \odot \mathbf{A}^{(L_N,N)} \odot \mathbf{S}, \quad (35)$$

where $\mathbf{G} = \mathbf{F}^{-1}$. It is explained in a constructive way in [24], [26] that there exist R symmetric matrices $\{\mathbf{M}^{(r)}\}$ built from (34), admitting the factorizations

$$\mathbf{M}^{(r)} = \mathbf{G}\Lambda^{(r)}\mathbf{G}^T, \quad r \in \{1,\ldots,R\}, \quad (36)$$

where $\Lambda^{(r)} \in \mathbb{C}^{R\times R}$ are diagonal matrices. In the exact case, the SD problem (36) can be solved by means of an eigenvalue decomposition. Remark that if the signal matrix $\mathbf{S}$ is uncorrelated such that $\mathbf{S}^H\mathbf{S}$ is a diagonal matrix, then $\mathbf{C}^{[L_1,\ldots,L_N]}$ in (35) is columnwise orthogonal. In that case $\mathbf{G}$ in (36) reduces to a unitary matrix [25] (up to irrelevant column scaling factors).

After $\mathbf{F} = \mathbf{G}^{-1}$ has been found from (36) we compute $\mathbf{H} = \mathbf{U}\Sigma\mathbf{F}$ and partition it as follows

$$\mathbf{H} = \left[\mathbf{H}^{(1)T},\ldots,\mathbf{H}^{(N)T}\right]^T,\ \mathbf{H}^{(n)} \in \mathbb{C}^{2(\prod_{p=1,p\neq n}^{N} K_p)(K_n-1)\times R}. \quad (37)$$

The matrices $\mathbf{B}^{[n,K_1,\ldots,K_N]}$ and $\mathbf{A}^{(2,n)}$ can now be determined via the rank-1 approximations

$$\min_{\mathbf{b}_r^{[n,K_1,\ldots,K_N]},\mathbf{a}_r^{(2,n)}} \left\| \mathbf{h}_r^{(n)} - \mathbf{b}_r^{[n,K_1,\ldots,K_N]} \otimes \mathbf{a}_r^{(2,n)} \right\|_F^2, \quad (38)$$

where $r \in \{1,\ldots,R\}$ and $n \in \{1,\ldots,N\}$. The generators $\{z_{r,n}\}$ are readily obtained from the Vandermonde vectors in which they appear; in the exact case they are explicitly given by $z_{r,n} = a_{2r}^{(2,n)}/a_{1r}^{(2,n)}$, $r \in \{1,\ldots,R\}$, $n \in \{1,\ldots,N\}$.

Once the generators $\{z_{r,n}\}$ have been obtained, $\mathbf{S}$ follows immediately from (35). More specifically, we first build $\mathbf{a}_r^{(L_n,n)} = \left[1, z_{r,n}, z_{r,n}^2, \ldots, z_{r,n}^{L_n-1}\right]^T$ and compute $\mathbf{N} = \mathbf{V}^*\mathbf{G}^T$. The rth column of $\mathbf{S}$ is now given by

$$\begin{aligned} \mathbf{s}_r &= \left(\bigotimes_{n=1}^{N} \frac{\mathbf{a}_r^{(L_n,n)H}}{\mathbf{a}_r^{(L_n,n)H}\mathbf{a}_r^{(L_n,n)}} \otimes \mathbf{I}_K \right) \mathbf{n}_r \\ &= \left(\prod_{n=1}^{N} \mathbf{a}_r^{(L_n,n)H}\mathbf{a}_r^{(L_n,n)} \right)^{-1} \left(\bigotimes_{n=1}^{N} \mathbf{a}_r^{(L_n,n)H} \otimes \mathbf{I}_K \right) \mathbf{n}_r . \end{aligned}$$

Note that Algorithm 1 does not impose any constraints on the location of the generators. On the hand, if the generators are located on the unit circle, then we first build the tensors $\{\widehat{\mathcal{X}}_{(n)}^{(K_1,\ldots,K_N)}\}_{n=1}^N$ with matrix representations (26). Algorithm 1 applied on $\{\widehat{\mathcal{X}}_{(n)}^{(K_1,\ldots,K_N)}\}_{n=1}^N$ will be referred to as FBA-ESPRIT. Similarly, if the generators are located on the unit circle and $\mathbf{S}$ is strictly NC, then we can build the tensors $\{\mathcal{Z}_{(n)}^{(K_1,\ldots,K_N)}\}_{n=1}^N$ with matrix representations (31). Algorithm 1 applied on $\{\mathcal{Z}_{(n)}^{(K_1,\ldots,K_N)}\}_{n=1}^N$ will be referred to as NC-ESPRIT.

Algorithm 1 ESPRIT for MHR (Proposition III.2).

Input: $\mathbf{X} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\mathbf{S}^T$

1. Choose pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$.
2. Build $\mathbf{Y}^{(K_1,\ldots,K_N)}$ in (21).
3. Compute SVD $\mathbf{Y}^{(K_1,\ldots,K_N)} = \mathbf{U}\Sigma\mathbf{V}^H$.
4. Determine R from Σ.
5. Build matrices $\{\mathbf{M}^{(r)}\}$ from (34) [24], [26].
6. Solve SD problem $\mathbf{M}^{(r)} = \mathbf{G}\Lambda^{(r)}\mathbf{G}^T$, $r \in \{1,\ldots,R\}$.
7. Compute $\mathbf{H} = \mathbf{U}\Sigma\mathbf{G}^{-1}$ and $\mathbf{C}^{[L_1,\ldots,L_N]} = \mathbf{V}^*\mathbf{G}^T$.
8. Partition $\mathbf{H}$ according to (37).
9. Solve rank-1 approximation problems (38).
10. Obtain $\{z_{r,n}\}$.

Output: $\{z_{r,n}\}$

B. Unitary ESPRIT for MHR

Based on an isomorphism between centro-Hermitian[2] matrices and real matrices [12] more accurate and efficient implementations of ESPRIT for the case where the generators are located on the unit circle have been obtained (e.g. [7]). In accordance with [7] we call them unitary ESPRIT. Here we propose a unitary ESPRIT method for MHR.

1) Phase centered Vandermonde vectors: W.l.o.g. the phase reference of the Vandermonde vectors in (14) with unit norm generators can be changed from the first element to the center element. Indeed, if $I_n = 2p$ for some $p \in \mathbb{N}$, then the phase ambiguity in (14) can be exploited to obtain

$$\begin{aligned} \mathbf{a}_r^{(n)} &= e^{-i(p-\frac{1}{2})\theta_{r,n}} \left[1, e^{i\theta_{r,n}}, e^{i2\theta_{r,n}}, \ldots, e^{i(I_n-1)\theta_{r,n}}\right]^T \\ &= \left[e^{-i(p-\frac{1}{2})\theta_{r,n}}, \ldots, e^{-i\frac{1}{2}\theta_{r,n}}, e^{i\frac{1}{2}\theta_{r,n}}, \ldots, e^{i(p-\frac{1}{2})\theta_{r,n}}\right]^T \\ &= \left[\check{\mathbf{a}}_r^{(n)T}, \check{\mathbf{a}}_r^{(n)H}\mathbf{J}_p^T\right]^T \in \mathbb{C}^{I_n}, \quad \theta_{r,n} \in \mathbb{R},\ \check{\mathbf{a}}_r^{(n)} \in \mathbb{C}^p. \quad (39) \end{aligned}$$

[2] A matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ is called centro-Hermitian if $\mathbf{A} = \mathbf{J}_m\mathbf{A}^*\mathbf{J}_n$.

Note that the center element has been chosen as the phase reference center. Similarly, if $I_n = 2p+1$ for some $p \in \mathbb{N}$, then $\mathbf{a}_r^{(n)}$ in (14) can take the form

$$\begin{aligned}\mathbf{a}_r^{(n)} &= e^{-ip\theta_{r,n}}\left[1, e^{i\theta_{r,n}}, e^{i2\theta_{r,n}}, \ldots, e^{i(I_n-1)\theta_{r,n}}\right]^T \\ &= \left[e^{-ip\theta_{r,n}}, \ldots, e^{-i\theta_{r,n}}, 1, e^{i\theta_{r,n}}, \ldots, e^{ip\theta_{r,n}}\right]^T \\ &= \left[\check{\mathbf{a}}_r^{(n)T}, 1, \check{\mathbf{a}}_r^{(n)H}\mathbf{J}_p^T\right]^T \in \mathbb{C}^{I_n}, \quad \theta_{r,n} \in \mathbb{R},\ \check{\mathbf{a}}_r^{(n)} \in \mathbb{C}^p. \end{aligned} \quad (40)$$

Remark that phase-centered Vandermonde vectors of the form (39)/(40) satisfy the relation $\mathbf{J}_{I_n}\mathbf{a}_r^{(n)*} = \mathbf{a}_r^{(n)}$. Define $\mathbf{D}^{(n)} = D_1([e^{i\phi_{1,n}}, \ldots, e^{i\phi_{R,n}}])$ in which $\phi_{r,n} = (p-\frac{1}{2})\theta_{r,n}$ if $I_n = 2p+1$ for some $p \in \mathbb{N}$ and $\phi_{r,n} = p\theta_{r,n}$ if $I_n = 2p$ for some $p \in \mathbb{N}$. We now consider the following phase-centered expression of (15):

$$\mathbf{X} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\mathbf{D}\mathbf{S}^T \in \mathbb{C}^{(\prod_{n=1}^N I_n)\times K}, \quad (41)$$

where the columns of $\{\mathbf{A}^{(n)}\}_{n=1}^N$ are of the form (39)/(40) and

$$\mathbf{D} = \prod_{n=1}^N \mathbf{D}^{(n)} = D_1([e^{i(\sum_{n=1}^N \phi_{1,n})}, \ldots, e^{i(\sum_{n=1}^N \phi_{R,n})}]). \quad (42)$$

2) Spatial smoothing: In the case of phase centered Vandermonde matrices, spatial smoothing yields (similar to (3)/(4)) factor matrices

$$\mathbf{A}^{(L_n,n)} = \left[\mathbf{a}_1^{(L_n,n)}, \ldots, \mathbf{a}_R^{(L_n,n)}\right],\ \mathbf{a}_r^{(L_n,n)} = \left[z_{r,n}^{(1-L_n)/2}, \ldots, z_{r,n}^{(L_n-1)/2}\right]^T,$$
$$\mathbf{A}^{(K_n,n)} = \left[\mathbf{a}_1^{(K_n,n)}, \ldots, \mathbf{a}_R^{(K_n,n)}\right],\ \mathbf{a}_r^{(K_n,n)} = \left[z_{r,n}^{(1-K_n)/2}, \ldots, z_{r,n}^{(K_n-1)/2}\right]^T$$

with $z_{r,n} = e^{i\theta_{r,n}}$ and $K_n + L_n = I_n + 1$. Spatial smoothing generates tensors $\mathcal{X}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\times_{p=1}^{n-1}K_p)\times(K_n-1)\times 2\times(\times_{q=n+1}^N K_q)\times(\times_{s=1}^N L_s)\times K}$, $n \in \{1,\ldots,N\}$, for which (18) still holds, i.e.,

$$\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} = \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\mathbf{C}^{[L_1,\ldots,L_N]T}, \quad (43)$$

provided $\mathbf{C}^{[L_1,\ldots,L_N]} = \bigodot_{n=1}^N \mathbf{A}^{(L_n,n)} \odot \mathbf{S}\mathbf{D}$ is substituted and provided phase centered Vandermonde matrices $\{\mathbf{A}^{(n)}\}$ are used.

3) From complex to real SD: Let $p \in \mathbb{N}$ and consider the unitary matrices

$$\mathbf{Q}_{2p} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_p & i\mathbf{I}_p \\ \mathbf{J}_p & -i\mathbf{J}_p \end{bmatrix}, \quad \mathbf{Q}_{2p+1} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_p & \mathbf{0}_p & i\mathbf{I}_p \\ \mathbf{0}_p^T & \sqrt{2} & \mathbf{0}_p^T \\ \mathbf{J}_p & \mathbf{0}_p & -i\mathbf{J}_p \end{bmatrix}.$$

It was observed in [9], [13] that these unitary matrices transform a phase centered Vandermonde matrix into a real matrix. Namely, if $K_n = 2p$, then

$$\mathbf{d}_r^{(K_n,n)} = \mathbf{Q}_{K_n}^H\mathbf{a}_r^{(K_n,n)} = \mathbf{Q}_{K_n}^H\begin{bmatrix} \check{\mathbf{a}}_r^{(K_n,n)} \\ \mathbf{J}_p\check{\mathbf{a}}_r^{(K_n,n)*} \end{bmatrix} = \frac{2}{\sqrt{2}}\begin{bmatrix} \mathrm{Re}\left\{\check{\mathbf{a}}_r^{(K_n,n)}\right\} \\ \mathrm{Im}\{\check{\mathbf{a}}_r^{(K_n,n)}\} \end{bmatrix} \quad (44)$$

with $\mathrm{Re}\left\{\check{\mathbf{a}}_r^{(K_n,n)}\right\} = \left[\cos((2p-1)\theta_{r,n}/2), \ldots, \cos(\theta_{r,n}/2)\right]^T$ and $\mathrm{Im}\{\check{\mathbf{a}}_r^{(K_n,n)}\} = -\left[\sin((2p-1)\theta_{r,n}/2), \ldots, \sin(\theta_{r,n}/2)\right]^T$.

Similarly, if $K_n = 2p+1$, then

$$\mathbf{d}_r^{(K_n,n)} = \mathbf{Q}_{K_n}^H\mathbf{a}_r^{(K_n,n)} = \mathbf{Q}_{K_n}^H\begin{bmatrix} \check{\mathbf{a}}_r^{(K_n,n)} \\ 1 \\ \mathbf{J}_p\check{\mathbf{a}}_r^{(K_n,n)*} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}}\mathrm{Re}\left\{\check{\mathbf{a}}_r^{(K_n,n)}\right\} \\ 1 \\ \frac{2}{\sqrt{2}}\mathrm{Im}\{\check{\mathbf{a}}_r^{(K_n,n)}\} \end{bmatrix} \quad (45)$$

with $\mathrm{Re}\left\{\check{\mathbf{a}}_r^{(K_n,n)}\right\} = \left[\cos(p\theta_{r,n}), \ldots, \cos(2\theta_{r,n}), \cos(\theta_{r,n})\right]^T$ and $\mathrm{Im}\{\check{\mathbf{a}}_r^{(K_n,n)}\} = -\left[\sin(p\theta_{r,n}), \ldots, \sin(2\theta_{r,n}), \sin(\theta_{r,n})\right]^T$. Denote

$$\mathbf{Q}^{[n,K_1,\ldots,K_N]} = \left(\bigotimes_{p=1}^{n-1}\mathbf{Q}_{K_p}\right) \otimes \mathbf{Q}_{K_n-1} \otimes \left(\bigotimes_{p=n+1}^{N}\mathbf{Q}_{K_p}\right). \quad (46)$$

We can now transform (43) into $(\mathbf{Q}^{[n,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} = (\mathbf{D}^{[n,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,n)})\mathbf{C}^{[L_1,\ldots,L_N]T}$, where

$$\mathbf{D}^{[n,K_1,\ldots,K_N]} = \left(\bigodot_{p=1}^{n-1}\mathbf{D}^{(K_p,p)}\right) \odot \mathbf{D}^{(K_n-1,n)} \odot \left(\bigodot_{p=n+1}^{N}\mathbf{D}^{(K_p,p)}\right), \quad (47)$$

$$\mathbf{D}^{(2,n)} = \frac{2}{\sqrt{2}}\begin{bmatrix} \cos(\theta_{1,n}/2), & \cdots, & \cos(\theta_{R,n}/2) \\ -\sin(\theta_{1,n}/2), & \cdots, & -\sin(\theta_{R,n}/2) \end{bmatrix}, \quad (48)$$

and $\mathbf{D}^{(K_n,n)} = \mathbf{Q}_{K_n}^H\mathbf{A}^{(K_n,n)}$ with columns (44)/(45). In other words, the complex phase-centered Vandermonde matrices $\{\mathbf{A}^{(K_n,n)}, \mathbf{A}^{(2,n)}\}_{n=1}^N$ have been transformed into the real valued matrices $\{\mathbf{D}^{(K_n,n)}, \mathbf{D}^{(2,n)}\}_{n=1}^N$. However, since $\mathbf{C}^{[L_1,\ldots,L_N]}$ is complex, $(\mathbf{Q}^{[n,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)}$ is still complex. We will now capitalize on the relation between centro-Hermitian and real matrices. It is well-known that the matrix $[\mathbf{A}, \mathbf{J}_m\mathbf{A}^*\mathbf{J}_n]$ is centro-Hermitian for any $\mathbf{A} \in \mathbb{C}^{m\times n}$. This motivates the construction of

$$\begin{aligned}\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)} &= \left[\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)},\ \mathbf{J}_\alpha\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)*}\mathbf{J}_{2K\prod_{n=1}^N L_n}\right] \\ &= \widehat{\mathbf{X}}_{(n)}^{(K_1,\ldots,K_N)}\mathbf{D}_{\mathbf{W}}, \quad n \in \{1,\ldots,N\},\end{aligned} \quad (49)$$

where $\alpha = 2(\prod_{p=1,p\neq n}^N K_p)(K_n-1)$, $\mathbf{D}_{\mathbf{W}} = \begin{bmatrix} \mathbf{I}_{2K\prod_{n=1}^N L_n} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{2K\prod_{n=1}^N L_n} \end{bmatrix}$ and $\widehat{\mathbf{X}}_{(n)}^{(K_1,\ldots,K_N)}$ is of the form (26) with $\widehat{\mathbf{C}}^{[L_1,\ldots,L_N]} = \mathbf{A}^{(L_1,1)} \odot \cdots \odot \mathbf{A}^{(L_N,N)} \odot \begin{bmatrix} \mathbf{S}\mathbf{D} \\ \mathbf{S}^*\mathbf{D}^* \end{bmatrix}$. Recall that since (49) is centro-Hermitian, the matrix $\mathbf{A}_1^H\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)}\mathbf{A}_2^*$ is real for any $\mathbf{J}$-real[3] matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ [12]. Since the matrices $\mathbf{Q}^{[n,K_1,\ldots,K_N]} \otimes \mathbf{Q}_2$ and $\mathbf{Q}_{2K\prod_{n=1}^N L_n}$ are $\mathbf{J}$-real we obtain the real tensors $\{\mathcal{W}_{(1)}^{(K_1,\ldots,K_N)}, \ldots, \mathcal{W}_{(N)}^{(K_1,\ldots,K_N)}\}$ and a real coupled decomposition of $\mathbf{W}^{(K_1,\ldots,K_N)} = [\mathbf{W}_{(1)}^{(K_1,\ldots,K_N)T}, \ldots, \mathbf{W}_{(N)}^{(K_1,\ldots,K_N)T}]^T$:

$$\begin{aligned}\mathbf{W}^{(K_1,\ldots,K_N)} &= \begin{bmatrix} (\mathbf{Q}^{[1,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\mathbf{Z}_{(1)}^{(K_1,\ldots,K_N)} \\ \vdots \\ (\mathbf{Q}^{[N,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\mathbf{Z}_{(N)}^{(K_1,\ldots,K_N)} \end{bmatrix}\mathbf{Q}_{2K\prod_{n=1}^N L_n}^* \\ &= \begin{bmatrix} \mathbf{D}^{[1,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,1)} \\ \vdots \\ \mathbf{D}^{[N,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,N)} \end{bmatrix}\widehat{\mathbf{C}}^{[L_1,\ldots,L_N]T}\mathbf{D}_{\mathbf{W}}\mathbf{Q}_{2K\prod_{n=1}^N L_n}^*.\end{aligned} \quad (50)$$

[3] A matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ is called $\mathbf{J}$-real if $\mathbf{J}_m\mathbf{A}^* = \mathbf{A}$.

Note that the unitary transformations $\mathbf{Q}^{[n,K_1,\ldots,K_N]} \otimes \mathbf{Q}_2$, $\mathbf{D_W}$ and $\mathbf{Q}_{2K\prod_{n=1}^N L_n}$ do not affect the coupled rank nor the uniqueness of the coupled CPD. Assume that the conditions stated in Proposition III.4 are satisfied. Then $\mathbf{W}^{(K_1,\ldots,K_N)}$ in (50) has full column rank [23]. Thus,

$$\text{range}\left(\mathbf{W}^{(K_1,\ldots,K_N)}\right) = \text{range}\left(\begin{bmatrix} \mathbf{D}^{[1,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,1)} \\ \vdots \\ \mathbf{D}^{[N,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,N)} \end{bmatrix}\right),$$

the coupled rank of $\{\mathcal{W}_{(n)}^{(K_1,\ldots,K_N)}\}$ is R and the coupled CPD of $\{\mathcal{W}_{(n)}^{(K_1,\ldots,K_N)}\}$ is unique. We can now proceed with $\mathbf{W}^{(K_1,\ldots,K_N)}$ as with $\mathbf{Y}^{(K_1,\ldots,K_N)}$ in subsection IV-A. All matrices involved are real. The generators $\{z_{r,n}\}$ are readily obtained from the Vandermonde vectors in which they appear; in the exact case they are explicitly given by $z_{r,n} = e^{-i\cdot 2\tan^{-1}(d_{2r}^{(2,n)}/d_{1r}^{(2,n)})}$, $r \in \{1,\ldots,R\}$, $n \in \{1,\ldots,N\}$. The unitary ESPRIT procedure for MHR is summarized as Algorithm 2.

Remark that if $\mathbf{S}^H\mathbf{S}$ is diagonal, then $\mathbf{G}$ in Algorithm 2 reduces to a real orthogonal matrix. Once the generators have been obtained, then $\mathbf{S}$ can be found via a similar procedure as described in subsection IV-A.

Algorithm 2 Unitary ESPRIT for MHR (Prop. III.4).

Input: $\mathbf{X} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\mathbf{S}^T$

1. Choose pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$.
2. Build real matrix $\mathbf{W}^{(K_1,\ldots,K_N)}$ in (50).
3. Compute real SVD $\mathbf{W}^{(K_1,\ldots,K_N)} = \mathbf{U}\Sigma\mathbf{V}^T$.
4. Determine R from Σ.
5. Build real matrices $\{\mathbf{M}^{(r)}\}$ from $\mathbf{U}\Sigma$ [24], [26].
6. Solve real SD problem $\mathbf{M}^{(r)} = \mathbf{G}\Lambda^{(r)}\mathbf{G}^T$, $r \in \{1,\ldots,R\}$, where $\Lambda^{(r)} \in \mathbb{R}^{R\times R}$ is diagonal and $\mathbf{G} \in \mathbb{R}^{R\times R}$ is nonsingular.
7. Compute $\mathbf{H} = \mathbf{U}\Sigma\mathbf{G}^{-1}$ and $\widetilde{\mathbf{C}}^{[L_1,\ldots,L_N]} = \mathbf{D_W}\mathbf{Q}_{2K\prod_{n=1}^N L_n}\mathbf{V}\mathbf{G}^T$.
8. Partition $\mathbf{H} = \left[\mathbf{H}^{(1)T},\ldots,\mathbf{H}^{(N)T}\right]^T$, $\mathbf{H}^{(n)} \in \mathbb{R}^{2(\prod_{p=1,p\neq n}^N K_p)(K_n-1)\times R}$.
9. Solve rank-1 approximation problems: $\min_{\mathbf{d}_r^{[n,K_1,\ldots,K_N]},\mathbf{d}_r^{(2,n)}} \left\|\mathbf{h}_r^{(n)} - \mathbf{d}_r^{[n,K_1,\ldots,K_N]} \otimes \mathbf{d}_r^{(2,n)}\right\|_F^2$, $\forall r, n$.
10. Obtain $\{z_{r,n}\}$.

Output: $\{z_{r,n}\}$

C. Unitary NC-ESPRIT for MHR

Here we develop, as an alternative to unitary ESPRIT, the unitary NC-ESPRIT method for the case of strictly NC signals. In the same way as in subsection IV-B we use spatial smoothing to generate $\mathcal{X}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\times_{p=1}^{n-1}K_p)\times(K_n-1)\times 2\times(\times_{q=n+1}^N K_q)\times(\times_{s=1}^N L_s)\times K}$ with matrix representation (43). Using $\mathbf{S} = \mathbf{S}_0\mathbf{D}_S$ we obtain

$$\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)} = \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\mathbf{D}\mathbf{D}_S\mathbf{C}_0^{[L_1,\ldots,L_N]T}, \quad (51)$$

where $\mathbf{D}$ is given by (42). Analogous to subsection III-D we obtain from (51) the additional tensors $\mathcal{Y}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\times_{p=1}^{n-1}K_p)\times(K_n-1)\times 2\times(\times_{q=n+1}^N K_q)\times(\times_{s=1}^N L_s)\times K}$, $n \in \{1,\ldots,N\}$, subject to $K_n + L_n = I_n + 1$ and with matrix representation $\mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)} \in \mathbb{C}^{(\prod_{p=1,p\neq n}^N K_p)(K_n-1)\times 2\times(\prod_{q=1}^N L_q)K}$:

$$\begin{aligned}\mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)} &= (\mathbf{J}_{\prod_{p=1,p\neq n}^N K_p(K_n-1)} \otimes \mathbf{J}_2)\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)*}\left(\mathbf{J}_{\prod_{n=1}^N L_n} \otimes \mathbf{I}_K\right)^T \\ &= \left(\mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\mathbf{D}^*\mathbf{D}_S^*\mathbf{C}_0^{[L_1,\ldots,L_N]T}, \quad (52)\end{aligned}$$

where we exploited that $\mathbf{J}_{\prod_{p=1,p\neq n}^N K_p(K_n-1)}\mathbf{B}^{[n,K_1,\ldots,K_N]*} = \mathbf{B}^{[n,K_1,\ldots,K_N]}$, $\mathbf{J}_2\mathbf{A}^{(2,n)*} = \mathbf{A}^{(2,n)}$ and $(\mathbf{J}_{\prod_{n=1}^N L_n} \otimes \mathbf{I}_K)\mathbf{C}_0^{[L_1,\ldots,L_N]*} = \mathbf{C}_0^{[L_1,\ldots,L_N]}$ in the case of phase-centered Vandermonde matrices. Stacking $\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)}$ and $\mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)}$ yields a tensor $\mathcal{Z}_{(n)}^{(K_1,\ldots,K_N)}$ with matrix representation

$$\begin{aligned}\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)} &= \left[\mathbf{X}_{(n)}^{(K_1,\ldots,K_N)T}, \mathbf{Y}_{(n)}^{(K_1,\ldots,K_N)T}\right]^T \\ &= \left(\widetilde{\mathbf{D}} \odot \mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)}\right)\mathbf{C}_0^{[L_1,\ldots,L_N]T}, \quad (53)\end{aligned}$$

where $\widetilde{\mathbf{D}} = \begin{bmatrix} \text{Vecd}(\mathbf{D_S}\mathbf{D})^T \\ \text{Vecd}\left(\mathbf{D_S}^*\right)^T \end{bmatrix} = \begin{bmatrix} e^{i\psi_1}, & \cdots, & e^{i\psi_R} \\ e^{-i\psi_1}, & \cdots, & e^{-i\psi_R} \end{bmatrix} \in \mathbb{C}^{2\times R}$ with $\psi_r = \varphi_r + \sum_{n=1}^N \phi_{r,n}$ in which the structure of $\mathbf{D}$ in (42) and $\mathbf{D_S} = D_1\left([e^{i\varphi_1},\ldots,e^{i\varphi_R}]\right)$ have been taken into account. Similar to unitary ESPRIT, we compute $(\mathbf{Q}_2^H \otimes \mathbf{Q}^{[n,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)} = (\mathbf{E} \odot \mathbf{D}^{[n,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,n)})\mathbf{C}_0^{[L_1,\ldots,L_N]T}$, where $\mathbf{Q}^{[n,K_1,\ldots,K_N]}$, $\mathbf{D}^{[n,K_1,\ldots,K_N]}$ and $\mathbf{D}^{(2,n)}$ are given by (46)–(48) and $\mathbf{E} = \mathbf{Q}_2^H\widetilde{\mathbf{D}} = \frac{2}{\sqrt{2}}\begin{bmatrix} \cos(\psi_1), & \cdots, & \cos(\psi_R) \\ \sin(\psi_1), & \cdots, & \sin(\psi_R) \end{bmatrix}$. From (53) we obtain the centro-Hermitian matrices

$$\begin{aligned}\widetilde{\mathbf{Z}}_{(n)}^{(K_1,\ldots,K_N)} &= [\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)}, \mathbf{J}_\alpha\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)*}\mathbf{J}_{K\prod_{n=1}^N L_n}] \\ &= (\widetilde{\mathbf{D}} \odot \mathbf{B}^{[n,K_1,\ldots,K_N]} \odot \mathbf{A}^{(2,n)})\widetilde{\mathbf{C}}_0^{[L_1,\ldots,L_N]T},\end{aligned}$$

where $\alpha = 4(\prod_{p=1,p\neq n}^N K_p)(K_n - 1)$ and $\widetilde{\mathbf{C}}_0^{[L_1,\ldots,L_N]} = [\mathbf{C}_0^{[L_1,\ldots,L_N]T}, \mathbf{C}_0^{[L_1,\ldots,L_N]T}(\mathbf{I}_{\prod_{n=1}^N L_n} \otimes \mathbf{J}_K)]$. In addition to the relations exploited in (52), we also used $(\mathbf{J}_2 \otimes \mathbf{J}_{(\prod_{p=1,p\neq n}^N K_p)(K_n-1)} \otimes \mathbf{J}_2)\mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)*}(\mathbf{J}_{\prod_{n=1}^N L_n} \otimes \mathbf{I}_K) = \mathbf{Z}_{(n)}^{(K_1,\ldots,K_N)}$, $(\mathbf{J}_{\prod_{n=1}^N L_n} \otimes \mathbf{J}_K)\mathbf{C}_0^{[L_1,\ldots,L_N]*} = (\mathbf{I}_{\prod_{n=1}^N L_n} \otimes \mathbf{J}_K)\mathbf{C}_0^{[L_1,\ldots,L_N]}$, $\mathbf{J}_K = \mathbf{J}_K^T$ and the commutativity in (1). Since $\mathbf{Q}_2 \otimes \mathbf{Q}^{[n,K_1,\ldots,K_N]} \otimes \mathbf{Q}_2$ and $\mathbf{Q}_{2K\prod_{n=1}^N L_n}$ are $\mathbf{J}$-real we obtain the real coupled decomposition of the tensors $\{\mathcal{W}_{(n)}^{(K_1,\ldots,K_N)}\}_{n=1}^N$ with matrix representation $\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)} = [\mathbf{W}_{(1)}^{(K_1,\ldots,K_N)T},\ldots,\mathbf{W}_{(N)}^{(K_1,\ldots,K_N)T}]^T$:

$$\begin{aligned}\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)} &= \begin{bmatrix} (\mathbf{Q}_2^H \otimes \mathbf{Q}^{[1,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\widetilde{\mathbf{Z}}_{(1)}^{(K_1,\ldots,K_N)} \\ \vdots \\ (\mathbf{Q}_2^H \otimes \mathbf{Q}^{[N,K_1,\ldots,K_N]H} \otimes \mathbf{Q}_2^H)\widetilde{\mathbf{Z}}_{(N)}^{(K_1,\ldots,K_N)} \end{bmatrix}\mathbf{Q}_\beta^* \\ &= \begin{bmatrix} \mathbf{E}^{[1,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,1)} \\ \vdots \\ \mathbf{E}^{[N,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,N)} \end{bmatrix}\widetilde{\mathbf{C}}_0^{[L_1,\ldots,L_N]T}\mathbf{Q}_\beta^*, \quad (54)\end{aligned}$$

where $\mathbf{E}^{[n,K_1,\ldots,K_N]} = \mathbf{E} \odot \mathbf{D}^{[n,K_1,\ldots,K_N]}$ and $\beta = 2K\prod_{n=1}^N L_n$. The unitary transformations $\mathbf{Q}_2 \otimes \mathbf{Q}^{[n,K_1,\ldots,K_N]} \otimes \mathbf{Q}_2$ and $\mathbf{Q}_{2K\prod_{n=1}^N L_n}$ do not affect the coupled rank nor the uniqueness of the coupled CPD. Note also that the rank of $\widetilde{\mathbf{C}}_0^{[L_1,\ldots,L_N]}$ is equal to the rank of $\mathbf{C}_0^{[L_1,\ldots,L_N]}$. Taking these considerations into account, it can be proven that under

the conditions in Proposition III.5, $\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)}$ in (54) has full column rank [23]. Hence,

$$\text{range}\left(\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)}\right) = \text{range}\left(\begin{bmatrix} \mathbf{E}^{[1,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,1)} \\ \vdots \\ \mathbf{E}^{[N,K_1,\ldots,K_N]} \odot \mathbf{D}^{(2,N)} \end{bmatrix}\right),$$

the coupled rank of $\{\mathcal{W}_{(n)}^{(K_1,\ldots,K_N)}\}$ is R and the coupled CPD of $\{\mathcal{W}_{(n)}^{(K_1,\ldots,K_N)}\}$ is unique. We can now proceed with $\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)}$ as with $\mathbf{Y}^{(K_1,\ldots,K_N)}$ in subsection IV-A. All matrices involved are real. The procedure is summarized as Algorithm 3.

Remark that if $\mathbf{S}_0^H\mathbf{S}_0$ is diagonal, then $\mathbf{G}$ in Algorithm 3 reduces to a real orthogonal matrix. Once the generators have been obtained, $\mathbf{S}$ can be found via a similar procedure as described in subsection IV-A.

Algorithm 3 Unitary NC-ESPRIT for MHR (Prop. III.5).

Input: $\mathbf{X} = \left(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(N)}\right)\mathbf{S}^T$

1. Choose pairs $\{(K_n, L_n)\}$ subject to $K_n + L_n = I_n + 1$.
2. Build real matrix $\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)}$ in (54).
3. Compute real SVD $\widetilde{\mathbf{W}}^{(K_1,\ldots,K_N)} = \mathbf{U}\Sigma\mathbf{V}^T$.
4. Determine R from Σ.
5. Build real matrices $\{\mathbf{M}^{(r)}\}$ from $\mathbf{U}\Sigma$ [24], [26].
6. Solve real SD problem $\mathbf{M}^{(r)} = \mathbf{G}\Lambda^{(r)}\mathbf{G}^T$, $r \in \{1,\ldots,R\}$, where $\Lambda^{(r)} \in \mathbb{R}^{R\times R}$ is diagonal and $\mathbf{G} \in \mathbb{R}^{R\times R}$ is nonsingular.
7. Compute $\mathbf{H} = \mathbf{U}\Sigma\mathbf{G}^{-1}$ and $\widetilde{\mathbf{C}}_0^{[L_1,\ldots,L_N]} = \mathbf{Q}_{2K\prod_{n=1}^N L_n}\mathbf{V}\mathbf{G}^T$.
8. Partition $\mathbf{H} = \left[\mathbf{H}^{(1)T},\ldots,\mathbf{H}^{(N)T}\right]^T$, $\mathbf{H}^{(n)} \in \mathbb{R}^{4(\prod_{p=1,p\neq n}^N K_p)(K_n-1)\times R}$.
9. Solve rank-1 approximation problems: $\min_{\mathbf{e}_r^{[n,K_1,\ldots,K_N]},\mathbf{d}_r^{(2,n)}} \|\mathbf{h}_r^{(n)} - \mathbf{e}_r^{[n,K_1,\ldots,K_N]} \otimes \mathbf{d}_r^{(2,n)}\|_F^2, \ \forall r, n.$
10. Obtain $\{z_{r,n}\}$.

Output: $\{z_{r,n}\}$

V. Numerical experiments

Consider the PD of $\mathcal{X}$ in (14). The goal is to estimate the generators $\{z_{r,n}\}$ from $\mathcal{T} = \mathcal{X} + \beta\mathcal{N}$, where $\mathcal{N}$ is an unstructured perturbation tensor and $\beta \in \mathbb{R}$ controls the noise level. The generators $\{z_{r,n}\}$ are randomly chosen on the unit circle. The real and imaginary entries of the perturbation tensors are randomly drawn from a Gaussian distribution with zero mean and unit variance.

The following Signal-to-Noise Ratio (SNR) measure will be used: SNR [dB] $= 10\log(\|\mathbf{X}_{(1)}\|_F^2/\|\beta\mathbf{N}_{(1)}\|_F^2)$. The performance evaluation will be based on the distance between the true generators $\{z_{r,n}\}$ and their estimates $\{\hat{z}_{r,n}\}$. The distance is measured according to

$$\text{E}_{\text{RMS}} = \min_{\sigma\in\Lambda_R} \sqrt{\frac{1}{NR}\sum_{n=1}^{N}\sum_{r=1}^{R}\left(z_{r,n} - \hat{z}_{\sigma(r),n}\right)^*\left(z_{r,n} - \hat{z}_{\sigma(r),n}\right)},$$

where Λ_R denotes the set of $R!$ possible permutations of the elements $\{1,\ldots,R\}$ and $\sigma(r)$ denotes the rth element of the corresponding permuted set. For the SD step in Algorithms 1–3 we use a simultaneous generalized Schur decomposition method [21].

A. Single snapshot ($K = 1$) 2D HR

We compare the FBA-ESPRIT and unitary ESPRIT methods with the MultiDimensional Folding (MDF) method developed in [16] for single-snapshot MHR.[4] The model parameters are $I_1 = 6$, $I_2 = 6$, $K = 1$ and $R = 6$. Since $K = 1$ the choice of spatial smoothing parameters K_1 and K_2 in Algorithms 1 and 2 is critical. We try out all possible pairs (K_1, K_2) and retain the pair that yields the best least-squares fit to $\mathcal{T}$. The mean E_{RMS} value over 200 trials as a function of SNR can be seen in figure 1. The FBA-ESPRIT and unitary ESPRIT methods yield the same performance. On average $(K_1, K_2) = (4, 4)$ produced the best fit. We also observe that Algorithms 1 and 2 perform better than the CPD based MDF method.

B. Multiple snapshot ($K > 1$) 2D HR and BPSK signals

We compare the NC-ESPRIT, unitary NC-ESPRIT and unitary ESPRIT methods in the case of BPSK signals, i.e., $\mathbf{S} = \mathbf{S}_0\mathbf{D}_\mathbf{S}$ with randomly drawn $\mathbf{S}_0 \in \{-1, 1\}^{K\times R}$ and randomly drawn unit norm diagonal elements of $\mathbf{D}_\mathbf{S}$. The model parameters are $I_1 = 3$, $I_2 = 3$, $K = 25$ and $R = 6$. Since K is large we fix $(K_1, K_2) = (3, 3)$ in Algorithms 1–3. The mean E_{RMS} value over 200 trials as a function of SNR can be seen in figure 2. We notice that the NC-ESPRIT and unitary NC-ESPRIT methods yield the same performance while the unitary ESPRIT method performs significantly worse. The reason is that the latter method ignores the structure of $\mathbf{S}$.

VI. Conclusion

MHR is an important problem in signal processing that is however not fully understood. We have first provided a link between the MHR problem and the coupled CPD. This has led to a new uniqueness condition for MHR. To the authors' knowledge, this is the most relaxed uniqueness condition for MHR that has been reported so far.

In the coupled CPD framework we have also derived a new algebraic method for MHR that under mild conditions is guaranteed to find the decomposition in the exact case. The algorithm can be interpreted as an MHR extension of the ESPRIT method. We have also developed MHR variants of unitary ESPRIT and unitary NC-ESPRIT via the coupled CPD framework.

We have limited the exposition to PD with Vandermonde factor matrices. The results can be extended to PD with generalized Vandermonde factor matrices representing exponential polynomials [1], [2].

References

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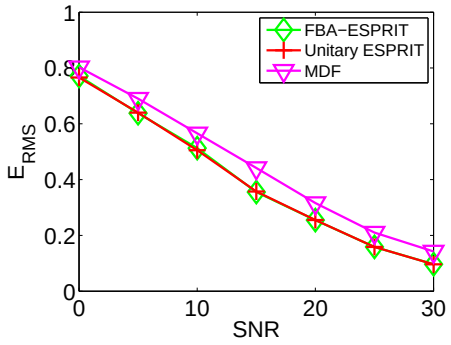

Fig. 1. Mean E_{RMS} for varying SNR, single snapshot case.

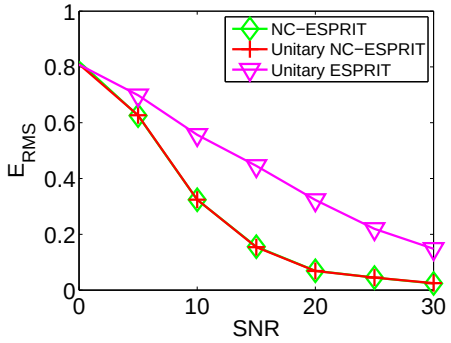

Fig. 2. Mean E_{RMS} for varying SNR, multiple snapshot case.

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