Tensor decompositions with a block-Toeplitz factor matrix

Mikael Sørensen and Lieven De Lathawuer

The intention of this report is to provide more details and discussion of the results presented in the manuscript [4]. Hence, references to equations and definitions in the manuscript [4] will be made throughout the paper.

1 From block-Toeplitz to coupled low-rank factorizations using subspace methods

In [4, Section III.A] we explained how to turn a block-Toeplitz factorization into coupled low-rank factorization. Here we provide more details on how this step can be implemented using subspace methods.

Recall that in [4] we considered factorizations of the form

$$\mathbf{X} = \mathbf{M}_{\mathrm{BT}}\mathbf{S}_{\mathrm{BT}}^{T}, \tag{1}$$

in which $\mathbf{M}_{\mathrm{BT}} \in \mathbb{C}^{I\times LR}$ is subject to a low-rank constraint and $\mathbf{S}_{\mathrm{BT}} \in \mathbb{C}^{K\times LR}$ is a block-Toeplitz matrix defined as

$$\mathbf{S}_{\mathrm{BT}} = \sum_{k=1}^{K+L-1} \mathbf{E}_k \otimes \mathbf{p}_k \in \mathbb{C}^{K\times LR}, \tag{2}$$

where $\mathbf{p}_k \in \mathbb{C}^{1\times R}$ and where $\mathbf{E}_k \in \mathbb{C}^{K\times L}$ are Toeplitz basis matrices of the form

$$(\mathbf{E}_k)_{ij} = \begin{cases} 1, & i = j + k - L, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The variables $\{\mathbf{p}_k\}$ in (2) can be stacked into the parameter matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_{K+L-1} \end{bmatrix} \in \mathbb{C}^{(K+L-1)\times R}. \tag{4}$$

Temporal smoothing applied on $\mathbf{X}$ yields (see [4, Section II] for details):

$$\mathbf{X}_{\mathrm{BT,sm}}^{(m)} = \mathbf{M}_{\mathrm{BT,sm}}^{(m)}\mathbf{S}_{\mathrm{SM,sm}}^{(m)T} \in \mathbb{C}^{Im\times(K-m+1)},$$

where $\mathbf{M}_{\mathrm{BT,sm}}^{(m)} \in \mathbb{C}^{Im\times(L+m-1)R}$ is defined by [4, eq. (20)] and where $\mathbf{S}_{\mathrm{SM,sm}}^{(m)} \in \mathbb{C}^{(K-m+1)\times(L+m-1)R}$ is defined by [4, eq. (21)]. The following subsections discuss subspace methods that can find $\mathbf{P}$ up to a nonsingular matrix from the knowledge of range $\left(\mathbf{X}_{\mathrm{SM,sm}}^{(m)T}\right)$.

1.1 Subspace method

Let $\mathbf{X}_{\mathrm{BT,sm}}^{(m)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{X}_{\mathrm{BT;sm}}^{(m)}$, where $\mathbf{V} \in \mathbb{C}^{(K-m+1)\times(L+m-1)R}$, then there exists a nonsingular matrix $\mathbf{N} \in \mathbb{C}^{(L+m-1)R\times(L+m-1)R}$ such that

$$\mathbf{S}_{\mathrm{BT,sm}}^{(m)} = \sum_{k=1}^{K+L-1} \mathbf{E}_k^{(m)} \otimes \mathbf{p}_k = \mathbf{V}^*\mathbf{N}, \tag{5}$$

where $\mathbf{E}_k^{(m)} \in \mathbb{C}^{(K-m+1)\times(L+m-1)}$, $k \in \{1,\ldots,K+L-1\}$, are the temporally smoothed Toeplitz basis matrices of the form

$$(\mathbf{E}_k^{(m)})_{ij} = \begin{cases} 1, & i = j+k-L-m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Partition $\mathbf{N}$ into the following blocks

$$\mathbf{N} = [\mathbf{N}_1,\ldots,\mathbf{N}_{L+m-1}]\,, \quad \mathbf{N}_l \in \mathbb{C}^{(L+m-1)R\times R}, \quad l \in \{1,\ldots,L+m-1\}. \tag{6}$$

Relation (5) can be written as

$$\left[\mathbf{I}_{L+m-1}\otimes\mathbf{V}^*, -\mathbf{E}_{\mathrm{tot}}^{(m)}\right]\begin{bmatrix} \mathbf{N}_1 \\ \vdots \\ \mathbf{N}_{L+m-1} \\ \mathbf{P} \end{bmatrix} = \mathbf{0}, \tag{7}$$

where $\mathbf{E}_{\mathrm{tot}}^{(m)} = \left[\mathrm{Vec}\left(\mathbf{E}_1^{(m)}\right),\ldots,\mathrm{Vec}\left(\mathbf{E}_{K+L-1}^{(m)}\right)\right] \in \mathbb{C}^{(K-m+1)(L+m-1)\times(K+L-1)}$. Due to [4, Lemma II.4] we know that the solution to (7) is unique up to a nonsingular matrix $\mathbf{G}^{-1} \in \mathbb{C}^{R\times R}$. Let $\{\widehat{\mathbf{N}}_1,\ldots,\widehat{\mathbf{N}}_{L+m-1},\widehat{\mathbf{P}}\}$ denote a solution to (7). Note that the solution has the property $\mathbf{P} = \widehat{\mathbf{P}}\mathbf{G}^{-1}$. Construct $\widehat{\mathbf{S}}_{\mathrm{BT}} = \sum_{k=1}^{K+L-1} \mathbf{E}_k \otimes \widehat{\mathbf{P}}_k$, where $\mathbf{E}_k \in \mathbb{C}^{K\times L}$ is given by (3). Then (1) can be reduced to

$$\mathbf{Y} = \left[\mathbf{Y}^{(1)},\ldots,\mathbf{Y}^{(L)}\right] = \mathbf{X}(\widehat{\mathbf{S}}_{\mathrm{BT}}^T)^\dagger = \mathbf{M}_{\mathrm{BT}}(\mathbf{I}_L\otimes\mathbf{G}^T), \tag{8}$$

where $\mathbf{Y}^{(l)} = \mathbf{M}_{\mathrm{BT}}^{(l)}\mathbf{G}^T \in \mathbb{C}^{I\times R}$. This leads to the coupled decomposition of $\mathbf{Y}_{\mathrm{tot}}$ in [4, eq. (29)] and repeated below

$$\mathbf{Y}_{\mathrm{tot}} = \begin{bmatrix} \mathbf{Y}^{(1)} \\ \vdots \\ \mathbf{Y}^{(L)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathrm{BT}}^{(1)} \\ \vdots \\ \mathbf{M}_{\mathrm{BT}}^{(L)} \end{bmatrix}\mathbf{G}^T. \tag{9}$$

1.2 Elimination of P

In this subsection we discuss an alternative subspace method developed in [1] that can be used to reduce a block-Toeplitz matrix factorization of the form (1) into the coupled decomposition (9) without the explicit determination of $\mathbf{P}$ in (7). Partition $\mathbf{S}_{\mathrm{BT,sm}}^{(m)}$ in (5) as follows

$$\mathbf{S}_{\mathrm{BT,sm}}^{(m)} = [\mathbf{S}_{\mathrm{BT,sm}}^{(1,m)}, \ldots, \mathbf{S}_{\mathrm{BT,sm}}^{(L+m-1,m)}], \quad \mathbf{S}_{\mathrm{BT,sm}}^{(l,m)} \in \mathbb{C}^{(K-m+1)\times R}.$$

Consider also the partitioning of $\mathbf{N}$ in (6). Using the property $\overline{\mathbf{S}}_{\mathrm{BT,sm}}^{(i,m)} = \underline{\mathbf{S}}_{\mathrm{BT,sm}}^{(i-1,m)} \Leftrightarrow \overline{\mathbf{V}}^* \mathbf{N}_i = \underline{\mathbf{V}}^* \mathbf{N}_{i-1}$, where $\overline{\mathbf{V}}^* \in \mathbb{C}^{(K-m)\times(L+m-1)R}$ and $\underline{\mathbf{V}}^* \in \mathbb{C}^{(K-m)\times(L+m-1)R}$, we obtain

$$\begin{bmatrix} \underline{\mathbf{V}}^* & -\overline{\mathbf{V}}^* & \mathbf{0}_{(K-m),L+m-1} & \cdots & \mathbf{0}_{(K-m),L+m-1} \\ \mathbf{0}_{(K-m),L+m-1} & \underline{\mathbf{V}}^* & -\overline{\mathbf{V}}^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0}_{(K-m),L+m-1} \\ \mathbf{0}_{(K-m),L+m-1} & \cdots & \mathbf{0}_{(K-m),L+m-1} & \underline{\mathbf{V}}^* & -\overline{\mathbf{V}}^* \end{bmatrix} \cdot \begin{bmatrix} \mathbf{N}_1 \\ \vdots \\ \mathbf{N}_{L+m-1} \end{bmatrix} = \mathbf{0}. \quad (10)$$

Due to [4, Lemma II.4], we can obtain $\mathbf{N}$ from (10) up to a block-diagonal matrix $\mathbf{I}_{L+m-1} \otimes \mathbf{G}^{-T}$, in which $\mathbf{G} \in \mathbb{C}^{R\times R}$ is nonsingular. In other words, if $\widehat{\mathbf{N}}$ is a solution to (10), then $\mathbf{N} = \widehat{\mathbf{N}}\left(\mathbf{I}_{L+m-1} \otimes \mathbf{G}^{-T}\right)$. By inserting $\widehat{\mathbf{N}}$ into (7), we obtain $\widehat{\mathbf{P}}$, corresponding to a solution of (7) where $\mathbf{N} = \widehat{\mathbf{N}}\left(\mathbf{I}_{L+m-1} \otimes \mathbf{G}^{-T}\right)$ and with property $\mathbf{P} = \widehat{\mathbf{P}}\mathbf{G}^{-1}$. This leads to the coupled decomposition of $\mathbf{Y}_{\mathrm{tot}}$ in (9).

1.3 Elimination of N

By utilizing the subspace method developed in [2] it is also possible to reduce the block-Toeplitz matrix factorization (1) into the coupled decomposition (9) without the explicit determination of $\mathbf{N}$ in (7). Let the column vectors of $\mathbf{V}^\perp = \left[\mathbf{v}_1^\perp, \ldots, \mathbf{v}_J^\perp\right] \in \mathbb{C}^{(K-m+1)\times J}$ constitute a basis for $\ker\left(\mathbf{X}_{\mathrm{BT,sm}}^{(m)*}\right) = \ker\left(\mathbf{S}_{\mathrm{BT,sm}}^{(m)H}\right)$, in which $J = (K-m+1) - (L+m-1)R$. For every $j \in \{1, \ldots, J\}$, we have

$$\mathbf{S}_{\mathrm{BT,sm}}^{(m)H}\mathbf{v}_j^\perp = \sum_{k=1}^{K+L-1} \left(\mathbf{E}_k^{(m)T} \otimes \mathbf{p}_k^H\right)\mathbf{v}_j^\perp = \sum_{k=1}^{K+L-1} \mathbf{E}_k^{(m)T}\mathbf{v}_j^\perp \otimes \mathbf{p}_k^H = \mathbf{0}_{(L+m-1)R}. \quad (11)$$

Let $\mathrm{Unvec}(\cdot)$ denote the linear operator with property $\mathrm{Unvec}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{b}\mathbf{a}^T$. Then

$$\mathrm{Unvec}\left(\mathbf{S}_{\mathrm{BT,sm}}^{(m)H}\mathbf{v}_j^\perp\right) = \sum_{k=1}^{K+L-1} \mathbf{p}_k^H(\mathbf{v}_j^\perp)^T\mathbf{E}_k^{(m)} = \mathbf{0}_{R,(L+m-1)}.$$

Relation (11) can therefore also be written as

$$\mathrm{Unvec}\left(\mathbf{S}_{\mathrm{BT,sm}}^{(m)H}\mathbf{v}_j^\perp\right)^H = \sum_{k=1}^{K+L-1} \mathbf{E}_k^{(m)T}(\mathbf{v}_j^\perp)^*\mathbf{p}_k = \mathbf{0}_{(L+m-1),R}. \quad (12)$$

Relation (12) can in turn be written more compactly as

$$\left[\mathbf{E}_1^{(m)T}, \ldots, \mathbf{E}_{K+L-1}^{(m)T}\right]\left(\mathbf{I}_{K+L-1} \otimes (\mathbf{v}_j^{\perp})^*\right)\mathbf{P} = \mathbf{0}_{(L+m-1),R}, \tag{13}$$

where $\mathbf{P} \in \mathbb{C}^{(K+L-1)\times R}$ is given by (4). Define

$$\mathbf{V}_j = \left[\mathbf{E}_1^{(m)T}(\mathbf{v}_j^{\perp})^*, \ldots, \mathbf{E}_{K+L-1}^{(m)T}(\mathbf{v}_j^{\perp})^*\right] \in \mathbb{C}^{(L+m-1)\times(K+L-1)}.$$

Define also

$$\mathbf{W} = \begin{bmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_J \end{bmatrix} \in \mathbb{C}^{J(L+m-1)\times(K+L-1)}.$$

Relation (13) can now be expressed as

$$\mathbf{W}\mathbf{P} = \mathbf{0}_{J(L+m-1),R}\,.$$

Again, if $\widehat{\mathbf{P}}$ denotes a solution to (14) then $\mathbf{P} = \widehat{\mathbf{P}}\mathbf{G}^{-1}$. The remaining reduction steps to (9) are now identical to the steps in Subsection 1.1.

2 General (column-by-column) low-rank mixing matrix structures related to Section III

Recall that in Section III of [4] we considered Toeplitz-block factorizations of the form

$$\mathbf{X} = \mathbf{M}_{\mathrm{TB}}^{(1)}\mathbf{S}_{\mathrm{TB}}^{(1)T} + \cdots + \mathbf{M}_{\mathrm{TB}}^{(R)}\mathbf{S}_{\mathrm{TB}}^{(R)T} = \mathbf{M}_{\mathrm{TB}}\mathbf{S}_{\mathrm{TB}}^{T} \in \mathbb{C}^{I\times K}, \tag{15}$$

in which the columns of the mixing matrix $\mathbf{M}_{\mathrm{TB}} = [\mathbf{M}_{\mathrm{TB}}^{(1)}, \ldots, \mathbf{M}_{\mathrm{TB}}^{(R)}] \in \mathbb{C}^{I\times LR}$ with $\mathbf{M}_{\mathrm{TB}}^{(r)} \in \mathbb{C}^{I\times L}$ are subject to low-rank structures. Recall also that L corresponds to the memory of the channel (i.e., the number of filter taps). The equivalent block-Toeplitz factorization of (15) is given by (1), and repeated below:

$$\mathbf{X} = \mathbf{M}_{\mathrm{TB}}\mathbf{\Pi}_{\mathbf{S}} \cdot \mathbf{\Pi}_{\mathbf{S}}^{T}\mathbf{S}_{\mathrm{TB}}^{T} = \mathbf{M}_{\mathrm{BT}}\mathbf{S}_{\mathrm{BT}}^{T}, \tag{16}$$

where $\mathbf{M}_{\mathrm{BT}} = [\mathbf{M}_{\mathrm{BT}}^{(1)}, \ldots, \mathbf{M}_{\mathrm{BT}}^{(L)}]$ with $\mathbf{M}_{\mathrm{BT}}^{(l)} = [\mathbf{m}_{\mathrm{TB},l}^{(1)}, \ldots, \mathbf{m}_{\mathrm{TB},l}^{(R)}] \in \mathbb{C}^{I\times R}$ and $\mathbf{\Pi}_{\mathbf{S}} \in \mathbb{C}^{RL\times RL}$ is the Toeplitz-block to block-Toeplitz conversion column permutation matrix.

In [4, Section III.B] we only considered the case where the columns of $\mathbf{M}_{\mathrm{TB}}$ (or equivalently $\mathbf{M}_{\mathrm{BT}}$) are subject to a Khatri-Rao structure. Here we briefly explain that the result can conveniently we extended to other related structures. The following Subsections 2.1–2.3 discuss examples in which the columns are subject to Khatri-Rao, coupled and BTD structures, respectively. Of course, the combinations of the mentioned structures are also possible (e.g., higher-order and coupled BTD structures). This will be briefly discussed in Subsection 2.4. The main point is that the low-rank structures are on a "column-by-column basis", i.e., the low-rank structures of the columns of $\mathbf{M}_{\mathrm{TB}}$ are independent of each other.

2.1 CPD structure

The most basic example is the case where the columns of $\mathbf{M}_{\mathrm{TB}}^{(r)}$ are vectorized rank-1 matrices ("Kronecker vectors"):

$$\mathbf{M}_{\mathrm{TB}}^{(r)} = [\mathbf{m}_{\mathrm{TB},1}^{(r)}, \ldots, \mathbf{m}_{\mathrm{TB},L}^{(r)}] = [\mathbf{a}_1^{(r)} \otimes \mathbf{b}_1^{(r)}, \ldots, \mathbf{a}_L^{(r)} \otimes \mathbf{b}_L^{(r)}] \in \mathbb{C}^{IJ \times L}, \tag{17}$$

where $\mathbf{a}_l^{(r)} \in \mathbb{C}^I$ and $\mathbf{b}_l^{(r)} \in \mathbb{C}^J$. For readability reasons, the row-dimension 'I' in (15) has changed to 'IJ' in (17). The related Khatri-Rao submatrix associated with the block-Toeplitz factorization is

$$\mathbf{M}_{\mathrm{BT}}^{(l)} = [\mathbf{m}_{\mathrm{TB},l}^{(1)}, \ldots, \mathbf{m}_{\mathrm{TB},l}^{(R)}] = [\mathbf{a}_l^{(1)} \otimes \mathbf{b}_l^{(1)}, \ldots, \mathbf{a}_l^{(R)} \otimes \mathbf{b}_l^{(R)}] \in \mathbb{C}^{IJ \times R}. \tag{18}$$

The CPD structure (17) (or equivalently (18)) serves as the foundational building block from which various structures can be constructed from.

2.2 Coupled CPD structure

Coupled CPD structures are also possible, i.e.,

$$\begin{aligned}\mathbf{M}_{\mathrm{TB}}^{(r)} = [\mathbf{m}_{\mathrm{TB},1}^{(r)}, \ldots, \mathbf{m}_{\mathrm{TB},L}^{(r)}] &= \begin{bmatrix} \mathbf{m}_{\mathrm{TB},1}^{(1,r)} & \cdots & \mathbf{m}_{\mathrm{TB},L}^{(1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{m}_{\mathrm{TB},1}^{(N,r)} & \cdots & \mathbf{m}_{\mathrm{TB},L}^{(N,r)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1^{(1,r)} \otimes \mathbf{b}_1^{(1,r)} & \cdots & \mathbf{a}_L^{(1,r)} \otimes \mathbf{b}_L^{(1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_1^{(N,r)} \otimes \mathbf{b}_1^{(N,r)} & \cdots & \mathbf{a}_L^{(N,r)} \otimes \mathbf{b}_L^{(N,r)} \end{bmatrix} \in \mathbb{C}^{(\prod_{n=1}^N I_n J_n) \times L},\end{aligned} \tag{19}$$

where $\mathbf{a}_l^{(n,r)} \in \mathbb{C}^{I_n}$ and $\mathbf{b}_l^{(n,r)} \in \mathbb{C}^{J_n}$ and where the new superscript 'n' indicates the "coupling mode", ranging from 1 to N. Similarly, the coupled CPD version of $\mathbf{M}_{\mathrm{BT}}$ has submatrices of the form

$$\begin{aligned}\mathbf{M}_{\mathrm{BT}}^{(l)} = [\mathbf{m}_{\mathrm{TB},l}^{(1)}, \ldots, \mathbf{m}_{\mathrm{TB},l}^{(R)}] &= \begin{bmatrix} \mathbf{m}_{\mathrm{TB},l}^{(1,1)} & \cdots & \mathbf{m}_{\mathrm{TB},l}^{(1,R)} \\ \vdots & \ddots & \vdots \\ \mathbf{m}_{\mathrm{TB},l}^{(N,1)} & \cdots & \mathbf{m}_{\mathrm{TB},l}^{(N,R)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_l^{(1,1)} \otimes \mathbf{b}_l^{(1,1)} & \cdots & \mathbf{a}_l^{(1,R)} \otimes \mathbf{b}_l^{(1,R)} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_l^{(N,1)} \otimes \mathbf{b}_l^{(N,1)} & \cdots & \mathbf{a}_l^{(N,R)} \otimes \mathbf{b}_l^{(N,R)} \end{bmatrix} \in \mathbb{C}^{(\prod_{n=1}^N I_n J_n) \times R}.\end{aligned} \tag{20}$$

In Subsection 3.3 we will explain how 'coupled' structures of the form (19) (or equivalently (20)) are related to the system studied in [4, Section IV.C.2].

2.3 BTD structures

We also consider BTD-type structures of the form

$$\mathbf{M}_{\mathrm{TB}}^{(r)} = [\mathbf{m}_{\mathrm{TB},1}^{(r)}, \ldots, \mathbf{m}_{\mathrm{TB},L}^{(r)}] = \left[\sum_{p=1}^{P} \mathbf{a}_1^{(p,r)} \otimes \mathbf{b}_1^{(p,r)}, \ldots, \sum_{p=1}^{P} \mathbf{a}_L^{(p,r)} \otimes \mathbf{b}_L^{(p,r)}\right]$$
$$= \left[\mathrm{Vec}\left(\mathbf{B}_1^{(r)}\mathbf{A}_1^{(r)T}\right), \ldots, \mathrm{Vec}\left(\mathbf{B}_L^{(r)}\mathbf{A}_L^{(r)T}\right)\right] \in \mathbb{C}^{IJ\times L}, \quad (21)$$

where $\mathbf{A}_l^{(r)} = [\mathbf{a}_l^{(1,r)}, \ldots, \mathbf{a}_l^{(P,r)}]$ and $\mathbf{B}_l^{(r)} = [\mathbf{b}_l^{(1,r)}, \ldots, \mathbf{b}_l^{(P,r)}]$. Similarly, we have

$$\mathbf{M}_{\mathrm{BT}}^{(l)} = [\mathbf{m}_{\mathrm{TB},l}^{(1)}, \ldots, \mathbf{m}_{\mathrm{TB},l}^{(R)}] = \left[\sum_{p=1}^{P} \mathbf{a}_l^{(p,1)} \otimes \mathbf{b}_l^{(p,1)}, \ldots, \sum_{p=1}^{P} \mathbf{a}_l^{(p,R)} \otimes \mathbf{b}_l^{(p,R)}\right] \in \mathbb{C}^{IJ\times R}. \quad (22)$$

In Subsections 3.1 and 3.2 we will explain how 'BTD' structures of the form (21) (or equivalently (22)) are a related to the system studied in [4, Sections IV.A, IV.B and IV.C.1].

2.4 Higher-order structures

The results can easily be extended to higher-order CPD structures:

$$\mathbf{M}_{\mathrm{TB}}^{(r)} = [\mathbf{m}_{\mathrm{TB},1}^{(r)}, \ldots, \mathbf{m}_{\mathrm{TB},L}^{(r)}] = [\mathbf{a}_1^{(1,r)} \otimes \cdots \otimes \mathbf{a}_1^{(N,r)}, \ldots, \mathbf{a}_L^{(1,r)} \otimes \cdots \otimes \mathbf{a}_L^{(N,r)}] \in \mathbb{C}^{(\prod_{n=1}^{N} I_n)\times L}, \quad (23)$$

where the row-dimension 'I' in (15) has changed to '$\prod_{n=1}^{N} I_n$' in (23) and the new superscript denotes the "tensor mode", ranging from 1 to N. The related higher-order Khatri-Rao submatrix associated with the block-Toeplitz factorization is

$$\mathbf{M}_{\mathrm{BT}}^{(l)} = [\mathbf{m}_{\mathrm{TB},l}^{(1)}, \ldots, \mathbf{m}_{\mathrm{TB},l}^{(R)}] = [\mathbf{a}_l^{(1,1)} \otimes \cdots \otimes \mathbf{a}_l^{(N,1)}, \ldots, \mathbf{a}_l^{(1,R)} \otimes \cdots \otimes \mathbf{a}_l^{(N,R)}] \in \mathbb{C}^{(\prod_{n=1}^{N} I_n)\times R}. \quad (24)$$

Higher-order structures, such as (23) (or equivalently (24)), can for instance be used to model systems enjoying multiple diversities.

2.4.1 Reduction to coupled CPD

Let us briefly demonstrate how to cope with higher-order and coupled PD versions of (16):

$$\mathbf{X}^{(n)} = \mathbf{M}_{\mathrm{BT}}^{(n)}\mathbf{S}_{\mathrm{BT}}^{T} = (\mathbf{A}^{(1,n)} \odot \cdots \odot \mathbf{A}^{(M_n,n)})\mathbf{S}_{\mathrm{BT}}^{T} \in \mathbb{C}^{I_n\times K}, \quad (25)$$

where $I_n = \prod_{m=1}^{M_n} I_{m,n}$, $n \in \{1, \ldots, N\}$, $\mathbf{A}^{(m,n)} \in \mathbb{C}^{I_{m,n}\times LR}$ and where $\mathbf{S}_{\mathrm{BT}} \in \mathbb{C}^{K\times LR}$ is a block-Toeplitz matrix of the form (2). The factorization in (25) admits several matrix representations of the form

$$\mathbf{X}^{n)} = \mathbf{M}_{\mathrm{BT}}^{(n)}\mathbf{S}_{\mathrm{BT}}^{T} = \left(\mathbf{A}^{(w,n,1)} \odot \mathbf{A}^{(w,n,2)}\right)\mathbf{S}_{\mathrm{BT}}^{T}, \quad (26)$$

where

$$\mathbf{A}^{(w,n,1)} = \bigodot_{p\in\Gamma_{w,n}} \mathbf{A}^{(p,n)} \in \mathbb{C}^{I_{w,n,1}\times LR} \text{ with } I_{w,n,1} = \prod_{p\in\Gamma_{w,n}} I_{p,n},$$
$$\mathbf{A}^{(w,n,2)} = \bigodot_{q\in\Upsilon_{w,n}} \mathbf{A}^{(q,n)} \in \mathbb{C}^{I_{w,n,2}\times LR} \text{ with } I_{w,n,2} = \prod_{q\in\Upsilon_{w,n}} I_{q,n},$$

in which the sets $\Gamma_{w,n}$ and $\Upsilon_{w,n}$ have properties $\Gamma_{w,n} \bigcup \Upsilon_{w,n} = \{1, 2, \ldots, M_n\}$ and $\Gamma_{w,n} \bigcap \Upsilon_{w,n} = \emptyset$. Let us work with W_n sets $\{\Gamma_{w,n}\}$ and $\{\Upsilon_{w,n}\}$ such that for each $n \in \{1, \ldots, N\}$ we have that $I_n = I_{w,n,1} I_{w,n,2}$.

We obtain the extended version of (16):

$$\mathbf{X}_{\text{tot}} = \begin{bmatrix} \mathbf{X}_{\text{tot}}^{(1)} \\ \vdots \\ \mathbf{X}_{\text{tot}}^{(N)} \end{bmatrix} \in \mathbb{C}^{(\sum_{n=1}^{N} I_n W_n)\times K}, \tag{27}$$

where

$$\mathbf{X}_{\text{tot}}^{(n)} = \begin{bmatrix} \mathbf{X}^{(1,n)} \\ \vdots \\ \mathbf{X}^{(W_n,n)} \end{bmatrix} \in \mathbb{C}^{I_n W_n \times K},$$

in which $\mathbf{X}^{(w,n)} \in \mathbb{C}^{I_n \times K}$ is of the form (16) and built from $\mathbf{X}^{(w,n)}$ in (26). Note that (27) admits the factorization (cf. (16)):

$$\mathbf{X}_{\text{tot}} = \mathbf{M}_{\text{tot}} \mathbf{S}_{\text{BT}}^{T},$$

where

$$\mathbf{M}_{\text{tot}} = \begin{bmatrix} \mathbf{M}_{\text{tot}}^{(1)} \\ \vdots \\ \mathbf{M}_{\text{tot}}^{(N)} \end{bmatrix} \in \mathbb{C}^{(\sum_{n=1}^{N} I_n W_n)\times LR}, \quad \mathbf{M}_{\text{tot}}^{(n)} = \begin{bmatrix} \mathbf{M}_{\text{BT}}^{(1,n)} \\ \vdots \\ \mathbf{M}_{\text{BT}}^{(W_n,n)} \end{bmatrix}.$$

For every pair (w, n) we obtain a higher-order and coupled version of (8):

$$\mathbf{Y}^{(w,n)} = \mathbf{X}^{(w,n)} (\widehat{\mathbf{S}}_{\text{BT}}^{T})^{\dagger} = \mathbf{M}_{\text{BT}}^{(w,n)} (\widehat{\mathbf{S}}^{T})^{\dagger} = \mathbf{M}_{\text{BT}}^{(w,n)} (\mathbf{I}_L \otimes \mathbf{G}^{T}) \in \mathbb{C}^{I_n \times LR}.$$

Consider the partitioning

$$\mathbf{Y}^{(w,n)} = [\mathbf{Y}_1^{(w,n)}, \ldots, \mathbf{Y}_L^{(w,n)}], \tag{28}$$

in which $\mathbf{Y}_l^{(w,n)} \in \mathbb{C}^{I_n \times R}$ for every $l \in \{1, \ldots, L\}$. Stacking yields a higher-order

and coupled version of (9):

$$\mathbf{Y}_{\text{tot}}^{(w,n)} = \begin{bmatrix} \mathbf{Y}_1^{(w,n)} \\ \vdots \\ \mathbf{Y}_L^{(w,n)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\text{BT},1}^{(w,n)} \\ \vdots \\ \mathbf{M}_{\text{BT},L}^{(w,n)} \end{bmatrix} \mathbf{G}^T$$
$$= \begin{bmatrix} \mathbf{A}^{(1,w,n,1)} \odot \mathbf{A}^{(1,w,n,2)} \\ \vdots \\ \mathbf{A}^{(L,w,n,1)} \odot \mathbf{A}^{(L,w,n,2)} \end{bmatrix} \mathbf{G}^T \in \mathbb{C}^{I_n L \times R}, \tag{29}$$

in which $\mathbf{M}_{\text{BT},l}^{(w,n)} = \mathbf{A}^{(l,w,n,1)} \odot \mathbf{A}^{(l,w,n,2)}$ where

$$\mathbf{A}^{(w,n,1)} = \left[\mathbf{A}^{(1,w,n,1)}, \ldots, \mathbf{A}^{(L,w,n,1)}\right] \in \mathbb{C}^{I_{w,n,1} \times LR}, \tag{30}$$
$$\mathbf{A}^{(w,n,2)} = \left[\mathbf{A}^{(1,w,n,2)}, \ldots, \mathbf{A}^{(L,w,n,2)}\right] \in \mathbb{C}^{I_{w,n,2} \times LR}, \tag{31}$$

with $\mathbf{A}^{(l,w,n,1)} \in \mathbb{C}^{I_{w,n,1} \times R}$ and $\mathbf{A}^{(l,w,n,2)} \in \mathbb{C}^{I_{w,n,2} \times R}$. Overall, we obtain

$$\mathbf{Y}_{\text{tot}} = \begin{bmatrix} \mathbf{Y}_{\text{tot}}^{(1)} \\ \vdots \\ \mathbf{Y}_{\text{tot}}^{(N)} \end{bmatrix} = \mathbf{M}_{\text{tot}} \mathbf{G}^T, \quad \mathbf{Y}_{\text{tot}}^{(n)} = \begin{bmatrix} \mathbf{Y}_{\text{tot}}^{(1,1)} \\ \vdots \\ \mathbf{Y}_{\text{tot}}^{(W_n,n)} \end{bmatrix},$$

where

$$\mathbf{M}_{\text{tot}} = \begin{bmatrix} \mathbf{M}_{\text{tot}}^{(1)} \\ \vdots \\ \mathbf{M}_{\text{tot}}^{(N)} \end{bmatrix}, \ \mathbf{F}_{\text{tot}}^{(n)} = \begin{bmatrix} \mathbf{M}_{\text{tot}}^{(1,1)} \\ \vdots \\ \mathbf{M}_{\text{tot}}^{(W_n,n)} \end{bmatrix}, \ \mathbf{M}_{\text{tot}}^{(w,n)} = \begin{bmatrix} \mathbf{M}_{\text{BT},1}^{(1,1)} \\ \vdots \\ \mathbf{M}_{\text{BT},L}^{(W_n,n)} \end{bmatrix}.$$

2.4.2 Reduction to coupled BTD

The above result can also be extended to mixing matrices with BTD structures. In short, replace the submatrices in $\mathbf{M}_{\text{BT}}^{(w,n)} = [\mathbf{M}_{\text{BT},1}^{(w,n)}, \ldots, \mathbf{M}_{\text{BT},L}^{(w,n)}]$ of the form $\mathbf{M}_{\text{BT},l}^{(w,n)} = [\mathbf{a}_1^{(l,w,n,1)} \otimes \mathbf{a}_1^{(l,w,n,2)}, \ldots, \mathbf{a}_R^{(l,w,n,1)} \otimes \mathbf{a}_R^{(l,w,n,2)}]$ with

$$\mathbf{M}_{\text{BT},l}^{(w,n)} = [\text{Vec}\left(\mathbf{A}_{P,1}^{(l,w,n,2)} \mathbf{A}_{P,1}^{(l,w,n,1)T}\right), \ldots, \text{Vec}\left(\mathbf{A}_{P,R}^{(l,w,n,2)} \mathbf{A}_{P,R}^{(l,w,n,1)T}\right)],$$

where $\mathbf{A}_{P,r}^{(l,w,n,1)} \in \mathbb{C}^{I_{w,n,1} \times P}$ and $\mathbf{A}_{P,r}^{(l,w,n,2)} \in \mathbb{C}^{I_{w,n,2} \times P}$. The BTD extensions of (30) and (31) are now given by

$$\mathbf{A}^{(w,n,1)} = \left[\mathbf{A}_P^{(1,w,n,1)}, \ldots, \mathbf{A}_P^{(L,w,n,1)}\right] \in \mathbb{C}^{I_{w,n,1} \times LPR},$$
$$\mathbf{A}^{(w,n,2)} = \left[\mathbf{A}_P^{(1,w,n,2)}, \ldots, \mathbf{A}_P^{(L,w,n,2)}\right] \in \mathbb{C}^{I_{w,n,2} \times LPR},$$

where

$$\mathbf{A}_P^{(l,w,n,1)} = [\mathbf{A}_{P,1}^{(l,w,n,1)}, \ldots, \mathbf{A}_{P,R}^{(l,w,n,1)}] \in \mathbb{C}^{I_{w,n,1} \times PR},$$
$$\mathbf{A}_P^{(l,w,n,2)} = [\mathbf{A}_{P,1}^{(l,w,n,2)}, \ldots, \mathbf{A}_{P,R}^{(l,w,n,2)}] \in \mathbb{C}^{I_{w,n,2} \times PR}.$$

Hence, the BTD extensions of $\mathbf{Y}_l^{(w,n)}$ in (28) and (29) are given by

$$\mathbf{Y}_l^{(w,n)} = \mathbf{M}_{\mathrm{BT},l}^{(w,n)} \mathbf{G}^T = [\mathrm{Vec}\left(\mathbf{A}_{P,1}^{(l,w,n,2)} \mathbf{A}_{P,1}^{(l,w,n,1)T}\right), \ldots, \mathrm{Vec}\left(\mathbf{A}_{P,R}^{(l,w,n,2)} \mathbf{A}_{P,R}^{(l,w,n,1)T}\right)] \mathbf{G}^T.$$

To summarize, mixing matrices with higher-order CPD/BTD structures can in many cases be reduced to the coupled CPD/BTD structures considered in [4, Section III].

3 Low-rank structures and collinearities (dependencies between the columns) related to Section IV

The general input-output formula for the systems considered in [4, Section IV] is:[1]

$$x_{(i-1)J+j,k}^{(n)} = \sum_{r=1}^{R} \sum_{p=1}^{P} a_i^{(n,p,r)} \sum_{l=1}^{L} b_{j+(l-1)J}^{(n,p,r)} s_{k-l+1}^{(r)}, \quad n \in \{1, \ldots, N\}, \tag{32}$$

where N denotes the number of widely separated antenna arrays, P the number of dominant paths, and L the filter order. The associated submatrices of the mixing matrix of (32) is given by (see [4, Section IV] for details):

$$\begin{aligned} \mathbf{M}_{\mathrm{TB}}^{(n,r)} &= \left[\sum_{p=1}^{P} \mathbf{a}^{(n,p,r)} \otimes \mathbf{b}_1^{(n,p,r)}, \ldots, \sum_{p=1}^{P} \mathbf{a}^{(n,p,r)} \otimes \mathbf{b}_L^{(n,p,r)}\right] = \sum_{p=1}^{P} \mathbf{a}^{(n,p,r)} \otimes \mathbf{B}^{(n,p,r)} \\ &= \left[\mathrm{Vec}\left(\mathbf{B}_1^{(n,r)} \mathbf{A}^{(n,r)T}\right), \ldots, \mathrm{Vec}\left(\mathbf{B}_L^{(n,r)} \mathbf{A}^{(n,r)T}\right)\right] \in \mathbb{C}^{I_n J_n \times L}, \end{aligned} \tag{33}$$

where $\mathbf{a}^{(n,p,r)} \in \mathbb{C}^{I_n}$ and $\mathbf{B}^{(n,p,r)} = [\mathbf{b}_1^{(n,p,r)}, \ldots, \mathbf{b}_L^{(n,p,r)}] \in \mathbb{C}^{J_n \times L}$ and where $\mathbf{A}^{(n,r)} = [\mathbf{a}^{(n,1,r)}, \ldots, \mathbf{a}^{(n,P,r)}] \in \mathbb{C}^{I_n \times P}$ and $\mathbf{B}_l^{(n,r)} = [\mathbf{b}_l^{(n,1,r)}, \ldots, \mathbf{b}_l^{(n,P,r)}] \in \mathbb{C}^{J_n \times P}$. Note that the collinearity in (33) implies that $\mathbf{a}^{(n,p,r)} := \mathbf{a}_1^{(n,p,r)} = \cdots = \mathbf{a}_L^{(n,p,r)}$ and that $\mathbf{A}^{(n,r)} = [\mathbf{a}^{(n,1,r)}, \ldots, \mathbf{a}^{(n,P,r)}] := \mathbf{A}_1^{(n,r)} = \cdots = \mathbf{A}_L^{(n,r)}$. Likewise, we have

$$\mathbf{F}_{\mathrm{BT}}^{(n,l)} = [\mathbf{m}_{\mathrm{TB,l}}^{(n,1)}, \ldots, \mathbf{m}_{\mathrm{TB},l}^{(n,R)}] = \left[\sum_{p=1}^{P} \mathbf{a}^{(n,p,1)} \otimes \mathbf{b}_l^{(n,p,1)}, \ldots, \sum_{p=1}^{P} \mathbf{a}^{(n,p,R)} \otimes \mathbf{b}_l^{(n,p,R)}\right] \in \mathbb{C}^{I_n J_n \times R}. \tag{34}$$

[1] Relation (32) can also be expressed in tensor format as:

$$\mathcal{X}^{(n)} = \sum_{r=1}^{R} \sum_{p=1}^{P} \sum_{l=1}^{L} \mathbf{a}^{(n,p,r)} \otimes \mathbf{b}_l^{(n,p,r)} \otimes \mathbf{s}_{\mathrm{TB},l}^{(r)} = \sum_{r=1}^{R} \sum_{p=1}^{P} \mathbf{a}^{(n,p,r)} \otimes \left(\mathbf{B}^{(n,p,r)} \mathbf{S}_{\mathrm{TB}}^{(r)T}\right), \quad n \in \{1, \ldots, N\},$$

where $\mathbf{B}^{(n,p,r)} = \left[\mathbf{b}_1^{(n,p,r)}, \ldots, \mathbf{b}_L^{(n,p,r)}\right] \in \mathbb{C}^{J \times L}$ is unstructured and $\mathbf{S}_{\mathrm{TB}}^{(r)} = \left[\mathbf{s}_{\mathrm{TB},1}^{(r)}, \ldots, \mathbf{s}_{\mathrm{TB},L}^{(r)}\right] \in \mathbb{C}^{K \times L}$ is Toeplitz.

By inspection of (33) (or equivalently (34)) it becomes apparent that the difference between the mixing matrices considered in [4, Section III] (e.g., those discussed in Section 2 in this document) and the mixing matrices considered in [4, Section IV] is that there is now an additional dependency between the columns of the factor matrix "$\mathbf{A}$". More precisely, some of the columns are constrained to be collinear.

Note also that it does not matter whether we investigate the structures of the mixing matrices $\mathbf{M}_{\mathrm{TB}}$ or $\mathbf{M}_{\mathrm{BT}}$. For this reason, only the structure of $\mathbf{M}_{\mathrm{TB}}$ will be discussed in the rest of this section.

3.1 Connection between Section III and Section IV.A

In [4, Section IV.A] we considered a system of the form (32) with $N = 1$:

$$x_{(i-1)J+j,k} = \sum_{r=1}^{R} \sum_{p=1}^{P} a_i^{(p,r)} \sum_{l=1}^{L} b_{j+(l-1)J}^{(p,r)} s_{k-l+1}^{(r)},$$

where the superscript 'n' has been dropped. This means that (33) reduces to

$$\begin{aligned} \mathbf{M}_{\mathrm{TB}}^{(r)} &= \left[\sum_{p=1}^{P} \mathbf{a}^{(p,r)} \otimes \mathbf{b}_1^{(p,r)}, \ldots, \sum_{p=1}^{P} \mathbf{a}^{(p,r)} \otimes \mathbf{b}_L^{(p,r)} \right] = \sum_{p=1}^{P} \mathbf{a}^{(p,r)} \otimes \mathbf{B}^{(p,r)} \\ &= \left[\mathrm{Vec}\left(\mathbf{B}_1^{(r)} \mathbf{A}^{(r)T} \right), \ldots, \mathrm{Vec}\left(\mathbf{B}_L^{(r)} \mathbf{A}^{(r)T} \right) \right] \in \mathbb{C}^{IJ \times L}. \end{aligned} \tag{35}$$

Comparing (21) with (35) it is clear that the system in [4, Section IV.A] is a special case of the BTD-type structured mixing system discussed in [4, Section III] with the additional constraint $\mathbf{A}^{(r)} := \mathbf{A}_1^{(r)} = \cdots = \mathbf{A}_L^{(r)}$. In other words, if this collinearity structure is ignored, then the decomposition (35) is equal to (21).

3.2 Connection between Section III and Section IV.C.1

The system in [4, Section IV.C.1] is just a special case of the system in [4, Section IV.A] with $J = 1$, i.e., (35) reduces to

$$\begin{aligned} \mathbf{M}_{\mathrm{TB}}^{(r)} &= \left[\sum_{p=1}^{P} \mathbf{a}^{(p,r)} b_1^{(p,r)}, \ldots, \sum_{p=1}^{P} \mathbf{a}^{(p,r)} b_L^{(p,r)} \right] = \sum_{p=1}^{P} \mathbf{a}^{(p,r)} \mathbf{b}^{(p,r)T} \\ &= \left[\mathbf{A}^{(r)} \mathbf{b}_1^{(r)}, \ldots, \mathbf{A}^{(r)} \mathbf{b}_L^{(r)} \right] \in \mathbb{C}^{I \times L}, \end{aligned}$$

where $\mathbf{b}^{(p,r)} = [b_1^{(p,r)}, \ldots, b_L^{(p,r)}]^T \in \mathbb{C}^L$ and where $\mathbf{b}_l^{(r)} = [b_l^{(1,r)}, \ldots, b_l^{(P,r)}]^T \in \mathbb{C}^P$. Consequently, this is again a special case of (21) with an additional collinearity structure.

3.3 Connection between Section III and Section IV.C.2

In [4, Section IV.C.2] we considered a system of the form (32) in which $N > 1$ but now $P = 1$ and $J = 1$:

$$x_{i,k}^{(n)} = \sum_{r=1}^{R} a_i^{(n,r)} \sum_{l=1}^{L} b_l^{(n,r)} s_{k-l+1}^{(r)}, \quad n \in \{1, \ldots, N\}.$$

This means that (33) reduces to

$$\mathbf{M}_{\mathrm{TB}}^{(n,r)} = [\mathbf{m}_{\mathrm{TB},1}^{(n,r)}, \ldots, \mathbf{m}_{\mathrm{TB},L}^{(n,r)}] = \left[\mathbf{a}^{(n,r)} b_1^{(n,r)}, \ldots, \mathbf{a}^{(n,r)} b_L^{(n,r)}\right] \in \mathbb{C}^{I_n \times L}.$$

Stacking yields

$$\mathbf{M}_{\mathrm{TB}}^{(r)} = \begin{bmatrix} \mathbf{M}_{\mathrm{TB}}^{(1,r)} \\ \vdots \\ \mathbf{M}_{\mathrm{TB}}^{(N,r)} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_{\mathrm{TB},1}^{(1,r)} & \cdots & \mathbf{m}_{\mathrm{TB},L}^{(1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{m}_{\mathrm{TB},1}^{(N,r)} & \cdots & \mathbf{m}_{\mathrm{TB},L}^{(N,r)} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{(1,r)} b_1^{(1,r)} & \cdots & \mathbf{a}^{(1,r)} b_L^{(1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{a}^{(N,r)} b_1^{(N,r)} & \cdots & \mathbf{a}^{(N,r)} b_L^{(N,r)} \end{bmatrix} \in \mathbb{C}^{(\prod_{n=1}^{N} I_n) \times L}. \tag{36}$$

Clearly, (36) is a special case of (19) with $J_n = 1$, $\forall n \in \{1, \ldots, N\}$ and with the additional collinearity structural constraint $\mathbf{a}^{(n,r)} := \mathbf{a}_1^{(n,r)} = \cdots = \mathbf{a}_L^{(n,r)}$, $\forall n \in \{1, \ldots, N\}$.

4 Signal matrices with low-rank and block-Toeplitz structures

Collinearity and Toeplitz structures can also be combined into an augmented block-Toeplitz matrix. It will be explained that it can be used in cases where the dimension K is small, and used by the authors in [3]. As an illustrative example, consider the system (32) with $N = 1$ and $P = 1$. The outputs can be stored in the matrix $\mathbf{Y} \in \mathbb{C}^{IK \times J}$ with factorization

$$\begin{aligned} \mathbf{Y} &= \sum_{r=1}^{R} \left((\mathbf{1}_L^T \otimes \mathbf{a}^{(r)}) \odot \mathbf{S}_{\mathrm{TB}}^{(r)} \right) \mathbf{H}^{(r)T} = \sum_{r=1}^{R} \left((\mathbf{1}_L^T \otimes \mathbf{a}^{(r)}) \odot \mathbf{S}_{\mathrm{TB}}^{(r)} \right) \mathbf{\Pi}_{\mathbf{S}} \cdot \mathbf{\Pi}_{\mathbf{S}}^T \mathbf{H}^{(r)T} \\ &= \sum_{l=1}^{L} \left(\mathbf{A} \odot \mathbf{S}_{\mathrm{BT}}^{(l)} \right) \widetilde{\mathbf{H}}^T = \left((\mathbf{1}_L^T \otimes \mathbf{A}) \odot \mathbf{S}_{\mathrm{BT}} \right) \widetilde{\mathbf{H}}^T, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \mathbf{A} &= \left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(R)}\right] \in \mathbb{C}^{I \times R}, \\ \widetilde{\mathbf{H}} &= \left[\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(R)}\right] \mathbf{\Pi}_{\mathbf{S}} \in \mathbb{C}^{J \times RL}, \\ \mathbf{S}_{\mathrm{BT}} &= \left[\mathbf{S}_{\mathrm{BT}}^{(1)}, \ldots, \mathbf{S}_{\mathrm{BT}}^{(L)}\right] \in \mathbb{C}^{K \times RL}, \end{aligned}$$

and we recall that $\mathbf{\Pi_S} \in \mathbb{C}^{RL \times RL}$ is the column permutation matrix that links the Toeplitz-block matrix $\mathbf{S}_{\mathrm{TB}} = \left[\mathbf{S}_{\mathrm{TB}}^{(1)}, \ldots, \mathbf{S}_{\mathrm{TB}}^{(R)}\right]$ to the block-Toeplitz matrix $\mathbf{S}_{\mathrm{BT}}$ via the identity $\mathbf{S}_{\mathrm{BT}} = \mathbf{S}_{\mathrm{TB}}\mathbf{\Pi_S}$.

Since $\mathbf{A} \odot \overline{\mathbf{S}}_{\mathrm{BT}}^{(l)} = \mathbf{A} \odot \underline{\mathbf{S}}_{\mathrm{BT}}^{(l-1)}$, the matrix $(\mathbf{1}_L^T \otimes \mathbf{A}) \odot \mathbf{S}_{\mathrm{BT}}$ is block-Toeplitz. Consequently, (37) corresponds to a block-Toeplitz factorization. Denote

$$\mathbf{G} = \left[\mathbf{A} \odot \overline{\mathbf{S}}_{\mathrm{BT}}^{(1)}, \mathbf{A} \odot \overline{\mathbf{S}}_{\mathrm{BT}}^{(2)}, \ldots, \mathbf{A} \odot \overline{\mathbf{S}}_{\mathrm{BT}}^{(L)}, \mathbf{A} \odot \underline{\mathbf{S}}_{\mathrm{BT}}^{(L)}\right] \in \mathbb{C}^{I(K-1) \times R(L+1)}. \tag{38}$$

Due to [4, Lemma 2.4], we now know that if

$$\begin{cases} \widetilde{\mathbf{H}} \text{ in (37) has full column rank,} \\ \mathbf{G} \text{ in (38) has full column rank,} \end{cases}$$

then the block-Toeplitz factorization (37) can be reduced to (e.g., using the subspace methods discussed in Section 1):

$$\mathbf{Z} = [\mathbf{Z}^{(l)}, \ldots, \mathbf{Z}^{(L)}] = \left((\mathbf{1}_L^T \otimes \mathbf{A}) \odot \mathbf{S}_{\mathrm{BT}}\right)(\mathbf{I}_L \otimes \mathbf{G}^T), \tag{39}$$

where $\mathbf{Z}^{(l)} = \mathbf{A} \odot \mathbf{S}_{\mathrm{BT}}^{(l)} \in \mathbb{C}^{IK \times R}$ and where $\mathbf{G} \in \mathbb{C}^{R \times R}$ is nonsingular. We note in passing, that since $\mathbf{Y}$ corresponds to a block-Toeplitz factorization, temporal smoothing can be used in a preprocessing step, e.g., if $\widetilde{\mathbf{H}}$ does not have full column rank.

Relation (39) is equal to

$$\begin{bmatrix} \mathbf{Z}^{(1)} \\ \vdots \\ \mathbf{Z}^{(L)} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \odot \mathbf{S}_{\mathrm{BT}}^{(1)} \\ \vdots \\ \mathbf{A} \odot \mathbf{S}_{\mathrm{BT}}^{(L)} \end{bmatrix} \mathbf{G}^T. \tag{40}$$

Define $\mathbf{d}(i,k) = \mathrm{Vec}\left(\mathbf{e}_i^{(I)} \otimes \mathbf{E}^{(k)}\right) \in \mathbb{C}^{IKL}$, where $\mathbf{E}^{(k)} \in \mathbb{C}^{K \times L}$ is a basis matrix of the form (3). Construct $\mathbf{D} \in \mathbb{C}^{IKL \times I(K+L-1)}$ as follows

$$\mathbf{D} = [\mathbf{d}(1,1), \ldots, \mathbf{d}(1,K+L-1), \ldots, \mathbf{d}(I,1), \ldots, \mathbf{d}(I,K+L-1)]\,.$$

Then (40) can also be written as

$$\begin{bmatrix} \mathbf{Z}^{(1)} \\ \vdots \\ \mathbf{Z}^{(L)} \end{bmatrix} = \mathbf{D}\,(\mathbf{A} \odot \mathbf{P})\,\mathbf{G}^T, \tag{41}$$

where $\mathbf{P} \in \mathbb{C}^{(K+L-1) \times R}$ is the parameter matrix defined by (4). Note that (41) takes the collinearity constraint $\mathbf{A} := \mathbf{A}^{(1)} = \cdots = \mathbf{A}^{(L)}$ in (40) into account. Note also that since $K(L-1) > L-1 \Leftrightarrow KL > K+L-1$, the basis vectors in the set

$\{\mathbf{d}(i,k)\}$ all contain a unique non-zero row-entry, meaning that $\mathbf{D}^\dagger$ exists and it is unique. Thus, from (41) we obtain

$$\mathbf{D}^\dagger \begin{bmatrix} \mathbf{Z}^{(1)} \\ \vdots \\ \mathbf{Z}^{(L)} \end{bmatrix} = (\mathbf{A} \odot \mathbf{P})\, \mathbf{G}^T ,$$

which clearly corresponds to a CPD. To summarize, collinearity and Toeplitz structured factorizations of form (37), in which the dimension K is small while the dimension J is sufficiently large, can be addressed as a decomposition problem where the involved collinear, low-rank and Toeplitz structures is collected into the factor matrix $(\mathbf{1}_L^T \otimes \mathbf{A}) \odot \mathbf{S}_{\mathrm{BT}}$.

5 Mixing matrices with low-rank and block-Hankel structures

In the previous section we considered a system in which the signal matrix is both low-rank and block-Toepitz. In this section we consider an example in which the mixing matrix is both low-rank and block-Hankel. More precisely, we assume that receiver is equipped with I receive antennas and R transmitted signals are passing through a convolutive channel so that output of the ith receive antenna and the nth sampling period is

$$x_i(n) = \sum_{r=1}^{R} a_i^{(r)} \sum_{l=0}^{L} b^{(r)}(l) s^{(r)}(n-l) , \tag{42}$$

where $a_i^{(r)}$ denotes the gain between the ith antenna and the rth signal, $\{s^{(r)}(n)\}$ are the unknown transmitted signals, $\{b^{(r)}(l)\}$ are the unknown channel coefficients, and L denotes the filter order. Note that (42) corresponds to (32) with $P = 1$, $J = 1$ and $N = 1$.

Assume that the transmitted signals $\{s^{(r)}(n)\}$ are mutually statistically independent from each other. Then the antenna coefficients $\{a_i^{(r)}\}$ and the channel coefficients $\{b^{(r)}(l)\}$ can be recovered using higher-order statistics. Let us work with the spatio-temporal fourth-order cumulants

$$\begin{aligned} C_{i_1 i_2 i_3 i_4}(l_1, l_2, l_3) &:= \mathrm{Cum}\left[x_{i_1}^*(n) x_{i_2}(n+l_1) x_{i_3}^*(n+l_2) x_{i_4}(n+l_3)\right] \\ &= \sum_{r=1}^{R} \gamma_r a_{i_1}^{(r)*} a_{i_2}^{(r)} a_{i_3}^{(r)*} a_{i_4}^{(r)} \sum_{l=0}^{L} b^{(r)*}(l) b^{(r)}(l+l_1) b^{(r)*}(l+l_2) b^{(r)}(l+l_3) , \end{aligned} \tag{43}$$

where $i_1, i_2, i_3, i_4 \in \{1, \ldots, I\}$ and $l_1, l_2, l_3 \in \{-L, \ldots, L\}$, $b^{(r)}(l) = 0$ for every $l > L$ or $-l < -L$, and $\gamma_r = \mathrm{Cum}\left[s^{(r)*}(n) s^{(r)}(n) s^{(r)*}(n) s^{(r)}(n)\right]$ denotes the kurtosis of

the rth source. Let us store the channel and antenna gain coefficients into the parameter matrices

$$\mathbf{P}^{(r)} = \left[\mathbf{p}_0^{(r)}, \ldots, \mathbf{p}_L^{(r)}\right] = \mathbf{a}^{(r)}\mathbf{b}^{(r)T} \in \mathbb{C}^{I\times(L+1)}, \quad r \in \{1, \ldots, R\},$$

where $\mathbf{a}^{(r)} = \left[a_1^{(r)}, \ldots, a_I^{(r)}\right]^T \in \mathbb{C}^I$ and $\mathbf{b}^{(r)} = \left[b_0^{(r)}, \ldots, b_L^{(r)}\right]^T \in \mathbb{C}^{L+1}$. Note that $\mathrm{Vec}\left(\mathbf{P}^{(r)}\right) = \mathbf{b}^{(r)} \otimes \mathbf{a}^{(r)}$. The goal is now to find the parameter matrices $\mathbf{P}^{(r)}$ from the cumulants (5). Let us store them into the tensor $\mathcal{T} \in \mathbb{C}^{I\times I(2L+1)\times I(2L+1)\times I(2L+1)}$ such that

$$\mathcal{T}_{i_1,(l_1+L)I+i_2,(l_2+L)I+i_3,(l_3+L)I+i_4} = C_{i_1 i_2 i_3 i_4}(l_1, l_2, l_3).$$

It can be verified that $\mathcal{T}$ admits the factorization

$$\mathcal{T} = \sum_{r=1}^{R} \gamma_r \sum_{l=0}^{L} \mathbf{p}_r^* \otimes \mathbf{h}_{\mathrm{HB},l}^{(r)} \otimes \mathbf{h}_{\mathrm{HB},l}^{(r)*} \otimes \mathbf{h}_{\mathrm{HB},l}^{(r)},$$

where $\mathbf{h}_{\mathrm{HB},l}^{(r)}$ denotes the lth column of

$$\mathbf{H}_{\mathrm{HB}}^{(r)} = \begin{bmatrix} \mathbf{0}_I & \cdots & \mathbf{0}_I & \mathbf{p}_0^{(r)} \\ \vdots & \iddots & \iddots & \vdots \\ \mathbf{0}_I & \mathbf{p}_0^{(r)} & \cdots & \mathbf{p}_{L-1}^{(r)} \\ \mathbf{p}_0^{(r)} & \mathbf{p}_1^{(r)} & \cdots & \mathbf{p}_L^{(r)} \\ \vdots & \vdots & \iddots & \mathbf{0}_I \\ \mathbf{p}_{L-1}^{(r)} & \mathbf{p}_L^{(r)} & \iddots & \vdots \\ \mathbf{p}_L^{(r)} & \mathbf{0}_I & \ldots & \mathbf{0}_I \end{bmatrix} = \mathbf{C}_{\mathrm{HB}}^{(r)} \otimes \mathbf{a}^{(r)} \in \mathbb{C}^{I(2L+1)\times(L+1)},$$

with banded Hankel matrices of the form

$$\mathbf{C}_{\mathrm{HB}}^{(r)} = \begin{bmatrix} 0 & \cdots & 0 & b_0^{(r)} \\ \vdots & \iddots & \iddots & \vdots \\ 0 & b_0^{(r)} & \ldots & b_{L-1}^{(r)} \\ b_0^{(r)} & b_1^{(r)} & \ldots & b_L^{(r)} \\ \vdots & \vdots & \iddots & 0 \\ b_{L-1}^{(r)} & b_L^{(r)} & \iddots & \vdots \\ b_L^{(r)} & 0 & \ldots & 0 \end{bmatrix} = \sum_{l=0}^{L} \mathbf{E}_l b_l^{(r)} \in \mathbb{C}^{(2L+1)\times(L+1)},$$

where $\mathbf{E}_l \in \mathbb{C}^{(2L+1)\times(L+1)}$ are basis matrices, defined according to

$$(\mathbf{E}_l)_{ij} = \begin{cases} 1, & i = L + l - j, \\ 0, & \text{otherwise.} \end{cases} \tag{44}$$

Let the matrix $\mathbf{T}^{(i_1,i_2)} \in \mathbb{C}^{I(2L+1)\times I(2L+1)}$ be constructed according to the relation $\mathbf{T}^{(i_1,i_2)}_{i_4 i_3} = \mathcal{T}_{i_1 i_2 i_3 i_4}$. Then for every $i_1 \in \{1,\dots,I\}$ and $i_2 \in \{1,\dots,I(2L+1)\}$ we have that

$$\mathbf{T}^{(i_1,i_2)} = \sum_{r=1}^{R} \gamma_r \mathbf{H}^{(r)}_{\mathrm{HB}} D_{i_1}\left(\mathbf{P}^{(r)*}\right) D_{i_2}\left(\mathbf{H}^{(r)}_{\mathrm{HB}}\right) \mathbf{H}^{(r)H}_{\mathrm{HB}}$$

and that

$$\mathbf{T}^{(i_1,i_2)H} = \sum_{r=1}^{R} \gamma_r \mathbf{H}^{(r)}_{\mathrm{HB}} D_{i_1}\left(\mathbf{P}^{(r)}\right) D_{i_2}\left(\mathbf{H}^{(r)*}_{\mathrm{HB}}\right) \mathbf{H}^{(r)H}_{\mathrm{HB}}.$$

Overall, we obtain

$$\begin{aligned}\mathbf{Y} := \Big[&\mathrm{Vec}\left(\mathbf{T}^{(1,1)}\right), \mathrm{Vec}\left(\mathbf{T}^{(2,1)}\right), \dots, \mathrm{Vec}\left(\mathbf{T}^{(I,I(2L_1+1))}\right), \\ &\mathrm{Vec}\left(\mathbf{T}^{(1,1)H}\right), \mathrm{Vec}\left(\mathbf{T}^{(2,1)H}\right), \dots, \mathrm{Vec}\left(\mathbf{T}^{(I,I(2L+1))H}\right)\Big] \\ &= \sum_{r=1}^{R} \gamma_r \left(\mathbf{H}^{(r)*}_{\mathrm{HB}} \odot \mathbf{H}^{(r)}_{\mathrm{HB}}\right) \mathbf{D}^{(r)T}_{\mathrm{HB}} \in \mathbb{C}^{I^2(2L_1+1)^2 \times 2I^2(2L+1)}, \end{aligned} \tag{45}$$

where

$$\mathbf{D}^{(r)}_{\mathrm{HB}} = \begin{bmatrix} \mathbf{P}^{(r)*} \odot \mathbf{H}^{(r)}_{\mathrm{HB}} \\ \mathbf{P}^{(r)} \odot \mathbf{H}^{(r)*}_{\mathrm{HB}} \end{bmatrix} \in \mathbb{C}^{2I^2(2L+1)\times(L+1)R}.$$

Observe that $\mathbf{H}_{\mathrm{HB}} = \left[\mathbf{H}^{(1)}_{\mathrm{HB}}, \dots, \mathbf{H}^{(R)}_{\mathrm{HB}}\right] \in \mathbb{C}^{I(2L+1)\times(L+1)R}$ is a structured Hankel-block matrix with $\mathbf{H}^{(r)}_{\mathrm{HB}} = \mathbf{C}^{(r)}_{\mathrm{HB}} \otimes \mathbf{a}^{(r)}$. Let $\mathbf{\Pi}_{\mathbf{H}} \in \mathbb{C}^{(L+1)R\times(L+1)R}$ denote the column permutation matrix that transforms the Hankel-block matrix $\mathbf{H}_{\mathrm{HB}}$ into the block-Hankel matrix

$$\mathbf{H}_{\mathrm{BH}} = \left[\mathbf{H}^{(0)}_{\mathrm{BH}}, \dots, \mathbf{H}^{(L)}_{\mathrm{BH}}\right] = \mathbf{H}_{\mathrm{HB}}\mathbf{\Pi}_{\mathbf{H}} \in \mathbb{C}^{I(2L+1)\times(L+1)R}, \tag{46}$$

with block-matrices $\mathbf{H}^{(l)}_{\mathrm{BH}} = [\mathbf{h}^{(1)}_{\mathrm{HB},l}, \dots, \mathbf{h}^{(R)}_{\mathrm{HB},l}] = \mathbf{C}^{(l)}_{\mathrm{BH}} \odot \mathbf{A} \in \mathbb{C}^{I(2L+1)\times R}$ in which $\mathbf{C}^{(l)}_{\mathrm{BH}} = [\mathbf{c}^{(1)}_{\mathrm{HB},l}, \dots, \mathbf{c}^{(R)}_{\mathrm{HB},l}]$ and $\mathbf{A} = [\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(R)}] \in \mathbb{C}^{I\times R}$. Define also $\mathbf{D}_{\mathrm{HB},\gamma} = [\gamma_1 \mathbf{D}^{(1)T}_{\mathrm{HB}}, \dots, \gamma_R \mathbf{D}^{(R)T}_{\mathrm{HB}}]^T$. Then relation (45) can be written as

$$\begin{aligned}\mathbf{Y} &= \sum_{r=1}^{R} \gamma_r \left(\mathbf{H}^{(r)*}_{\mathrm{HB}} \odot \mathbf{H}^{(r)}_{\mathrm{HB}}\right) \mathbf{D}^{(r)}_{\mathrm{HB}} \\ &= \left(\mathbf{H}^{*}_{\mathrm{HB}} \odot \mathbf{H}_{\mathrm{HB}}\right)\mathbf{\Pi}_{\mathbf{H}} \cdot \mathbf{\Pi}^{T}_{\mathbf{H}} \mathbf{D}^{T}_{\mathrm{HB},\gamma} \\ &= \left(\mathbf{H}^{*}_{\mathrm{BH}} \odot \mathbf{H}_{\mathrm{BH}}\right) \cdot \mathbf{D}^{T}_{\mathrm{BH},\gamma}, \end{aligned} \tag{47}$$

where $\mathbf{D}_{\mathrm{BH},\gamma} = \mathbf{D}_{\mathrm{HB},\gamma}\mathbf{\Pi}_{\mathbf{H}}$ and $\mathbf{H}_{\mathrm{BH}}$ is given by (46).

Define

$$\mathbf{G} = \left[\underline{\underline{\mathbf{H}}}_{\mathrm{BH}}^{(0)} \odot \mathbf{A}, \ldots, \underline{\underline{\mathbf{H}}}_{\mathrm{BH}}^{(L)} \odot \mathbf{A}, \overline{\overline{\mathbf{H}}}_{\mathrm{BH}}^{(L)} \odot \mathbf{A}\right].$$

Thanks to the relation $(\underline{\underline{\mathbf{H}}}_{\mathrm{BH}}^{(l)} \odot \mathbf{A}) \odot (\underline{\underline{\mathbf{H}}}_{\mathrm{BH}}^{(l)} \odot \mathbf{A}) = (\overline{\overline{\mathbf{H}}}_{\mathrm{BH}}^{(l+1)} \odot \mathbf{A}) \odot (\overline{\overline{\mathbf{H}}}_{\mathrm{BH}}^{(l+1)} \odot \mathbf{A})$ and due to [4, Lemma II.4], we know that if

$$\begin{cases} \mathbf{G}^* \odot \mathbf{G} \text{ has full column rank,} \\ \mathbf{D}_{\mathrm{BH},\gamma} \text{ has full column rank,} \end{cases}$$

then we can find a block-Hankel matrix $\mathbf{U}_{\mathrm{BH}}$ from range $(\mathbf{Y})$ so that the block-Hankel factorization (47) can be reduced to (e.g., using subspace methods such as those discussed in Section 1):

$$\mathbf{U}_{\mathrm{BH}} = \left(\mathbf{H}_{\mathrm{BH}}^* \odot \mathbf{H}_{\mathrm{BH}}\right)(\mathbf{I}_{L+1} \otimes \mathbf{F}^T), \tag{48}$$

where $\mathbf{F} \in \mathbb{C}^{R \times R}$ is a nonsingular matrix. Consider the partitioning $\mathbf{U}_{\mathrm{BH}} = [\mathbf{U}_{\mathrm{BH}}^{(0)}, \ldots, \mathbf{U}_{\mathrm{BH}}^{(L)}]$. From (48) we obtain the coupled decomposition

$$\begin{bmatrix} \mathbf{U}_{\mathrm{BH}}^{(0)} \\ \vdots \\ \mathbf{U}_{\mathrm{BH}}^{(L)} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\mathrm{BH}}^{(0)*} \odot \mathbf{H}_{\mathrm{BH}}^{(0)} \\ \vdots \\ \mathbf{H}_{\mathrm{BH}}^{(L)*} \odot \mathbf{H}_{\mathrm{BH}}^{(L)} \end{bmatrix} \mathbf{F}^T = \begin{bmatrix} (\mathbf{B}_{\mathrm{BH}}^{(0)} \odot \mathbf{A})^* \odot (\mathbf{B}_{\mathrm{BH}}^{(0)} \odot \mathbf{A}) \\ \vdots \\ (\mathbf{B}_{\mathrm{BH}}^{(L)} \odot \mathbf{A})^* \odot (\mathbf{B}_{\mathrm{BH}}^{(L)} \odot \mathbf{A}) \end{bmatrix} \mathbf{F}^T. \tag{49}$$

Let $\mathbf{d}(l,i) = \mathrm{Vec}\left(\mathbf{E}_l \otimes \mathbf{e}_i^{(I)}\right) \in \mathbb{C}^{(2L+1)(L+1)I}$, where $\mathbf{E}_l \in \mathbb{C}^{(2L+1)\times(L+1)}$ is a basis matrix of the form (44) and $\mathbf{e}_i^{(I)} \in \mathbb{C}^I$ is a unit vector with entry at position i and zero elsewhere. Construct

$$\mathbf{D} = [\mathbf{d}(0,1), \ldots, \mathbf{d}(L,1), \ldots, \mathbf{d}(L,I)] \in \mathbb{C}^{(2L+1)(L+1)I \times (L+1)I},$$

then (49) is equal to

$$\begin{bmatrix} \mathbf{U}_{\mathrm{BH}}^{(0)} \\ \vdots \\ \mathbf{U}_{\mathrm{BH}}^{(L)} \end{bmatrix} = (\mathbf{D} \otimes \mathbf{I}_I) \otimes (\mathbf{D} \otimes \mathbf{I}_I)\left((\mathbf{B} \odot \mathbf{A})^* \odot (\mathbf{B} \odot \mathbf{A})\right)\mathbf{F}^T \tag{50}$$

where $\mathbf{B} = [\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(R)}] \in \mathbb{C}^{(L+1)\times R}$ is the channel coefficient matrix and we recall that $\mathbf{A} = [\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(R)}] \in \mathbb{C}^{I \times R}$ is the antenna array response matrix. Note that the basis vectors $\{\mathbf{d}(l,i)\}$ each contain a unique non-zero row-entry which means that $\mathbf{D}^\dagger$ exists and it is also unique. Thus, from (50) we obtain

$$\mathbf{X} = ((\mathbf{D} \otimes \mathbf{I}_I) \otimes (\mathbf{D} \otimes \mathbf{I}_I))^\dagger \begin{bmatrix} \mathbf{U}_{\mathrm{BH}}^{(0)} \\ \vdots \\ \mathbf{U}_{\mathrm{BH}}^{(L)} \end{bmatrix} = \left((\mathbf{B} \odot \mathbf{A})^* \odot (\mathbf{B} \odot \mathbf{A})\right)\mathbf{F}^T.$$

In words, the problem of finding $\mathbf{A}$ and $\mathbf{B}$ has been reduced to the CPD problem $\mathbf{X} = (\mathbf{B}^* \odot \mathbf{A}^* \odot \mathbf{B} \odot \mathbf{A})\,\mathbf{F}^T$. It is well-known that this CPD is unique under mild conditions (e.g. if $C_2(\mathbf{B}^* \odot \mathbf{A}^*) \odot C_2(\mathbf{B} \odot \mathbf{A})$ has full column rank) and can be computed via algebraic methods. We can now conclude that $\mathbf{A}$ and $\mathbf{B}$ can be obtained (up to the intrinsic column scaling ambiguities) from fourth-order cumulants using a convolutive extension of CPD.

References

[1] H. Liu and G. Xu, "Closed-form blind symbol estimation in digital communication," *IEEE Trans. Signal Process.*, vol. 43, no. 11, pp. 2714–2723, Nov. 1995.

[2] E. Moulines, P. Duhamel, J.-F. Cardoso, and S. Mayargue, "Subspace methods for the blind identification of multichannel FIR filters," *IEEE Trans. Signal Processing*, vol. 43, no. 2, pp. 516–525, Feb. 1995.

[3] M. Sørensen and L. De Lathauwer, "Tensor Decompositions with Block-Toeplitz Structure and Applications in Signal Processing," in *Proc. Asilomar*, Nov. 6-9, Monterey, CA, USA, 2011.

[4] M. Sørensen, F. Van Eeghem, and L. De Lathauwer, "Blind multichannel deconvolution and convolutive extensions of canonical polyadic and block term decompositions," ESAT-STADIUS, KU Leuven, Belgium, Tech. Rep. 16-73, 2016.