SIMULTANEOUS GENERALIZED SCHUR DECOMPOSITION METHODS FOR COMPUTING THE CANONICAL POLYADIC DECOMPOSITION

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Abstract. The aim of this paper is to compute a Canonical Polyadic Decomposition (CPD) of a tensor using a Simultaneous Generalized Schur Decomposition (SGSD). First we propose new methods to compute a CPD with triangular factor matrices. This leads to more relaxed and robust SGSD methods. Next, we propose new ways to deal with partially (Hermitian) symmetric CPDs via the SGSD approach. For the case when the CPD contains a partial Hermitian symmetry we also provide a new uniqueness result. Finally, numerical experiments are reported.

Key words. tensor, polyadic decomposition, canonical decomposition (CANDECOMP), parallel factor (PARAFAC), simultaneous generalized Schur decomposition, partially symmetric tensor, triangular matrices.

1. Introduction. A Simultaneous Generalized Schur Decomposition (SGSD) approach for computing a third-order Canonical Polyadic Decomposition (CPD) with a partial symmetry was proposed in [20]. It was later adapted to the general unsymmetric third-order CPD case in [5]. The SGSD method has been applied in blind separation of communication signals [20, 7], in multidimensional harmonic retrieval [11, 1] and in blind identification of underdetermined mixtures [8]. The SGSD method requires that two of the factor matrices of the CPD have full column rank. However, a CPD problem with one full column rank factor matrix can often be converted to a CPD problem with all factor matrices having full column rank [6].

The original SGSD method [20], here called SGSD1-CP, for computing a third-order CPD requires that all three of the involved factor matrices have full column rank, as will be explained in section 4. Moreover, it does not fully take the structure among the involved triangular matrices into account, as will also be explained in section 4. We present new methods for computing a CPD with triangular factor matrices. Based on these results we first propose a new SGSD method, called SGSD2-CP, for the computation of a third-order CPD with three full column rank factor matrices. It attempts to take the structure among the involved triangular matrices into account and can be understood as an improved and generalized version of the procedure for computing the parameters of a structured tensor [4]. Second, we present a more relaxed SGSD method, called SGSD3-CP, which only requires that

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‡Research supported by: (1) Research Council KU Leuven: GOA-Ambiorics, GOA-MaNet, CoE EF/05/006 Optimization in Engineering (OPTEC), CIF1, STRT1/08/23, (2) F.W.O.: (a) project G.0427.10N, (b) Research Communities ICCoS, ANMMM and MLDM, (3) the Belgian Federal Science Policy Office: IUAP P7 (DYSCO II, Dynamical systems, control and optimization, 2012-2017), (4) EU: ERNSI.

The work of Mikael Sørensen was supported by the EU by a Marie-Curie Fellowship under contract No MEST- CT-2005-021175.

two of the matrix factors of the given CPD have full column rank. The proposed SGSD2-CP and SGSD3-CP methods mainly differ from the existing methods [20, 5] in the way the involved triangular matrices are computed. A part of this work has been presented in the conference paper [18]. CPDs containing partial (Hermitian) symmetries are common in many practical problems such as blind source separation based on simultaneous matrix diagonalization [3, 15, 8]. Hence, the third contribution of this paper are new SGSD methods for computing a CPD containing partial (Hermitian) symmetries. For the case when the CPD contains a partial Hermitian symmetry we also provide a new uniqueness result.

The paper is organized as follows. The rest of the introduction will present our notation followed by a quick review of the CPD and SGSD. In section 2 we discuss the relation between the SGSD and CPD. Next, in section 3 we present a generalized version of the extended QZ method for computing the SGSD. Section 4 will review the original SGSD1-CP method. Next, in sections 5 and 6 we propose the two new SGSD2-CP and SGSD3-CP methods for computing a CPD. Section 7 will discuss new SGSD approaches for the computation of a CPD containing partial (Hermitian) symmetries while section 8 presents a new uniqueness result for a CPD with a partial Hermitian symmetry. Section 9 will explain how to extend the SGSD methods to Nth-order tensors. Finally, in section 10 we conclude the paper.

1.1. Notation. Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao products, defined as

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad , \quad \mathbf{A} \odot \mathbf{B} \triangleq \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \mathbf{a}_2 \otimes \mathbf{b}_2 & \dots \end{bmatrix},$$

in which $(\mathbf{A})_{mn} = a_{mn}$ and $\mathbf{a}_r$ and $\mathbf{b}_r$ denote the rth column vector of $\mathbf{A}$ and $\mathbf{B}$, respectively. Throughout the paper $\mathbf{a}_r$ denotes the rth column vector of the matrix $\mathbf{A}$. Furthermore, the outer product of N vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ is denoted by $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ and it satisfies the relation $\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)}\right)_{i_1 i_2 \dots i_N} = a^{(1)}_{i_1} a^{(2)}_{i_2} \cdots a^{(N)}_{i_N}$.

Further, $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^\dagger$, $\|\cdot\|_F$, $|\cdot|$, $\operatorname{Col}(\cdot)$, $\operatorname{Null}(\cdot)$, $(\cdot)^\perp$, $\operatorname{Re}\{\cdot\}$ and $\operatorname{Im}\{\cdot\}$ denote the transpose, conjugate, conjugate-transpose, Moore-Penrose pseudo-inverse, Frobenius norm, determinant, column space, null space, complementary subspace, real part and imaginary part of a matrix, respectively.

Let δ_{ij} be the Dirac delta function which is zero except when the two indices are equal, in which case $\delta_{ii} = 1$. The identity matrix and all-ones vector are denoted by $\mathbf{I}_R \in \mathbb{C}^{R \times R}$ and $\mathbf{1}_R = [1, \dots, 1]^T \in \mathbb{C}^R$, respectively. Let $\mathbf{0}_{M,N} \in \mathbb{C}^{M \times N}$ and $\mathbf{0}_M \in \mathbb{C}^M$ denote an all-zero matrix and an all-zero vector, respectively. The vector $\mathbf{e}^{(R)}_p \in \mathbb{C}^R$ is equal to one at entry p and zero elsewhere. Matlab index notation will be used to denote submatrices of a given matrix. For example, $\mathbf{A}(1:k,:)$ denotes the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k.

Let $\operatorname{Blockdiag}(\mathbf{X}_1, \dots, \mathbf{X}_R)$ denote a block-diagonal matrix holding the matrices $\{\mathbf{X}_r\}$ on its block-diagonal. The notation $\operatorname{triu}(\cdot)$ and $\operatorname{diag}(\cdot)$ is used to denote the operator that sets the strictly lower triangular elements and off-diagonal elements of a matrix equal to zero, respectively. Let $\mathbf{A} \in \mathbb{C}^{J \times K}$ with $J, K \geq R$ and $x \in \mathbb{R}$, then

$\mathrm{triu}_R(\mathbf{A}) \in \mathbb{C}^{R\times R}$ and $\mathrm{sign}(x)$ are equal to

$$\mathrm{triu}_R(\mathbf{A}) = \mathrm{triu}(\mathbf{A}(1:R,1:R)), \quad \mathrm{sign}(x) = \begin{cases} 1 & , x>0 \\ 0 & , x=0 \\ -1 & , x<0 \end{cases}.$$

Moreover, let $\mathbf{A} \in \mathbb{C}^{I\times J}$, then $\mathrm{Vec}(\mathbf{A}) \in \mathbb{C}^{IJ}$ denotes the column vector defined by $(\mathrm{Vec}(\mathbf{A}))_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $\mathbf{a} \in \mathbb{C}^{IJ}$, then the reverse operation is $\mathrm{Unvec}(\mathbf{a}) = \mathbf{A} \in \mathbb{C}^{I\times J}$ such that $(\mathbf{a})_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $\mathbf{A} \in \mathbb{C}^{I\times I}$, then $\mathrm{Vecd}(\mathbf{A}) \in \mathbb{C}^{I}$ denotes the column vector defined by $(\mathrm{Vecd}(\mathbf{A}))_i = (\mathbf{A})_{ii}$. Let $\mathbf{A} \in \mathbb{C}^{I\times J}$, then $D_k(\mathbf{A}) \in \mathbb{C}^{J\times J}$ denotes the diagonal matrix holding row k of $\mathbf{A}$ on its diagonal.

The k-rank of a matrix $\mathbf{A}$ is denoted by $k(\mathbf{A})$. It is equal to the largest integer $k(\mathbf{A})$ such that every subset of $k(\mathbf{A})$ columns of $\mathbf{A}$ is linearly independent.

1.2. Canonical Polyadic Decomposition (CPD). A Nth-order rank-1 tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ is defined as the outer product of non-zero vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$, $n \in \{1,\ldots,N\}$, such that $\mathcal{X}_{i_1\ldots i_N} = \prod_{n=1}^{N} a^{(n)}_{i_n}$. The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}^{(1)}_r \circ \cdots \circ \mathbf{a}^{(N)}_r, \tag{1.1}$$

where $\mathbf{a}^{(n)}_r \in \mathbb{C}^{I_n}$. In this paper decomposition (1.1) will be referred to as the Canonical Polyadic Decomposition (CPD) of $\mathcal{X}$. Let us stack the vectors $\{\mathbf{a}^{(n)}_r\}$ into the matrices

$$\mathbf{A}^{(n)} = \left[\begin{array}{ccc} \mathbf{a}^{(n)}_1, & \cdots & ,\mathbf{a}^{(n)}_R \end{array}\right] \in \mathbb{C}^{I_n\times R}, \quad n \in \{1,\ldots,N\}. \tag{1.2}$$

The matrices $\mathbf{A}^{(n)}$ in (1.2) will be referred to as the CPD factor matrices. In this paper we say that the CPD of the tensor $\mathcal{X}$ in (1.1) is partially symmetric if $\mathbf{A}^{(m)} = \mathbf{A}^{(n)}$ or $\mathbf{A}^{(m)} = \mathbf{A}^{(n)*}$ for some $m \neq n$.

The CPD of $\mathcal{X}$ in (1.1) is said to be unique if all the N-tuplets $\left(\overline{\mathbf{A}}^{(1)},\ldots,\overline{\mathbf{A}}^{(N)}\right)$ satisfying (1.1) are related via $\overline{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)}\Pi\Delta_{\mathbf{A}^{(n)}}$, $\forall n \in \{1,\ldots,N\}$, where Π is a permutation matrix and $\{\Delta_{\mathbf{A}^{(n)}}\}$ are diagonal matrices satisfying $\prod_{n=1}^{N}\Delta_{\mathbf{A}^{(n)}} = \mathbf{I}_R$.

The following three matrix representations of a CPD of a third-order tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ will be used throughout the paper. Let $\mathbf{X}^{(i_1\cdot\cdot)} \in \mathbb{C}^{I_2\times I_3}$ denote the matrix such that $\mathbf{X}^{(i_1\cdot\cdot)}_{i_2i_3} = \mathcal{X}_{i_1i_2i_3}$, then $\mathbf{X}^{(i_1\cdot\cdot)} = \mathbf{A}^{(2)}D_{i_1}\left(\mathbf{A}^{(1)}\right)\mathbf{A}^{(3)T}$ and

$$\mathbb{C}^{I_1I_2\times I_3} \ni \mathbf{X}_{(1)} \triangleq \left[\begin{array}{c} \mathbf{X}^{(1\cdot\cdot)} \\ \vdots \\ \mathbf{X}^{(I_1\cdot\cdot)} \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^{(2)}D_1\left(\mathbf{A}^{(1)}\right) \\ \vdots \\ \mathbf{A}^{(2)}D_{I_1}\left(\mathbf{A}^{(1)}\right) \end{array}\right]\mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)}\odot\mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T}.$$

Let the matrices $\mathbf{X}^{(\cdot i_2\cdot)} \in \mathbb{C}^{I_3\times I_1}$ be constructed such that $\mathbf{X}^{(\cdot i_2\cdot)}_{i_3i_1} = \mathcal{X}_{i_1i_2i_3}$, then $\mathbf{X}^{(\cdot i_2\cdot)} = \mathbf{A}^{(3)}D_{i_2}\left(\mathbf{A}^{(2)}\right)\mathbf{A}^{(1)T}$ and

$$\mathbb{C}^{I_2I_3\times I_1} \ni \mathbf{X}_{(2)} \triangleq \left[\begin{array}{c} \mathbf{X}^{(\cdot 1\cdot)} \\ \vdots \\ \mathbf{X}^{(\cdot I_2\cdot)} \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^{(3)}D_1\left(\mathbf{A}^{(2)}\right) \\ \vdots \\ \mathbf{A}^{(3)}D_{I_2}\left(\mathbf{A}^{(2)}\right) \end{array}\right]\mathbf{A}^{(1)T} = \left(\mathbf{A}^{(2)}\odot\mathbf{A}^{(3)}\right)\mathbf{A}^{(1)T}.$$

Let $\mathbf{X}^{(\cdot\cdot i_3)} \in \mathbb{C}^{I_1 \times I_2}$ satisfy $\mathbf{X}^{(\cdot\cdot i_3)}_{i_1 i_2} = \mathcal{X}_{i_1 i_2 i_3}$, then $\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right) \mathbf{A}^{(2)T}$ and

$$\mathbb{C}^{I_1 I_3 \times I_2} \ni \mathbf{X}_{(3)} \triangleq \begin{bmatrix} \mathbf{X}^{(\cdot\cdot 1)} \\ \vdots \\ \mathbf{X}^{(\cdot\cdot I_3)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} D_1\left(\mathbf{A}^{(3)}\right) \\ \vdots \\ \mathbf{A}^{(1)} D_{I_3}\left(\mathbf{A}^{(3)}\right) \end{bmatrix} \mathbf{A}^{(2)T} = \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(1)}\right) \mathbf{A}^{(2)T}.$$

In practice, we are usually only interested in a low-rank approximation of $\mathcal{X}$. Typically the least-squares cost function

$$f\left(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)}\right) = \sum_{i_3=1}^{I_3} \left\| \mathbf{X}^{(\cdot\cdot i_3)} - \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right) \mathbf{A}^{(2)T} \right\|_F^2 \tag{1.3}$$

is used. Unfortunately a minimizer of (1.3) may not exist [10, 14]. That is, instead of a minimizer the cost function has an infimum and the minimization may yield diverging components [14].

1.3. Simultaneous Generalized Schur Decomposition (SGSD). Given a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ of rank R and $I_1, I_2 \geq R$, a SGSD of $\mathcal{X}$ is defined by

$$\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{Q}\mathbf{R}^{(i_3)}\mathbf{Z}^T, \quad \forall i_3 \in \{1, \ldots, I_3\}, \tag{1.4}$$

where $\mathbf{Q} \in \mathbb{C}^{I_1 \times I_1}$ and $\mathbf{Z} \in \mathbb{C}^{I_2 \times I_2}$ are unitary matrices and

$$\mathbf{R}^{(i_3)} = \begin{bmatrix} \overline{\mathbf{R}}^{(i_3)} & \mathbf{0}_{R, I_2 - R} \\ \mathbf{0}_{I_1 - R, R} & \mathbf{0}_{I_1 - R, I_2 - R} \end{bmatrix} \in \mathbb{C}^{I_1 \times I_2} \tag{1.5}$$

in which $\overline{\mathbf{R}}^{(i_3)} \in \mathbb{C}^{R \times R}$ are upper triangular matrices. In practice, the rank of $\mathcal{X}$ may be larger than R and an approximate SGSD is of interest. The unknown matrices $\mathbf{Q}$, $\mathbf{Z}$ and $\{\mathbf{R}^{(i_3)}\}$ will be found by minimizing the least-squares cost function

$$g\left(\mathbf{Q}, \mathbf{Z}, \{\mathbf{R}^{(i_3)}\}\right) = \sum_{i_3=1}^{I_3} \left\| \mathbf{X}^{(\cdot\cdot i_3)} - \mathbf{Q}\mathbf{R}^{(i_3)}\mathbf{Z}^T \right\|_F^2, \tag{1.6}$$

where $\mathbf{Q} \in \mathbb{C}^{I_1 \times I_1}$ and $\mathbf{Z} \in \mathbb{C}^{I_2 \times I_2}$ are unitary matrices and $\mathbf{R}^{(i_3)} \in \mathbb{C}^{I_1 \times I_2}$ is of the form (1.5).

It can be shown in a similar way as in [1], that the problem of minimizing the cost function (1.6) is equivalent to the maximization of the cost function

$$h\left(\mathbf{Q}, \mathbf{Z}\right) = \sum_{i_3=1}^{I_3} \left\| \mathrm{triu}_R\left(\mathbf{Q}^H \mathbf{X}^{(\cdot\cdot i_3)} \mathbf{Z}^*\right) \right\|_F^2. \tag{1.7}$$

Let us stack the entries of the upper triangular part in

$$\mathbf{r}^{(i_3)} = \left[\mathbf{R}^{(i_3)}(1,1), \mathbf{R}^{(i_3)}(1:2,2), \ldots, \mathbf{R}^{(i_3)}(1:R,R)\right]^T \in \mathbb{C}^{\frac{R(R+1)}{2}}.$$

We have $\mathbf{r}^{(i_3)} = \mathbf{W}^H \mathrm{Vec}\left(\mathbf{X}^{(\cdot\cdot i_3)}\right)$ with $\mathbf{W}$ given by

$$\begin{aligned} \mathbf{Q} &= \left[\mathbf{q}_1, \ldots, \mathbf{q}_{I_1}\right] \in \mathbb{C}^{I_1 \times I_1} \\ \mathbf{Z} &= \left[\mathbf{z}_1, \ldots, \mathbf{z}_{I_2}\right] \in \mathbb{C}^{I_2 \times I_2} \\ \mathbf{W} &= \left[\mathbf{z}_1 \otimes \mathbf{q}_1, \mathbf{z}_2 \otimes \mathbf{q}_1, \mathbf{z}_2 \otimes \mathbf{q}_2, \ldots, \mathbf{z}_R \otimes \mathbf{q}_1, \ldots, \mathbf{z}_R \otimes \mathbf{q}_R\right] \in \mathbb{C}^{I_1 I_2 \times \frac{R(R+1)}{2}}. \end{aligned}$$

Due to the reformulation of the SGSD problem from (1.6) into (1.7), as suggested in [1], it is clear that an optimal solution for the SGSD of $\mathcal{X}$ exists. Indeed, due to the continuity of the function (1.7) and the compactness of its domain one can conclude that an optimal solution to (1.7) always exists which implies that an optimal solution to (1.6) also always exists.

2. Link between SGSD and CPD. The computation of CPD via SGSD was proposed in [5, 20]. In this section we will formulate more explicitly than in [5, 20] how the two decompositions are connected. The derivation will make use of the following lemma.

LEMMA 2.1. *Consider $\boldsymbol{A} \in \mathbb{C}^{I\times R}$ with $k(\boldsymbol{A}) \geq 2$. Then there exists a nonsingular matrix $\boldsymbol{N} \in \mathbb{C}^{I\times I}$ for which $\overline{\boldsymbol{A}} = \boldsymbol{N}\boldsymbol{A}$ is such that $D_{j_1}\left(\overline{\boldsymbol{A}}\right)^{-1} D_{j_2}\left(\overline{\boldsymbol{A}}\right)$ exists and has distinct non-zero diagonal elements for some $j_1, j_2 \in \{1, \ldots, I\}$.*

Proof. A proof can for instance be found in [19]. □

In the following proposition a generalized eigenvalue is a scalar $\lambda \in \mathbb{C}$ which reduces the rank of a pencil, say $\mathbf{X}^{(\cdot\cdot 1)} - \lambda\mathbf{X}^{(\cdot\cdot 2)} \in \mathbb{C}^{I_1\times I_2}$.

PROPOSITION 2.2. *Given a rank-R tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ of which the CPD involves the factor matrices $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n\times R}$, $n \in \{1, 2, 3\}$, such that it has matrix representations*

$$\boldsymbol{X}^{(\cdot\cdot i_3)} = \boldsymbol{A}^{(1)} D_{i_3}\left(\boldsymbol{A}^{(3)}\right)\boldsymbol{A}^{(2)T}, \quad i_3 \in \{1, \ldots, I_3\}. \tag{2.1}$$

Assume that $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ have full column rank and $k\left(\boldsymbol{A}^{(3)}\right) \geq 2$. If there exist unitary matrices $\boldsymbol{Q} \in \mathbb{C}^{I_1\times I_1}$ and $\boldsymbol{Z} \in \mathbb{C}^{I_2\times I_2}$ such that

$$\boldsymbol{R}^{(\cdot\cdot i_3)} = \boldsymbol{Q}^H \boldsymbol{X}^{(\cdot\cdot i_3)} \boldsymbol{Z}^* = \begin{bmatrix} \overline{\boldsymbol{R}}^{(\cdot\cdot i_3)} & \boldsymbol{0}_{R,I_2-R} \\ \boldsymbol{0}_{I_1-R,R} & \boldsymbol{0}_{I_1-R,I_2-R} \end{bmatrix} \in \mathbb{C}^{I_1\times I_2},\ i_3 \in \{1, \ldots, I_3\}, \tag{2.2}$$

where $\overline{\boldsymbol{R}}^{(\cdot\cdot i_3)} \in \mathbb{C}^{R\times R}$ are upper triangular matrices, then there exists a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{R\times R}$ and a diagonal matrix $\boldsymbol{D} \in \mathbb{C}^{R\times R}$ such that

$$\boldsymbol{A}^{(1)}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}, \quad \boldsymbol{A}^{(2)}\boldsymbol{P} = \boldsymbol{Z}\boldsymbol{L}, \quad \boldsymbol{A}^{(3)}\boldsymbol{P}\boldsymbol{D} = \begin{bmatrix} Vecd\left(\overline{\boldsymbol{R}}^{(\cdot\cdot 1)}\right)^T \\ Vecd\left(\overline{\boldsymbol{R}}^{(\cdot\cdot 2)}\right)^T \\ \vdots \\ Vecd\left(\overline{\boldsymbol{R}}^{(\cdot\cdot I_3)}\right)^T \end{bmatrix}, \tag{2.3}$$

in which

$$\boldsymbol{R} = \begin{bmatrix} \overline{\boldsymbol{R}} \\ \boldsymbol{0}_{I_1-R,R} \end{bmatrix} \in \mathbb{C}^{I_1\times R}, \quad \overline{\boldsymbol{R}} \in \mathbb{C}^{R\times R} \text{ is upper triangular,} \tag{2.4}$$

$$\boldsymbol{L} = \begin{bmatrix} \overline{\boldsymbol{L}} \\ \boldsymbol{0}_{I_2-R,R} \end{bmatrix} \in \mathbb{C}^{I_2\times R}, \quad \overline{\boldsymbol{L}} \in \mathbb{C}^{R\times R} \text{ is lower triangular.} \tag{2.5}$$

The other way around, if $\boldsymbol{A}^{(1)}$ admits the QR-factorization $\boldsymbol{A}^{(1)} = \boldsymbol{Q}\boldsymbol{R}$ and $\boldsymbol{A}^{(2)}$ the QL-factorization $\boldsymbol{A}^{(2)} = \boldsymbol{Z}\boldsymbol{L}$, then (2.2) holds with $\{\overline{\boldsymbol{R}}^{(\cdot\cdot i_3)}\}$ upper triangular.

Proof. We first show the second part of the proposition. Substitution of $\mathbf{A}^{(1)} = \mathbf{Q}\mathbf{R}$ and $\mathbf{A}^{(2)} = \mathbf{Z}\mathbf{L}$ in (2.1) immediately yields (2.2) with $\overline{\mathbf{R}}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$ upper triangular.

We now show the first part of the proposition. From Lemma 2.1 we know that $k\left(\mathbf{A}^{(3)}\right) \geq 2$ implies that there exists a nonsingular matrix $\mathbf{N} \in \mathbb{C}^{I_3 \times I_3}$ for which $\overline{\mathbf{A}}^{(3)} = \mathbf{N}\mathbf{A}^{(3)}$ is such that $D_{j_1}\left(\overline{\mathbf{A}}^{(3)}\right)^{-1} D_{j_2}\left(\overline{\mathbf{A}}^{(3)}\right)$ exists and has distinct non-zero diagonal entries for some $j_1, j_2 \in \{1, \ldots, I_3\}$. We denote $\overline{\mathbf{X}}_{(1)} = \mathbf{X}_{(1)}\mathbf{N}^T$. We have

$$\overline{\mathbf{X}}^{(\cdot\cdot j_1)} = \mathbf{A}^{(1)} D_{j_1}\left(\overline{\mathbf{A}}^{(3)}\right) \mathbf{A}^{(2)T} \tag{2.6}$$

$$\overline{\mathbf{X}}^{(\cdot\cdot j_2)} = \mathbf{A}^{(1)} D_{j_2}\left(\overline{\mathbf{A}}^{(3)}\right) \mathbf{A}^{(2)T}. \tag{2.7}$$

The distinct diagonal entries of $D_{j_1}\left(\overline{\mathbf{A}}^{(3)}\right)^{-1} D_{j_2}\left(\overline{\mathbf{A}}^{(3)}\right)$ are the generalized eigenvalues of the pencil $\overline{\mathbf{X}}^{(\cdot\cdot j_1)} - \lambda\overline{\mathbf{X}}^{(\cdot\cdot j_2)}$. The matrices of right and left eigenvectors are $\left(\mathbf{A}^{(2)T}\right)^{\dagger}$ and $\mathbf{A}^{(1)\dagger}$, respectively.

Unitary matrices $\mathbf{Q} \in \mathbb{C}^{I_1 \times I_1}$ and $\mathbf{Z} \in \mathbb{C}^{I_2 \times I_2}$ satisfying (2.2) yield a Generalized Schur Decomposition of the pair $(\overline{\mathbf{X}}^{(\cdot\cdot j_1)}, \overline{\mathbf{X}}^{(\cdot\cdot j_2)})$:

$$\mathbf{R}^{(\cdot\cdot j_1)} = \mathbf{Q}^H \overline{\mathbf{X}}^{(\cdot\cdot j_1)} \mathbf{Z}^*, \quad \overline{\mathbf{R}}^{(\cdot\cdot j_1)} = \mathrm{triu}_R\left(\mathbf{R}^{(\cdot\cdot j_1)}\right) \tag{2.8}$$

$$\mathbf{R}^{(\cdot\cdot j_2)} = \mathbf{Q}^H \overline{\mathbf{X}}^{(\cdot\cdot j_2)} \mathbf{Z}^*, \quad \overline{\mathbf{R}}^{(\cdot\cdot j_2)} = \mathrm{triu}_R\left(\mathbf{R}^{(\cdot\cdot j_2)}\right), \tag{2.9}$$

where $\overline{\mathbf{R}}^{(\cdot\cdot j_1)}$ and $\overline{\mathbf{R}}^{(\cdot\cdot j_2)}$ are upper triangular. The diagonal entries of the diagonal matrix $\mathrm{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot j_1)}\right)\mathrm{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot j_2)}\right)^{-1}$ also correspond to the generalized eigenvalues of the pencil $\overline{\mathbf{X}}^{(\cdot\cdot j_1)} - \lambda\overline{\mathbf{X}}^{(\cdot\cdot j_2)}$. There exists a permutation matrix $\mathbf{P} \in \mathbb{R}^{R \times R}$ such that $\mathbf{P}^T D_{j_1}\left(\overline{\mathbf{A}}^{(3)}\right)^{-1} D_{j_2}\left(\overline{\mathbf{A}}^{(3)}\right)\mathbf{P} = \mathrm{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot j_1)}\right)\mathrm{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot j_2)}\right)^{-1}$. The pencil of upper triangular matrices $\overline{\mathbf{R}}^{(\cdot\cdot j_1)} - \lambda\overline{\mathbf{R}}^{(\cdot\cdot j_2)}$ has upper triangular matrices of right and left eigenvectors, which we denote by $\overline{\mathbf{E}}_R \in \mathbb{C}^{R \times R}$ and $\overline{\mathbf{E}}_L \in \mathbb{C}^{R \times R}$, respectively. Consider

$$\mathbf{E}_R = \begin{bmatrix} \overline{\mathbf{E}}_R \\ \mathbf{0}_{I_1 - R, R} \end{bmatrix} \in \mathbb{C}^{I_1 \times R}, \quad \mathbf{E}_L = \begin{bmatrix} \overline{\mathbf{E}}_L \\ \mathbf{0}_{I_2 - R, R} \end{bmatrix} \in \mathbb{C}^{I_2 \times R},$$

then $\mathbf{E}_R$ and $\mathbf{E}_L$ are matrices of right and left eigenvectors, respectively, of the pencil $\mathbf{R}^{(\cdot\cdot j_1)} - \lambda\mathbf{R}^{(\cdot\cdot j_2)}$. Combining (2.6)–(2.7) with (2.8)–(2.9) we obtain as in (2.3) that

$$\left(\mathbf{P}^T \mathbf{A}^{(2)T}\right)^{\dagger} = \mathbf{Z}^* \mathbf{E}_R \Leftrightarrow \mathbf{A}^{(2)}\mathbf{P} = \mathbf{Z}\left(\mathbf{E}_R^{\dagger}\right)^T = \mathbf{Z}\mathbf{L}, \tag{2.10}$$

$$\mathbf{P}^T \mathbf{A}^{(1)\dagger} = \mathbf{E}_L \mathbf{Q}^H \Leftrightarrow \mathbf{A}^{(1)}\mathbf{P} = \mathbf{Q}\left(\mathbf{E}_L\right)^{\dagger} = \mathbf{Q}\mathbf{R}, \tag{2.11}$$

where

$$\mathbf{R} = \begin{bmatrix} \overline{\mathbf{R}} \\ \mathbf{0}_{I_1 - R, R} \end{bmatrix} = \left(\mathbf{E}_L\right)^{\dagger}, \quad \mathbf{L} = \begin{bmatrix} \overline{\mathbf{L}} \\ \mathbf{0}_{I_2 - R, R} \end{bmatrix} = \left(\mathbf{E}_R^{\dagger}\right)^T$$

in which $\overline{\mathbf{R}}$ and $\overline{\mathbf{L}}^T$ are upper triangular matrices. Since $\mathbf{E}_R$ and $\mathbf{E}_L$ have full column rank, the matrices $\mathbf{L}$ and $\mathbf{R}$ must also have full column rank. By inserting (2.10) and (2.11) into (2.2) we get $\mathbf{R}^{(\cdot\cdot i_3)} = \mathbf{R}\mathbf{P}D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{P}^T\mathbf{L}^T$. Let $\overline{\mathbf{R}}^{(\cdot\cdot i_3)} =$

$\mathbf{R}^{(\cdot\cdot i_3)}(1:R,1:R)$ and $\mathbf{D} = \operatorname{diag}\left(\overline{\mathbf{R}}\right)\operatorname{diag}\left(\overline{\mathbf{L}}\right)$ and recall that $\mathbf{P}^T D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{P} = D_{i_3}\left(\mathbf{A}^{(3)}\mathbf{P}\right)$, then

$$\begin{aligned}\operatorname{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot i_3)}\right) &= \operatorname{diag}\left(\overline{\mathbf{R}}\mathbf{P}^T D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{P}\overline{\mathbf{L}}^T\right)\\ &= \mathbf{P}^T D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{P}\operatorname{diag}\left(\overline{\mathbf{R}}\right)\operatorname{diag}\left(\overline{\mathbf{L}}\right)\\ &= D_{i_3}\left(\mathbf{A}^{(3)}\mathbf{P}\right)\mathbf{D}, \quad \forall i_3 \in \{1,\ldots,I_3\}.\end{aligned}$$

We finally obtain $\mathbf{D}\mathbf{P}^T\mathbf{A}^{(3)T} = \left[\operatorname{Vecd}\left(\overline{\mathbf{R}}^{(\cdot\cdot 1)}\right),\ldots,\operatorname{Vecd}\left(\overline{\mathbf{R}}^{(\cdot\cdot I_3)}\right)\right]$. □

Recall that the factor matrices $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ of the (approximate) CPD of the tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ are typically found by minimizing the cost function

$$f\left(\mathbf{A}^{(1)},\mathbf{A}^{(2)},\mathbf{A}^{(3)}\right) = \sum_{i_3=1}^{I_3}\left\|\mathbf{X}^{(\cdot\cdot i_3)} - \mathbf{A}^{(1)}D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{A}^{(2)T}\right\|_F^2. \tag{2.12}$$

Let $\mathbf{A}^{(1)} = \mathbf{Q}\mathbf{R}$ be the QR factorization of $\mathbf{A}^{(1)}$ where $\mathbf{Q} \in \mathbb{C}^{I_1\times I_1}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{I_1\times R}$ is a matrix of the form (2.4). Similarly, let $\mathbf{A}^{(2)} = \mathbf{Z}\mathbf{L}$ be the QL factorization of $\mathbf{A}^{(2)}$ where $\mathbf{Z} \in \mathbb{C}^{I_2\times I_2}$ is a unitary matrix and $\mathbf{L} \in \mathbb{C}^{I_2\times R}$ is a matrix of the form (2.5). Due to Proposition 2.2 we can replace the matrices $\{\mathbf{R}D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{L}^T\}$ in (2.12) by the upper triangular matrices $\{\mathbf{R}^{(\cdot\cdot i_3)}\}$ and still preserve the uniqueness of the solution. This means that the SGSD method will attempt to minimize the cost function

$$g\left(\mathbf{Q},\mathbf{Z},\{\mathbf{R}^{(\cdot\cdot i_3)}\}\right) = \sum_{i_3=1}^{I_3}\left\|\mathbf{X}^{(\cdot\cdot i_3)} - \mathbf{Q}\mathbf{R}^{(\cdot\cdot i_3)}\mathbf{Z}^T\right\|_F^2. \tag{2.13}$$

We know from subsection 1.3 that minimizing the cost function (2.13) is equivalent to maximizing the cost function

$$h\left(\mathbf{Q},\mathbf{Z}\right) = \sum_{i_3=1}^{I_3}\left\|\operatorname{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)\right\|_F^2. \tag{2.14}$$

Thus, the SGSD-CP method first computes the unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ that make the matrices $\{\overline{\mathbf{X}}^{(\cdot\cdot i_3)}\}$ as upper triangular as possible by maximizing (2.14). This means that we have gone from the original non-unitary optimization problem (2.12), which may not have an optimal solution, to the unitary optimization problem (2.14) in which an optimal solution always exists.

3. Generalized version of the extended QZ method. Various methods have been proposed to numerically maximize the cost function (2.14). For instance, a two-sided Jacobi method for the case when the CPD is real valued has been proposed in [5]. Moreover, a one-sided Jacobi method has been proposed in [21] which is also valid for complex data. In [20] the extended QZ method was proposed for the case when $I_1 = I_2 = R$. In this section we generalize the extended QZ method to the case $I_1, I_2 \geq R$. The unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ are generated as

$$\mathbf{Q} = \prod_{s=1}^{S}\prod_{r=0}^{I_1-1}\mathbf{Q}[r,s] \quad , \quad \mathbf{Z} = \prod_{s=1}^{S}\prod_{r=0}^{I_2-1}\mathbf{Z}[r,s],$$

where S denotes the number of executed sweeps,

$$\mathbf{Q}[r,s]^H = \begin{bmatrix} \mathbf{I}_{r-1} & \mathbf{0}_{r-1,I_1-r+1} \\ \mathbf{0}_{I_1-r+1,r-1} & \mathbf{U}_s \end{bmatrix} \in \mathbb{C}^{I_1 \times I_1}$$
$$\mathbf{Z}[r,s] = \begin{bmatrix} \mathbf{A}_s & \mathbf{0}_{r,R-r} & \mathbf{C}_s \\ \mathbf{0}_{R-r,r} & \mathbf{I}_{R-r} & \mathbf{0}_{R-r,I_2-R} \\ \mathbf{B}_s & \mathbf{0}_{I_2-R,R-r} & \mathbf{D}_s \end{bmatrix} \in \mathbb{C}^{I_2 \times I_2},$$

where $\mathbf{U}_s \in \mathbb{C}^{(I_1-r+1)\times(I_1-r+1)}$ is unitary and $\mathbf{A}_s \in \mathbb{C}^{r\times r}$, $\mathbf{B}_s \in \mathbb{C}^{(I_2-R)\times r}$, $\mathbf{C}_s \in \mathbb{C}^{r\times(I_2-R)}$ and $\mathbf{D}_s \in \mathbb{C}^{(I_2-R)\times(I_2-R)}$ are such that $\mathbf{Z}[r,s]$ is unitary. In the extended QZ method the matrices $\mathbf{Q}$ and $\mathbf{Z}$ are updated in alternating manner. The updating rules are $\mathbf{X}^{(\cdot\cdot i_3)} \leftarrow \mathbf{Q}[r,s]^H \mathbf{X}^{(\cdot\cdot i_3)}$, $\mathbf{Q} \leftarrow \mathbf{Q}\cdot\mathbf{Q}[r,s]$ and $\mathbf{X}^{(\cdot\cdot i_3)} \leftarrow \mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}[r,s]^*$, $\mathbf{Z} \leftarrow \mathbf{Z}\cdot\mathbf{Z}[r,s]$.

The matrix $\mathbf{Q}[r,s]^H$ is designed to minimize the below-diagonal part of the rth column of the matrices $\{\mathbf{X}^{(\cdot\cdot i_3)}\}$. Let $\widetilde{\mathbf{X}}^{(r)} \in \mathbb{C}^{(I_1-r+1)\times I_3}$ be defined by

$$\widetilde{\mathbf{X}}^{(r)} = \left[\mathbf{X}^{(\cdot\cdot 1)}(r:I_1,r), \cdots, \mathbf{X}^{(\cdot\cdot I_3)}(r:I_1,r)\right], \quad r \in \{1,\ldots,\min(R, I_1-1)\},$$

and let $\widetilde{\mathbf{X}}^{(r)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\widetilde{\mathbf{X}}^{(r)}$. A unitary matrix $\mathbf{U}_s$ that minimizes the Frobenius norm of $\left(\mathbf{U}_s\widetilde{\mathbf{X}}^{(r)}\right)(2:I_1-r+1,:)$ is $\mathbf{U}_s = \mathbf{U}^H$.

The matrix $\mathbf{Z}[r,s]$ is designed to minimize the part of the matrices $\{\mathbf{X}^{(\cdot\cdot i_3)}\}$ that corresponds to the vectors $\{\mathbf{X}^{(\cdot\cdot i_3)}(r,1:r-1)\}$ and $\{\mathbf{X}^{(\cdot\cdot i_3)}(r,R+1:I_2)\}$. Let $\widehat{\mathbf{X}}^{(r)} \in \mathbb{C}^{I_3\times(r+I_2-R)}$ be defined by

$$\widehat{\mathbf{X}}^{(r)} = \begin{bmatrix} \mathbf{X}^{(\cdot\cdot 1)}(r,1:r), \mathbf{X}^{(\cdot\cdot 1)}(r,R+1:I_2) \\ \vdots \\ \mathbf{X}^{(\cdot\cdot I_3)}(r,1:r), \mathbf{X}^{(\cdot\cdot I_3)}(r,R+1:I_2) \end{bmatrix}, \quad r \in \{2-\mathrm{sign}(I_2-R),\ldots,R\},$$

and let $\widehat{\mathbf{X}}^{(r)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\widehat{\mathbf{X}}^{(r)}$. A unitary matrix

$$\mathbf{Z}_s^* = \begin{bmatrix} \mathbf{V}_s & \mathbf{X}_s \\ \mathbf{W}_s & \mathbf{Y}_s \end{bmatrix}$$

that minimizes the Frobenius norm of $\left(\widehat{\mathbf{X}}^{(r)}\begin{bmatrix} \mathbf{V}_s & \mathbf{X}_s \\ \mathbf{W}_s & \mathbf{Y}_s \end{bmatrix}\right)(:,[1:r-1,r+1:r+I_2-R])$ is $\mathbf{Z}_s^* = [\mathbf{V}(:,2:r),\mathbf{V}(:,1),\mathbf{V}(:,r+1:r+I_2-R)]$.

An outline of the generalized version of the extended QZ procedure is presented as Algorithm 1.

The following sections 4, 5 and 6 will discuss methods to find $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ once $\mathbf{Q}$ and $\mathbf{Z}$ have been found.

4. SGSD1-CP. The original SGSD1-CP method proposed in [20] will be reviewed in this section. The working assumption is that the factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n\times R}$, $n \in \{1,2,3\}$ have full column rank. Let $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ be a tensor of rank R constructed from the full column rank factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n\times R}$, $n \in \{1,2,3\}$, such that

$$\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{A}^{(2)T}, \quad i_3 \in \{1,\ldots,I_3\}.$$

Algorithm 1 Outline of the extended QZ method when $I_1, I_2 \geq R$.

Initialize: $\mathbf{Q}$ and $\mathbf{Z}$
Repeat until convergence
for $r = 1$ to $\min(R, I_1 - 1)$ **do**
$\widetilde{\mathbf{X}}^{(r)} = \left[\mathbf{X}^{(\cdot\cdot 1)}(r : I_1, r), \cdots, \mathbf{X}^{(\cdot\cdot I_3)}(r : I_1, r)\right]$
$\mathbf{U}\Sigma\mathbf{V}^H = \text{svd}\left(\widetilde{\mathbf{X}}^{(r)}\right)$
$\mathbf{X}^{(\cdot\cdot i_3)}(r : I_1, :) \leftarrow \mathbf{U}^H\mathbf{X}^{(\cdot\cdot i_3)}(r : I_1, :)$
$\mathbf{Q}(:, r : I_1) \leftarrow \mathbf{Q}(:, r : I_1)\mathbf{U}$
end for
for $r = R$ to $\max(1, R - I_2 + 2)$ **do**
$$\widehat{\mathbf{X}}^{(r)} = \begin{bmatrix} \mathbf{X}^{(\cdot\cdot 1)}(r, 1 : r)\,,\ \mathbf{X}^{(\cdot\cdot 1)}(r, R+1 : I_2) \\ \vdots \\ \mathbf{X}^{(\cdot\cdot I_3)}(r, 1 : r)\,,\ \mathbf{X}^{(\cdot\cdot I_3)}(r, R+1 : I_2) \end{bmatrix}$$
$\mathbf{U}\Sigma\mathbf{V}^H = \text{svd}\left(\widehat{\mathbf{X}}^{(r)}\right)$
$\mathbf{V} \leftarrow [\mathbf{V}(:, 2 : r), \mathbf{V}(:, 1), \mathbf{V}(:, r + 1 : r + I_2 - R)]$
$\mathbf{X}^{(\cdot\cdot i_3)}(:, [1 : r, R + 1 : I_2]) \leftarrow \mathbf{X}^{(\cdot\cdot i_3)}(:, [1 : r, R + 1 : I_2])\mathbf{V}$
$\mathbf{Z}(:, [1 : r, R + 1 : I_2]) \leftarrow \mathbf{Z}(:, [1 : r, R + 1 : I_2])\mathbf{V}^*$
end for

Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found and let $\overline{\mathbf{R}}^{(\cdot\cdot i_3)} = \text{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)$. We know from Proposition 2.2 that $\mathbf{A}^{(3)} = \left[\text{Vecd}\left(\overline{\mathbf{R}}^{(\cdot\cdot 1)}\right), \ldots, \text{Vecd}\left(\overline{\mathbf{R}}^{(\cdot\cdot I_3)}\right)\right]^T$ up to the inherent column scaling and permutation ambiguities. Assume that $\mathbf{A}^{(3)}$ has been found and compute $\mathbf{F} = \mathbf{X}_{(1)}\left(\mathbf{A}^{(3)T}\right)^\dagger$. In the exact case, we know that $\mathbf{F} = \mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}$ and therefore $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ follow from the R decoupled best rank-1 approximation problems

$$\min_{\mathbf{a}_r^{(1)}, \mathbf{a}_r^{(2)}} \left\|\mathbf{f}_r - \mathbf{a}_r^{(1)} \otimes \mathbf{a}_r^{(1)}\right\|_F^2 = \min_{\mathbf{a}_r^{(1)}, \mathbf{a}_r^{(2)}} \left\|\text{Unvec}\left(\mathbf{f}_r\right) - \mathbf{a}_r^{(2)}\mathbf{a}_r^{(1)T}\right\|_F^2, \quad r \in \{1, \ldots, R\}.$$

In the presence of noise a refinement of the factor matrices $\{\mathbf{A}^{(n)}\}$ is often necessary. The subsequent refinement step should be implemented by a method which fully takes the CPD structure of the problem into account. For optimization methods, see [17] and references therein.

5. SGSD2-CP. In this section the SGSD2-CP method for computing a third-order CPD will be presented. This method attempts to take the relations among the triangular matrices $\{\mathbf{R}^{(\cdot\cdot i_3)}\}$ into account when computing $\mathbf{A}^{(3)}$ whereas the SGSD1-CP method only uses the diagonals of the triangular matrices $\{\mathbf{R}^{(\cdot\cdot i_3)}\}$ when computing $\mathbf{A}^{(3)}$. As in the SGSD1-CP method, the working assumption of the SGSD2-CP method is that all three factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1, 2, 3\}$ have full column rank.

Let the third-order tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ of rank R be constructed from the full

column rank factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1,2,3\}$, such that

$$\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{A}^{(2)T} \in \mathbb{C}^{I_1 \times I_2}, \quad \forall i_3 \in \{1,\ldots,I_3\}.$$

Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found. Construct the tensor $\mathcal{R} \in \mathbb{C}^{R\times R\times I_3}$ such that $\mathcal{R}_{i_1 i_2 i_3} = \mathbf{R}_{i_1 i_2}^{(\cdot\cdot i_3)} = \mathrm{triu}_R\left(\mathbf{Q}^H \mathbf{X}^{(\cdot\cdot i_3)} \mathbf{Z}^*\right)_{i_1 i_2}$, then $\mathbf{R}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$. Moreover, let $\mathbf{R}^{(i_1\cdot\cdot)} \in \mathbb{C}^{R\times I_3}$ satisfy the relation $\mathbf{R}_{i_2 i_3}^{(i_1\cdot\cdot)} = \mathcal{R}_{i_1 i_2 i_3}$, then $\mathbf{R}^{(i_1\cdot\cdot)} = \overline{\mathbf{L}} D_{i_1}\left(\overline{\mathbf{R}}\right)\mathbf{A}^{(3)T}$. Stack the matrices $\{\mathbf{R}^{(i_1\cdot\cdot)}\}$ into the matrix

$$\mathbb{C}^{R^2\times I_3} \ni \mathbf{R}_{(1)} = \begin{bmatrix} \mathbf{R}^{(1\cdot\cdot)} \\ \vdots \\ \mathbf{R}^{(R\cdot\cdot)} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{L}} D_1\left(\overline{\mathbf{R}}\right) \\ \vdots \\ \overline{\mathbf{L}} D_R\left(\overline{\mathbf{R}}\right) \end{bmatrix} \mathbf{A}^{(3)T} = \left(\overline{\mathbf{R}} \odot \overline{\mathbf{L}}\right)\mathbf{A}^{(3)T}.$$

We will now present an algorithm to compute the factor matrices from $\mathrm{Col}\left(\mathbf{R}_{(1)}\right)$. Let $\mathbf{R}_{(1)} = \mathbf{U}\Sigma\mathbf{W}^H$ denote the compact SVD of $\mathbf{R}_{(1)}$, where $\mathbf{U} \in \mathbb{C}^{R^2\times R}$ and $\mathbf{W} \in \mathbb{C}^{I_3\times R}$ are columnwise orthonormal matrices and $\Sigma \in \mathbb{C}^{R\times R}$ is a positive diagonal matrix. The Khatri-Rao product of two full column rank matrices has full column rank and consequently the matrix $\overline{\mathbf{R}} \odot \overline{\mathbf{L}}$ has full column rank. Since the matrix $\mathbf{A}^{(3)}$ has full column rank by assumption we can now conclude that there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ such that

$$\mathbf{W}^*\mathbf{M}^{-T} = \mathbf{A}^{(3)} \tag{5.1}$$

$$\mathbf{U}\Sigma\mathbf{M} = \overline{\mathbf{R}} \odot \overline{\mathbf{L}} \Leftrightarrow \mathbf{U}\Sigma\mathbf{m}_r = \overline{\mathbf{r}}_r \otimes \overline{\mathbf{l}}_r, \quad r \in \{1,\ldots,R\}. \tag{5.2}$$

Let $\mathbf{a}_r = \overline{\mathbf{r}}_r(1:r)$ and $\mathbf{b}_r = \overline{\mathbf{l}}_r(r:R)$, then (5.2) is equivalent to

$$\mathbf{U}\Sigma\mathbf{m}_r = \sum_{1\le p\le r}\ \sum_{r\le q\le R}\left(\mathbf{e}_p^{(R)} \otimes \mathbf{e}_q^{(R)}\right)(\mathbf{a}_r \otimes \mathbf{b}_r) = \mathbf{E}^{(r)}(\mathbf{a}_r \otimes \mathbf{b}_r), \quad r \in \{1,\ldots,R\}, \tag{5.3}$$

where

$$\mathbf{E}^{(r)} = [\mathbf{e}_1 \otimes \mathbf{e}_r, \ldots, \mathbf{e}_1 \otimes \mathbf{e}_R, \ldots, \mathbf{e}_r \otimes \mathbf{e}_r, \ldots, \mathbf{e}_r \otimes \mathbf{e}_R] \in \mathbb{C}^{R^2\times r(R-r+1)}., \tag{5.4}$$

Equation (5.3) can be written as

$$\mathbf{U}\Sigma\mathbf{m}_r = \mathbf{E}^{(r)}(\mathbf{a}_r \otimes \mathbf{b}_r) \Leftrightarrow \left[\mathbf{U}\Sigma, -\mathbf{E}^{(r)}\right]\begin{bmatrix} \mathbf{m}_r \\ \mathbf{a}_r \otimes \mathbf{b}_r \end{bmatrix} = \mathbf{0}_{R^2}, \quad r \in \{1,\ldots,R\}. \tag{5.5}$$

Proposition 5.2 will show that (5.5) has a unique solution. Moreover, it will provide us with a method to solve the system (5.5). In order to prove it we need the following lemma.

LEMMA 5.1. *Let $\boldsymbol{R} \in \mathbb{C}^{R\times R}$ be a nonsingular upper triangular matrix and let $\boldsymbol{L} \in \mathbb{C}^{R\times R}$ be a nonsingular lower triangular matrix. Then the matrix $\boldsymbol{S} = \boldsymbol{R} \odot \boldsymbol{L} \in \mathbb{C}^{R^2\times R}$ has the following property*

$$\boldsymbol{S}(r + (r-1)R, :) = \alpha_r \boldsymbol{e}_r^{(R)T}, \quad \alpha_r \in \mathbb{C}, \forall r \in \{1,\ldots,R\}.$$

Proof. The proof is straightforward, noting that

$$\mathbf{S}\left(t+(t-1)R,u\right)=\left(\mathbf{R}\right)_{tu}\left(\mathbf{L}\right)_{tu} \tag{5.6}$$

and

$$\begin{cases}\left(\mathbf{R}\right)_{tu}\in\mathbb{C}\,, & u<t\\ \left(\mathbf{R}\right)_{tu}\neq 0\,, & u=t\\ \left(\mathbf{R}\right)_{tu}=0\,, & u>t\end{cases} \tag{5.7}$$

$$\begin{cases}\left(\mathbf{L}\right)_{tu}=0\,, & u<t\\ \left(\mathbf{L}\right)_{tu}\neq 0\,, & u=t\\ \left(\mathbf{L}\right)_{tu}\in\mathbb{C}\,, & u>t\end{cases} \tag{5.8}$$

By combining (5.6), (5.7) and (5.8) we can conclude that

$$\mathbf{S}\left(r+(r-1)R,:\right)=\alpha_r\mathbf{e}_r^{(R)T},\quad \alpha_r\in\mathbb{C},\quad \forall r\in\{1,\dots,R\}\,.$$

□

PROPOSITION 5.2. *Let $\boldsymbol{R}=[\boldsymbol{r}_1,\dots,\boldsymbol{r}_R]\in\mathbb{C}^{R\times R}$ be a nonsingular upper triangular matrix and let $\boldsymbol{L}=[\boldsymbol{l}_1,\dots,\boldsymbol{l}_R]\in\mathbb{C}^{R\times R}$ be a nonsingular lower triangular matrix. Let $\boldsymbol{E}^{(r)}$ be of the form* (5.4)*. If the column vectors of $\boldsymbol{U}\in\mathbb{C}^{R^2\times R}$ constitute a basis for $Col\left(\boldsymbol{R}\odot\boldsymbol{L}\right)$, then the matrix*

$$\left[\boldsymbol{U},-\boldsymbol{E}^{(r)}\right]\in\mathbb{C}^{R^2\times(R+r(R-r+1))} \tag{5.9}$$

has rank $R+r(R-r+1)-1$.

Proof. Consider the basis $\{\mathbf{r}_m\otimes\mathbf{l}_m\}$ for $\mathrm{Col}\left(\mathbf{R}\odot\mathbf{L}\right)$. On one hand, $\mathbf{r}_r\otimes\mathbf{l}_r$ is a linear combination of the columns of $\mathbf{E}^{(r)}$. On the other hand, by Lemma 5.1, vector $\mathbf{r}_n\otimes\mathbf{l}_n$, with $n\neq r$ has a non-zero at position $n+(n-1)R$, while vectors $\mathbf{r}_p\otimes\mathbf{l}_p$, $n\neq p\neq r$, and the columns of $\mathbf{E}^{(r)}$ are zero at that position. Hence, matrix

$$\left[\mathbf{R}\odot\mathbf{L},-\mathbf{E}^{(r)}\right]$$

has rank $r(R-r+1)-1$. In conclusion, the matrix (5.9) has rank $R+r(R-r+1)-1$. □

Proposition 5.2 tells us that we may ignore the Kronecker product structure $\mathbf{a}_r\otimes\mathbf{b}_r$ when solving (5.5) since the matrix $\left[\mathbf{U}\Sigma,-\mathbf{E}^{(r)}\right]$ has a one-dimensional kernel. We now reduce the number of equations by projecting on the orthogonal complement of $\mathrm{Col}\left(\mathbf{E}^{(r)}\right)$.

COROLLARY 5.3. *Let $\boldsymbol{R}=[\boldsymbol{r}_1,\dots,\boldsymbol{r}_R]\in\mathbb{C}^{R\times R}$ and $\boldsymbol{L}=[\boldsymbol{l}_1,\dots,\boldsymbol{l}_R]\in\mathbb{C}^{R\times R}$ be a nonsingular upper and lower triangular matrix, respectively. Define the diagonal projection matrix*

$$\mathbb{C}^{R^2\times R^2}\ni\boldsymbol{D}^{(r)}=\boldsymbol{I}_{R^2}-\boldsymbol{E}^{(r)}\boldsymbol{E}^{(r)T}\,, \tag{5.10}$$

where $\boldsymbol{E}^{(r)}\in\mathbb{C}^{R^2\times r(R-r+1)}$ is of the form (5.4)*. If the column vectors of $\boldsymbol{U}\in\mathbb{C}^{R^2\times R}$ constitute a basis for $Col\left(\boldsymbol{R}\odot\boldsymbol{L}\right)$, then the matrix*

$$\mathbb{C}^{R^2\times R}\ni\boldsymbol{U}^{(r)}=\boldsymbol{D}^{(r)}\boldsymbol{U} \tag{5.11}$$

has rank $R-1$, $\forall r \in \{1,\ldots,R\}$.

Proof. Consider the columns $\{\mathbf{r}_s \otimes \mathbf{l}_s\}$ of $\mathbf{R} \odot \mathbf{L}$. On one hand, $\mathbf{r}_r \otimes \mathbf{l}_r$ is a linear combination of the columns of $\mathbf{E}^{(r)}$ and is therefore set equal to zero by $\mathbf{D}^{(r)}$. On the other hand, by Lemma 5.1, vector $\mathbf{r}_n \otimes \mathbf{l}_n$, with $n \neq r$ has a non-zero at position $n+(n-1)R$, while vectors $\mathbf{r}_p \otimes \mathbf{l}_p$, $n \neq p \neq r$, are zero at that position. The entries at position $n+(n-1)R$, $1 \leq n \neq r \leq R$, are not set equal to zeros by $\mathbf{D}^{(r)}$. We conclude that $\mathbf{D}^{(r)}(\mathbf{R} \odot \mathbf{L})$ has rank $R-1$. Hence, any matrix $\mathbf{U}$ of which the columns form a basis for $\mathrm{Col}(\mathbf{R} \odot \mathbf{L})$ yields a matrix $\mathbf{D}^{(r)}\mathbf{U}$ of rank $R-1$. ∎

By projecting (5.5) on the orthogonal complement of $\mathbf{E}^{(r)}$ we obtain

$$\mathbf{U}^{(r)}\mathbf{m}_r = \mathbf{0}_{R^2}\,, \quad r \in \{1,\ldots,R\} \tag{5.12}$$

with $\mathbf{U}^{(r)} = \mathbf{D}^{(r)}\mathbf{U}\Sigma$ and $\mathbf{D}^{(r)}$ defined in (5.10). According to Corollary 5.3, $\mathbf{U}^{(r)}$ still has a one-dimensional kernel. In practice, we first remove the all-zero row-vectors of $\mathbf{U}^{(r)}$ to obtain $\widetilde{\mathbf{U}}^{(r)} \in \mathbb{C}^{\left(R^2-r(R-r+1)\right)\times R}$, and we solve

$$\widetilde{\mathbf{U}}^{(r)}\mathbf{m}_r = \mathbf{0}_{R^2-r(R-r+1)}\,, \quad r \in \{1,\ldots,R\}\,. \tag{5.13}$$

Note that the set of R^2 equations in $R + r(R-r+1)$ unknowns (5.5) has been reduced to a set of $R^2 - r(R-r+1)$ homogeneous equations in R unknowns. In the case of an approximate tensor decomposition, we set $\mathbf{m}_r$ equal to the singular vector associated with the smallest singular value of $\widetilde{\mathbf{U}}^{(r)}$. Once $\mathbf{M} = [\mathbf{m}_1,\ldots,\mathbf{m}_R]$ has been obtained from (5.12), then from (5.1) we get $\mathbf{A}^{(3)} = \mathbf{W}^*\mathbf{M}^{-T}$. Compute $\mathbf{F} = \mathbf{X}_{(1)}\left(\mathbf{A}^{(3)T}\right)^\dagger = \mathbf{X}_{(1)}\mathbf{M}^T\mathbf{W}^T$, then as in the SGSD1-CP method, the matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ now follow from the R decoupled best rank-1 approximation problems

$$\min_{\mathbf{a}_r^{(1)},\mathbf{a}_r^{(2)}} \left\| \mathrm{Unvec}\left(\mathbf{f}_r\right) - \mathbf{a}_r^{(2)}\mathbf{a}_r^{(1)T} \right\|_F^2, \quad r \in \{1,\ldots,R\}\,.$$

Finally, we obtain $\mathbf{A}^{(3)}$ via relation $\mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)^\dagger \mathbf{X}_{(1)}$.

6. SGSD3-CP. In this section we present the SGSD3-CP method for computing a third-order CPD where only two of the factor matrices are required to have full column rank. This is in contrast to the SGSD1-CP and SGSD2-CP methods which both require that all three of the involved factor matrices have full column rank. In subsection 6.1 we first consider the case when $k\left(\mathbf{A}^{(3)}\right) \geq 2$ while in subsection 6.2 we relax this constraint to $k\left(\mathbf{A}^{(3)}\right) \geq 1$.

6.1. SGSD3-CP with $k\left(\mathbf{A}^{(3)}\right) \geq 2$. Let the third-order tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ of rank R be constructed from the two full column rank factor matrices $\mathbf{A}^{(1)} \in \mathbb{C}^{I_1\times R}$, $\mathbf{A}^{(2)} \in \mathbb{C}^{I_2\times R}$ and the factor matrix $\mathbf{A}^{(3)} \in \mathbb{C}^{I_3\times R}$ with $k\left(\mathbf{A}^{(3)}\right) \geq 2$ such that

$$\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{A}^{(1)}D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{A}^{(2)T}, \quad i_3 \in \{1,\ldots,I_3\}\,.$$

Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found and denote $\mathbf{R}^{(\cdot\cdot i_3)} = \mathrm{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)$. As in the SGSD1-CP method, we find $\mathbf{A}^{(3)}$ from the relation

$$\mathbf{A}^{(3)T} = \left[\mathrm{Vecd}\left(\mathbf{R}^{(\cdot\cdot 1)}\right),\ldots,\mathrm{Vecd}\left(\mathbf{R}\right)^{(\cdot\cdot I_3)}\right]\,.$$

Let the tensor $\mathcal{R} \in \mathbb{C}^{R\times R\times I_3}$ be constructed such that $\mathcal{R}_{i_1 i_2 i_3} = \mathbf{R}^{(\cdot\cdot i_3)}_{i_1 i_2}$, then $\mathbf{R}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$. Moreover, let $\mathbf{R}^{(\cdot i_2 \cdot)} \in \mathbb{C}^{I_3\times R}$ satisfy the relation $\mathbf{R}^{(\cdot i_2 \cdot)}_{i_3 i_1} = \mathcal{R}_{i_1 i_2 i_3}$, then $\mathbf{R}^{(\cdot i_2 \cdot)} = \mathbf{A}^{(3)} D_{i_2}\left(\overline{\mathbf{L}}\right)\overline{\mathbf{R}}^T$. Stack the matrices $\{\mathbf{R}^{(\cdot i_2 \cdot)}\}$ into the matrix

$$\mathbb{C}^{RI_3\times R} \ni \mathbf{R}_{(2)} = \begin{bmatrix} \mathbf{R}^{(\cdot 1\cdot)} \\ \vdots \\ \mathbf{R}^{(\cdot R\cdot)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(3)} D_1\left(\overline{\mathbf{L}}\right) \\ \vdots \\ \mathbf{A}^{(3)} D_R\left(\overline{\mathbf{L}}\right) \end{bmatrix} \overline{\mathbf{R}}^T = \left(\overline{\mathbf{L}} \odot \mathbf{A}^{(3)}\right)\overline{\mathbf{R}}^T.$$

Let $\mathbf{R}_{(2)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{R}_{(2)}$. Since $\overline{\mathbf{L}}$ has full column rank, the matrix $\overline{\mathbf{L}} \odot \mathbf{A}^{(3)}$ has full column rank. This implies that there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ such that

$$\mathbf{UM} = \overline{\mathbf{L}} \odot \mathbf{A}^{(3)} \Leftrightarrow \mathbf{U}\mathbf{m}_r = \bar{\mathbf{l}}_r \otimes \mathbf{a}^{(3)}_r, \quad r \in \{1,\dots,R\}. \tag{6.1}$$

Let $\mathbf{b}_r = \bar{\mathbf{l}}_r\,(r:R) \in \mathbb{C}^{R-r+1}$ and

$$\mathbf{E}^{(r)} = \left[\mathbf{e}^{(R)}_r, \mathbf{e}^{(R)}_{r+1}, \dots, \mathbf{e}^{(R)}_R\right] \in \mathbb{C}^{R\times(R-r+1)}, \tag{6.2}$$

then $\bar{\mathbf{l}}_r = \mathbf{E}^{(r)}\mathbf{b}_r$ and

$$\bar{\mathbf{l}}_r \otimes \mathbf{a}^{(3)}_r = \left(\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right)\mathbf{b}_r. \tag{6.3}$$

By inserting (6.3) into (6.1) we obtain

$$\mathbf{U}\mathbf{m}_r = \left(\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right)\mathbf{b}_r \Leftrightarrow \left[\mathbf{U}, -\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right]\begin{bmatrix} \mathbf{m}_r \\ \mathbf{b}_r \end{bmatrix} = \mathbf{0}_{R^2},\ r \in \{1,\dots,R\}. \tag{6.4}$$

Due to the following Proposition 6.1 the linear system (6.4) has a unique solution. The least-squares solution to (6.4) is given by the right singular vector of the matrix $\left[\mathbf{U}, -\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right]$ associated with its smallest singular value.

PROPOSITION 6.1. *Given $\boldsymbol{A}^{(3)} \in \mathbb{C}^{I_3\times R}$ with $k\left(\boldsymbol{A}^{(3)}\right) \geq 2$ and let $\boldsymbol{L} \in \mathbb{C}^{R\times R}$ be a nonsingular lower triangular matrix. Let $\boldsymbol{E}^{(r)}$ be of the form* (6.2)*. If the column vectors of $\boldsymbol{U} \in \mathbb{C}^{RI_3\times R}$ constitute a basis for Col$\left(\boldsymbol{L} \odot \boldsymbol{A}^{(3)}\right)$, then the matrix*

$$\left[\boldsymbol{U}, -\boldsymbol{E}^{(r)} \otimes \boldsymbol{a}^{(3)}_r\right] \in \mathbb{C}^{RI_3\times(2R-r+1)} \tag{6.5}$$

has rank $2R-r$.

Proof. Consider the columns $\{\mathbf{l}_r \otimes \mathbf{a}^{(3)}_r\}$ of Col$\left(\mathbf{L} \odot \mathbf{A}^{(3)}\right)$, then

$$\mathbf{U} = [\mathbf{u}_1,\dots,\mathbf{u}_R] = \left[\mathbf{l}_1 \otimes \mathbf{a}^{(3)}_1, \dots, \mathbf{l}_R \otimes \mathbf{a}^{(3)}_R\right] = \begin{bmatrix} l_{11}\mathbf{a}^{(3)}_1 & \mathbf{0}_{I_3} & \cdots & \mathbf{0}_{I_3} \\ l_{21}\mathbf{a}^{(3)}_1 & l_{22}\mathbf{a}^{(3)}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathbf{0}_{I_3} \\ l_{R1}\mathbf{a}^{(3)}_1 & l_{R2}\mathbf{a}^{(3)}_2 & \cdots & l_{RR}\mathbf{a}^{(3)}_R \end{bmatrix}.$$

We also have

$$\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r = \left[\mathbf{e}^{(R)}_r \otimes \mathbf{a}^{(3)}_r, \dots, \mathbf{e}^{(R)}_R \otimes \mathbf{a}^{(3)}_r\right].$$

Since

$$\mathbf{u}_r = \mathbf{l}_r \otimes \mathbf{a}_r^{(3)} = \sum_{q=r}^{R} l_{qr} \mathbf{e}_q^{(R)} \otimes \mathbf{a}_r^{(3)}$$

the problem of determining the rank of

$$\left[\mathbf{U}, -\mathbf{E}^{(r)} \otimes \mathbf{a}_r^{(3)}\right] \in \mathbb{C}^{RI_3 \times (2R-r+1)}$$

reduces to finding the rank of

$$\left[\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_R, \mathbf{e}_r^{(R)} \otimes \mathbf{a}_r^{(3)}, \mathbf{e}_{r+1}^{(R)} \otimes \mathbf{a}_r^{(3)}, \ldots, \mathbf{e}_R^{(R)} \otimes \mathbf{a}_r^{(3)}\right] \in \mathbb{C}^{RI_3 \times (2R-r)}. \tag{6.6}$$

By reorganizing the column vectors of (6.6) we obtain

$$\left[\begin{array}{ccc|c|cc|c|cc} \mathbf{u}_1 & \cdots & \mathbf{u}_{r-1} & \mathbf{e}_r^{(R)} \otimes \mathbf{a}_r^{(3)} & \mathbf{u}_{r+1} & \mathbf{e}_{r+1}^{(R)} \otimes \mathbf{a}_r^{(3)} & \cdots & \mathbf{u}_R & \mathbf{e}_R^{(R)} \otimes \mathbf{a}_r^{(3)} \end{array}\right] =$$

$$\left[\begin{array}{ccc|c|cc|c|cc} l_{1,1}\mathbf{a}_1^{(3)} & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ l_{r-2,1}\mathbf{a}_1^{(3)} & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} \\ \hline l_{r-1,1}\mathbf{a}_1^{(3)} & l_{r-1,r-1}\mathbf{a}_{r-1}^{(3)} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} \\ l_{r,1}\mathbf{a}_1^{(3)} & l_{r,r-1}\mathbf{a}_{r-1}^{(3)} & \mathbf{a}_r^{(3)} & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} & & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} \\ l_{r+1,1}\mathbf{a}_1^{(3)} & l_{r+1,r-1}\mathbf{a}_{r-1}^{(3)} & \mathbf{0}_{I_3} & l_{r+1,r+1}\mathbf{a}_{r+1}^{(3)} & \mathbf{a}_r^{(3)} & & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} \\ \hline l_{r+2,1}\mathbf{a}_1^{(3)} & l_{r+2,r-1}\mathbf{a}_{r-1}^{(3)} & \mathbf{0}_{I_3} & l_{r+2,r+1}\mathbf{a}_{r+1}^{(3)} & \mathbf{0}_{I_3} & & & \mathbf{0}_{I_3} & \mathbf{0}_{I_3} \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \vdots & \vdots \\ l_{R,1}\mathbf{a}_1^{(3)} & l_{R,r-1}\mathbf{a}_{r-1}^{(3)} & \mathbf{0}_{I_3} & l_{R,r+1}\mathbf{a}_{r+1}^{(3)} & \mathbf{0}_{I_3} & & & l_{R,R}\mathbf{a}_R^{(3)} & \mathbf{a}_r^{(3)} \end{array}\right]. \tag{6.7}$$

The matrix (6.7) is lower block-triangular. Since $k\left(\mathbf{A}^{(3)}\right) \geq 2$ and $\mathbf{L}$ is nonsingular the matrices on the block-diagonal of (6.7) have full column rank which means that the matrix (6.7) has full column rank. Hence, the matrix (6.5) has rank $2R - r$. Consequently any matrix $\mathbf{U}$ of which the columns form a basis for $\text{Col}\left(\mathbf{L} \odot \mathbf{A}^{(3)}\right)$ yields a matrix of rank $2R - r$. □

After $\overline{\mathbf{L}} \in \mathbb{C}^{R \times R}$ has been computed, then via $\mathbf{b}_r$ in (6.4) we obtain $\mathbf{L} = [\overline{\mathbf{L}}^T, \mathbf{0}_{I_2-R,R}^T]^T \in \mathbb{C}^{I_2 \times R}$ and $\mathbf{A}^{(2)} = \mathbf{Z}^* \mathbf{L}$. Finally, we compute $\mathbf{A}^{(1)}$ via relation $\mathbf{A}^{(1)T} = \left(\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right)^\dagger \mathbf{X}_{(2)}$.

6.2. SGSD3-CP with $k\left(\mathbf{A}^{(3)}\right) = 1$. Assume now that the third-order tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ of rank R is constructed from the two full column rank factor matrices $\mathbf{A}^{(1)} \in \mathbb{C}^{I_1 \times R}$ and $\mathbf{A}^{(2)} \in \mathbb{C}^{I_2 \times R}$ but the factor matrix $\mathbf{A}^{(3)} \in \mathbb{C}^{I_3 \times R}$ now only has $k\left(\mathbf{A}^{(3)}\right) = 1$. In this case $\mathbf{A}^{(3)}$ can be written as

$$\mathbf{A}^{(3)} = \left[\mathbf{1}_{L_1}^T \otimes \mathbf{c}_1, \ldots, \mathbf{1}_{L_Q}^T \otimes \mathbf{c}_Q\right], \quad \mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_Q] \in \mathbb{C}^{I_3 \times Q} \tag{6.8}$$

up to a permutation and scaling of the columns of $\mathbf{A}^{(3)}$ and $R = \sum_{q=1}^{Q} L_q$. We can write the decomposition of $\mathcal{X}$ as follows

$$\mathcal{X} = \sum_{q=1}^{Q} \left(\mathbf{A}_q \mathbf{B}_q^T \right) \circ \mathbf{c}_q \,, \tag{6.9}$$

where

$$\begin{aligned} \mathbf{A}^{(1)} &= [\mathbf{A}_1, \ldots, \mathbf{A}_Q] \,, \quad \mathbf{A}_q \in \mathbb{C}^{I_1 \times L_q} \\ \mathbf{A}^{(2)} &= [\mathbf{B}_1, \ldots, \mathbf{B}_Q] \,, \quad \mathbf{B}_q \in \mathbb{C}^{I_2 \times L_q} \,. \end{aligned}$$

The decomposition (6.9) is known as the $(L_q, L_q, 1)$-Block Term Decomposition (BTD) of $\mathcal{X}$ [9]. Since $\mathbf{A}_q \mathbf{B}_q^T = \mathbf{A}_q \mathbf{F}_q \mathbf{F}_q^{-1} \mathbf{B}_q^T$ for any nonsingular matrix $\mathbf{F}_q \in \mathbb{C}^{L_q \times L_q}$ we can from $\mathcal{X}$ in (6.9) only find $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ up to a column permuted block-diagonal matrix. We now explain that the SGSD3-CP method is still valid even if $k\left(\mathbf{A}^{(3)}\right) = 1$. It will based on generalizations of Propositions 2.2 and 6.1.

PROPOSITION 6.2. *Given a rank-R tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ of which the $(L_q, L_q, 1)$-BTD involves the factor matrices $\boldsymbol{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1, 2, 3\}$, such that it has matrix representations*

$$\boldsymbol{X}^{(\cdot\cdot i_3)} = \boldsymbol{A}^{(1)} D_{i_3}\left(\boldsymbol{A}^{(3)}\right) \boldsymbol{A}^{(2)T} \,, \quad i_3 \in \{1, \ldots, I_3\} \,. \tag{6.10}$$

Assume that $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ have full column rank and $\boldsymbol{A}^{(3)}$ is of the form (6.8) with $k\,(\boldsymbol{C}) \geq 2$. If there exist unitary matrices $\boldsymbol{Q} \in \mathbb{C}^{I_1 \times I_1}$ and $\boldsymbol{Z} \in \mathbb{C}^{I_2 \times I_2}$ such that

$$\boldsymbol{R}^{(\cdot\cdot i_3)} = \boldsymbol{Q}^H \boldsymbol{X}^{(\cdot\cdot i_3)} \boldsymbol{Z}^* = \begin{bmatrix} \overline{\boldsymbol{R}}^{(\cdot\cdot i_3)} & \boldsymbol{0}_{R, I_2 - R} \\ \boldsymbol{0}_{I_1 - R, R} & \boldsymbol{0}_{I_1 - R, I_2 - R} \end{bmatrix} \in \mathbb{C}^{I_1 \times I_2}, \; i_3 \in \{1, \ldots, I_3\}, \tag{6.11}$$

where $\overline{\boldsymbol{R}}^{(\cdot\cdot i_3)} \in \mathbb{C}^{R \times R}$ are upper triangular matrices, then there exists a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{R \times R}$, a diagonal matrix $\boldsymbol{D} \in \mathbb{C}^{R \times R}$ and nonsingular matrices $\boldsymbol{F}_q \in \mathbb{C}^{L_q \times L_q}$ such that

$$\boldsymbol{A}^{(1)} \boldsymbol{F} \boldsymbol{P} = \boldsymbol{Q} \boldsymbol{R} \,, \quad \boldsymbol{A}^{(2)} \boldsymbol{F}^{-1} \boldsymbol{P} = \boldsymbol{Z} \boldsymbol{L} \,, \quad \boldsymbol{A}^{(3)} \boldsymbol{P} \boldsymbol{D} = \begin{bmatrix} \mathit{Vecd}\left(\overline{\boldsymbol{R}}^{(\cdot\cdot 1)}\right)^T \\ \mathit{Vecd}\left(\overline{\boldsymbol{R}}^{(\cdot\cdot 2)}\right)^T \\ \vdots \\ \mathit{Vecd}\left(\overline{\boldsymbol{R}}^{(\cdot\cdot I_3)}\right)^T \end{bmatrix}, \tag{6.12}$$

in which $\boldsymbol{F} = \mathit{Blockdiag}\,(\boldsymbol{F}_1, \ldots, \boldsymbol{F}_Q)$, $\boldsymbol{R} \in \mathbb{C}^{I_1 \times R}$ and $\boldsymbol{L} \in \mathbb{C}^{I_2 \times R}$ are of the forms (2.4) and (2.5), respectively. Conversely, if $\boldsymbol{A}^{(1)}$ admits the QR-factorization $\boldsymbol{A}^{(1)} = \boldsymbol{Q}\boldsymbol{R}$ and $\boldsymbol{A}^{(2)}$ the QL-factorization $\boldsymbol{A}^{(2)} = \boldsymbol{Z}\boldsymbol{L}$, then (6.11) holds with $\{\overline{\boldsymbol{R}}^{(\cdot\cdot i_3)}\}$ upper triangular.

Proof. The second part of the proposition is obvious. Indeed, by substituting $\mathbf{A}^{(1)} = \mathbf{Q}\mathbf{R}$ and $\mathbf{A}^{(2)} = \mathbf{Z}\mathbf{L}$ in (2.1) immediately yields (6.11) with $\overline{\mathbf{R}}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right) \overline{\mathbf{L}}^T$ upper triangular.

We now show the first part of the proposition. Since $k\,(\mathbf{C}) \geq 2$ we know from Lemma 2.1 that there exists a nonsingular matrix $\mathbf{N} \in \mathbb{C}^{I_3 \times I_3}$ for which $\overline{\mathbf{C}} = \mathbf{N}\mathbf{C}$

is such that $D_{j_1}\left(\overline{\mathbf{C}}\right)^{-1} D_{j_2}\left(\overline{\mathbf{C}}\right)$ exists and has distinct non-zero diagonal entries for some $j_1, j_2 \in \{1, \ldots, Q\}$. We denote $\overline{\mathbf{X}}_{(1)} = \mathbf{X}_{(1)}\mathbf{N}^T$. We have

$$\overline{\mathbf{X}}^{(\cdot\cdot j_1)} = \mathbf{A}^{(1)} D_{j_1}\left(\overline{\mathbf{A}}^{(3)}\right)\mathbf{A}^{(2)T} \tag{6.13}$$

$$\overline{\mathbf{X}}^{(\cdot\cdot j_2)} = \mathbf{A}^{(1)} D_{j_2}\left(\overline{\mathbf{A}}^{(3)}\right)\mathbf{A}^{(2)T}. \tag{6.14}$$

The diagonal entry of $D_{j_1}\left(\overline{\mathbf{A}}^{(3)}\right)^{-1} D_{j_2}\left(\overline{\mathbf{A}}^{(3)}\right)$ denoted by λ_q is the generalized eigenvalues of the pencil $\overline{\mathbf{X}}^{(\cdot\cdot j_1)} - \lambda_q \overline{\mathbf{X}}^{(\cdot\cdot j_2)}$ with algebraic multiplicity L_q. The matrices of right and left eigenvectors are $\left(\mathbf{A}^{(2)T}\right)^{\dagger}$ and $\mathbf{A}^{(1)\dagger}$, respectively. Since $L_q \geq 1$ the right and left eigenvectors can only be unique up to a column permuted block-diagonal matrix of the form of $\mathbf{F} = \text{Blockdiag}\left(\mathbf{F}_1, \ldots, \mathbf{F}_Q\right)$ in which $\mathbf{F}_q \in \mathbb{C}^{L_q \times L_q}$.

Unitary matrices $\mathbf{Q} \in \mathbb{C}^{I_1 \times I_1}$ and $\mathbf{Z} \in \mathbb{C}^{I_2 \times I_2}$ satisfying (6.11) yield a Generalized Schur Decomposition of the pair $(\overline{\mathbf{X}}^{(\cdot\cdot j_1)}, \overline{\mathbf{X}}^{(\cdot\cdot j_2)})$:

$$\mathbf{R}^{(\cdot\cdot j_1)} = \mathbf{Q}^H \overline{\mathbf{X}}^{(\cdot\cdot j_1)} \mathbf{Z}^*, \quad \overline{\mathbf{R}}^{(\cdot\cdot j_1)} = \text{triu}_R\left(\mathbf{R}^{(\cdot\cdot j_1)}\right) \tag{6.15}$$

$$\mathbf{R}^{(\cdot\cdot j_2)} = \mathbf{Q}^H \overline{\mathbf{X}}^{(\cdot\cdot j_2)} \mathbf{Z}^*, \quad \overline{\mathbf{R}}^{(\cdot\cdot j_2)} = \text{triu}_R\left(\mathbf{R}^{(\cdot\cdot j_2)}\right) \tag{6.16}$$

where $\overline{\mathbf{R}}^{(\cdot\cdot j_1)}$ and $\overline{\mathbf{R}}^{(\cdot\cdot j_2)}$ are upper triangular. The diagonal entries of the diagonal matrices $\text{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot j_1)}\right)\text{diag}\left(\overline{\mathbf{R}}^{(\cdot\cdot j_2)}\right)^{-1}$ also correspond to the generalized eigenvalues of the pencil $\overline{\mathbf{X}}^{(\cdot\cdot j_1)} - \lambda \overline{\mathbf{X}}^{(\cdot\cdot j_2)}$. Combining (6.13)–(6.14) with (6.15)–(6.16) we obtain the first part of the proposition in the same way as explained in Proposition 2.2. □

PROPOSITION 6.3. *Given*

$$\boldsymbol{A}^{(3)} = \left[\boldsymbol{1}_{L_1}^T \otimes \boldsymbol{c}_1, \ldots, \boldsymbol{1}_{L_Q}^T \otimes \boldsymbol{c}_Q\right] \in \mathbb{C}^{I_3 \times R}, \quad \boldsymbol{C} = [\boldsymbol{c}_1, \ldots, \boldsymbol{c}_Q] \in \mathbb{C}^{I_3 \times Q} \tag{6.17}$$

with $k\left(\boldsymbol{C}\right) \geq 2$ *and* $R = \sum_{q=1}^{Q} L_q$. *Let* $\boldsymbol{L} \in \mathbb{C}^{R \times R}$ *be a nonsingular lower triangular matrix and let* $\boldsymbol{E}^{(r)}$ *be of the form* (6.2). *Let the column vectors of* $\boldsymbol{U} \in \mathbb{C}^{RI_3 \times R}$ *constitute a basis for* $Col\left(\boldsymbol{L} \odot \boldsymbol{A}^{(3)}\right)$. *Denote* $L_{(q)} = \sum_{p=1}^{q} L_p$, *then the matrix*

$$\boldsymbol{G}^{(r)} = \left[\boldsymbol{U}, -\boldsymbol{E}^{(r)} \otimes \boldsymbol{a}_r^{(3)}\right] \in \mathbb{C}^{RI_3 \times (2R-r+1)} \tag{6.18}$$

has rank $2R - r + L_q$ *when* $r \in \{L_{(q-1)} + 1, \ldots, L_{(q)}\}$. *Furthermore, when* $r \in \{L_{(q-1)} + 1, \ldots, L_{(q)}\}$, *then the column vectors of* $\boldsymbol{V}^{(r)} \in \mathbb{C}^{(2R-r+1) \times (L_{(q)} - r + 1)}$ *given by*

$$\boldsymbol{V}^{(r)} = \left[\begin{bmatrix} \boldsymbol{m}_{L_{(q-1)}+r} \\ \boldsymbol{b}_{L_{(q-1)}+r} \end{bmatrix}, \begin{bmatrix} \boldsymbol{m}_{L_{(q-1)}+r+1} \\ 0 \\ \boldsymbol{b}_{L_{(q-1)}+r+1} \end{bmatrix}, \cdots, \begin{bmatrix} \boldsymbol{m}_{L_{(q)}} \\ \boldsymbol{0}_{L_{(q)}-r} \\ \boldsymbol{b}_{L_{(q)}} \end{bmatrix} \right] \tag{6.19}$$

constitute a basis for $Null\left(\boldsymbol{G}^{(r)}\right)$, *where the vectors* $\boldsymbol{m}_p \in \mathbb{C}^R$ *ensures that* $\boldsymbol{G}^{(r)}\boldsymbol{V}^{(r)} = \boldsymbol{0}_{RI_3, (L_{(q-1)}-r+1)}$ *and* $\boldsymbol{b}_p = \boldsymbol{r}_p(1:p)$.

Proof. The proof is analogous to Proposition 6.1. Consider the columns $\{\mathbf{l}_r \otimes \mathbf{a}_r^{(3)}\}$ of $\mathbf{L} \odot \mathbf{A}^{(3)}$, then

$$\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_R] = \left[\mathbf{l}_1 \otimes \mathbf{a}_1^{(3)}, \ldots, \mathbf{l}_R \otimes \mathbf{a}_R^{(3)}\right].$$

Since $\mathbf{a}^{(3)}_{L_{(q-1)}+1} = \mathbf{a}^{(3)}_{L_{(q-1)}+2} = \cdots = \mathbf{a}^{(3)}_{L_{(q)}}$ we know that when $r \in \{L_{(q-1)} + 1, \ldots, L_{(q)}\}$ then the column vectors of $\left[\mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_{L_{(q)}}\right]$ can be written as linear combinations of the column vectors of $\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r$. Hence, when $r \in \{L_{(q-1)} + 1, \ldots, L_{(q)}\}$ the problem of determining the rank of

$$\left[\mathbf{U}, -\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right] \in \mathbb{C}^{RI_3 \times (2R-r+1)} \tag{6.20}$$

reduces to finding the rank of $\mathbf{H}^{(r)} \in \mathbb{C}^{RI_3 \times (2R-r+L_q)}$ given by

$$\mathbf{H}^{(r)} = \left[\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \mathbf{u}_{L_{(q)}+1}, \ldots, \mathbf{u}_R, \mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right] . \tag{6.21}$$

As observed in the proof of Proposition 6.1, the matrix (6.21) is a column permuted lower block-triangular with nonsingular matrices on its block-diagonal. Hence, the matrix (6.20) has rank $2R - r + L_q$ when $r \in \{L_{(q-1)} + 1, \ldots, L_{(q)}\}$. Consequently any matrix $\mathbf{U}$ of which the columns form a basis for Col $\left(\mathbf{L} \odot \mathbf{A}^{(3)}\right)$ yields a matrix $\mathbf{G}^{(r)}$ given by (6.18) of rank $2R - r + L_q$ when $r \in \{L_{(q-1)} + 1, \ldots, L_{(q)}\}$.

It is also clear from (6.21) that the column vectors of $\mathbf{V}^{(r)}$ in (6.19) form a basis for Null $\left(\mathbf{G}^{(r)}\right)$. □

Due to Propositions 6.2 and 6.3 we can still compute the CPD of a tensor via the SGSD3-CP procedure up to its inherent ambiguities even if $k\left(\mathbf{A}^{(3)}\right) = 1$.

7. CPD with a partial symmetry. In many applications the CPD of a tensor has a partial symmetry. This is for instance the case when dealing with a tensor in which covariance matrices are stacked [3, 15, 8]. As a further motivation, let us consider the unsymmetric CPD in which one of the factor matrices, say $\mathbf{A}^{(3)}$, has full column rank. For instance, let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_1 \times I_3}$ of rank R be constructed from the factor matrices $\mathbf{A}^{(1)} \in \mathbb{C}^{I_1 \times R}$, $\mathbf{A}^{(2)} \in \mathbb{C}^{I_2 \times R}$ and full column rank $\mathbf{A}^{(3)} \in \mathbb{C}^{I_3 \times R}$. It has the following matrix representation

$$\mathbb{C}^{I_1 I_2 \times I_3} \ni \mathbf{X}_{(1)} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right) \mathbf{A}^{(3)T}. \tag{7.1}$$

A mild uniqueness condition that applies in this case is stated next.

THEOREM 7.1. *Consider the tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R and matrix representation $\boldsymbol{X}_{(1)} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(2)}\right) \boldsymbol{A}^{(3)T}$. Define $\boldsymbol{C} \in \mathbb{C}^{I_1^2 I_2^2 \times R(R-1)/2}$ by*

$$c_{(i_1-1)(I_1 I_2^2)+(j_1-1)I_2^2+(i_2-1)I_2+j_2, \frac{(r_2-2)(r_2-1)}{2}+r_1} = \begin{vmatrix} a^{(1)}_{i_1,r_1} & a^{(1)}_{i_1,r_2} \\ a^{(1)}_{j_1,r_1} & a^{(1)}_{j_1,r_2} \end{vmatrix} \cdot \begin{vmatrix} a^{(2)}_{i_2,r_1} & a^{(2)}_{i_2,r_2} \\ a^{(2)}_{j_2,r_1} & a^{(2)}_{j_2,r_2} \end{vmatrix},$$

where $1 \leq i_1, j_1 \leq I_1$, $1 \leq i_2, j_2 \leq I_2$ and $1 \leq r_1 < r_2 \leq R$. If $\boldsymbol{A}^{(3)}$ and $\boldsymbol{C}$ have full column rank, then the CPD of $\mathcal{X}$ is unique [6, 13]. Generically[1]*, this is true if [6]*

$$2R(R-1) \leq I_1(I_1-1)I_2(I_2-1) \quad \textit{and} \quad R \leq I_3\,. \tag{7.2}$$

Condition (7.2) is also sufficient for generic uniqueness when $\mathbf{A}^{(1)} = \mathbf{A}^{(2)*}$, with $\mathbf{A}^{(1)}$ generic complex. The condition $R \leq I_3$ is meant to ensure that $\mathbf{A}^{(3)}$ has full column rank.

[1] A CPD property is called generic if it holds with probability one when the entries of the factor matrices are drawn from absolutely continuous probability density functions.

Assume that the conditions stated in Theorem 7.1 are satisfied. Let $\mathbf{X}_{(1)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{X}_{(1)}$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ such that $\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} = \mathbf{U}\Sigma\mathbf{M}$ and $\mathbf{A}^{(3)} = \mathbf{M}^{-1}\mathbf{V}^*$. It was explained in [6] that the CPD problem (7.1) can be transformed into the partially symmetric CPD problem

$$\mathbb{C}^{R^2\times R} \ni \mathbf{Y}_{(1)} = (\mathbf{M} \odot \mathbf{M})\,\mathbf{D}^T\,, \tag{7.3}$$

where $\mathbf{D} \in \mathbb{C}^{R\times R}$, see [6] for the construction of $\mathbf{Y}_{(1)}$. Hence, the problem of computing a CPD with a partial symmetry must be considered as a core problem in numerical multilinear algebra.

In subsection 7.1 we review how a partial symmetry was dealt with in the SGSD1-CP method in [20]. Next, in subsections 7.2 and 7.3 we explain how to incorporate a partial symmetry into the SGSD2-CP and SGSD3-CP methods. We limit the discussion to the case where $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$, but the results are also valid when $\mathbf{A}^{(1)} = \mathbf{A}^{(2)*}$.

7.1. SGSD1-CP with a partial symmetry. Consider a tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_1\times I_3}$ with rank R and matrix representation

$$\mathbb{C}^{I_1^2\times I_3} \ni \mathbf{X}_{(1)} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(1)}\right)\mathbf{A}^{(3)T}. \tag{7.4}$$

Assume that the factor matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(3)}$ in (7.4) have full column rank. Then in [20] the computation of a CPD with the partial symmetry $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$ was done via the SGSD1-CP method. This was done by ignoring the relation between the factor matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ and by just treating the problem as an unsymmetric CPD problem.

7.2. SGSD2-CP with a partial symmetry. Assume that the factor matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(3)}$ in (7.4) have full column rank. Instead of just ignoring the partial symmetry, we present a partially symmetric version of the SGSD2-CP method.

Let $\mathbf{A}^{(1)} = \mathbf{QR}$ be the QR factorization of $\mathbf{A}^{(1)}$ where $\mathbf{Q} \in \mathbb{C}^{I_1\times I_1}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{I_1\times R}$ is a matrix of the form (2.4). Similarly, let $\mathbf{A}^{(2)} = \mathbf{ZL}$ be the QL factorization of $\mathbf{A}^{(2)}$ where $\mathbf{Z} \in \mathbb{C}^{I_1\times I_1}$ is a unitary matrix and $\mathbf{L} \in \mathbb{C}^{I_1\times R}$ is a matrix of the form (2.5). Since $\mathbf{A}^{(1)} = \mathbf{QR} = \mathbf{ZL} = \mathbf{A}^{(2)}$ we can express $\overline{\mathbf{L}}$ in terms of $\overline{\mathbf{R}}$ and vice-versa:

$$\overline{\mathbf{L}} = \mathbf{W}\overline{\mathbf{R}}\,, \qquad\qquad \overline{\mathbf{R}} = \mathbf{W}^H\overline{\mathbf{L}}\,, \tag{7.5}$$

in which $\mathbf{W} = \mathbf{V}(1:R, 1:R)$ and $\mathbf{V} = \mathbf{Q}^H\mathbf{Z}$.

We first determine $\mathbf{Q}$ and $\mathbf{Z}$ by maximizing

$$h\left(\mathbf{Q},\mathbf{Z}\right) = \sum_{i_3=1}^{I_3} \left\|\mathrm{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)\right\|_F^2.$$

Assume $\mathbf{Q}$ and $\mathbf{Z}$ have been found. Let the tensor $\mathcal{R} \in \mathbb{C}^{R\times R\times I_3}$ be constructed such that $\mathcal{R}_{i_1i_2i_3} = \mathbf{R}_{i_1i_2}^{(\cdot\cdot i_3)} = \mathrm{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)_{i_1i_2}$, then $\mathbf{R}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}}D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$. Moreover, let $\mathbf{R}^{(i_1\cdot\cdot)} \in \mathbb{C}^{R\times I_3}$ satisfy the relation $\mathbf{R}_{i_2i_3}^{(i_1\cdot\cdot)} = \mathcal{R}_{i_1i_2i_3}$, then $\mathbf{R}^{(i_1\cdot\cdot)} = \overline{\mathbf{L}}D_{i_1}\left(\overline{\mathbf{R}}\right)\mathbf{A}^{(3)T}$. Stack the matrices $\{\mathbf{R}^{(i_1\cdot\cdot)}\}$ into the matrix

$$\mathbb{C}^{R^2\times I_3} \ni \mathbf{R}_{(1)} = \begin{bmatrix} \mathbf{R}^{(1\cdot\cdot)} \\ \vdots \\ \mathbf{R}^{(R\cdot\cdot)} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{L}}D_1\left(\overline{\mathbf{R}}\right) \\ \vdots \\ \overline{\mathbf{L}}D_R\left(\overline{\mathbf{R}}\right) \end{bmatrix}\mathbf{A}^{(3)T} = \left(\overline{\mathbf{R}} \odot \overline{\mathbf{L}}\right)\mathbf{A}^{(3)T}. \tag{7.6}$$

Let us first use (7.5) to express (7.6) in terms of $\mathbf{R}$ only. We left-multiply (7.6) by $(\mathbf{I}_R \otimes \mathbf{W})$ and work with $(\mathbf{I}_R \otimes \mathbf{W})\,\mathbf{R}_{(1)}$ instead of $\mathbf{R}_{(1)}$. This in turn means that (7.6) can be expressed in terms of $\overline{\mathbf{R}}$ as follows $\mathbf{S}_{(1)} = (\mathbf{I}_R \otimes \mathbf{W})\,\mathbf{R}_{(1)} = \left(\overline{\mathbf{R}} \odot \overline{\mathbf{R}}\right)\mathbf{A}^{(3)T}$. We will now present an algorithm to compute the factor matrices from $\mathrm{Col}\left(\mathbf{S}_{(1)}\right)$. Let $\mathbf{S}_{(1)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{S}_{(1)}$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ such that

$$\mathbf{U}\mathbf{M} = \overline{\mathbf{R}} \odot \overline{\mathbf{R}} \Leftrightarrow \mathbf{U}\mathbf{m}_r = \overline{\mathbf{r}}_r \otimes \overline{\mathbf{r}}_r\,, \quad r \in \{1,\ldots,R\}\,. \tag{7.7}$$

Let $\mathbf{a}_r = \overline{\mathbf{r}}_r(1:r)$, then (7.7) is equivalent to

$$\mathbf{U}\mathbf{m}_r = \sum_{1\le p\le r}\sum_{1\le q\le r}\left(\mathbf{e}_p^{(R)} \otimes \mathbf{e}_q^{(R)}\right)(\mathbf{a}_r \otimes \mathbf{a}_r) = \mathbf{E}^{(r)}\,(\mathbf{a}_r \otimes \mathbf{a}_r)\;,\; r \in \{1,\ldots,R\}, \tag{7.8}$$

where $\mathbf{E}^{(r)} \in \mathbb{C}^{R^2\times r^2}$ is given by

$$\mathbf{E}^{(r)} = \left[\mathbf{e}_1^{(R)} \otimes \mathbf{e}_1^{(R)}, \ldots, \mathbf{e}_1^{(R)} \otimes \mathbf{e}_r^{(R)}, \ldots, \mathbf{e}_r^{(R)} \otimes \mathbf{e}_1^{(R)}, \ldots, \mathbf{e}_r^{(R)} \otimes \mathbf{e}_r^{(R)}\right]. \tag{7.9}$$

Equation (7.8) can be written as

$$\left[\mathbf{U}\Sigma, -\mathbf{E}^{(r)}\right]\begin{bmatrix}\mathbf{m}_r\\ \mathbf{a}_r \otimes \mathbf{a}_r\end{bmatrix} = \mathbf{0}_{R+r^2}\,, \quad r \in \{1,\ldots,R\}. \tag{7.10}$$

Proposition 7.3 tells us first that $\left[\mathbf{U}\Sigma, -\mathbf{E}^{(r)}\right]$ in (7.10) has an r-dimensional kernel. Second, it states that if we impose in (7.10) that $\mathbf{a}_n$, $1 \le n \le R$, are the columns of a nonsingular upper triangular matrix, then we can determine $\mathbf{a}_r$ in a unique manner. In order to prove it we need the following lemma.

LEMMA 7.2. *Let $\boldsymbol{R} \in \mathbb{C}^{R\times R}$ be a nonsingular upper triangular matrix, then the matrix $\boldsymbol{S} = \boldsymbol{R} \odot \boldsymbol{R} \in \mathbb{C}^{R^2\times R}$ has the following property*

$$\boldsymbol{S}(r + (r-1)R, 1:r) = \alpha_r \boldsymbol{e}_r^{(r)T}, \quad \alpha_r \in \mathbb{C},\, r \in \{1,\ldots,R\}\,. \tag{7.11}$$

Proof. The proof is straightforward, noting that

$$\mathbf{S}\,(t + (t-1)R, u) = (\mathbf{R})_{tu}\,(\mathbf{R})_{tu} \tag{7.12}$$

and

$$\begin{cases}(\mathbf{R})_{tu} \in \mathbb{C}\,, & u < t\\ (\mathbf{R})_{tu} \ne 0\,, & u = t\\ (\mathbf{R})_{tu} = 0\,, & u > t\end{cases} \tag{7.13}$$

By combining (7.12) and (7.13) we get (7.11). ∎

PROPOSITION 7.3. *Let $\boldsymbol{R} = [\boldsymbol{r}_1,\ldots,\boldsymbol{r}_R] \in \mathbb{C}^{R\times R}$ be a nonsingular upper triangular matrix and let $\boldsymbol{E}^{(r)} \in \mathbb{C}^{R^2\times r^2}$ be of the form (7.9). If the column vectors of $\boldsymbol{U} \in \mathbb{C}^{R^2\times R}$ constitute a basis for $Col(\boldsymbol{R} \odot \boldsymbol{R})$, then the matrix*

$$\boldsymbol{G}^{(r)} = \left[\boldsymbol{U}, -\boldsymbol{E}^{(r)}\right] \in \mathbb{C}^{R^2\times\left(R+r^2\right)} \tag{7.14}$$

has rank $R + r(r-1)$. *Furthermore, let the column vectors of* $\boldsymbol{V}^{(r)} \in \mathbb{C}^{\left(R+r^2\right)\times r}$ *constitute a basis for Null*$\left(\boldsymbol{G}^{(r)}\right)$, *then* $\boldsymbol{W}^{(r)} = \boldsymbol{V}^{(r)}\left(R+(r-1)r+1 : R+r^2, :\right) \in \mathbb{C}^{r\times r}$ *has rank* 1 *and Col*$\left(\boldsymbol{W}^{(r)}\right) = Col\left(\boldsymbol{a}_r\right)$, *in which* $\boldsymbol{a}_r = \boldsymbol{r}_r(1:r) \in \mathbb{C}^r$.

Proof. Let us first prove that $\mathbf{G}^{(r)}$ in (7.14) has an r-dimensional kernel. Consider the columns $\{\mathbf{r}_n \otimes \mathbf{r}_n\}$ of $\mathrm{Col}\,(\mathbf{R} \odot \mathbf{R})$. On one hand, the vectors $\{\mathbf{r}_n \otimes \mathbf{r}_n\}_{n\leq r}$ are linear combinations of the columns of $\mathbf{E}^{(r)}$. On the other hand, by Lemma 7.2, vector $\mathbf{r}_n \otimes \mathbf{r}_n$, with $n > r$ has a non-zero entry at position $n+(n-1)R$, while vectors $\mathbf{r}_p \otimes \mathbf{r}_p$, $p \leq r$, and the columns of $\mathbf{E}^{(r)}$ are zero at that position. Furthermore, due to the property (7.11) we also know that the vectors $\{\mathbf{r}_p \otimes \mathbf{r}_p\}_{p>r}$ are linearly independent. Consequently, the matrix $\left[\mathbf{R} \odot \mathbf{R}, -\mathbf{E}^{(r)}\right]$ has rank $R+r(r-1)$. In conclusion, matrix (7.14) has rank $R + r(r-1)$.

Let us now prove that $\mathbf{W}^{(r)}$ has rank 1 and $\mathrm{Col}\left(\mathbf{W}^{(r)}\right) = \mathrm{Col}\,(\mathbf{a}_r)$, $1 \leq r \leq R$. Note that $\mathbf{E}^{(r)}\,(\mathbf{b}_p \otimes \mathbf{b}_p) = \mathbf{r}_p \otimes \mathbf{r}_p$ when $\mathbf{b}_p = \begin{bmatrix} \mathbf{a}_p \\ \mathbf{0}_{r-p} \end{bmatrix}$, and therefore $\mathbf{x}_p = \begin{bmatrix} \mathbf{m}_p \\ \mathbf{b}_p \otimes \mathbf{b}_p \end{bmatrix} \in \mathrm{Null}\left(\mathbf{G}^{(r)}\right)$, $1 \leq p \leq r$, where the vectors $\mathbf{m}_p \in \mathbb{C}^R$ are chosen such that $\mathbf{G}^{(r)}\mathbf{x}_p = \mathbf{0}_{R^2}$, $1 \leq p \leq r$. Due to Lemma 7.2 we know that the vectors $\{\mathbf{b}_p \otimes \mathbf{b}_p\}_{p=1}^r$ and hence the vectors $\{\mathbf{x}_p\}_{p=1}^r$ are linearly independent. Consequently, $\{\mathbf{x}_p\}_{p=1}^r$ is a basis for $\mathrm{Null}\left(\mathbf{G}^{(r)}\right)$. Let the column vectors of $\mathbf{V}^{(r)} \in \mathbb{C}^{\left(R+r^2\right)\times r}$ constitute an alternative basis for $\mathrm{Null}\left(\mathbf{G}^{(r)}\right)$, then there exists a nonsingular matrix $\mathbf{N} \in \mathbb{C}^{r\times r}$ such that

$$\mathbf{V}^{(r)} = [\mathbf{x}_1, \ldots, \mathbf{x}_r]\,\mathbf{N} = \begin{bmatrix} \mathbf{m}_1 & & \mathbf{m}_r \\ \mathbf{b}_1 \otimes \mathbf{b}_1 & , \cdots , & \mathbf{b}_r \otimes \mathbf{b}_r \end{bmatrix} \mathbf{N}\,. \tag{7.15}$$

From (7.15) we observe that the column vectors of

$$\mathbf{W}^{(r)} = \mathbf{V}^{(r)}\left((r-1)\,R+1 : (r-1)\,R+r, :\right) \in \mathbb{C}^{r\times r}$$

are all proportional to $\mathbf{a}_r = \mathbf{b}_r$, i.e., the sth column vector of $\mathbf{W}^{(r)}$ satisfies $\mathbf{w}_s^{(r)} = \alpha_s \mathbf{a}_r$ for some $\alpha_s \in \mathbb{C}$. Thus, we can conclude that any matrix $\mathbf{U}$ of which the columns form a basis for $\mathrm{Col}\,(\mathbf{R} \odot \mathbf{R})$ yields a rank-1 matrix $\mathbf{W}^{(r)}$ of which the column space $\mathrm{Col}\left(\mathbf{W}^{(r)}\right) = \mathrm{Col}\,(\mathbf{a}_r)$, $r \in \{1, \ldots, R\}$. □

The following Corollary 7.4 tells us that by projecting on the orthogonal complement of $\mathrm{Col}\left(\mathbf{E}^{(r)}\right)$ we may reduce the number of equations in (7.10).

Corollary 7.4. *Let* $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_R] \in \mathbb{C}^{R\times R}$ *be a nonsingular upper triangular matrix and define the diagonal projection matrix*

$$\mathbb{C}^{R^2\times R^2} \ni \boldsymbol{D}^{(r)} = \boldsymbol{I}_{R^2} - \boldsymbol{E}^{(r)}\boldsymbol{E}^{(r)T}\,, \tag{7.16}$$

where $\boldsymbol{E}^{(r)} \in \mathbb{C}^{R^2\times r^2}$ *is of the form* (7.9). *If the column vectors of* $\boldsymbol{U} \in \mathbb{C}^{R^2\times R}$ *constitute a basis for Col*$(\boldsymbol{R} \odot \boldsymbol{R})$, *then the matrix*

$$\mathbb{C}^{R^2\times R} \ni \boldsymbol{U}^{(r)} = \boldsymbol{D}^{(r)}\,\boldsymbol{U} \tag{7.17}$$

has rank $R-r$, $r \in \{1, \ldots, R\}$. *Moreover, let the column vectors of* $\boldsymbol{V}^{(r)} \in \mathbb{C}^{R\times r}$ *constitute a basis for Null*$\left(\boldsymbol{U}^{(r)}\right)$ *and* $\mathbb{C}^{R^2\times r} \ni \boldsymbol{W}^{(r)} = \boldsymbol{U}\boldsymbol{V}^{(r)}$, *then the matrix* $\boldsymbol{Z}^{(r)} =$

$\boldsymbol{W}^{(r)}\left((r-1)\,R+1:(r-1)\,R+r,:\right) \in \mathbb{C}^{r\times r}$ *has rank* 1 *and* $Col\left(\boldsymbol{Z}^{(r)}\right) = Col(\boldsymbol{r}_r)$, $r \in \{1,\ldots,R\}$.

Proof. Let us first prove that $\mathbf{U}^{(r)} = \mathbf{D}^{(r)}\mathbf{U}$ has rank $R-r$, $r \in \{1,\ldots,R\}$. Consider the columns $\{\mathbf{r}_n \otimes \mathbf{r}_n\}$ of $\mathbf{R}\odot\mathbf{R}$. On one hand, the vectors in the set $\{\mathbf{r}_n \otimes \mathbf{r}_n\}_{n\leq r}$ can be written as a linear combination of the column vectors $\mathbf{E}^{(r)}$. On the other hand, by Lemma 7.2, vector $\mathbf{r}_n \otimes \mathbf{r}_n$, with $n > r$ has a non-zero entry at position $n+(n-1)R$, that is not set to zero by $\mathbf{D}^{(r)}$. Furthermore, due to property (7.11) we know that the vectors $\{\mathbf{r}_n \otimes \mathbf{r}_n\}_{n>r}$ are linearly independent. Hence, we can conclude that any matrix $\mathbf{U}$ of which the columns form a basis for $\mathrm{Col}\,(\mathbf{R}\odot\mathbf{R})$ yields a matrix $\mathbf{U}^{(r)} = \mathbf{D}^{(r)}\mathbf{U}$ of rank $R-r$, $r \in \{1,\ldots,R\}$.

Let us now prove that $\mathbf{W}^{(r)}$ has rank 1 and $\mathrm{Col}\left(\mathbf{W}^{(r)}\right) = \mathrm{Col}\,(\mathbf{a}_r)$, $1 \leq r \leq R$. First note that $\mathbf{U}^{(r)} = \mathbf{D}^{(r)}\mathbf{U} = \mathbf{D}^{(r)}\,(\mathbf{R}\odot\mathbf{R})\,\mathbf{F}$ for some nonsingular matrix $\mathbf{F} \in \mathbb{C}^{R\times R}$. Since $\mathbf{D}^{(r)}$ sets the non-zero entries of $\{\mathbf{r}_p \otimes \mathbf{r}_p\}_{p\leq r}$ equal to zero while the vectors $\{\mathbf{r}_p \otimes \mathbf{r}_p\}_{p>r}$ are still linearly independent we conclude that $\{\mathbf{e}_p^{(R)}\}_{p\leq r}$ is a basis for $\mathrm{Null}\left(\mathbf{U}^{(r)}\right)$. This means that

$$\begin{aligned}\mathbf{W}^{(r)} &= (\mathbf{R}\odot\mathbf{R})\,\mathbf{F}\underline{\mathbf{E}}^{(r)}\mathbf{N}\\ &= ([\mathbf{r}_1,\ldots,\mathbf{r}_r]\odot[\mathbf{r}_1,\ldots,\mathbf{r}_r])\,\mathbf{M}\,.\end{aligned} \tag{7.18}$$

for some nonsingular matrices $\mathbf{M},\mathbf{N} \in \mathbb{C}^{r\times r}$, where $\underline{\mathbf{E}}^{(r)} = \left[\mathbf{e}_1^{(R)},\ldots,\mathbf{e}_r^{(R)}\right] \in \mathbb{C}^{R\times r}$. From (7.18) we observe that the column vectors of

$$\mathbf{Z}^{(r)} = \mathbf{W}^{(r)}\left((r-1)\,R+1:(r-1)\,R+r,:\right)$$

are all proportional to $\mathbf{r}_r$, that is $\mathbf{z}_s^{(r)} = \alpha_s\mathbf{r}_r$ for some $\alpha_s \in \mathbb{C}$. Thus, we can conclude that any matrix $\mathbf{U}$ of which the columns form a basis for $\mathrm{Col}\,(\mathbf{R}\odot\mathbf{R})$ yields a matrix $\mathbf{Z}^{(r)}$ with rank 1 and $\mathrm{Col}\left(\mathbf{Z}^{(r)}\right) = \mathrm{Col}\,(\mathbf{r}_r)$, $r \in \{1,\ldots,R\}$. □

By projecting (7.10) on the orthogonal complement of $\mathbf{E}^{(r)}$ we obtain

$$\mathbf{U}^{(r)}\mathbf{m}_r = \mathbf{0}_{R^2}\,, \quad r \in [1,\ldots,R\} \tag{7.19}$$

with $\mathbf{U}^{(r)} = \mathbf{D}^{(r)}\mathbf{U}\Sigma$ and $\mathbf{D}^{(r)}$ defined in (7.16). Due to Corollary 7.4, $\mathbf{U}^{(r)}$ still has an r-dimensional kernel. Corollary 7.4 also tells us that we can find $\mathbf{r}_r$ from $\mathrm{Null}\left(\mathbf{U}^{(r)}\right)$. We first remove the all-zero row-vectors of $\mathbf{U}^{(r)}$, obtaining $\widetilde{\mathbf{U}}^{(r)} \in \mathbb{C}^{\left(R^2-r^2\right)\times R}$, so that (7.19) reduces to

$$\widetilde{\mathbf{U}}^{(r)}\mathbf{m}_r = \mathbf{0}_{R^2-r^2}\,, \quad r \in [1,\ldots,R\}\,. \tag{7.20}$$

Note that set (7.10) of R^2 equations in $R+r^2$ unknowns has been reduced to a set of R^2-r^2 equations in R unknowns. Next, we set the column vectors of $\mathbf{V}^{(r)}$ equal to the right singular vectors associated with the r smallest singular values of $\widetilde{\mathbf{U}}^{(r)}$. Construct $\mathbf{W}^{(r)} = \mathbf{U}^{(r)}\mathbf{V}^{(r)}$ and set $\mathbf{a}_r$ equal to the singular vector associated with the largest singular value of $\mathbf{Z}^{(r)} = \mathbf{W}^{(r)}\left((r-1)\,R+1:(r-1)\,R+r,:\right)$. The vector $\mathbf{r}_r$ is now given by $\mathbf{r}_r = \left[\mathbf{a}_r^T,\mathbf{0}_{R-r}^T\right]^T \in \mathbb{C}^R$. Once the vectors $\{\mathbf{r}_r\}$ have been found, we obtain $\mathbf{A}^{(1)}$ from the relation $\mathbf{A}^{(1)} = \mathbf{QR}$.

So far, we have been working with $\mathbf{S}_{(1)} = \left(\overline{\mathbf{R}} \odot \overline{\mathbf{R}}\right) \mathbf{A}^{(3)T}$. As an alternative, we may also use (7.5) to express (7.6) in terms of $\mathbf{L}$ only. We form $\mathbf{T}_{(1)} = \left(\mathbf{W}^H \otimes \mathbf{I}_R\right) \mathbf{R}_{(1)} = \left(\overline{\mathbf{L}} \odot \overline{\mathbf{L}}\right) \mathbf{A}^{(3)T}$ and work as follows. Let $\mathbf{T}_{(1)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{T}_{(1)}$, then there exists a nonsingular matrix $\mathbf{N} \in \mathbb{C}^{R\times R}$ such that

$$\mathbf{U}\mathbf{n}_r = \overline{\mathbf{l}}_r \otimes \overline{\mathbf{l}}_r = \sum_{r\leq p\leq R} \sum_{r\leq q\leq R} \left(\mathbf{e}_p^{(R)} \otimes \mathbf{e}_q^{(R)}\right) (\mathbf{b}_r \otimes \mathbf{b}_r) = \overline{\mathbf{E}}^{(r)} (\mathbf{b}_r \otimes \mathbf{b}_r) \,, \quad (7.21)$$

where $\mathbf{b}_r = \overline{\mathbf{l}}_r(r:R)$, and $\overline{\mathbf{E}}^{(r)} \in \mathbb{C}^{R^2\times(R-r+1)^2}$ is given by

$$\overline{\mathbf{E}}^{(r)} = \left[\mathbf{e}_r^{(R)} \otimes \mathbf{e}_r^{(R)}, \ldots, \mathbf{e}_r^{(R)} \otimes \mathbf{e}_{r+1}^{(R)}, \ldots, \mathbf{e}_R^{(R)} \otimes \mathbf{e}_r^{(R)}, \ldots, \mathbf{e}_R^{(R)} \otimes \mathbf{e}_R^{(R)}\right] . \quad (7.22)$$

As in the upper triangular case, we first determine $\mathbf{n}_r$ by solving a homogeneous linear system. Let $\mathbb{C}^{R^2\times R} \ni \mathbf{U}^{(r)} = \overline{\mathbf{D}}^{(r)}\mathbf{U}$ in which $\mathbb{C}^{R^2\times R^2} \ni \overline{\mathbf{D}}^{(r)} = \mathbf{I}_{R^2} - \overline{\mathbf{E}}^{(r)}\overline{\mathbf{E}}^{(r)T}$. Let the column vectors of $\mathbf{V}^{(r)} \in \mathbb{C}^{R\times(R-r+1)}$ constitute a basis for Null $\left(\mathbf{U}^{(r)}\right)$, $\mathbf{W}^{(r)} = \mathbf{U}\mathbf{V}^{(r)}$ and

$$\mathbf{Z}^{(r)} = \mathbf{W}^{(r)}(1:(R-r+1),:) \in \mathbb{C}^{(R-r+1)\times(R-r+1)}. \quad (7.23)$$

As in the upper triangular case, the vector $\mathbf{b}_r$ now follows from the best rank-1 approximation of $\mathbf{Z}^{(r)}$ in (7.23).

We overall find a column of $\mathbf{A}^{(1)}$ from the corresponding column of $\overline{\mathbf{R}}$ (via $\mathbf{S}_{(1)}$) or $\overline{\mathbf{L}}$ (via $\mathbf{T}_{(1)}$), depending on which has the least number of nonzero entries. This reduces the computational complexity. Consequently, if $\mathbf{a}_r^{(1)}$ denotes the rth column vector of $\mathbf{A}^{(1)}$, then when $r \leq \lfloor \frac{R+1}{2} \rfloor$ we construct $\mathbf{a}_r^{(1)}$ from $\overline{\mathbf{R}}$. On the other hand, when $r \geq \lfloor \frac{R+1}{2} \rfloor + 1$ we construct $\mathbf{a}_r^{(1)}$ from $\overline{\mathbf{L}}$. Thus, we obtain

$$\mathbf{A}^{(1)} = \left[\mathbf{Q}\cdot\mathbf{R}\left(:, 1:\left\lfloor\frac{R+1}{2}\right\rfloor\right), \mathbf{Z}\cdot\mathbf{L}\left(:, \left\lfloor\frac{R+1}{2}\right\rfloor + 1 : R\right)\right] .$$

The matrix $\mathbf{A}^{(3)}$ now follows via relation $\mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(1)}\right)^{\dagger} \mathbf{X}_{(1)}$.

7.3. SGSD3-CP with a partial symmetry. In this subsection we adapt the SGSD3-CP method to the case where a CPD has a partial symmetry. The method assumes that $\mathbf{A}^{(1)}$ has full column rank and $k\left(\mathbf{A}^{(3)}\right) \geq 1$. This is in contrast with the previously discussed SGSD1-CP and SGSD2-CP methods for computing a CPD with a partial symmetry which require that both $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(3)}$ have full column rank.

Consider the QR factorization $\mathbf{A}^{(1)} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q} \in \mathbb{C}^{I_1\times I_1}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{I_1\times R}$ is a matrix of the form (2.4). Consider also the QL factorization $\mathbf{A}^{(1)} = \mathbf{Z}\mathbf{L}$, where $\mathbf{Z} \in \mathbb{C}^{I_1\times I_1}$ is a unitary matrix and $\mathbf{L} \in \mathbb{C}^{I_1\times R}$ is a matrix of the form (2.5). Due to Proposition 2.2 or 6.2, the first step is to compute the unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ that make the matrices $\{\mathbf{X}^{(\cdot\cdot i_3)}\}$ as upper triangular as possible by maximizing $f(\mathbf{Q},\mathbf{Z}) = \sum_{i_3=1}^{I_3} \left\|\mathrm{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)\right\|_F^2$. Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found and construct the tensor $\mathcal{R} \in \mathbb{C}^{R\times R\times I_3}$ according to $\mathcal{R}_{i_1i_2i_3} = \mathbf{R}_{i_1i_2}^{(\cdot\cdot i_3)} = \mathrm{triu}_R\left(\mathbf{Q}^H\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}^*\right)_{i_1i_2}$. As in the SGSD1-CP method with

partial symmetry we find $\mathbf{A}^{(3)}$ from $\mathbf{A}^{(3)T} = \left[\text{Vecd}\left(\mathbf{R}^{(\cdot\cdot 1)}\right), \ldots, \text{Vecd}\left(\mathbf{R}^{(\cdot\cdot I_3)}\right)\right]$. Next, we determine $\mathbf{A}^{(1)}$. Let $\mathbf{R}^{(\cdot i_2 \cdot)} \in \mathbb{C}^{I_3 \times R}$ satisfy the relation $\mathbf{R}^{(\cdot i_2 \cdot)}_{i_3 i_1} = \mathcal{R}_{i_1 i_2 i_3}$, then $\mathbf{R}^{(\cdot i_2 \cdot)} = \mathbf{A}^{(3)} D_{i_2}\left(\overline{\mathbf{L}}\right)\overline{\mathbf{R}}^T$. Stack the matrices $\{\mathbf{R}^{(\cdot i_2 \cdot)}\}$ into the matrix

$$\mathbb{C}^{RI_3 \times R} \ni \mathbf{R}_{(2)} = \begin{bmatrix} \mathbf{R}^{(\cdot 1 \cdot)} \\ \vdots \\ \mathbf{R}^{(\cdot R \cdot)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(3)} D_1\left(\overline{\mathbf{L}}\right) \\ \vdots \\ \mathbf{A}^{(3)} D_R\left(\overline{\mathbf{L}}\right) \end{bmatrix} \overline{\mathbf{R}}^T = \left(\overline{\mathbf{L}} \odot \mathbf{A}^{(3)}\right)\overline{\mathbf{R}}^T.$$

Let $\mathbf{R}_{(2)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{R}_{(2)}$ and let $\mathbf{M} \in \mathbb{C}^{R \times R}$ be a nonsingular matrix such that

$$\mathbf{UM} = \overline{\mathbf{L}} \odot \mathbf{A}^{(3)} \Leftrightarrow \mathbf{U}\mathbf{m}_r = \bar{\mathbf{l}}_r \otimes \mathbf{a}^{(3)}_r, \quad r \in \{1, \ldots, R\}. \tag{7.24}$$

Furthermore, let $\mathbf{b}_r = \bar{\mathbf{l}}_r(r : R) \in \mathbb{C}^{R-r+1}$, then due to Proposition 6.1 or 6.3 we know that we can obtain $\overline{\mathbf{L}}$ by solving the homogeneous linear systems

$$\left[\mathbf{U}, -\mathbf{E}^{(r)} \otimes \mathbf{a}^{(3)}_r\right] \begin{bmatrix} \mathbf{m}_r \\ \mathbf{b}_r \end{bmatrix} = \mathbf{0}_{I_3 R}, \quad r \in \{1, \ldots, R\}, \tag{7.25}$$

where $\mathbf{E}^{(r)} \in \mathbb{C}^{R \times (R-r+1)}$ is of the form (6.2).

Alternatively, let $\mathbf{R}^{(\cdot\cdot i_3)} \in \mathbb{C}^{R \times R}$ satisfy the relation $\mathbf{R}^{(\cdot\cdot i_3)}_{i_1 i_2} = \mathcal{R}_{i_1 i_2 i_3}$, then $\mathbf{R}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$. Stack the matrices $\{\mathbf{R}^{(\cdot\cdot i_3)}\}$ into the matrix

$$\mathbb{C}^{RI_3 \times R} \ni \mathbf{R}_{(3)} = \begin{bmatrix} \mathbf{R}^{(\cdot\cdot 1)} \\ \vdots \\ \mathbf{R}^{(\cdot\cdot I_3)} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{R}} D_1\left(\mathbf{A}^{(3)}\right) \\ \vdots \\ \overline{\mathbf{R}} D_{I_3}\left(\mathbf{A}^{(3)}\right) \end{bmatrix} \overline{\mathbf{L}}^T = \left(\mathbf{A}^{(3)} \odot \overline{\mathbf{R}}\right)\overline{\mathbf{L}}^T.$$

Compute the compact SVD $\mathbf{R}_{(3)} = \mathbf{S}\Sigma\mathbf{V}^H$ and let $\mathbf{N} \in \mathbb{C}^{R \times R}$ be a nonsingular matrix such that

$$\mathbf{SN} = \mathbf{A}^{(3)} \odot \overline{\mathbf{R}} \Leftrightarrow \mathbf{S}\mathbf{n}_r = \mathbf{a}^{(3)}_r \otimes \bar{\mathbf{r}}_r, \quad r \in \{1, \ldots, R\}. \tag{7.26}$$

Furthermore, let $\mathbf{a}_r = \bar{\mathbf{r}}_r(1 : r) \in \mathbb{C}^r$, then again due to Proposition 6.1 or 6.3 we know that we can obtain $\overline{\mathbf{R}}$ by solving the homogeneous linear systems

$$\left[\mathbf{S}, -\mathbf{a}^{(3)}_r \otimes \mathbf{F}^{(r)}\right] \begin{bmatrix} \mathbf{n}_r \\ \mathbf{a}_r \end{bmatrix} = \mathbf{0}_{I_3 R}, \quad r \in \{1, \ldots, R\}, \tag{7.27}$$

where

$$\mathbf{F}^{(r)} = \left[\mathbf{e}^{(R)}_1, \mathbf{e}^{(R)}_2, \ldots, \mathbf{e}^{(R)}_r\right] \in \mathbb{C}^{R \times r}. \tag{7.28}$$

The influence of perturbation noise can be reduced by to simultaneously make use of relations (7.25) and (7.27). Recall from (7.5) that $\overline{\mathbf{L}} = \mathbf{W}\overline{\mathbf{R}}$ and $\overline{\mathbf{R}} = \mathbf{W}^H\overline{\mathbf{L}}$, in which $\mathbf{W} = \mathbf{V}(1 : R, 1 : R)$ and $\mathbf{V} = \mathbf{Q}^H\mathbf{Z}$. From (7.26) and relation $\overline{\mathbf{L}} = \mathbf{W}\overline{\mathbf{R}}$ we obtain

$$\left(\mathbf{I}_{I_3} \otimes \mathbf{W}\right)\mathbf{S}\mathbf{n}_r = \mathbf{a}^{(3)}_r \otimes \bar{\mathbf{l}}_r = \left(\mathbf{a}^{(3)}_r \otimes \mathbf{E}^{(r)}\right)\mathbf{b}_r, \tag{7.29}$$

where $\mathbf{b}_r = \overline{\mathbf{l}}_r\,(r:R) \in \mathbb{C}^{R-r+1}$ and $\mathbf{E}^{(r)} \in \mathbb{C}^{R\times(R-r+1)}$ is given by (6.2). By combining (7.25) and (7.29) we get

$$\begin{bmatrix} \mathbf{U} & \mathbf{0}_{I_3R,R} & -\mathbf{E}^{(r)}\otimes\mathbf{a}_r^{(3)} \\ \mathbf{0}_{I_3R,R} & (\mathbf{I}_{I_3}\otimes\mathbf{W})\,\mathbf{S} & -\mathbf{a}_r^{(3)}\otimes\mathbf{E}^{(r)} \end{bmatrix} \begin{bmatrix} \mathbf{m}_r \\ \mathbf{n}_r \\ \mathbf{b}_r \end{bmatrix} = \mathbf{0}_{2I_3R}\,,\; r\in\{1,\ldots,R\}\,. \tag{7.30}$$

Similarly, from (7.24) and relation $\overline{\mathbf{R}} = \mathbf{W}^H\overline{\mathbf{L}}$ we obtain

$$\left(\mathbf{W}^H\otimes\mathbf{I}_{I_3}\right)\mathbf{U}\mathbf{m}_r = \overline{\mathbf{r}}_r\otimes\mathbf{a}_r^{(3)} = \left(\mathbf{F}^{(r)}\otimes\mathbf{a}_r^{(3)}\right)\mathbf{a}_r\,, \tag{7.31}$$

where $\mathbf{a}_r = \overline{\mathbf{r}}_r\,(1:r) \in \mathbb{C}^r$ and $\mathbf{F}^{(r)} \in \mathbb{C}^{R\times r}$ is given by (7.28). By combining (7.27) and (7.31) we get

$$\begin{bmatrix} \left(\mathbf{W}^H\otimes\mathbf{I}_{I_3}\right)\mathbf{U} & \mathbf{0}_{I_3R,R} & -\mathbf{F}^{(r)}\otimes\mathbf{a}_r^{(3)} \\ \mathbf{0}_{I_3R,R} & \mathbf{S} & -\mathbf{a}_r^{(3)}\otimes\mathbf{F}^{(r)} \end{bmatrix} \begin{bmatrix} \mathbf{m}_r \\ \mathbf{n}_r \\ \mathbf{a}_r \end{bmatrix} = \mathbf{0}_{2I_3R}\,,\; r\in\{1,\ldots,R\}. \tag{7.32}$$

We overall find a column of $\mathbf{A}^{(1)}$ from the corresponding column of $\overline{\mathbf{R}}$ (via (7.30)) or $\overline{\mathbf{L}}$ (via (7.32)), depending on which has the least number of nonzero entries. Consequently, if $\mathbf{a}_r^{(1)}$ denotes the rth column vector of $\mathbf{A}^{(1)}$, then when $r \leq \lfloor\frac{R+1}{2}\rfloor$ we construct $\mathbf{a}_r^{(1)}$ from $\overline{\mathbf{R}}$. On the other hand, when $r \geq \lfloor\frac{R+1}{2}\rfloor + 1$ we construct $\mathbf{a}_r^{(1)}$ from $\overline{\mathbf{L}}$. Thus, we get

$$\mathbf{A}^{(1)} = \left[\mathbf{Q}\cdot\mathbf{R}\left(:,1:\left\lfloor\frac{R+1}{2}\right\rfloor\right),\, \mathbf{Z}\cdot\mathbf{L}\left(:,\left\lfloor\frac{R+1}{2}\right\rfloor+1:R\right)\right].$$

8. CPD with a partial Hermitian symmetry. In this section we consider the partial Hermitian symmetry $\mathbf{A}^{(2)} = \mathbf{A}^{(1)*}$. This is for instance relevant for applications involving covariance matrices of complex data [3, 8, 2]. Let us define the tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times I_1\times I_3}$ by $y_{i_1i_2i_3} = x^*_{i_2i_1i_3}$ for all indices. Let us now stack $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{Z} \in \mathbb{C}^{I_1\times I_1\times 2I_3}$ in such a way that $\mathbf{Z}_{(1)} = \left[\mathbf{X}_{(1)}, \mathbf{Y}_{(1)}\right] \in \mathbb{C}^{I_1^2\times 2I_3}$. In the case of partial Hermitian symmetry, the first factor matrices of $\mathcal{X}$ and $\mathcal{Y}$ are both equal to $\mathbf{A}^{(1)}$, and the second factor matrices of $\mathcal{X}$ and $\mathcal{Y}$ are both equal to $\mathbf{A}^{(1)*}$. We have

$$\mathbf{Z}_{(1)} = \left(\mathbf{A}^{(1)}\odot\mathbf{A}^{(1)*}\right)\left[\mathbf{A}^{(3)}, \mathbf{A}^{(3)*}\right]^T. \tag{8.1}$$

We may now study the uniqueness of the CPD of $\mathcal{Z}$ in order to obtain uniqueness results for $\mathcal{X}$. We formulate as Theorem 8.3 a relaxation of Theorem 7.1 for the case with partial Hermitian symmetry. The proof makes use of Lemma 8.2 which in turn makes use of the following result.

LEMMA 8.1. *Given an analytic function $f : \mathbb{C}^n \to \mathbb{C}$. If there exists a vector $\boldsymbol{x}\in\mathbb{C}^n$ such that $f(\boldsymbol{x}) \neq 0$, then the set $\{\,\boldsymbol{x}\,|\,f(\boldsymbol{x}) = 0\,\}$ is of Lebesgue measure zero.*

Proof. For a proof, see for instance [12]. □

LEMMA 8.2. *Let $\boldsymbol{P} \in \mathbb{C}^{I\times R}$, then $\left[\boldsymbol{P}^T, \boldsymbol{P}^H\right]^T \in \mathbb{C}^{2I\times R}$ generically has rank $\min(2I, R)$.*

Proof. The proof is straightforward, simply notice that

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{P}^* \end{bmatrix} = \begin{bmatrix} \mathbf{I}_I & \sqrt{-1}\cdot\mathbf{I}_I \\ \mathbf{I}_I & -\sqrt{-1}\cdot\mathbf{I}_I \end{bmatrix} \begin{bmatrix} \mathrm{Re}\,\{\mathbf{P}\} \\ \mathrm{Im}\,\{\mathbf{P}\} \end{bmatrix} = \mathbf{T}\begin{bmatrix} \mathrm{Re}\,\{\mathbf{P}\} \\ \mathrm{Im}\,\{\mathbf{P}\} \end{bmatrix}. \tag{8.2}$$

If $2I \geq R$, then we let $\left[\mathrm{Re}\left\{\mathbf{P}\right\}^T, \mathrm{Im}\left\{\mathbf{P}\right\}^T\right]^T = \left[\mathbf{I}_R^T, \mathbf{0}_{2I-R,R}^T\right]^T$ and if $2I < R$, then we let $\left[\mathrm{Re}\left\{\mathbf{P}\right\}^T, \mathrm{Im}\left\{\mathbf{P}\right\}^T\right]^T = \left[\mathbf{I}_{2I}^T, \mathbf{0}_{2I,R-2I}^T\right]^T$. Since $\mathbf{T} \in \mathbb{C}^{2I\times 2I}$ in (8.2) is nonsingular we know that for this particular case the rank of $\left[\mathbf{P}^T, \mathbf{P}^H\right]^T$ is equal to $\min(2I, R)$. By invoking Lemma 8.1 we can now conclude that $\left[\mathbf{P}^T, \mathbf{P}^H\right]^T$ generically has rank $\min(2I, R)$. □

THEOREM 8.3. *Consider the tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_1\times I_3}$ with rank R and matrix representation $\mathbb{C}^{I_1^2\times I_3} \ni \boldsymbol{X}_{(1)} = \left(\boldsymbol{A}^{(1)} \odot \boldsymbol{A}^{(1)*}\right)\boldsymbol{A}^{(3)T}$. Define $\boldsymbol{C} \in \mathbb{C}^{I_1^4\times R(R-1)/2}$ by*

$$c_{(i_1-1)I_1^3+(j_1-1)I_1^2+(i_2-1)I_1+j_2,\frac{(r_2-2)(r_2-1)}{2}+r_1} = \begin{vmatrix} a^{(1)}_{i_1,r_1} & a^{(1)}_{i_1,r_2} \\ a^{(1)}_{j_1,r_1} & a^{(1)}_{j_1,r_2} \end{vmatrix} \cdot \begin{vmatrix} a^{(1)*}_{i_2,r_1} & a^{(1)*}_{i_2,r_2} \\ a^{(1)*}_{j_2,r_1} & a^{(1)*}_{j_2,r_2} \end{vmatrix},$$

where $1 \leq i_1, j_1, i_2, j_2 \leq I_1$ and $1 \leq r_1 < r_2 \leq R$. If $\left[\boldsymbol{A}^{(3)T}, \boldsymbol{A}^{(3)H}\right]^T$ and $\boldsymbol{C}$ have full column rank, then the CPD of $\mathcal{X}$ is unique. Generically, this is true if

$$2R(R-1) \leq I_1^2(I_1-1)^2 \quad \textit{and} \quad R \leq 2I_3\,. \tag{8.3}$$

Proof. By exploiting the partial symmetry $\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{A}^{(1)}D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{A}^{(1)H}$ we can build the I_3 additional matrix representations

$$\mathbf{Y}^{(\cdot\cdot i_3)} = \mathbf{X}^{(\cdot\cdot i_3)H} = \mathbf{A}^{(1)}D_{i_3}\left(\mathbf{A}^{(3)}\right)^*\mathbf{A}^{(1)H}, \quad i_3 \in \{1,\ldots,I_3\}$$

and obtain $\mathbf{Z}_{(1)} \in \mathbb{C}^{I_1^2\times 2I_3}$ given by

$$\mathbf{Z}_{(1)} = \left[\mathbf{X}_{(1)}, \mathrm{Vec}\left(\mathbf{Y}^{(\cdot\cdot 1)}\right), \ldots, \mathrm{Vec}\left(\mathbf{Y}^{(\cdot\cdot I_3)}\right)\right] = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(1)*}\right)\left[\mathbf{A}^{(3)T}, \mathbf{A}^{(3)H}\right].$$

The theorem now follows directly from Theorem 7.1 and Lemma 8.2. □

In the case of complex Hermitian symmetry one may also resort to Theorem 7.1. Generic bound (7.2) also applies, with $I_1 = I_2$. Comparing condition (7.2) with (8.3) it is clear that Theorem 8.3 provides a more relaxed bound. In applications such as [3, 8, 2] this means that fewer covariance matrices are needed.

Under the conditions in Theorem 8.3 we may compute the CPD of $\mathcal{X}$ through the CPD of $\mathcal{Z}$ as in [6] and briefly described in section 7. Let $\mathbf{Z}_{(1)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{Z}_{(1)}$ in (8.1), then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ such that $\mathbf{A}^{(1)} \odot \mathbf{A}^{(1)*} = \mathbf{U}\Sigma\mathbf{M}$ and $\left[\mathbf{A}^{(3)T}, \mathbf{A}^{(3)H}\right]^T = \mathbf{M}^{-1}\mathbf{V}^*$. The CPD of $\mathcal{X}$ follows from the partially symmetric CPD (7.3) also in case where $\mathbf{A}^{(3)}$ is rank deficient. Once $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(1)*}$ have been obtained from $\mathbf{M}$, we obtain $\mathbf{A}^{(3)}$ via relation

$$\mathbf{A}^{(3)T} = \begin{bmatrix} \mathbf{A}^{(1)} \odot \mathbf{A}^{(1)*} \\ \mathbf{A}^{(1)*} \odot \mathbf{A}^{(1)} \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{X}_{(1)} \\ \mathbf{Y}^*_{(1)} \end{bmatrix}.$$

Moreover, if the CPD contains the partial Hermitian symmetry $\mathbf{A}^{(1)} = \mathbf{A}^{(2)*}$, then due to (8.1) we know that as long as $\mathbf{A}^{(1)}$ and $\left[\mathbf{A}^{(3)T}, \mathbf{A}^{(3)H}\right]^T$ have full column rank the SGSD1-CP and SGSD2-CP methods can still be used, i.e., the rank constraint on $\mathbf{A}^{(3)}$ has been relaxed to a rank constraint on $\left[\mathbf{A}^{(3)T}, \mathbf{A}^{(3)H}\right]^T$.

9. Extension to Nth-order tensors. In this section we briefly explain how to extend the SGSD-CP methods to Nth-order tensors. Given a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with rank R constructed from the factor matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R}$, $n \in \{1, \ldots, N\}$, such that

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(N)} . \tag{9.1}$$

By appropriate reshaping the tensor (9.1) has the following matrix representation

$$\mathbf{X} = (\mathbf{A} \odot \mathbf{B})\, \mathbf{C}^T, \tag{9.2}$$

where

$$\begin{aligned} &\mathbf{A} = \mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(P)} \in \mathbb{C}^{I \times R}, && I = \prod_{n=1}^{P} I_n \\ &\mathbf{B} = \mathbf{A}^{(P+1)} \odot \cdots \odot \mathbf{A}^{(Q)} \in \mathbb{C}^{J \times R}, && J = \prod_{n=P+1}^{Q} I_n \\ &\mathbf{C} = \mathbf{A}^{(Q+1)} \odot \cdots \odot \mathbf{A}^{(N)} \in \mathbb{C}^{K \times R}, && K = \prod_{n=Q+1}^{N} I_n \end{aligned}$$

in which $1 \leq P < Q \leq N$. By ignoring the structural relations among the matrices in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, then (9.2) reduces to a third-order CPD problem. Assume that $\mathbf{A}$ and $\mathbf{B}$ have full column rank and $k(\mathbf{C}) \geq 1$, then in order to compute $\mathbf{A}$ and $\mathbf{B}$ from (9.2) the discussed SGSD-CP methods can be applied. Let $\mathbf{A} = \mathbf{QR}$ be the QR factorization of $\mathbf{A}$, where $\mathbf{Q} \in \mathbb{C}^{I \times I}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{I \times R}$ is a matrix of the form (2.4) and $\mathbf{A} = \mathbf{ZL}$ be the QL factorization of $\mathbf{A}$ where $\mathbf{Z} \in \mathbb{C}^{J \times J}$ is a unitary matrix and $\mathbf{L} \in \mathbb{C}^{J \times R}$ is a matrix of the form (2.5), then cost function (2.14) is equal to

$$g(\mathbf{Q}, \mathbf{Z}) = \sum_{k=1}^{K} \left\| \mathrm{triu}_R \left(\mathbf{Q}^H \mathbf{X}^{(\cdot\cdot k)} \mathbf{Z}^* \right) \right\|_F^2 , \tag{9.3}$$

where

$$\mathbf{X}^{(\cdot\cdot k)} = \mathbf{A} D_k(\mathbf{C})\, \mathbf{B}^T , \quad k \in \{1, \ldots, K\} .$$

Assume that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have been found via one of the proposed SGSD methods. The factor matrices now follow from best rank-1 tensor approximation problems. For instance, from $\mathbf{A}$ we obtain

$$\min_{\mathbf{a}_r^{(1)}, \ldots, \mathbf{a}_r^{(P)}} \left\| \mathbf{a}_r - \mathbf{a}_r^{(1)} \otimes \cdots \otimes \mathbf{a}_r^{(P)} \right\|_F^2 , \quad r \in \{1, \ldots, R\} . \tag{9.4}$$

Notice that when $P = 2$, then (9.4) reduces to best rank-1 matrix approximation problems which can be solved by standard numerical linear algebra methods. By similar procedure we obtain $\{\mathbf{A}^{(n)}\}_{n=P+1}^{Q}$ and $\{\mathbf{A}^{(n)}\}_{n=Q+1}^{N}$ from rank-1 approximation problems via $\mathbf{B}$ and $\mathbf{C}$, respectively.

If $\mathbf{A} = \mathbf{B}^*$, then we work with the tensor

$$\mathbf{Z} = (\mathbf{A} \odot \mathbf{A}^*) \begin{bmatrix} \mathbf{C} \\ \mathbf{C}^* \end{bmatrix}^T \in \mathbb{C}^{I^2 \times 2K} \tag{9.5}$$

obtained via the procedure described in section 8.

To summarize, we can compute the CPD of an Nth-order tensor by treating it as a third-order tensor. Note that a third-order tensor is usually highly overdetermined and therefore we do not expect to lose any significant accuracy by combing the modes of an Nth-order tensor.

10. Conclusion. Based on a link between the SGSD and CPD it is possible to address some CPD problems as SGSD problems. We reviewed the original SGSD-CP method which was called SGSD1-CP in this paper. It was originally designed to only jointly diagonalize a set of square matrices. Hence, we first generalized the extended QZ method to the non-square case.

The SGSD1-CP method does not fully take the relations among the involved triangular matrices into account. This motivated us to propose the SGSD2-CP which attempts to take the relations among the involved triangular matrices into account. In the derivation of SGSD2-CP we showed that it is possible to deduce the involved triangular matrices from the column space of a matrix representation of the tensor. Moreover, we provided a numerical method to compute the triangular factor matrices from the knowledge of the column space of the matrix representation of the tensor of interest.

Another drawback of the original SGSD1-CP method is that it requires that all three of the involved factor matrices have full column rank. This motivated us to develop the SGSD3-CP method which only requires that two of the involved factor matrix to have full column rank. This is in fact the necessary condition for the SGSD approach to the CPD problem. Again, we provided a numerical method to compute the involved triangular factor matrices.

In many practical problems the CPD has a partial (Hermitian) symmetry. We explained how to incorporate such a constraint into the SGSD2-CP and SGSD3-CP methods. For the case when the CPD having a partial Hermitian symmetry we also proved a new uniqueness result. Finally, we explained how to extend the SGSD-CP methods for computing a Nth-order CPD with possibly also partial (Hermitian) symmetries.

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